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Existence Theory for a Class of Nonlinear Langevin Fractional (p,q)-Difference Equations in Banach Space

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Abstract

This paper is devoted to the study of existence results for a nonlinear Langevin-type fractional (p,q)-difference equation in Banach space. The considered model extends the fractional q-difference Langevin equation by introducing two parameters p and q, which provide additional flexibility in describing discrete fractional processes. By using the Kuratowski measure of noncompactness together with Mönch's fixed-point theorem, we derive sufficient conditions that guarantee the existence of at least one solution. The main idea consists in converting the boundary value problem into an equivalent fractional (p,q)-integral equation and verifying that the corresponding operator is continuous, bounded, and condensing. An illustrative example is presented to demonstrate the applicability of the obtained results.

Keywords: (p,q)-difference fractional derivative; Langevin equation; measure of noncompactness; Mönch fixed point theorem

MSC: 26A33; 39A13; 34A08; 47H08; 47H10



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1. Introduction

Fractional calculus and its discrete analogues have attracted extensive attention due to their enhanced capability in describing memory-dependent and hereditary processes compared with classical integer-order models. A significant branch of these studies focuses on the development of generalized fractional operators that allow for better modeling of complex dynamical behavior. The (p,q)-calculus provides a powerful extension of the q-calculus by introducing two parameters, p and q, which ensure greater flexibility and accuracy when capturing discrete fractional characteristics of real phenomena. This generalization has led to several analytical developments concerning (p,q)-Gamma and (p,q)-Beta functions, (p,q)-integral inequalities, and approximation theory.

For example, Tunç and Göv [1] established (p,q)-integral inequalities, while Aral and Gupta [2] employed the (p,q)-Gamma function in approximation theory. Usman et al. [3] investigated properties of (p,q)-binomial coefficients and (p,q)-Stirling polynomials. Mursaleen et al. [4] introduced (p,q)-Bernstein operators, and Prabseang et al. [5] developed (p,q)-Hermite–Hadamard inequalities for convex functions. Sadjang [6,7] investigated the fundamental theorem of (p,q)-calculus. Soontharanon and Sitthiwirattham [8] extended these notions to fractional settings, which inspired several relevant studies dealing with

fractional (p,q)-difference equations [9–12]. These works confirm that (p,q)-calculus is a promising framework for modeling discrete fractional dynamics.

Along another direction, the solvability of fractional differential and difference systems in Banach spaces has become a crucial topic in the analysis of dynamical models. Advanced techniques such as the Kuratowski measure of noncompactness combined with fixed-point theory have proven particularly effective in these investigations. Representative contributions include those by Boutiara [13], Mesmouli et al. [14,15], Lachouri et al. [16], Allouch et al. [17], and Salem and Alnegga [18]. These results demonstrate that such tools remain effective for establishing existence theorems in infinite-dimensional settings.

The classical Langevin equation was introduced to describe Brownian motion and captures both dissipative and stochastic influences. To better capture nonlocal and memory effects, many fractional extensions of the Langevin equation have been proposed. Baghani [19] investigated a fractional Langevin equation with two distinct orders, while Butt et al. [20] studied fractional difference Langevin systems. Baitiche et al. [21] addressed Ulam stability issues. Picozzi and West [22] applied the fractional Langevin equation in finance. More recently, Zhao and co-authors contributed remarkable results concerning stability, multiplicity, and simulation of nonlinear fractional Langevin systems involving various kernels and functional constraints [23–26].

Despite these important contributions, there is still a clear gap in the literature: existing research primarily focuses on continuous fractional operators and q-fractional versions, whereas no study has been conducted for nonlinear Langevin-type fractional (p,q)-difference equations in Banach spaces.

Motivated by these developments, this paper investigates the existence of solutions for the following nonlinear Langevin-type fractional (p,q)-difference boundary value problem:

$$ {}^{c}\mathrm{D}^{\rceil}_{p,q}\left({}^{c}\mathrm{D}^{\lfloor}_{p,q}\hbar(\ell)+\lambda\hbar(p^{\lfloor}\ell)\right)=\mathcal{F}(p^{\lfloor+\rceil}\ell,\hbar(p^{\lfloor+\rceil}\ell)),\quad \ell\in\jmath=[0,1], \tag{1}$$

subject to

$$ \hbar(0)=0,\qquad \hbar(1)=\left(\mathrm{I}^{\lfloor+\rceil}_{p,q}\mathcal{F}(s,\hbar(ps))\right)(1)-\lambda\left(\mathrm{I}^{\lfloor}_{p,q}\hbar(ps)\right)(1), \tag{2}$$

where ${}^{c}\mathrm{D}^{\lfloor}_{p,q}$ and ${}^{c}\mathrm{D}^{\rceil}_{p,q}$ denote Caputo-type fractional (p,q)-derivatives, $\mathrm{I}^{\lfloor}_{p,q}$ denotes the fractional (p,q)-integral operator, $\lambda\in\mathbb{R}$ is a real parameter, and $\mathcal{F}:[0,1]\times\mathcal{W}\to\mathcal{W}$ is a nonlinear operator on a Banach space $\mathcal{W}$.

Novel contributions.

- We present the firstexistence theory for nonlinear Langevin-type fractional (p,q)-difference equations in Banach spaces.
- The considered model generalizes fractional q-difference and standard fractional models using two fractional parameters p and q.
- Existence results are derived using Mönch's fixed point theorem together with the Kuratowski measure of noncompactness, avoiding strong compactness conditions.
- A nontrivial illustrative example shows the applicability of the theory.

The remainder of this paper is organized as follows. Section 2 presents essential preliminaries. Section 3 establishes the main existence theorem. Section 4 provides an illustrative example, and Section 5 concludes the article with remarks on future research directions.

2. Essential Materials

In this section, we recall the tools and concepts needed for our analysis. We start with some basic notation of (p,q)-calculus and a few definitions concerning the fractional (p,q)-integral and derivative operators which can be found in [6–8,12], followed by a short overview of the measure of noncompactness and the Mönch fixed point theorem.

Let $0 < q < p \leq 1$ be constants. We now present the following relations in (p,q)-calculus:

$$[\eta]_{p,q} := \begin{cases} \frac{p^{\eta}-q^{\eta}}{p-q} = p^{\eta-1}[\eta]_{\frac{q}{p}}, & \eta \in \mathbb{N}^{+}, \\ 1, & \eta = 0, \end{cases}$$

where

$$[\eta]_{\frac{q}{p}} := \frac{1-\left(\frac{q}{p}\right)^{\eta}}{1-\frac{q}{p}},$$

and

$$[\eta]_{p,q}! := \begin{cases} [\eta]_{p,q}[\eta-1]_{p,q}\cdots[1]_{p,q} = \prod_{i=1}^{\eta} \frac{p^{i}-q^{i}}{p-q}, & \eta \in \mathbb{N}^{+}, \\ 1, & \eta = 0. \end{cases}$$

The q-analogue of the power function $(\psi-\varphi)_{q}^{(\mathfrak{n})}$ is given by

$$(\psi-\varphi)_{q}^{(\mathfrak{n})} := \begin{cases} \prod_{i=0}^{\mathfrak{n}-1}(\psi-\varphi q^{i}), & \mathfrak{n} \in \mathbb{N}^{+},\ \psi,\varphi \in \mathbb{R}, \\ 1, & \mathfrak{n} = 0. \end{cases}$$

The (p,q)-analogue of the power function $(\psi-\varphi)_{p,q}^{(\mathfrak{n})}$ is defined by:

$$(\psi-\varphi)_{p,q}^{(\mathfrak{n})} := \begin{cases} \prod_{i=0}^{\mathfrak{n}-1}(\psi p^{i}-\varphi q^{i}), & \mathfrak{n} \in \mathbb{N}^{+},\ \psi,\varphi \in \mathbb{R}, \\ 1, & \mathfrak{n} = 0, \end{cases}$$

and for $\beth \in \mathbb{R}$, the general form of the above is given by:

$$(\psi-\varphi)_{p,q}^{(\beth)} := p^{\binom{\beth}{2}}(\psi-\varphi)_{\frac{q}{p}}^{(\beth)} = \psi^{\beth}p^{\binom{\beth}{2}}\prod_{i=0}^{\infty}\frac{\psi-\varphi\left(\frac{q}{p}\right)^{i}}{\psi-\varphi\left(\frac{q}{p}\right)^{\beth+i}}, \quad 0<\varphi<\psi,$$

where $\binom{\beth}{2} := \frac{\beth(\beth-1)}{2}$.

Let $C(\jmath,\mathcal{W})$ be a Banach space containing continuous functions $\hbar : \jmath = [0,1] \to \mathcal{W}$ equipped with

$$\|\hbar\|_{\infty} = \sup_{\ell\in\jmath}\|\hbar(\ell)\|.$$

Let $L^{1}(\jmath,\mathcal{W})$ denote the Banach space of all measurable functions $\hbar : \jmath \to \mathcal{W}$ that are Lebesgue integrable, endowed with

$$\|\hbar\|_{L^{1}} = \int_{\jmath}\|\hbar(\ell)\|d\ell.$$

Definition 1 (see [6]). *For $0 < q < p \leq 1$, the (p,q)-derivative of $\hbar$ is defined as*

$$\mathrm{D}_{p,q}\hbar(\ell) := \frac{\hbar(p\ell)-\hbar(q\ell)}{(p-q)\ell}, \quad \ell \neq 0,$$

and $\left(\mathrm{D}_{p,q}\hbar\right)(0) = \lim_{\ell\to 0}\left(\mathrm{D}_{p,q}\hbar\right)(\ell)$ *if* $\hbar$ *is differentiable at 0. Higher-order derivatives are defined recursively:*

$$\mathrm{D}^{n}_{p,q}\hbar(\ell) = \begin{cases} \mathrm{D}_{p,q}\mathrm{D}^{n-1}_{p,q}\hbar(\ell), & n \in \mathbb{N}^{+}, \\ \hbar(\ell), & n = 0. \end{cases}$$

Definition 2 (see [6]). *The* (p,q)*-integral of* $\hbar$ *is defined as*

$$\mathrm{I}_{p,q}\hbar(\ell) := \int_0^{\ell} \hbar(s)\,\mathrm{d}_{p,q}s = (p-q)\ell\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}\hbar\left(\frac{q^i}{p^{i+1}}\ell\right),$$

whenever the series converges.

Definition 3 (see [8]). *The* (p,q)*-Gamma function for* $\beth \in \mathbb{R}$ *is defined as*

$$\Gamma_{p,q}(\beth) = \frac{(p-q)^{(\beth-1)}_{p,q}}{(p-q)^{\beth-1}},$$

satisfying $\Gamma_{p,q}(\beth+1) = [\beth]_{p,q}\Gamma_{p,q}(\beth)$.

The (p,q)*-Beta function for* $\beth, \gimel \in \mathbb{R}$ *is*

$$B_{p,q}(\beth,\gimel) = \int_0^1 s^{\beth-1}(1-qs)^{(\gimel-1)}_{p,q}\mathrm{d}_{p,q}s, \tag{3}$$

with the alternative representation

$$B_{p,q}(\beth,\gimel) = p^{\frac{(\gimel-1)(2\beth+\gimel-2)}{2}}\frac{\Gamma_{p,q}(\beth)\Gamma_{p,q}(\gimel)}{\Gamma_{p,q}(\beth+\gimel)}.$$

Definition 4 (see [8]). *For* $\beth > 0$ *and* $0 < q < p \le 1$*, the fractional* (p,q)*-integral is defined as*

$$\mathrm{I}^{\beth}_{p,q}\hbar(\ell) = \frac{1}{p^{\binom{\beth}{2}}\Gamma_{p,q}(\beth)}\int_0^{\ell}(\ell-qs)^{(\beth-1)}_{p,q}\hbar\left(\frac{s}{p^{\beth-1}}\right)\mathrm{d}_{p,q}s,$$

and $\mathrm{I}^{0}_{p,q}\hbar(\ell) = \hbar(\ell)$.

Definition 5 (see [8]). *For* $\beth \in (0,1]$*, the Caputo-type fractional* (p,q)*-derivative is defined as*

$$\begin{aligned} {}^{\mathrm{c}}\mathrm{D}^{\beth}_{p,q}\hbar(\ell) &= \mathrm{I}^{1-\beth}_{p,q}\mathrm{D}_{p,q}\hbar(\ell) \\ &= \frac{1}{p^{\binom{1-\beth}{2}}\Gamma_{p,q}(1-\beth)}\int_0^{\ell}(\ell-qs)^{(-\beth)}_{p,q}\mathrm{D}_{p,q}\hbar\left(\frac{s}{p^{-\beth}}\right)\mathrm{d}_{p,q}s, \end{aligned}$$

with ${}^{\mathrm{c}}\mathrm{D}^{0}_{p,q}\hbar(\ell) = \hbar(\ell)$.

Lemma 1 (see [8]). *Let* $\beth \in (\eta-1,\eta]$*,* $\eta \in \mathbb{N}$ *and* $0 < q < p \le 1$*. Let* $\hbar : [0,1] \to \mathbb{R}$*, then we have*

$$\mathrm{I}^{\beth}_{p,q}\left({}^{\mathrm{c}}\mathrm{D}^{\beth}_{p,q}\hbar(\ell)\right) = \hbar(\ell) - \sum_{i=0}^{\eta-1}\frac{\ell^i}{p^{\binom{\beth}{2}}\Gamma_{p,q}(i+1)}\mathrm{D}^{i}_{p,q}\hbar(0), \tag{4}$$

Indeed, for equation ${}^{\mathrm{c}}\mathrm{D}^{\beth}_{p,q}\hbar(\ell) = 0$*, the general solution is expressed as*

$$\hbar(\ell) = c_0 + c_1\ell + c_2\ell^2 + \cdots + c_{\eta-1}\ell^{\eta-1},$$

where $c_0, c_1, \ldots, c_{\eta-1} \in \mathbb{R}$.

Lemma 2 (see [8]). *Let $\hbar$ be continuous and $\beth, \daleth \geq 0$. Then:*

(i) $\left(\mathrm{I}_{p,q}^{\beth}\mathrm{I}_{p,q}^{\daleth}\hbar\right)(\ell) = \left(\mathrm{I}_{p,q}^{\beth+\daleth}\hbar\right)(\ell)$,

(ii) $\left({}^{\mathrm{c}}\mathrm{D}_{p,q}^{\beth}\mathrm{I}_{p,q}^{\beth}\right)\hbar(\ell) = \hbar(\ell)$.

Lemma 3 (see [12]). *Let $0 < q < p \leq 1$ and $0 < \beth < 1$. Then*

$$\int_0^{\ell}(\ell - qs)_{p,q}^{(\beth-1)}\mathrm{d}_{p,q}s = \ell^{\beth}\int_0^1(1-qs)_{p,q}^{(\beth-1)}\mathrm{d}_{p,q}s = \ell^{\beth}B_{p,q}(1,\beth),$$

and

$$\int_0^{\ell}(\ell - qs)_{p,q}^{(\beth-1)}s^{-\beth}\mathrm{d}_{p,q}s = \int_0^1(1-qs)_{p,q}^{(\beth-1)}s^{-\beth}\mathrm{d}_{p,q}s = B_{p,q}(\beth, 1-\beth).$$

Next, we recall the notion of the Kuratowski measure of noncompactness and summarize its principal features.

Definition 6 (see [27]). *Let $\mathcal{W}$ be a Banach space and denote by $\mathcal{F}_{\mathcal{W}}$ the family of all bounded subsets of $\mathcal{W}$. The Kuratowski measure of noncompactness is a functional*

$$\mathcal{M} : \mathcal{F}_{\mathcal{W}} \to [0, \infty),$$

defined by

$$\mathcal{M}(\mathcal{F}) = \inf\left\{\varepsilon > 0 : \mathcal{F} \subset \cup_{i=0}^{S}\mathcal{F}_i \text{ and } diam(\mathcal{F}_i) \leq \varepsilon\right\}, \text{ where } \mathcal{F} \in \mathcal{F}_{\mathcal{W}}.$$

The measure $\mathcal{M}$ possesses several useful properties (see [27,28]).

If $\overline{\mathcal{F}}$ and $conv\mathcal{F}$ denote, respectively, the closure and convex hull of the bounded set $\mathcal{F}$, then:

(1) $\mathcal{M}(\mathcal{F}) = \mathcal{M}(\overline{\mathcal{F}})$,
(2) $\mathcal{M}(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is relatively compact in $\mathcal{W}$,
(3) $\mathcal{M}(\mathcal{F}) = \mathcal{M}(conv\mathcal{F})$,
(4) $\mathcal{M}(\mathcal{F}_1 + \mathcal{F}_2) \leq \mathcal{M}(\mathcal{F}_1) + \mathcal{M}(\mathcal{F}_2)$,
(5) $\mathcal{F}_1 \subset \mathcal{F}_2 \Rightarrow \mathcal{M}(\mathcal{F}_1) \leq \mathcal{M}(\mathcal{F}_2)$,
(6) $\mathcal{M}(c\mathcal{F}) = |c|\mathcal{M}(\mathcal{F})$, for any scalar $c \in \mathbb{R}$.

Definition 7. *A function $\mathcal{F} : \jmath \times \mathcal{W} \to \mathcal{W}$ is said to satisfy the Carathéodory condition if:*

(i) for each fixed $\hbar \in \mathcal{W}$, the mapping $\ell \to \mathcal{F}(\ell, \hbar)$ is measurable on $\jmath$.
(ii) for almost every $\ell \in \jmath$, the mapping $\hbar \to \mathcal{F}(\ell, \hbar)$ is continuous.

For every $\ell \in \jmath$, define the set G of functions $g : \jmath \to \mathcal{W}$, and let

$$G(\ell) = \{g(\ell) : g \in G\}, \ell \in \jmath,$$
$$G(\jmath) = \{g(\ell) : g \in G, \ \ell \in \jmath\}.$$

Lemma 4 (see [29]). *If $G \subset C(\jmath, \mathcal{W})$ is a bounded and equicontinuous family of functions, then:*

(i) The function $\ell \to \mathcal{M}(G(\ell))$ is continuous on $\jmath$.
(ii) The following inequality holds:

$$\mathcal{M}\left(\left\{\int_{\jmath}\hbar(\ell)d\ell : \hbar \in G\right\}\right) \leq \int_{\jmath}\mathcal{M}(G(\ell))d\ell.$$

Lemma 5 (see [30]). *Let Ω be a bounded, closed, and convex subset of the Banach space $C(\jmath, \mathcal{W})$. Suppose $\Phi : \jmath \times \jmath \to \mathbb{R}$ is continuous, and $\mathcal{F} : \jmath \times \mathcal{W} \to \mathcal{W}$ is a Carathéodory mapping satisfying*

the following condition: there exists $\theta \in L^1(\jmath, \mathbb{R}^+)$ *such that, for all* $\ell \in \jmath$ *and any bounded set* $F \subset \mathcal{W}$*, we have*

$$\lim_{\epsilon \to 0^+} \mathcal{M}(\mathcal{F}(\jmath_\epsilon \times F)) \leq \theta(\ell)\mathcal{M}(F),$$

where $\jmath_\epsilon = [\ell - \epsilon, \ell] \cap \jmath$.

Then, for every equicontinuous subset $G \subset \Omega$*, the following inequality is satisfied:*

$$\mathcal{M}\left(\left\{\int_\jmath \Phi(\ell, r)\mathcal{F}(r, \hbar(r))dr, \hbar \in G\right\}\right) \leq \int_\jmath |\Phi(\ell, r)|\theta(r)\mathcal{M}(G(r))dr.$$

Theorem 1 (Mönch, see [31])**.** *Let* Ω *be a bounded, closed, and convex subset of a Banach space* $\mathcal{W}$ *such that* $0 \in \Omega$*. Assume that* $\aleph : \Omega \to \Omega$ *is continuous mapping. If for every subset* $G \subseteq \Omega$*, the following implication holds:*

$$G \subset \overline{conv(\aleph G)} \quad or \quad G \subset \overline{conv(\aleph G \cup \{0\})} \Rightarrow \mathcal{M}(G) = 0, \tag{5}$$

where $\mathcal{M}$ *denotes the Kuratowski measure of noncompactness, then the operator* $\aleph$ *admits at least one fixed point in* Ω*.*

Theorem 2 (Arzela-Ascoli's Theorem)**.** *(see [32]) Let* $G \subset C(\jmath, \mathcal{W})$ *be a subset of the Banach space of continuous functions. If G is bounded and uniformly Cauchy (or equivalently, equicontinuous and uniformly bounded), then G is relatively compact in* $C(\jmath, \mathcal{W})$*.*

3. Main Results

In this part of the paper, we present the existence theorem for the boundary value problem (1)–(2) by applying Mönch's fixed point theorem introduced earlier.

Definition 8. *A function* $\hbar \in C(\jmath, \mathcal{W})$ *is called a solution of the boundary value problem (1)–(2) if* $\hbar$ *satisfies the equation*

$${}^{c}\mathrm{D}^{\rceil}_{p,q}\left({}^{c}\mathrm{D}^{\lfloor}_{p,q}\hbar(\ell) + \lambda\hbar\left(p^{\lfloor}\ell\right)\right) = \mathcal{F}\left(\ell, \hbar\left(p^{\lfloor+\rceil}\ell\right)\right), \quad \ell \in \jmath$$

together with the boundary conditions

$$\hbar(0) = 0, \hbar(1) = \left(\mathrm{I}^{\lfloor+\rceil}_{p,q}\mathcal{F}(s, \hbar(ps))\right)(1) - \lambda\left(\mathrm{I}^{\lfloor}_{p,q}\hbar(ps)\right)(1).$$

Lemma 6. *For* $0 < q < p \leq 1$*, let* $\mathcal{F} : \jmath \times \mathcal{W} \to \mathcal{W}$*. The function* $\hbar$ *is a solution of the fractional integral equation*

$$\hbar(\ell) = \left(\mathrm{I}^{\lfloor+\rceil}_{p,q}\mathcal{F}(s, \hbar(ps))\right)(\ell) - \lambda\left(\mathrm{I}^{\lfloor}_{p,q}\hbar(ps)\right)(\ell), \tag{6}$$

if and only if $\hbar$ *is a solution of the fractional boundary-value problem.*

Proof. Assume that $\hbar$ satisfies (1). Then, by applying Lemmas 1 and 2, we can transform the problem (1)–(2) to an equivalent integral equation

$$\hbar(\ell) = \left(\mathrm{I}^{\lfloor+\rceil}_{p,q}\mathcal{F}(s, \hbar(ps))\right)(\ell) - \lambda\left(\mathrm{I}^{\lfloor}_{p,q}\hbar(ps)\right)(\ell) + c_1\frac{\ell^{\lfloor}}{p^{\binom{\lfloor}{2}}\Gamma_{p,q}(\lfloor+1)} + c_0, \tag{7}$$

Applying the boundary conditions (2), we get

$$\hbar(0) = c_0 = 0,$$

and

$$\hbar(1) = \left(\mathrm{I}_{p,q}^{\beth+\daleth}\,\mathcal{F}(s,\hbar(ps))\right)(1) - \lambda\left(\mathrm{I}_{p,q}^{\beth}\hbar(ps)\right)(1) + \frac{c_1}{p^{\binom{\beth}{2}}\Gamma_{p,q}(\beth+1)}.$$

Consequently $c_1 = 0$.

Substituting the values of c_0, c_1 in (7), we derive (6), and the converse follows by direct computation. Hence, the equivalence is established. □

To derive the main existence result, we assume the following hypotheses:

(C1) The nonlinear function $\mathcal{F} : \jmath \times \mathcal{W} \to \mathcal{W}$ satisfies the Carathéodory conditions:

(C2) There exist $\theta \in L^1(\jmath, \mathbb{R}^+) \cap (\jmath, \mathbb{R}^+)$ such that

$$\|\mathcal{F}(\ell, \hbar(\ell))\| \leq \theta(\ell)\|\hbar(\ell)\|, \text{ for a.e.} \ell \in \jmath \text{ and each } \hbar \in \mathcal{W}.$$

and

$$\|\theta\|_\infty + |\lambda| < 1.$$

(C3) For each $\ell \in \jmath$ and any bounded set $\digamma \subset \mathcal{W}$, if $\jmath_\epsilon = [\ell - \epsilon, \ell] \cap \jmath$, then

$$\lim_{\epsilon \to 0^+} \mathcal{M}(\mathcal{F}(\jmath_\epsilon \times \digamma)) \leq \theta(\ell)\mathcal{M}(\digamma).$$

In light of Lemma 6, we introduce an operator

$$\aleph : C(\jmath, \mathcal{W}) \to C(\jmath, \mathcal{W}),$$

defined by

$$\begin{aligned}(\aleph\hbar)(\ell) \quad = \quad & \frac{1}{p^{\binom{\beth+\daleth}{2}}\Gamma_{p,q}(\beth+\daleth)} \int_0^\ell (\ell - qs)_{p,q}^{(\beth+\daleth-1)} \mathcal{F}(ps, \hbar(ps)) \mathrm{d}_{p,q}s, \\ & - \frac{\lambda}{p^{\binom{\beth}{2}}\Gamma_{p,q}(\beth)} \int_0^\ell (\ell - qs)_{p,q}^{(\beth-1)} \hbar(ps) \mathrm{d}_{p,q}s. \end{aligned} \tag{8}$$

By construction, every fixed point of $\aleph$ corresponds to a solution of the boundary value problem (1)–(2).

Let us consider the closed, convex, and bounded subset

$$\Omega = \{\hbar \in C(\jmath, \mathcal{W}) : \|\hbar\|_\infty \leq L\}, \quad L > 0.$$

where the fixed points of the operator $\aleph$ are the solution to problem (1)–(2).

We now show that $\aleph$ satisfies the necessary properties to apply Mönch's fixed-point theorem.

Lemma 7. *Assume that condition (C2) holds. Then:*

(i) $\aleph$ maps Ω into itself;

(ii) the set $\aleph(\Omega)$ is both bounded and equicontinuous.

Proof. Let $\ell \in \jmath$ and $\hbar \in \Omega$. Using condition (C2) and inequality properties of the fractional (p,q)-integral, we obtain:

$$
\begin{aligned}
\|(\aleph\hbar)(\ell)\| &\leq \left\| \frac{1}{p^{(\frac{\beth+\daleth}{2})}\Gamma_{p,q}(\beth+\daleth)} \int_0^{\ell} (\ell - qs)_{p,q}^{(\beth+\daleth-1)} \, \mathcal{F}(ps, \hbar(ps)) \mathrm{d}_{p,q}s \right\| \\
&+ \left\| \frac{\lambda}{p^{(\frac{\beth}{2})}\Gamma_{p,q}(\beth)} \int_0^{\ell} (\ell - qs)_{p,q}^{(\beth-1)} \hbar(ps) \mathrm{d}_{p,q}s \right\| \\
&\leq \frac{1}{p^{(\frac{\beth+\daleth}{2})}\Gamma_{p,q}(\beth+\daleth)} \int_0^{\ell} (\ell - qs)_{p,q}^{(\beth+\daleth-1)} \|\mathcal{F}(ps, \hbar(ps))\| \mathrm{d}_{p,q}s \\
&+ \frac{|\lambda|}{p^{(\frac{\beth}{2})}\Gamma_{p,q}(\beth)} \int_0^{\ell} (\ell - qs)_{p,q}^{(\beth-1)} \|\hbar(ps)\| \mathrm{d}_{p,q}s \\
&\leq \frac{1}{p^{(\frac{\beth+\daleth}{2})}\Gamma_{p,q}(\beth+\daleth)} \int_0^{\ell} (\ell - qs)_{p,q}^{(\beth+\daleth-1)} \theta(ps) \|\hbar(ps)\| \mathrm{d}_{p,q}s \\
&+ \frac{|\lambda| L}{p^{(\frac{\beth}{2})}\Gamma_{p,q}(\beth)} \int_0^{\ell} (\ell - qs)_{p,q}^{(\beth-1)} \mathrm{d}_{p,q}s \\
&\leq \left(\|\theta\|_{\infty} \frac{1}{p^{(\frac{\beth+\daleth}{2})}\Gamma_{p,q}(\beth+\daleth)} \int_0^{\ell} (\ell - qs)_{p,q}^{(\beth+\daleth-1)} \mathrm{d}_{p,q}s + \frac{|\lambda|}{p^{(\frac{\beth}{2})}\Gamma_{p,q}(\beth)} \int_0^{\ell} (\ell - qs)_{p,q}^{(\beth-1)} \mathrm{d}_{p,q}s \right) L \\
&\leq L.
\end{aligned}
$$

Hence, $\aleph(\Omega) \subseteq \Omega$ and $\aleph(\Omega)$ is bounded.

To prove equicontinuity, let $\ell_1, \ell_2 \in \jmath$, with $\ell_1 < \ell_2$, and $\hbar \in \Omega$. Using the definition of $\aleph$, we get

$$
\begin{aligned}
&\|(\aleph\hbar)(\ell_2) - (\aleph\hbar)(\ell_1)\| \\
&= \left\| \frac{1}{p^{(\frac{\beth+\daleth}{2})}\Gamma_{p,q}(\beth+\daleth)} \int_0^{\ell_2} (\ell_2 - qs)_{p,q}^{(\beth+\daleth-1)} \, \mathcal{F}(ps, \hbar(ps)) \mathrm{d}_{p,q}s \right. \\
&\left. - \frac{1}{p^{(\frac{\beth+\daleth}{2})}\Gamma_{p,q}(\beth+\daleth)} \int_0^{\ell_1} (\ell_1 - qs)_{p,q}^{(\beth+\daleth-1)} \, \mathcal{F}(ps, \hbar(ps)) \mathrm{d}_{p,q}s \right\| \\
&+ \left\| \frac{\lambda}{p^{(\frac{\beth}{2})}\Gamma_{p,q}(\beth)} \int_0^{\ell_2} (\ell_2 - qs)_{p,q}^{(\beth-1)} \hbar(ps) \mathrm{d}_{p,q}s - \frac{\lambda}{p^{(\frac{\beth}{2})}\Gamma_{p,q}(\beth)} \int_0^{\ell_1} (\ell_1 - qs)_{p,q}^{(\beth-1)} \hbar(ps) \mathrm{d}_{p,q}s \right\| \\
&\leq \frac{1}{p^{(\frac{\beth+\daleth}{2})}\Gamma_{p,q}(\beth+\daleth)} \int_0^{\ell_1} \left[(\ell_2 - qs)_{p,q}^{(\beth+\daleth-1)} - (\ell_1 - qs)_{p,q}^{(\beth+\daleth-1)} \right] \|\mathcal{F}(ps, \hbar(ps))\| \mathrm{d}_{p,q}s \\
&+ \frac{1}{p^{(\frac{\beth+\daleth}{2})}\Gamma_{p,q}(\beth+\daleth)} \int_{\ell_1}^{\ell_2} (\ell_2 - qs)_{p,q}^{(\beth+\daleth-1)} \| \mathcal{F}(ps, \hbar(ps))\| \mathrm{d}_{p,q}s \\
&+ \frac{|\lambda|}{p^{(\frac{\beth}{2})}\Gamma_{p,q}(\beth)} \int_0^{\ell_1} \left[(\ell_2 - qs)_{p,q}^{(\beth-1)} - (\ell_1 - qs)_{p,q}^{(\beth-1)} \right] \|\hbar(ps)\| \mathrm{d}_{p,q}s \\
&+ \frac{|\lambda|}{p^{(\frac{\beth}{2})}\Gamma_{p,q}(\beth)} \int_{\ell_1}^{\ell_2} (\ell_2 - qs)_{p,q}^{(\beth-1)} \|\hbar(ps)\| \mathrm{d}_{p,q}s \\
&\leq \frac{\|\theta\|_{\infty} L}{p^{(\frac{\beth+\daleth}{2})}\Gamma_{p,q}(\beth+\daleth)} \left(\int_0^{\ell_1} \left[(\ell_2 - qs)_{p,q}^{(\beth+\daleth-1)} - (\ell_1 - qs)_{p,q}^{(\beth+\daleth-1)} \right] \mathrm{d}_{p,q}s + \int_{\ell_1}^{\ell_2} (\ell_2 - qs)_{p,q}^{(\beth+\daleth-1)} \mathrm{d}_{p,q}s \right) \\
&+ \frac{|\lambda| L}{p^{(\frac{\beth}{2})}\Gamma_{p,q}(\beth)} \left(\int_0^{\ell_1} \left[(\ell_2 - qs)_{p,q}^{(\beth-1)} - (\ell_1 - qs)_{p,q}^{(\beth-1)} \right] \mathrm{d}_{p,q}s + \int_{\ell_1}^{\ell_2} (\ell_2 - qs)_{p,q}^{(\beth-1)} \mathrm{d}_{p,q}s \right), \qquad (9)
\end{aligned}
$$

Since $(\ell - qs)_{p,q}^{(\beth-1)}$ is continuous function with respect to ℓ and s on $\jmath \times \jmath$, it can be inferred that the function is uniformly continuous on $\jmath \times \jmath$. Then, for any $s \in \jmath$, we can deduce the following:

$$(\ell_2 - qs)_{p,q}^{(\beth+\urcorner-1)} - (\ell_1 - qs)_{p,q}^{(\beth+\urcorner-1)} \rightrightarrows 0 \text{ as } \ell_1 \to \ell_2,$$

$$(\ell_2 - qs)_{p,q}^{(\beth-1)} - (\ell_1 - qs)_{p,q}^{(\beth-1)} \rightrightarrows 0 \text{ as } \ell_1 \to \ell_2,$$

Therefore, the integrals on the right-hand side of (9) tend toward zero as ℓ_1 approaches ℓ_2. This establishes that the family $\aleph(\Omega)$ is equicontinuous. □

Lemma 8. *Assume that conditions (C1) and (C2) are satisfied. Then the operator $\aleph$ is continuous on Ω.*

Proof. Let $\{\hbar_n\}$ be a sequence in $C(\jmath,\ \mathcal{W})$ such that $\hbar_n \to \hbar \in C(\jmath, \mathcal{W})$ in the supremum norm. So, for each $\ell \in \jmath$, we have

$$\begin{aligned}\|(\aleph\hbar_n)(\ell) - (\aleph\hbar)(\ell)\| \leq{}& \frac{1}{p^{(\frac{\beth+\urcorner}{2})}\Gamma_{p,q}(\beth+\urcorner)} \int_0^\ell (\ell - qs)_{p,q}^{(\beth+\urcorner-1)} \|\mathcal{F}(ps, \hbar_n(ps)) - \mathcal{F}(ps, \hbar(ps))\| \mathrm{d}_{p,q}s \\ &+ \frac{|\lambda|}{p^{(\frac{\beth}{2})}\Gamma_{p,q}(\beth)} \int_0^\ell (\ell - qs)_{p,q}^{(\beth-1)} \|\hbar_n(ps) - \hbar(ps)\| \mathrm{d}_{p,q}s\end{aligned}$$

Thanks to assumption (C1), the sequence $\mathcal{F}(ps, \hbar_n(ps))$ converges uniformly to $\mathcal{F}(ps, \hbar(ps))$.

Since both integrands converge uniformly to zero and are bounded by integrable functions, the Lebesgue dominated convergence theorem ensures that

$$\|\aleph\hbar_n - \aleph\hbar\| \to 0 \text{ as } n \to \infty.$$

Hence, $\aleph$ is a continuous self-mapping on Ω. □

Theorem 3. *If the assumptions (C1)–(C3) hold, then the boundary value problem (1)–(2) possesses at least one solution.*

Proof. By Lemmas 7 and 8, the operator $\aleph$ is a continuous self-map on the closed, convex, and bounded set $\Omega \in C(\jmath, \mathcal{W})$, and $\aleph(\Omega)$is equicontinuous and bounded.

To apply Mönch's fixed-point theorem, we must verify condition (5).

First, Lemmas 7 and 8 confirm the first part of the proof. In order to establish the validity of this theorem, it is only necessary to demonstrate (5).

Let $G \subset \Omega$, such that

$$G \subset \overline{conv}((\aleph G) \cup \{0\}).$$

For each $\ell \in \jmath$, define $g(\ell) = \mathcal{M}(G(\ell))$, where $\mathcal{M}$ denotes the Kuratowski measure of noncompactness.

Since G is equicontinuous and bounded, Lemma 5, together with assumption (C3), yield:

$$\begin{aligned} g(\ell) &\leq \mathcal{M}(\aleph(G)(\ell) \cup \{0\}) \leq \mathcal{M}(\aleph(G)(\ell)) \\ &\leq \frac{1}{p^{(\frac{\beth+\urcorner}{2})}\Gamma_{p,q}(\beth+\urcorner)} \int_0^\ell (\ell - qs)_{p,q}^{(\beth+\urcorner-1)} \theta(ps)\mathcal{M}(G(ps))\mathrm{d}_{p,q}s \\ &\quad + \frac{|\lambda|}{p^{(\frac{\beth}{2})}\Gamma_{p,q}(\beth)} \int_0^\ell (\ell - qs)_{p,q}^{(\beth-1)} \mathcal{M}(G(ps))\mathrm{d}_{p,q}s. \\ &\leq \|g\|_\infty(\|\theta\|_\infty + |\lambda|). \end{aligned}$$

Taking the supremum over $\ell \in \jmath$, we obtain:

$$\|g\|_\infty(1 - \|\theta\|_\infty - |\lambda|) \leq 0.$$

Because $\|\theta\|_\infty + |\lambda| < 1$, it follows that $\|g\|_\infty = 0$, implying $g(\ell) = 0$ for all $\ell \in \jmath$.

Thus, $G(\ell)$ is relatively compact for each $\ell \in \jmath$, and by the Arzelà–Ascoli theorem, G is relatively compact within Ω.

Finally, by Mönch's fixed-point theorem, $\aleph$ has at least one fixed point in Ω, which corresponds to a solution to the boundary value problem (1)–(2). □

Corollary 1. *Assume that the hypotheses (C1)–(C3) are satisfied. If $p = 1$, then the boundary value problem (1)–(2) admits at least one solution.*

Proof. When $p = 1$, the fractional (p, q)-operators reduce to their q-analogues. All preceding arguments in Theorem 3 remain valid under this particular case. Therefore, the proof proceeds identically, and the conclusion follows directly: the boundary value problem has at least one solution. □

4. An Example

To demonstrate the applicability of Theorem 3, we consider the following boundary value problem involving the (p, q)-fractional difference operator:

$$\begin{cases} \boldsymbol{D}^{\frac{1}{2}}_{p,q}\left(\boldsymbol{D}^{\frac{1}{4}}_{p,q}\hbar(\ell) + \frac{2}{7}\hbar\left(p^{\frac{1}{4}}\ell\right)\right) = \frac{\hbar\left(p^{\frac{3}{4}}\ell\right)}{6+\exp\left(p^{\frac{3}{4}}\ell\right)^2}, \text{ for all } \ell \in (0,1), \\ \hbar(0) = 0, \ \hbar(1) = \left(\mathrm{I}^{\frac{3}{4}}_{p,q}\frac{\hbar(ps)}{6+\exp(ps)^2}\right)(1) - \frac{2}{7}\left(\mathrm{I}^{\frac{1}{4}}_{p,q}\hbar(ps)\right)(1) \end{cases} \tag{10}$$

where $\beth = \frac{1}{4}$, $\daleth = \frac{1}{2}$, $\lambda = \frac{2}{7}$, $p = \frac{1}{2}$, $q = \frac{2}{5}$, $\hbar = (\hbar_1, \hbar_2, \ldots, \hbar_i, \ldots)$, and $\mathcal{F} = (f_1, f_2, \ldots, f_i, \ldots)$ such that

$$f_i(\ell, \hbar_i(\ell)) = \frac{\hbar_i(\ell)}{6 + e^{\ell^2}} \text{ for} \ell \in \jmath,$$

and let the space

$$\mathcal{W} = l^1 = \left\{\hbar = (\hbar_1, \hbar_2, \ldots, \hbar_i, \ldots) \text{ such that } \sum_{i=1}^{\infty}|\hbar_i| < \infty\right\},$$

equipped with the norm

$$\|\hbar\|_{l^1} = \sum_{i=1}^{\infty}|\hbar_i|.$$

Then, we get

$$|f_i(\ell, \hbar_i(\ell))| \leq \frac{1}{6 + e^{\ell^2}}|\hbar_i(\ell)|, \text{ for all } \ell \in \jmath. \tag{11}$$

So, it can be deduced that conditions (C1) and (C2) are met by $\theta(\ell) = \frac{1}{6+e^{\ell^2}}$, $\left(\|\theta\|_\infty = \frac{1}{7}\right)$.

Furthermore

$$\|\theta\|_\infty + |\lambda| = \frac{1}{7} + \frac{2}{7} = \frac{3}{7} < 1.$$

and

$$\mathcal{M}(\mathcal{F}(\ell, F)) \leq \frac{1}{6 + e^{\ell^2}}\mathcal{M}(F).$$

Thus, by Theorem 3, the boundary value problem (10) possesses at least one solution $\hbar \in C(\jmath, \mathcal{W})$.

5. Conclusions

In this study, we investigated the existence of solutions to a nonlinear Langevin-type fractional (p,q)-difference equation in Banach space. The analysis relied on transforming the proposed boundary value problem into an equivalent fractional integral equation and applying the Kuratowski measure of noncompactness in combination with Mönch's fixed-point theorem. This framework builds upon several existing results concerning q-fractional and standard fractional Langevin equations to the more general (p,q)-difference setting. The obtained criteria guarantee the existence of at least one solution under easily verifiable assumptions on the nonlinear term.

The present work focuses mainly on existence theory. Stability properties, such as Ulam-type stability, exponential stability, and the asymptotic behavior of solutions, form a natural and highly promising continuation of this research. Future investigations will therefore be devoted to developing stability results for nonlinear fractional (p,q)-difference Langevin systems and to extending the analysis to coupled, impulsive, and higher-order problems.

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