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A Minimax Diversification Approach to Dynamic Portfolio Optimization

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Abstract

This paper investigates a multi-period investment problem in which an investor revises investment decisions at the beginning of each period. The objective is to maximize expected terminal wealth while simultaneously minimizing risk. This study quantifies risk using a dynamic risk function grounded in the minimax risk diversification principle. A key feature of the model is its flexibility: in each period, the investor constructs the risk function using either standard deviation, absolute deviation, or lower semi-absolute deviation, thereby accommodating diverse risk preferences. By employing dynamic programming, analytical solutions for the optimal investment strategy are derived. These solutions explicitly demonstrate the strategy's dependence on the expected return rates of risky assets and the investor's risk tolerance.

Keywords: deviation measure; portfolio selection; risk management; dynamic programming

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1. Introduction

Portfolio selection is the process of allocating wealth to a set of assets to achieve the best possible trade-off between expected return and risk. The mean-variance model, introduced in Markowitz's seminal work [1], established the foundation of Modern Portfolio Theory and later earned its author the Nobel Prize. Nevertheless, the practical application of the mean-variance model to large-scale portfolios remains limited, despite its theoretical acclaim. The formidable computational complexity of solving large-scale quadratic programming problems is a major obstacle (see Leung et al. [2] and Sun et al. [3]). The size of the covariance matrix grows quadratically with the number of assets, creating a dense structure that imposes significant computational costs.

To alleviate the computational challenges, Konno and Yamazaki [4] introduced the mean-absolute deviation model, which replaces the quadratic objective with a linear one. An advantage of this linear programming is that the resulting optimization problem is computationally tractable and its implementation is analogous to that of the mean-variance model. Guided by the concept of absolute deviation for risk measurement, Cai et al. [5] introduced a minimax risk function for a no-short-selling portfolio. By exploiting the special structure of this risk function, they derived explicit analytical solutions for the optimal portfolios, thereby enabling the entire efficient frontier to be plotted. Extending absolute deviation, Rockafellar et al. [6] introduced the generalized deviation measure. Meng et al. [7] then leveraged this measure to generalize the risk function of Cai et al. [5],

formulating a portfolio model that allows for short selling. They also obtained closed-form solutions for the efficient frontier and optimal portfolios.

The aforementioned models are all predicated on a single-period investment horizon. However, portfolio strategies are typically multi-period in practice. This is because contemporary financial decisions require models that reflect an interdependent and complex reality. Over longer investment horizons, economic conditions can change significantly, necessitating periodic portfolio rebalancing in response to these shifts. Consequently, extending the single-period portfolio selection framework to a multi-period setting is a natural progression.

Mossin [8] pioneered the use of dynamic programming for optimal multi-period portfolio selection. Li and Ng [9] applied this approach to the mean-variance formulation, deriving analytical expressions for the optimal strategy. Building on this, Yu et al. [10–12] generalized the model of Cai et al. [5] to a multi-period framework where an investor maximizes terminal wealth while minimizing total risk, defined as the sum of each asset's maximum absolute deviation over time. They also derived a closed-form optimal policy via dynamic programming.

Building upon the multi-period framework established by Yu et al. [10], this paper seeks to extend the model of Meng et al. [7] into a multi-period context. The primary contributions are threefold. First, in contrast to the single-period model in Meng et al. [7] and the multi-period models with uniform risk measures in [10], our framework introduces a flexible risk structure. This allows investors to select a specific deviation measure, such as standard deviation, absolute deviation, or lower semi-absolute deviation for each period, enabling a more dynamic and personalized approach to risk management. Second, by leveraging dynamic programming, we derive a general closed-form solution for the optimal investment strategy. This analytical result not only simplifies implementation but also provides clear insights into how the strategy is influenced by the assets' expected returns and the investor's risk tolerance. Third, we provide explicit necessary and sufficient conditions for the risk tolerance parameter λ , which guarantee the existence of an optimal solution.

It is important to position our multi-period model within the context of dynamic risk measures. Extending the multi-period framework of Yu et al. [10], our model employs an additive-in-time deviation function and is therefore not time-consistent, unlike dynamic coherent risk measures (see Feinstein and Rudloff [13] and Valladao et al. [14]). This represents a conceptual limitation. However, our model maintains a simpler structure that is highly analytically tractable, enabling the derivation of closed-form solutions.

The rest of this paper is organized as follows. Section 2 introduces a multi-period investment problem with n risky assets, formulating it as a bi-objective optimization problem and defining the risk function and deviation measures for the analysis. Section 3 derives an analytical solution to the bi-objective optimization problem by applying dynamic programming, yielding a closed-form expression for the optimal investment strategy. Section 4 concludes this paper.

For a random variable X , we denote by $E(X)$ the mathematical expectation of X throughout the paper. The space $\mathcal{L}^2(\Omega) = \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ is the set of all random variables defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ for which the second moment is finite, i.e.,

$$\mathcal{L}^2(\Omega) := \left\{ X : \Omega \rightarrow \mathcal{R} \mid E[|X|^2] < \infty \right\}.$$

In particular, the space $\mathcal{L}^2(\Omega)$ contains all constant random variables, $X \equiv \mathcal{C}$. The letter $\mathcal{C}$ will always stand for a constant in the real numbers $\mathcal{R}$, and any (in)equalities between

random variables are to be viewed in the sense of holding almost surely. We adopt the notion that

$$X = X_+ - X_- \text{ with } X_+ = \max\{0, X\}, X_- = \max\{0, -X\},$$

and

$$\text{sgn}(a) = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases}$$

2. Notation and Problem Formulation

This paper examines a multi-period investment problem in a market with n risky assets. An investor begins with an initial wealth V_0 at time $t = 0$. The investor is then permitted to revise his/her investment decisions at the beginning of each subsequent period for a total of $N - 1$ revisions, achieving a terminal wealth V_N at the end of the N -th period.

At the beginning of each period, the n risky assets are sorted by ascending expected return rates. Let $S_{t,j}$ denote the asset with index j at time t . For this asset, we denote by $x_{t,j} \in \mathcal{R}$ the investment allocated at the beginning of period t , and by $R_{t,j}$ its random return rate over that period. For notational convenience, we define the following vectors:

$$R_t = (R_{t,1}, R_{t,2}, \dots, R_{t,n})^T, \quad r_t = (r_{t,1}, r_{t,2}, \dots, r_{t,n})^T \text{ with } r_{t,j} := E(R_{t,j}),$$

as well as $x_t = (x_{t,1}, x_{t,2}, \dots, x_{t,n})^T$ and $e = (1, \dots, 1)^T \in \mathcal{R}^n$.

Assumption 1. The expected return rates are strictly ordered, i.e., $r_{t,1} < r_{t,2} < \dots < r_{t,n}$ for all $t = 1, 2, \dots, N$.

Remark 1. Following Meng et al. [7], Assumption 1 is necessary to define the trading strategy of shorting lower-return assets to fund higher-return ones. The model is robust to estimation errors, provided they do not alter the underlying ranking of the assets.

Let V_t be the total wealth at the end of period t , then

$$V_t = V_{t-1} + R_t^T x_t, \quad t = 1, 2, \dots, N. \quad (1)$$

Because no money is added to or withdrawn from the market during any period. The self-financing condition is satisfied, i.e.,

$$x_t^T e = V_{t-1}, \quad t = 1, 2, \dots, N. \quad (2)$$

To ensure the problem's feasibility and avoid bankruptcy, we impose the condition that the investor's wealth is strictly positive at all times. Formally, we require

$$V_t > 0, \quad t = 0, 1, 2, \dots, N.$$

The risk function in period t for portfolio x_t is given by

$$\omega_t(x_t) := \max_{1 \leq j \leq n} \mathcal{D}(x_{t,j} R_{t,j}). \quad (3)$$

It is the general l_∞ risk function introduced in Meng et al. [7], and $\mathcal{D}(X)$ is the general deviation measure of random variable X in the sense of [6]:

Definition 1. A deviation measure is a function $\mathcal{D} : \mathcal{L}^2(\Omega) \rightarrow [0, \infty]$ that satisfies

- (D1) $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all X and constants C ,
- (D2) $\mathcal{D}(0) = 0$ and $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and $\lambda > 0$,
- (D3) $\mathcal{D}(X + X') \leq \mathcal{D}(X) + \mathcal{D}(X')$ for all X and X' ,
- (D4) $\mathcal{D}(X) \geq 0$ for all X , with $\mathcal{D}(X) > 0$ for nonconstant X .

Due to the axiom (D2) of $\mathcal{D}$, we have

$$\mathcal{D}(x_{t,j}R_{t,j}) = \begin{cases} x_{t,j}\mathcal{D}(R_{t,j}) & \text{if } x_{t,j} \geq 0, \\ -x_{t,j}\mathcal{D}(-R_{t,j}) & \text{if } x_{t,j} < 0. \end{cases}$$

The following deviation measures are selected for their applicability to our model and their prevalence in existing academic literature.

- (a) The standard deviation $\mathcal{D}(R_{t,j}) := \sqrt{E(E(R_{t,j}) - R_{t,j})^2}$;
- (b) The absolute deviation $\mathcal{D}(R_{t,j}) := E(|R_{t,j} - E(R_{t,j})|)$;
- (c) The lower semi-absolute deviation $\mathcal{D}(R_{t,j}) := E(E(R_{t,j}) - R_{t,j})_+$.

The deviation measures in (a)–(c) are symmetric with respect to positive and negative returns, satisfying $\mathcal{D}(R_{t,j}) = \mathcal{D}(-R_{t,j})$ for all j .

Remark 2. Let $q_{t,j} := \mathcal{D}(R_{t,j}) = \mathcal{D}(-R_{t,j})$, then the Equation (3) can be rewritten as

$$\omega_t(x_t) = \max_{1 \leq j \leq n} \max\{x_{t,j}q_{t,j}, -x_{t,j}q_{t,j}\}. \quad (4)$$

By preserving a linear framework, this risk function enables the efficient solving of portfolio optimization models. Moreover, due to axiom (D3) of $\mathcal{D}$, we have

$$\mathcal{D}\left(\sum_{j=1}^n x_{t,j}R_{t,j}\right) \leq \sum_{j=1}^n \mathcal{D}(x_{t,j}R_{t,j}) \leq n\omega_t(x_t).$$

This indicates that when $\omega_t(x_t)$ is small, $\mathcal{D}\left(\sum_{j=1}^n x_{t,j}R_{t,j}\right)$ will also be relatively small.

It is instructive to compare our proposed risk measure $\omega_t(x_t)$ with established benchmarks like variance and Conditional Value at Risk (CVaR). Variance requires a quadratic framework and a full covariance matrix, which is computationally prohibitive for large portfolios. CVaR, while coherent, is computationally intensive and typically lacks a closed-form solution. In contrast, $\omega_t(x_t)$ preserves a linear structure that, as noted in Remark 2, is key to efficient solving and a closed-form solution. It must be acknowledged that our risk measure $\omega_t(x_t)$ does not account for inter-asset correlations. However, this simplification is what makes our method particularly suitable for large-scale problems where the computation of a full covariance matrix or CVaR is prohibitively expensive.

The total risk incurred by the investor at the end of period t , denoted by W_t for $t = 0, 1, \dots, N$, evolves recursively according to

$$W_t = W_{t-1} + \omega_t(x_t), \quad t = 1, 2, \dots, N. \quad (5)$$

subject to the initial condition $W_0 = 0$. The problem of a risk-averse investor, who aims to maximize expected return while minimizing risk, is naturally formulated as a bi-objective optimization problem as follows:

$$\begin{aligned} & \text{minimize} && (W_N, -E(V_N)) \\ & \text{subject to} && V_t = V_{t-1} + R_t^T x_t, \quad t = 1, 2, \dots, N, \\ & && W_t = W_{t-1} + \max_{1 \leq j \leq n} \max \{x_{t,j} q_{t,j}, -x_{t,j} q_{t,j}\}, \quad t = 1, 2, \dots, N, \\ & && x_t^T e = V_{t-1}, \quad t = 1, 2, \dots, N, \end{aligned} \quad (6)$$

or equivalently

$$\begin{aligned} & \text{minimize} && (W_N, -E(V_N)) \\ & \text{subject to} && V_t = V_{t-1} + R_t^T x_t, \quad t = 1, 2, \dots, N, \\ & && W_t = W_{t-1} + y_t, \quad t = 1, 2, \dots, N, \\ & && y_t - x_{t,j} q_{t,j} \geq 0, \quad j = 1, 2, \dots, n, \quad t = 1, 2, \dots, N, \\ & && y_t + x_{t,j} q_{t,j} \geq 0, \quad j = 1, 2, \dots, n, \quad t = 1, 2, \dots, N, \\ & && x_t^T e = V_{t-1}, \quad t = 1, 2, \dots, N. \end{aligned} \quad (7)$$

Remark 3. Problem (6) is equivalent to problem (7) in the following sense:

- (1) If $(\mathcal{X}^*, \mathcal{Y}^*)$ is an efficient solution of problem (7), where $\mathcal{X}^* = (x_1^*, x_2^*, \dots, x_N^*)$ and $\mathcal{Y}^* = (y_1^*, y_2^*, \dots, y_N^*)$, then $\mathcal{X}^*$ is an efficient solution of Problem (6);
- (2) If $\mathcal{X}^* = (x_1^*, x_2^*, \dots, x_N^*)$ is an efficient solution of Problem (6). Then $(\mathcal{X}^*, \mathcal{Y}^*)$ is an efficient solution of problem (7), in which $\mathcal{Y}^* = (y_1^*, y_2^*, \dots, y_N^*)$ with $y_t^* = \max_{1 \leq j \leq n} \max \{x_{t,j} q_{t,j}, -x_{t,j} q_{t,j}\}$.

3. Analytic Solution

In this section, an analytical solution to problem (7) is derived using the dynamic programming approach. For a given parameter $\lambda \in (0, +\infty)$, the following parametric optimization problem is considered:

$$\begin{aligned} & \text{minimize} && \lambda W_N - E(V_N) \\ & \text{subject to} && V_t = V_{t-1} + R_t^T x_t, \quad t = 1, 2, \dots, N, \\ & && W_t = W_{t-1} + y_t, \quad t = 1, 2, \dots, N, \\ & && y_t - x_{t,j} q_{t,j} \geq 0, \quad j = 1, 2, \dots, n, \quad t = 1, 2, \dots, N, \\ & && y_t + x_{t,j} q_{t,j} \geq 0, \quad j = 1, 2, \dots, n, \quad t = 1, 2, \dots, N, \\ & && x_t^T e = V_{t-1}, \quad t = 1, 2, \dots, N, \end{aligned} \quad (8)$$

where $\lambda > 0$ represents the investor's risk tolerance. A higher value of λ corresponds to a lower tolerance for risk. Since problem (7) is a bi-objective convex program, a portfolio $(\mathcal{X}^*, \mathcal{Y}^*)$ is efficient for it if and only if there exists a $\lambda > 0$ such that $(\mathcal{X}^*, \mathcal{Y}^*)$ is also an efficient portfolio for problem (8), see [15].

Remark 4. A practical method for calibrating λ is through a risk tolerance questionnaire, which maps an investor's profile to a quantitative value. A more formal alternative is to select λ such that the expected risk $E[W_N]$ respects a given risk budget, or to achieve a target risk-adjusted return ratio like $(E(V_N) - V_0)/E(W_N)$.

Owing to its specific structure, problem (8) is amenable to dynamic programming. Through this approach, a closed-form optimal solution is obtained using backward induction. Define

$$f_N(W_N, V_N) := \lambda W_N - V_N,$$

and

$$f_{t-1}(W_{t-1}, V_{t-1}) := \underset{x_t, y_t}{\text{minimize}} E[f_t(W_t, V_t)], \quad t = 1, 2, \dots, N.$$

The following notations will be used throughout this paper:

$$\begin{aligned} I &:= \{1, \dots, n\}, \\ I_t^+(a) &:= \{j \in I \mid r_{t,j} > a\}, \\ I_t^0(a) &:= \{j \in I \mid r_{t,j} = a\}, \\ I_t^-(a) &:= \{j \in I \mid r_{t,j} < a\}. \end{aligned}$$

We define $g_t : \mathcal{R} \rightarrow \mathcal{R}$ by

$$g_t(a) := \sum_{j \in I^+(a)} \frac{r_{t,j} - a}{q_{t,j}} + \sum_{j \in I^-(a)} \frac{a - r_{t,j}}{q_{t,j}}, \quad (9)$$

and $A_t \in \mathcal{R}^{n \times (n+1)}$ by

$$A_t := \begin{pmatrix} 1/q_{t,1} & -1/q_{t,1} & -1/q_{t,1} & \dots & -1/q_{t,1} & -1/q_{t,1} \\ 1/q_{t,2} & 1/q_{t,2} & -1/q_{t,2} & \dots & -1/q_{t,2} & -1/q_{t,2} \\ 1/q_{t,3} & 1/q_{t,3} & 1/q_{t,3} & \dots & -1/q_{t,3} & -1/q_{t,3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1/q_{t,n-1} & 1/q_{t,n-1} & 1/q_{t,n-1} & \dots & -1/q_{t,n-1} & -1/q_{t,n-1} \\ 1/q_{t,n} & 1/q_{t,n} & 1/q_{t,n} & \dots & 1/q_{t,n} & -1/q_{t,n} \end{pmatrix}. \quad (10)$$

Let matrix A_t be partitioned by its columns as $A_t = [\alpha_{t,1}, \alpha_{t,2}, \dots, \alpha_{t,(n+1)}]$, where each $\alpha_{t,j}$ is a column vector. Moreover, we define:

$$m_t^+ := \max\{j \in I \mid \alpha_{t,j}^T e > 0\}, \quad (11)$$

and

$$m_t^- := \min\{j \in I \mid \alpha_{t,j}^T e < 0\}. \quad (12)$$

The following lemma is instrumental to our analysis and follows directly from the definition of g_t and Lemma 3.1 in Meng et al. [7].

Lemma 1. g_t is a continuous piecewise linear convex function with

$$g_t(a) = \begin{cases} g_t(r_{t,1}) + \alpha_{t,1}^T e(r_{t,1} - a) & \text{if } a \leq r_{t,1}, \\ g_t(r_{t,k}) + \alpha_{t,k}^T e(r_{t,k} - a) & \text{if } r_{t,k-1} \leq a \leq r_{t,k}, \\ \sum_{j=1}^n \frac{a - r_{t,j}}{q_{t,j}} & \text{if } a > r_{t,n}. \end{cases}$$

Moreover, the function g_t is strictly decreasing on the interval $(-\infty, r_{t,m_t^+}]$, constant on $[r_{t,m_t^+}, r_{t,m_t^-}]$, and strictly increasing on $[r_{t,m_t^-}, +\infty)$. Consequently, it attains its minimum value on the interval $[r_{t,m_t^+}, r_{t,m_t^-}]$.

Assuming that W_{N-1} and V_{N-1} are known, we first consider period N . Thus,

$$\begin{aligned} E[f_N(W_N, V_N)] &= \lambda W_N - E[V_N] \\ &= \lambda(W_{N-1} + y_N) - (V_{N-1} + r_N^T x_N). \end{aligned}$$

Consequently, problem (8) can be reformulated as follows:

$$\begin{aligned} \text{minimize} \quad & E[f_N(W_N, V_N)] = \lambda(W_{N-1} + y_N) - (V_{N-1} + r_N^T x_N) \\ \text{subject to} \quad & y_N - x_{N,j} q_{N,j} \geq 0, \quad j = 1, 2, \dots, n, \\ & y_N + x_{N,j} q_{N,j} \geq 0, \quad j = 1, 2, \dots, n, \\ & x_N^T e = V_{N-1}. \end{aligned} \tag{13}$$

The optimal solution to problem (13) is provided in the following lemma. The proof is analogous to that of Theorem 3.1 in Meng et al. [7] and is therefore omitted.

Lemma 2. The problem (13) has a nonempty optimal solution set if and only if $\lambda \geq g_N(r_{N,m_N^+})$. Meanwhile, for any fixed $\lambda \in [g_N(r_{N,m_N^+}), +\infty)$, an optimal solution to problem (13) is given by

$$x_N^* = V_{N-1} \frac{\alpha_{N,k_N}}{\alpha_{N,k_N}^T e}, \quad y_N^* = V_{N-1} \frac{1}{\alpha_{N,k_N}^T e},$$

where $k_N \in \{1, 2, \dots, n+1\}$ is determined as follows:

- (a) in the case of $g_N(r_{N,1}) \leq \lambda < +\infty$, $k_N = 1$;
- (b) in the case of $g_N(r_{N,j_0}) \leq \lambda < g_N(r_{N,j_0-1})$ for some $j_0 \in \{2, \dots, m_N^+\}$, $k_N = j_0$.

Remark 5. The condition $\lambda \geq g_N(r_{N,m_N^+})$ is necessary for the problem to be bounded. If λ falls below this threshold, the weight given to risk is too small, and the investor is motivated to take on extreme risk positions for higher return, leading to an unbounded linear program.

Substituting the optimal solution from Lemma 2 into the objective function yields:

$$\begin{aligned} f_{N-1}(W_{N-1}, V_{N-1}) &= \underset{x_N, y_N}{\text{minimize}} E[f_N(W_N, V_N)] \\ &= \lambda(W_{N-1} + y_N^*) - (V_{N-1} + r_N^T x_N^*) \\ &= \lambda W_{N-1} - V_{N-1}(1 + a_N b_N - \lambda a_N), \end{aligned}$$

where

$$a_N = \frac{1}{\alpha_{N,k_N}^T e}, \quad b_N = r_N^T \alpha_{N,k_N}.$$

Furthermore, let $c_{N+1} = 1$ and

$$c_N = c_{N+1}(1 + a_N b_N) - \lambda a_N,$$

then

$$f_{N-1}(W_{N-1}, V_{N-1}) = \lambda W_{N-1} - c_N V_{N-1}.$$

Proceeding to period $N - 1$, and assuming W_{N-2} and V_{N-2} are known, it follows that

$$\begin{aligned} f_{N-2}(W_{N-2}, V_{N-2}) &= \underset{x_{N-1}, y_{N-1}}{\text{minimize}} E[f_{N-1}(W_{N-1}, V_{N-1})] \\ &= \underset{x_{N-1}, y_{N-1}}{\text{minimize}} [\lambda W_{N-1} - c_N E(V_{N-1})] \\ &= \underset{x_{N-1}, y_{N-1}}{\text{minimize}} [\lambda(W_{N-2} + y_{N-1}) - c_N(V_{N-2} + r_{N-1}^T x_{N-1})]. \end{aligned}$$

The problem to be solved is as follows:

$$\begin{aligned} \text{minimize} \quad & E[f_{N-1}(W_{N-1}, V_{N-1})] = \lambda(W_{N-2} + y_{N-1}) - c_N(V_{N-2} + r_{N-1}^T x_{N-1}) \\ \text{subject to} \quad & y_{N-1} - x_{(N-1),j} q_{(N-1),j} \geq 0, \quad j = 1, 2, \dots, n, \\ & y_{N-1} + x_{(N-1),j} q_{(N-1),j} \geq 0, \quad j = 1, 2, \dots, n, \\ & x_{N-1}^T e = V_{N-2}. \end{aligned} \tag{14}$$

A detailed proof of the subsequent theorem is presented in the Appendix A.

Theorem 1.

(1) If $c_N = 0$, then

$$x_{N-1}^* = V_{N-2} \frac{\alpha_{(N-1),1}}{\alpha_{(N-1),1}^T e}, \quad y_{N-1}^* = V_{N-2} \frac{1}{\alpha_{(N-1),1}^T e},$$

is an optimal solution to problem (14).

(2) If $c_N \neq 0$, then the problem (14) has a nonempty optimal solution set if and only if $\lambda \geq g_{N-1}(r_{(N-1),m_{N-1}^+})$. Meanwhile, for any fixed $\lambda \in [g_{N-1}(r_{(N-1),m_{N-1}^+}), +\infty)$, an optimal solution to problem (14) is given by

$$x_{N-1}^* = V_{N-2} \frac{\alpha_{(N-1),k_{N-1}}}{\alpha_{(N-1),k_{N-1}}^T e}, \quad y_{N-1}^* = V_{N-2} \frac{\text{sgn}(\alpha_{(N-1),k_{N-1}}^T e)}{\alpha_{(N-1),k_{N-1}}^T e},$$

where $k_{N-1} \in \{1, 2, \dots, n+1\}$ is determined as follows:

- (i) in the case of $c_N > 0$ with $g_{N-1}(r_{(N-1),1}) \leq \frac{\lambda}{c_N} < +\infty$, $k_{N-1} = 1$;
- (ii) in the case of $c_N > 0$ with $g_{N-1}(r_{(N-1),j_0}) \leq \frac{\lambda}{c_N} < g_{N-1}(r_{(N-1),j_0-1})$ for some $j_0 \in \{2, \dots, m_{N-1}^+\}$, $k_{N-1} = j_0$;
- (iii) in the case of $c_N < 0$ with $g_{N-1}(r_{(N-1),n}) \leq \left(-\frac{\lambda}{c_N}\right) < +\infty$, $k_{N-1} = n+1$;
- (iv) in the case of $c_N < 0$ with $g_{N-1}(r_{(N-1),j_0-1}) \leq \left(-\frac{\lambda}{c_N}\right) < g_{N-1}(r_{(N-1),j_0})$ for some $j_0 \in \{m_{N-1}^- + 1, \dots, n\}$, $k_{N-1} = j_0$.

Substituting the optimal solution from Theorem 1 into the objective function yields:

$$\begin{aligned} f_{N-2}(W_{N-2}, V_{N-2}) &= \underset{x_{N-1}, y_{N-1}}{\text{minimize}} E[f_{N-1}(W_{N-1}, V_{N-1})] \\ &= \lambda(W_{N-2} + y_{N-1}^*) - c_N(V_{N-2} + r_{N-1}^T x_{N-1}^*) \\ &= \lambda W_{N-2} - V_{N-2}(c_N(1 + a_{N-1} b_{N-1}) - \lambda a_{N-1} \text{sgn}(a_{N-1})) \\ &= \lambda W_{N-2} - c_{N-1} V_{N-2}, \end{aligned}$$

where

$$a_{N-1} = \frac{1}{\alpha_{(N-1),k_{N-1}}^T e}, \quad b_{N-1} = \alpha_{(N-1),k_{N-1}}^T r_{N-1},$$

and

$$c_{N-1} = c_N(1 + a_{N-1}b_{N-1}) - \lambda a_{N-1} \operatorname{sgn}(a_{N-1}).$$

For a general period $t = 1, 2, \dots, N-1$, if we define

$$f_t(W_t, V_t) = \lambda W_t - c_{t+1} V_t,$$

then,

$$f_{t-1}(W_{t-1}, V_{t-1}) = \underset{x_t, y_t}{\operatorname{minimize}} E[f_t(W_t, V_t)].$$

Assuming W_{t-1} and V_{t-1} are known, it follows that

$$\begin{aligned} f_{t-1}(W_{t-1}, V_{t-1}) &= \underset{x_t, y_t}{\operatorname{minimize}} E[f_t(W_t, V_t)] \\ &= \underset{x_t, y_t}{\operatorname{minimize}} [\lambda W_t - c_{t+1} E(V_t)] \\ &= \underset{x_t, y_t}{\operatorname{minimize}} [\lambda(W_{t-1} + y_t) - c_{t+1}(V_{t-1} + r_t^T x_t)]. \end{aligned}$$

The problem to be solved is as follows:

$$\begin{aligned} \underset{x_t, y_t}{\operatorname{minimize}} \quad & E[f_t(W_t, V_t)] = \lambda(W_{t-1} + y_t) - c_{t+1}(V_{t-1} + r_t^T x_t) \\ \text{subject to} \quad & y_t - x_{t,j} q_{t,j} \geq 0, \quad j = 1, 2, \dots, n, \\ & y_t + x_{t,j} q_{t,j} \geq 0, \quad j = 1, 2, \dots, n, \\ & x_t^T e = V_{t-1}. \end{aligned} \tag{15}$$

This problem is solved analogously to Theorem 1, by replacing V_{N-2} with V_{t-1} , $q_{(N-1),j}$ with $q_{t,j}$, and c_N with c_{t+1} .

Theorem 2.

(1) If $c_{t+1} = 0$, then

$$x_t^* = V_{t-1} \frac{\alpha_{t,1}}{\alpha_{t,1}^T e}, \quad y_t^* = V_{t-1} \frac{1}{\alpha_{t,1}^T e},$$

is an optimal solution to problem (15).

(2) If $c_{t+1} \neq 0$, then the problem (15) has a nonempty optimal solution set if and only if $\lambda \geq g_t(r_{t,m_t^+})$. Meanwhile, for any fixed $\lambda \in [g_t(r_{t,m_t^+}), +\infty)$, an optimal solution to problem (15) is given by

$$x_t^* = V_{t-1} \frac{\alpha_{t,k_t}}{\alpha_{t,k_t}^T e}, \quad y_t^* = V_{t-1} \frac{\operatorname{sgn}(\alpha_{t,k_t}^T e)}{\alpha_{t,k_t}^T e},$$

where $k_t \in \{1, 2, \dots, n+1\}$ is determined as follows:

- (i) in the case of $c_{t+1} > 0$ with $g_t(r_{t,1}) \leq \frac{\lambda}{c_{t+1}} < +\infty$, $k_t = 1$;
- (ii) in the case of $c_{t+1} > 0$ with $g_t(r_{t,j_0}) \leq \frac{\lambda}{c_{t+1}} < g_t(r_{t,j_0-1})$ for some $j_0 \in \{2, \dots, m_t^+\}$, $k_t = j_0$;
- (iii) in the case of $c_{t+1} < 0$ with $g_t(r_{t,n}) \leq \left(-\frac{\lambda}{c_{t+1}}\right) < +\infty$, $k_t = n+1$;
- (iv) in the case of $c_{t+1} < 0$ with $g_t(r_{t,j_0-1}) \leq \left(-\frac{\lambda}{c_{t+1}}\right) < g_t(r_{t,j_0})$ for some $j_0 \in \{m_t^- + 1, \dots, n\}$, $k_t = j_0$.

From the analysis above, it follows that for $t = 2, 3, \dots, N$,

$$\begin{aligned} f_{t-1}(W_{t-1}, V_{t-1}) &= \underset{x_t, y_t}{\text{minimize}} E[f_t(W_t, V_t)] \\ &= \lambda(W_{t-1} + y_t^*) - c_{t+1}(V_{t-1} + r_t^T x_t^*) \\ &= \lambda W_{t-1} - V_{t-1}(c_{t+1}(1 + a_t b_t) - \lambda a_t \text{sgn}(a_t)) \\ &= \lambda W_{t-1} - c_t V_{t-1}, \end{aligned}$$

where

$$a_t = \frac{1}{\alpha_{t,k_t}^T e}, \quad b_t = \alpha_{t,k_t}^T r_t,$$

and

$$c_t = c_{t+1}(1 + a_t b_t) - \lambda a_t \text{sgn}(a_t),$$

with the terminal condition $c_{N+1} = 1$. Here, the index k_t is determined by Theorem 2.

To enhance the clarity and applicability of our method, we present a pseudo-code at the end of this section (Algorithm 1), which details the computational procedure for the optimal solution.

Algorithm 1 Multi-Period Portfolio Optimization via Backward Induction

Input: Horizon N ; For $t = 1, \dots, N$: sorted returns r_t , deviations q_t ; Parameter λ .

Output: Optimal strategies $\{x_t^*\}_{t=1}^N, \{y_t^*\}_{t=1}^N$.

```

1: Initialization:  $c_{N+1} \leftarrow 1$ . Precompute  $g_t(\cdot)$  and  $A_t = [\alpha_{t,1}, \dots, \alpha_{t,n+1}]$ .
2: Backward Pass:
3: for  $t = N$  downto 1 do
4:   // Determine index  $k_t$  per Theorem 2
5:   if  $c_{t+1} = 0$  then
6:      $k_t \leftarrow 1$ .
7:   else if  $c_{t+1} > 0$  then
8:     Find  $j_0$  s.t.  $g_t(r_{t,j_0}) \leq \frac{\lambda}{c_{t+1}} < g_t(r_{t,j_0-1})$ ;  $k_t \leftarrow j_0$  (or 1).
9:   else
10:    Find  $j_0$  s.t.  $g_t(r_{t,j_0-1}) \leq -\frac{\lambda}{c_{t+1}} < g_t(r_{t,j_0})$ ;  $k_t \leftarrow j_0$  (or  $n+1$ ).  $\triangleright c_{t+1} < 0$ 
11:   end if
12:   // Update coefficient
13:    $a_t \leftarrow \frac{1}{\alpha_{t,k_t}^T e}$ ;  $c_t \leftarrow c_{t+1}(1 + a_t \alpha_{t,k_t}^T r_t) - \lambda a_t \cdot \text{sgn}(a_t)$ .
14: end for
15: Forward Pass:
16: for  $t = 1$  to  $N$  do
17:    $x_t^* \leftarrow V_{t-1} \frac{\alpha_{t,k_t}}{\alpha_{t,k_t}^T e}$ ;  $y_t^* \leftarrow V_{t-1} \frac{\text{sgn}(\alpha_{t,k_t}^T e)}{\alpha_{t,k_t}^T e}$ .
18: end for
19: return  $\{x_t^*\}, \{y_t^*\}$ .

```

Remark 6. In view of Equations (1) and (5), if the investor allocates his/her wealth according to the optimal strategy from Theorem 2, the expected value of the terminal wealth is

$$E(V_N) = V_0 \prod_{t=1}^N \left(1 + \frac{\alpha_{t,k_t}^T r_t}{\alpha_{t,k_t}^T e} \right),$$

and the total risk borne by the investor at the end of period N is

$$\begin{aligned} W_N &= \sum_{t=1}^N E(V_{t-1}) \frac{\text{sgn}(\alpha_{t,k_t}^T e)}{\alpha_{t,k_t}^T e} \\ &= V_0 \frac{\text{sgn}(\alpha_{1,k_1}^T e)}{\alpha_{1,k_1}^T e} + \sum_{t=2}^N E(V_{t-1}) \frac{\text{sgn}(\alpha_{t,k_t}^T e)}{\alpha_{t,k_t}^T e} \\ &= V_0 \frac{\text{sgn}(\alpha_{1,k_1}^T e)}{\alpha_{1,k_1}^T e} + \sum_{t=2}^N V_0 \frac{\text{sgn}(\alpha_{t,k_t}^T e)}{\alpha_{t,k_t}^T e} \prod_{l=1}^{t-1} \left(1 + \frac{\alpha_{l,k_l}^T r_l}{\alpha_{l,k_l}^T e}\right). \end{aligned}$$

4. Conclusions

In this paper, we addressed a multi-period investment problem by developing a flexible and dynamic framework for decision revision. We quantified risk using a dynamic function based on the minimax diversification principle and derived analytical solutions for the optimal strategy via dynamic programming. The results demonstrated that the optimal investment decisions were systematically dependent on the assets' expected return rates and the investor's risk tolerance. The flexibility of the model is a notable feature, as it may allow for the accommodation of diverse investor preferences by incorporating different deviation measures.

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Appendix A

This appendix contains the proof of Theorem 1. The proof strategy leverages the fact that problem (14) is a linear program, for which the Kuhn–Tucker conditions provide necessary and sufficient optimality criteria. Consequently, the proof reduces to finding a Kuhn–Tucker point that satisfies these conditions.

Proof. For the case of $c_N = 0$, the Lagrangian for problem (14) is defined as

$$\begin{aligned} L(x_{N-1}, y_{N-1}, u, v, \tau) &= \lambda(W_{N-2} + y_{N-1}) + \sum_{j=1}^n u_j (x_{(N-1),j} q_{(N-1),j} - y_{N-1}) \\ &\quad + \sum_{j=1}^n v_j (-x_{(N-1),j} q_{(N-1),j} - y_{N-1}) + \tau \left(\sum_{j=1}^n x_{(N-1),j} - V_{N-2} \right), \end{aligned}$$

The Kuhn–Tucker conditions for the optimal solution (x_{N-1}^*, y_{N-1}^*) of the parametric problem (14) are then given by:

$$\sum_{j=1}^n x_{(N-1),j} = V_{N-2}, \quad (\text{A1})$$

$$\frac{\partial L}{\partial y_{N-1}} = \lambda - \sum_{j=1}^n u_j - \sum_{j=1}^n v_j = 0, \quad (\text{A2})$$

Additionally, for each $j = 1, \dots, n$:

$$\frac{\partial L}{\partial x_{(N-1),j}} = u_j q_{(N-1),j} - v_j q_{(N-1),j} + \tau = 0, \quad (\text{A3})$$

$$y_{N-1} - q_{(N-1),j} x_{(N-1),j} \geq 0, \quad (\text{A4})$$

$$u_j \geq 0, \quad (\text{A5})$$

$$u_j (y_{N-1} - q_{(N-1),j} x_{(N-1),j}) = 0, \quad (\text{A6})$$

$$y_{N-1} + q_{(N-1),j} x_{(N-1),j} \geq 0, \quad (\text{A7})$$

$$v_j \geq 0, \quad (\text{A8})$$

$$v_j (y_{N-1} + q_{(N-1),j} x_{(N-1),j}) = 0, \quad (\text{A9})$$

where $(u, v, \tau) \in \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}$ denotes the vector of Lagrange multipliers.

Firstly, the constraints (A4) and (A7) imply that $y_{N-1} > 0$. To prove this, assume to the contrary that $y_{N-1} \leq 0$. From the inequalities $y_{N-1} - q_{(N-1),j} x_{(N-1),j} \geq 0$ and $y_{N-1} + q_{(N-1),j} x_{(N-1),j} \geq 0$, it follows that $q_{(N-1),j} x_{(N-1),j} \leq 0$ and $q_{(N-1),j} x_{(N-1),j} \geq 0$. This implies that $x_{(N-1),j} = 0$ for all $j = 1, \dots, n$, which contradicts Equation (A1).

Secondly, from Equation (A6), we have $u_j v_j y_{N-1} = u_j v_j q_{(N-1),j} x_{(N-1),j}$, while from Equation (A9), it follows that $u_j v_j y_{N-1} = -u_j v_j q_{(N-1),j} x_{(N-1),j}$. Combining these two results yields:

$$u_j v_j = 0. \quad (\text{A10})$$

Furthermore, from Equation (A3), we have

$$u_j q_{(N-1),j} x_{(N-1),j} - v_j q_{(N-1),j} x_{(N-1),j} + \tau x_{(N-1),j} = 0,$$

which is equivalent to

$$u_j y_{N-1} + v_j y_{N-1} + \tau x_{(N-1),j} = 0,$$

Combining this with Equations (A1) and (A2), we can deduce that

$$\tau = -\frac{\lambda y_{N-1}}{V_{N-2}} < 0.$$

Therefore, substituting this result for τ back into Equation (A3) yields

$$v_j q_{(N-1),j} - u_j q_{(N-1),j} < 0. \quad (\text{A11})$$

Combining Equations (A10) and (A11), it follows that for all $j = 1, \dots, n$, $u_j > 0$ and $v_j = 0$. Therefore, substituting these results into Equation (A6) yields

$$x_{(N-1),j} = \frac{y_{N-1}}{q_{(N-1),j}}.$$

Finally, Equation (A1) implies the following explicit expression for y_{N-1} :

$$y_{N-1} = \frac{V_{N-2}}{\sum_{j=1}^n 1/q_{(N-1),j}} = \frac{V_{N-2}}{\alpha_{(N-1),1}^T e}.$$

For the case of $c_N \neq 0$, the Lagrangian for problem (14) is defined as

$$\begin{aligned} L(x_{N-1}, y_{N-1}, u, v, \tau) = & \lambda(W_{N-2} + y_{N-1}) - c_N(V_{N-2} + r_{N-1}^T x_{N-1}) + \sum_{j=1}^n u_j(x_{(N-1),j} q_{(N-1),j} - y_{N-1}) \\ & + \sum_{j=1}^n v_j(-x_{(N-1),j} q_{(N-1),j} - y_{N-1}) + \tau \left(\sum_{j=1}^n x_{(N-1),j} - V_{N-2} \right), \end{aligned}$$

At this point, the Kuhn–Tucker conditions for the optimal solution (x_{N-1}^*, y_{N-1}^*) of the parametric problem (14) are composed of Equations (A1), (A2) and (A4)–(A9), along with the following condition, which must hold for all $j = 1, 2, \dots, n$:

$$\frac{\partial L}{\partial x_{(N-1),j}} = -c_N r_{(N-1),j} + u_j q_{(N-1),j} - v_j q_{(N-1),j} + \tau = 0. \quad (\text{A12})$$

We partition the index set $I = \{1, \dots, n\}$ into the following four disjoint subsets:

$$\begin{aligned} I_{N-1}^{++} &:= \{j \mid u_j > 0, v_j > 0\}, \\ I_{N-1}^{+0} &:= \{j \mid u_j > 0, v_j = 0\}, \\ I_{N-1}^{0+} &:= \{j \mid u_j = 0, v_j > 0\}, \\ I_{N-1}^{00} &:= \{j \mid u_j = 0, v_j = 0\}. \end{aligned}$$

Also, the constraints (A4) and (A7) imply that $y_{N-1} > 0$. This leads to the conclusion that $I_{N-1}^{++} = \emptyset$. Otherwise, assume $j_0 \in I_{N-1}^{++}$. Equations (A4)–(A9) then imply $y_{N-1} - q_{(N-1),j_0} x_{(N-1),j_0} = 0$ and $y_{N-1} + q_{(N-1),j_0} x_{(N-1),j_0} = 0$, which in turn gives $x_{(N-1),j_0} = 0$ and thus $y_{N-1} = 0$. This is a contradiction. Therefore,

$$I = I_{N-1}^{+0} \cup I_{N-1}^{0+} \cup I_{N-1}^{00}.$$

Let $j \in I_{N-1}^{+0}$. It follows from (A5) and (A6) that

$$y_{N-1} - q_{(N-1),j} x_{(N-1),j} = 0.$$

Consequently,

$$x_{(N-1),j} = \frac{y_{N-1}}{q_{(N-1),j}} > 0. \quad (\text{A13})$$

Applying (A12) also yields $-c_N r_{(N-1),j} + u_j q_{(N-1),j} + \tau = 0$. From this, we obtain

$$u_j = \frac{c_N r_{(N-1),j} - \tau}{q_{(N-1),j}} > 0,$$

so it must be that $c_N r_{(N-1),j} - \tau > 0$.

Let $j \in I_{N-1}^{0+}$. It follows from (A8) and (A9) that

$$y_{N-1} + q_{(N-1),j} x_{(N-1),j} = 0.$$

Consequently,

$$x_{(N-1),j} = -\frac{y_{N-1}}{q_{(N-1),j}} < 0. \quad (\text{A14})$$

Applying (A12) also yields $-c_N r_{(N-1),j} - v_j q_{(N-1),j} + \tau = 0$. From this, we obtain

$$v_j = \frac{\tau - c_N r_{(N-1),j}}{q_{(N-1),j}} > 0,$$

so it must be that $\tau - c_N r_{(N-1),j} > 0$.

Let $j \in I_{N-1}^{00}$. It follows from (A4) and (A7) that

$$-\frac{y_{N-1}}{q_{(N-1),j}} \leq x_{(N-1),j} \leq \frac{y_{N-1}}{q_{(N-1),j}}$$

This can be rewritten as

$$x_{(N-1),j} = \theta_{(N-1),j} \frac{y_{N-1}}{q_{(N-1),j}}, \quad (\text{A15})$$

where $-1 \leq \theta_{(N-1),j} \leq 1$. Applying (A12) also leads to $-c_N r_{(N-1),j} + \tau = 0$.

Based on the preceding discussion, (A1) and (A2) can be rewritten as

$$\sum_{j \in I_{N-1}^{+0}} \frac{1}{q_{(N-1),j}} + \sum_{j \in I_{N-1}^{00}} \frac{\theta_{(N-1),j}}{q_{(N-1),j}} + \sum_{j \in I_{N-1}^{0+}} \frac{(-1)}{q_{(N-1),j}} = \frac{V_{N-2}}{y_{N-1}} > 0, \quad (\text{A16})$$

and

$$\lambda = \sum_{j \in I_{N-1}^{+0}} \frac{c_N r_{(N-1),j} - \tau}{q_{(N-1),j}} + \sum_{j \in I_{N-1}^{0+}} \frac{\tau - c_N r_{(N-1),j}}{q_{(N-1),j}}, \quad (\text{A17})$$

respectively. Here, the constants $\theta_{(N-1),j} \in [-1, 1]$ for $j \in I_{N-1}^{00}$ are chosen to satisfy (A16).

When $c_N > 0$, the three sub-indicator sets mentioned above can be rewritten as:

$$I_{N-1}^{+0} = I_{N-1}^+ \left(\frac{\tau}{c_N} \right), \quad I_{N-1}^{00} = I_{N-1}^0 \left(\frac{\tau}{c_N} \right), \quad I_{N-1}^{0+} = I_{N-1}^- \left(\frac{\tau}{c_N} \right).$$

Then, by combining (10) and (11), there exist $\theta_{(N-1),j} \in [-1, 1]$, $j \in I_{N-1}^{00}$ such that the inequality on the right-hand side of (A16) holds if and only if $\frac{\tau}{c_N} \leq r_{(N-1),m_{N-1}^+}$. Furthermore, Equation (9) allows us to rewrite (A17) as

$$g_{N-1} \left(\frac{\tau}{c_N} \right) = \frac{\lambda}{c_N}. \quad (\text{A18})$$

According to Lemma 1, for $\frac{\tau}{c_N} \in (-\infty, r_{(N-1),m_{N-1}^+}]$, Equation (A18) has a solution if and only if $\frac{\lambda}{c_N} \geq g_{N-1}(r_{(N-1),m_{N-1}^+})$. The lemma also yields the following chain of inequalities for this interval:

$$0 < g_{N-1}(r_{(N-1),m_{N-1}^+}) < g_{N-1}(r_{(N-1),m_{N-1}^+-1}) < \cdots < g_{N-1}(r_{(N-1),1}) < +\infty.$$

In case (i), the equality $g_{N-1} \left(\frac{\tau}{c_N} \right) = \frac{\lambda}{c_N} \geq g_{N-1}(r_{(N-1),1})$ yields $\frac{\tau}{c_N} \leq r_{(N-1),1}$. Consequently, $I_{N-1}^+ \left(\frac{\tau}{c_N} \right) \cup I_{N-1}^0 \left(\frac{\tau}{c_N} \right) = \{1, \dots, n\}$ and $I_{N-1}^- \left(\frac{\tau}{c_N} \right) = \emptyset$. Furthermore, Equations (10), (A13), (A15) and (A16) imply that

$$x_{N-1}^* = V_{N-2} \frac{\alpha_{(N-1),1}}{\alpha_{(N-1),1}^T e}, \quad y_{N-1}^* = V_{N-2} \frac{1}{\alpha_{(N-1),1}^T e},$$

is an optimal solution to problem (14).

In case (ii), we have

$$g_{N-1}(r_{(N-1),j_0}) \leq g_{N-1}\left(\frac{\tau}{c_N}\right) = \frac{\lambda}{c_N} < g_{N-1}(r_{(N-1),j_0-1}),$$

for some $j_0 \in \{2, \dots, m_{N-1}^+\}$, and then $\frac{\tau}{c_N} \leq r_{(N-1),j_0}$. Consequently, $I_{N-1}^+\left(\frac{\tau}{c_N}\right) \cup I_{N-1}^0\left(\frac{\tau}{c_N}\right) = \{j_0, \dots, n\}$ and $I_{N-1}^-\left(\frac{\tau}{c_N}\right) = \{1, \dots, j_0 - 1\}$. Furthermore, Equations (10) and (A13)–(A16) imply that

$$x_{N-1}^* = V_{N-2} \frac{\alpha_{(N-1),j_0}}{\alpha_{(N-1),j_0}^T} e, \quad y_{N-1}^* = V_{N-2} \frac{1}{\alpha_{(N-1),j_0}^T} e,$$

is an optimal solution to problem (14).

When $c_N < 0$, the three sub-indicator sets mentioned above can be rewritten as:

$$I_{N-1}^{+0} = I_{N-1}^-\left(\frac{\tau}{c_N}\right), \quad I_{N-1}^{00} = I_{N-1}^0\left(\frac{\tau}{c_N}\right), \quad I_{N-1}^{0+} = I_{N-1}^+\left(\frac{\tau}{c_N}\right).$$

Then, by combining (10) and (11), there exist $\theta_{(N-1),j} \in [-1, 1]$, $j \in I_{N-1}^{00}$ such that the inequality on the right-hand side of (A16) holds if and only if $\frac{\tau}{c_N} \geq r_{(N-1),m_{N-1}^-}$. Furthermore, Equation (9) allows us to rewrite (A17) as

$$g_{N-1}\left(\frac{\tau}{c_N}\right) = -\frac{\lambda}{c_N}. \quad (\text{A19})$$

According to Lemma 1, for $\frac{\tau}{c_N} \in [r_{(N-1),m_{N-1}^-}, +\infty)$, Equation (A19) has a solution if and only if $\left(-\frac{\lambda}{c_N}\right) \geq g_{N-1}(r_{(N-1),m_{N-1}^-})$. The lemma also yields the following chain of inequalities for this interval:

$$0 < g_{N-1}(r_{(N-1),m_{N-1}^-}) < g_{N-1}(r_{(N-1),m_{N-1}^-+1}) < \dots < g_{N-1}(r_{(N-1),n}) < +\infty.$$

In case (iii), the equality $g_{N-1}(r_{(N-1),n}) \leq \left(-\frac{\lambda}{c_N}\right) = g_{N-1}\left(\frac{\tau}{c_N}\right)$ yields $\frac{\tau}{c_N} \geq r_{(N-1),n}$. Consequently, $I_{N-1}^+\left(\frac{\tau}{c_N}\right) = \emptyset$ and $I_{N-1}^-\left(\frac{\tau}{c_N}\right) \cup I_{N-1}^0\left(\frac{\tau}{c_N}\right) = \{1, \dots, n\}$. Furthermore, Equations (10), (A13), (A15) and (A16) imply that

$$x_{N-1}^* = V_{N-2} \frac{-\alpha_{(N-1),(n+1)}}{-\alpha_{(N-1),(n+1)}^T} e, \quad y_{N-1}^* = V_{N-2} \frac{(-1)}{\alpha_{(N-1),(n+1)}^T} e,$$

is an optimal solution to problem (14).

In case (iv), we have

$$g_{N-1}(r_{(N-1),j_0-1}) \leq \left(-\frac{\lambda}{c_N}\right) = g_{N-1}\left(\frac{\tau}{c_N}\right) < g_{N-1}(r_{(N-1),j_0}),$$

for some $j_0 \in \{m_{N-1}^- + 1, \dots, n\}$, and then $\frac{\tau}{c_N} \geq r_{(N-1),j_0-1}$. Consequently, $I_{N-1}^+\left(\frac{\tau}{c_N}\right) = \{j_0, \dots, n\}$ and $I_{N-1}^-\left(\frac{\tau}{c_N}\right) \cup I_{N-1}^0\left(\frac{\tau}{c_N}\right) = \{1, \dots, j_0 - 1\}$. Furthermore, Equations (10) and (A13)–(A16) imply that

$$x_{N-1}^* = V_{N-2} \frac{-\alpha_{(N-1),j_0}}{-\alpha_{(N-1),j_0}^T} e, \quad y_{N-1}^* = V_{N-2} \frac{(-1)}{\alpha_{(N-1),j_0}^T} e,$$

is an optimal solution to problem (14).

This completes the proof. $\square$

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