mathematics

MDPI

Review

Magnetic Curves in Differential Geometry: A Comprehensive Survey

Bang-Yen Chen [1,*], Foued Aloui [2,*], Md Ajmal Khan [3] and Majid Ali Choudhary [3]

[1] Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
[2] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box 65892, Riyadh 11566, Saudi Arabia
[3] Department of Mathematics, Maulana Azad National Urdu University, Hyderabad 500032, India; mdajmalkhan1999@gmail.com (M.A.K.); majid_alichoudhary@yahoo.co.in (M.A.C.)
* Correspondence: chenb@msu.edu (B.-Y.C.); fyaloui@imamu.edu.sa (F.A.)

Abstract

The concept of "magnetic lines of force," or "magnetic curves", was introduced in the 1830s by Michael Faraday (1791–1867); his work provided the foundation for the modern understanding of magnetic fields. In differential geometry, a magnetic curve is a concept that arises from the intersection of geometry and physics. These curves represent the trajectories of a charged particle experiencing the Lorentz force as it travels through a magnetic field. These curves have garnered significant interest due to their intricate geometric properties and diverse applications. This paper provides a comprehensive exploration of magnetic curves, delving into their fundamental characteristics and classification.

Keywords: magnetic field; magnetic curve; Killing magnetic curve; Bishop frame; T-, N-, and B-magnetic curves; Hopf magnetic curve; slant magnetic curve; pseudo-Hermitian magnetic curve

MSC: 53C15; 53B21; 53A35; 53C25; 53B25; 53Z05; 53C80; 53C50; 53A10; 53C42; 58C20; 35A30



Academic Editor: Xinguang Zhang

Received: 22 October 2025
Revised: 23 November 2025
Accepted: 28 November 2025
Published: 1 December 2025

Citation: Chen, B.-Y.; Aloui, F.; Khan, M.A.; Choudhary, M.A. Magnetic Curves in Differential Geometry: A Comprehensive Survey. *Mathematics* **2025**, *13*, 3849. https://doi.org/10.3390/math13233849

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

The concept of magnetic lines of force, or magnetic curves, was introduced in the 1830s by English scientist Michael Faraday (1791–1867); his work provided the foundation for the modern understanding of magnetic fields (see [1]). The link between electricity and magnetic fields was first established experimentally a decade before Faraday, when it was shown that a current could produce a magnetic field [2]. This critical finding proved that the two phenomena were closely related. Building on [1], the author in [3] gave a rigorous mathematical framework to the concept of electromagnetic fields in the 1860s. The famous Maxwell equations unified the theories of electricity and magnetism, confirming Faraday's ideas.

While magnetic curves have their roots in classical physics, their exploration within the realm of differential geometry necessitates transcending Euclidean space. Flat geometries are insufficient to fully comprehend many phenomena related to the motion of charged particles. The curvature of the surrounding manifold introduces novel behaviors absent in $\mathbb{E}^n$. Thus, incorporating magnetic curves into complex geometric contexts, such as Riemannian, hyperbolic, and contact-type manifolds, is crucial to addressing these enriched geometric and physical dynamics.

In differential geometry, we assume (M, g) is a complete Riemannian manifold equipped with a closed 2-form F, termed a magnetic field. Then the Lorentz force associated with the magnetic structure (M, g, F) is depicted as a skew-symmetric tensor field Φ of type $(1, 1)$ satisfying

$$F(X, Y) = g(\Phi(X), Y), \quad \forall X, Y \in \Gamma(TM).$$

The path of a charged particle subjected to F is depicted by a smooth curve γ on M, meeting the requirements of the subsequent:

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma'),$$

where ∇ is the Levi–Civita connection associated to g. It is important to note that $\operatorname{div}\Phi = 0$. Since Φ is skew-symmetric, the magnetic trajectories are traversed at constant speed. A magnetic curve γ is termed a normal magnetic curve when it has unit speed [4].

Moreover, the selection of the frame for analyzing magnetic curves is more than just a technical consideration. The classical Frenet frame becomes indeterminate when curvature is zero, and often, this frame does not align with the surrounding structure in many geometric contexts. Consequently, alternative frames such as the Bishop frame or those adapted to specific structures in contact, almost contact, or Kähler geometries are essential to deriving meaningful and comprehensive results. Addressing this requirement from the outset helps the reader understand why various frameworks are employed in the literature.

The investigation of a magnetic force F and its associated magnetic curve on M, originally introduced in the realm of physics, has since evolved into a rich area of geometric study. The exploration of magnetic curves in a Riemannian manifold gained significant momentum in the early 1990s, although foundational contributions appeared much earlier (see Arnold's works [5,6]). Notably, Arnold posed several problems related to the behavior of charged particles within a Riemannian manifold of arbitrary dimension, which were later elaborated upon by Ginzburg in [7] and others.

The geodesic flow in a Riemannian manifold (M, g) describes the paths that minimize the action within a specific physical system residing in the manifold. These paths are characterized by second-order nonlinear differential equations, typically manifesting as Euler–Lagrange equations of motion. Magnetic curves extend the concept of geodesics. In the context of physics, such curves depict the paths taken by charged particles as they navigate the manifold, influenced by the magnetic force F.

Even with notable advancements, several issues persist within the theory of magnetic curves. These challenges include classification problems in manifolds of higher dimensions, comprehending magnetic paths in spaces with non-standard geometric structures, and broadening established findings to contexts where the surrounding manifold exhibits torsion, indefinite metrics, or distinct curvature characteristics. Emphasizing these challenges underscores the extensive scope of research covered in this review.

In recent years, phenomena associated with magnetic fields have garnered considerable attention in applied mathematical modeling, particularly within fluid dynamics and thermal transport. Research involving metachronal propulsion in non-Newtonian fluids, magnetohydrodynamic flow in advanced functional spaces, and heat and mass transfer in tri-layered systems influenced by magnetic and Marangoni effects illustrates the extensive impact of magnetic fields across various scientific fields. While these topics extend beyond the realm of differential geometry, they highlight the crucial role that magnetic effects play in both theoretical and applied settings, thereby encouraging further geometric exploration of magnetic curves.

A commonly studied example of a magnetic force F on a Riemannian 2-manifold involves scaling the surface's volume form by a constant factor q, often interpreted as the field's strength. The nature of magnetic curves under such a field depends on the curvature of the surface as follows:

- On $\mathbb{S}^2(c)$ with constant Gaussian curvature $c > 0$, magnetic curves follow small Euclidean circles. Each of these circles has a radius given by $(q^2 + c)^{-1/2}$.
- On the Euclidean plane $\mathbb{E}^2$, a magnetic curve is a perfect circle, and the motion is periodic with a period of $\frac{2\pi}{q}$.
- On the hyperbolic plane $\mathbb{H}^2(-c)$ with constant negative Gaussian curvature $-c < 0$, the behavior depends on the relation between $|q|$ and $\sqrt{c}$, namely,
  - If $|q| > \sqrt{c}$, magnetic curves are closed.
  - If $|q| = \sqrt{c}$, the normal trajectories are horocycles.
  - For $|q| < \sqrt{c}$, the paths are non-closed and extend infinitely.

For further details, see reference [8,9]. This research field has also been extended to various types of ambient spaces. In particular, within a complex space form, research focuses on Kählerian magnetic field (see [10]). For these fields, explicit magnetic trajectories have been determined in $\mathbb{CP}^n$ [11]. The Kählerian magnetic field is pivotal in both theoretical and mathematical physics.

The concept of magnetic curves is a fascinating topic in differential geometry, and many authors, inspired by this, have studied it in different ambient spaces. For instance, Romaniuc and Munteanu determined magnetic curves for the Killing vector field in a Euclidean 3-space, as detailed in [12]. The methods about N- and B-magnetic curves and novel approaches to analyzing magnetic flows tied to the Killing F in an oriented Riemannian 3-manifold are examined in [13]. Additionally, Ref. [14] discusses the N- and B-magnetic curves in 3-dimensional semi-Riemannian contexts. The concepts of T-, N_1-, and N_2-type magnetic curves concerning the Bishop frame within a Euclidean 3-space were introduced by Kazan and Karadağ in [15]. The authors in [16] investigated space-like and time-like magnetic curves within a para-Kähler manifold. The properties of magnetic curves in nearly Kähler 4-manifolds and characterizations of Kähler magnetic curves concerning Kähler forms' symplectic pairs are covered in [17].

Many researchers have studied these trajectories in a Sasakian manifold. Magnetic curves about the contact magnetic field in this manifold were investigated in [18] for arbitrary dimension, and magnetic curves in 3-dimensional quasi-para-Sasakian geometry were investigated in [19]. Further studies have been conducted on the magnetic curves in various settings (see, e.g., [20–23]).

Alternatively, the magnetic curves in contact magnetic force in cosymplectic manifolds were obtained in [24] for arbitrary dimension, and Erjavec and Inoguchi investigated the magnetic curves for the almost cosymplectic structure in [25]. The exploration of the magnetic curve associated with a Killing vector field in Galilean space, a concept often used in the context of classical mechanics, was conducted by Aydin in [4]. These trajectories were also studied in C-manifolds and Kenmotsu manifolds in [26,27], respectively. Lately, magnetic curves have been the focus of many studies across various ambient spaces (for example, refer to [28–30]). Moreover, a Killing magnetic curve is a specific class of magnetic curves generated by a Killing vector field, which is distinguished by its property of preserving the metric of the ambient manifold; this symmetry property significantly influences the nature of magnetic curves. In this context, Killing magnetic curves are not only geometrically significant but also exhibit profound physical implications, especially in spaces with high degrees of symmetry. The interaction of the Killing vector field with the magnetic force induces a flow that is often simpler and more structured, which leads to these

curves being integrable in many cases. The studies reviewed on Killing magnetic curves have focused on their behavior across distinct ambient settings, such as the product space $\mathbb{S}^2 \times \mathbb{R}$, Minkowski 3-space, and also almost paracontact 3-manifolds. In the comprehensive study in [31], the authors explored the intricate relationship between magnetic trajectories and the Killing vector field on $\mathbb{S}^2 \times \mathbb{R}$. Through meticulous analysis, they provided a complete classification of these magnetic curves, offering invaluable insights into their geometric properties and behavior. In the seminal work in [32], the authors made an important contribution to the field of magnetic curves by providing a comprehensive classification of magnetic curves in Minkowski 3-space generated by the Killing magnetic field. Calvaruso et al., in their work [33], conducted a comprehensive study of magnetic curves in 3-dimensional normal paracontact metric structures endowed with a Killing vector field, providing a complete classification. In [34], Erjavec extended this research field to the realm of $SL(2, \mathbb{R})$ geometry, further enriching our understanding of magnetic curves in various geometric settings.

Another fascinating category of magnetic curves is slant magnetic curves, which are characterized by the fact that they form a constant angle with a particular structure in the ambient space. The angle between magnetic curves and a chosen vector field or geometric structure has been extensively studied in S-manifolds and Sasakian Lorentzian 3-manifolds. Guvencc et al. [35] conducted a comprehensive investigation of slant normal magnetic curves within the framework of S-manifolds. Their research culminated in a significant result that classifies any γ as a slant normal magnetic curve in an S-manifold iff it adheres to a specific set of slant φ-curves that satisfy well-defined curvature equations. Lee [36] furthered the understanding of magnetic curves by exploring both slant magnetic curves and contact magnetic curves. The work expanded the scope of magnetic curve studies to encompass these particular geometric settings, providing valuable insights into their unique properties and characteristics.

The purpose of this review is, in part, to assist newcomers in navigating the swiftly expanding body of work on magnetic curves. By focusing on both traditional and novel directions, we aim to emphasize the ongoing influence of the interplay of geometry, physics, and frame theory in shaping contemporary research. This viewpoint is especially beneficial for researchers new to the field, as it elucidates why various geometric frameworks and methods naturally develop.

This review paper presents a comprehensive synthesis of existing research on magnetic curves across a diverse spectrum of manifolds. Our focus extends to both general properties and specific types of magnetic curves, including Killing and slant magnetic curves. By drawing from a broad range of studies, we aim to propose a clear and concise overview of the current state of the field, highlighting key advancements, unresolved questions, and promising avenues for future investigation. This paper is structured to initially delve into magnetic curves within classical geometries, such as semi-Riemannian and para-Kähler manifolds. Subsequently, we transition to more specialized spaces, including Sasakian and Kenmotsu manifolds, as well as ambient spaces with unique geometric structures, like the Heisenberg group and anti-de Sitter space. Through this review, we seek to provide a valuable resource for researchers seeking to navigate the intricacies of magnetic curves in various geometric contexts. By identifying knowledge gaps and offering insights from multiple perspectives, this paper contributes to the ongoing development of the field.

2. Preliminaries

2.1. Magnetic Fields and Magnetic Curves

Consider a Riemannian manifold (M, g) equipped with a metric g and a closed 2-form F. The $(1,1)$-tensor Φ is described as the Lorentz force related to the magnetic field F, defined by

$$F(X,Y) = g(\Phi X, Y), \quad \forall X, Y \in \mathfrak{X}(M).$$

Here F is termed the *magnetic field* (cf. [8,37,38]). Consider a smooth curve $\gamma : I \to M$. If γ parametrized by t meets

$$\nabla_{\gamma'(t)}\gamma'(t) = \Phi(\gamma'(t)), \tag{1}$$

then γ is described as a magnetic curve ($\mathcal{MC}$) of F. Equation (1) can be viewed as an extension of the geodesic equation. In particular, if $F = 0$, then the trajectories of the particles are geodesics. Thus, $\mathcal{MC}$s can be seen as a generalization of geodesics [18].

Given that the Lorentz force exhibits skew symmetry, one gets

$$\frac{d}{dt}g(\gamma', \gamma') = 2g(\nabla'_{\gamma}\gamma', \gamma') = 0,$$

Thus, the $\mathcal{MC}$ maintains a constant velocity $v(t) = \|\gamma'\| = v_0$. If $\gamma(t)$ is parameterized by arc-length function s, γ is referred to as a normal $\mathcal{MC}$ [12].

2.2. N-Magnetic Curves

Let $\gamma : I \subset \mathbb{R} \to M^3$ within (M^3, g). Then γ is an N-$\mathcal{MC}$ if its normal vector field N fulfills

$$\nabla_{\gamma'}N = \Phi(N) = V \times N,$$

where V stands for the magnetic vector field (see [13]).

2.3. B-Magnetic Curves

Let $\gamma : I \subset R \to M^3$ in (M^3, g). Then γ is a B-$\mathcal{MC}$ if its binormal vector field B fulfills (cf. [13])

$$\nabla_{\gamma'}B = \Phi(B) = V \times B.$$

According to [15], the B-$\mathcal{MC}$ is defined via a type-2 Bishop frame. Consider $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ in a 3-dimensional Euclidean setting, equipped with this frame, and let $F_V \in \mathbb{R}^3$. If B is associated with the frame and satisfies

$$\nabla_{\gamma'}B = \Phi(B) = V \times B,$$

γ is regarded as a B-$\mathcal{MC}$ relative to it.

2.4. Bishop Frame

The dynamic Frenet frame associated with a unit-speed curve γ is composed of three notably orthogonal vector fields: T, N, and B. These fields change along γ according to

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where

$$g(N,N) = g(B,B) = g(T,T) = 1,$$

$$g(B,T) = g(N,B) = g(T,N) = 0,$$

with κ and τ indicating the curvature and torsion of γ [39].

In 1975, Bishop [40] introduced a different moving frame, which is known as the Bishop frame. In $\mathbb{E}^3$, the Bishop frame is constructed by parallel-transporting an orthonormal frame along γ. Unlike the Frenet frame, the Bishop frame remains well-defined even when κ vanishes. It is characterized by having zero second curvature, and it is given by

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

where $\{N_1, N_2\}$ serves as any suitable, chosen basis for the rest of the frame. The set $\{T, N_1, N_2\}$ is referred to as the Bishop trihedra, while κ_1 and κ_2 are called the Bishop curvatures of γ.

The relationship between the Frenet frame and the Bishop frame is

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(t) & \sin\theta(t) \\ 0 & -\sin\theta(t) & \cos\theta(t) \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

where

$$\theta(t) = \arctan(\frac{\kappa_2}{\kappa_1}), \quad \tau(t) = \theta'(t) \text{ and}$$

$$\kappa(t) = \sqrt{(\kappa_1)^2 + (\kappa_2)^2}, \quad \kappa_1 = \kappa\cos\theta(t), \quad \kappa_2 = \kappa\sin\theta(t).$$

The type-2 Bishop frame is another adaptation that is substantially parallel and is described by

$$\begin{bmatrix} \xi_1' \\ \xi_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\varepsilon_1 \\ 0 & 0 & -\varepsilon_2 \\ \varepsilon_1 & \varepsilon_2 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix}.$$

The association between the Frenet frame and the type-2 Bishop frame is stated as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin\theta(t) & -\cos\theta(t) & 0 \\ \cos\theta(t) & \sin\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix},$$

where

$$\theta(t) = \arctan\frac{\varepsilon_2}{\varepsilon_1}, \quad \kappa(t) = \theta'(t), \quad \tau = \sqrt{(\varepsilon_1)^2 + (\varepsilon_2)^2}.$$

$\{\xi_1, \xi_2, B, \varepsilon_1, \varepsilon_2\}$ indicates the type-2 Bishop frame of $\gamma = \gamma(t)$, with the type-2 Bishop curvatures being given by $\varepsilon_1(t) = -\tau\cos\theta(t)$ and $\varepsilon_2(t) = -\tau\sin\theta(t)$. We refer the reader to [40] for further details.

2.5. Magnetic Curve with Respect to the Bishop Frame

Consider $\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ equipped with either the standard or type-2 Bishop frame and $F_V \in \mathbb{R}^3$; $\Phi(X) = V \times X$ denotes the Lorentz force map induced by a fixed V. The magnetic nature of γ is characterized by the vector of the frame that fulfills $\nabla_{\gamma'} X = \Phi(X)$. Various cases are described below (see [15]).

2.5.1. T-Magnetic Curves

If T satisfies $\nabla_{\gamma'} T = V \times T$, then γ is a T-$\mathcal{MC}$ in accordance with the Bishop frame.

2.5.2. N_1-Magnetic Curves

If N_1 satisfies $\nabla_{\gamma'} N_1 = V \times N_1$, then γ is referred to as an N_1-$\mathcal{MC}$ according to the Bishop frame.

2.5.3. N_2-Magnetic Curves

If N_2 satisfies $\nabla_{\gamma'} N_2 = V \times N_2$, then γ is an N_2-$\mathcal{MC}$ in the sense of the Bishop frame.

2.5.4. ξ_1-Magnetic Curves

In the instance of the type-2 Bishop framework, if ξ_1 satisfies $\nabla_{\gamma'} \xi_1 = V \times \xi_1$, then γ is called a ξ_1-$\mathcal{MC}$.

2.5.5. ξ_2-Magnetic Curves

If ξ_2 satisfies $\nabla_{\gamma'} \xi_2 = V \times \xi_2$, then γ is known as a ξ_2-$\mathcal{MC}$.

2.6. Conformable Curves in Euclidean 3-Space

Consider $\gamma = \gamma(s)$; if $\gamma \in (0, \infty) \to \mathbb{R}^3$ is an α-differentiable curve with $0 < \alpha < 1$, then γ is a conformable curve in a Euclidean 3-space [41].

Definition 1 ([42]). *Assume that $\gamma = \gamma(s)$ is a regular unit-speed conformable curve in (M^3, g). Furthermore, assume that $T = D_\alpha(\gamma)(s) s^{\alpha - 1}$, $N = D_\alpha(T)(s)/\|D_\alpha(T)(s)\|$, and $B = T \times N$. Let $\{T, N, B\}$ be the conformable Frenet frame of $\gamma(s)$. Then*

$$\begin{pmatrix} D_\alpha(T)(s) \\ D_\alpha(N)(s) \\ D_\alpha(B)(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\alpha(s) & 0 \\ -\kappa_\alpha(s) & 0 & \tau_\alpha(s) \\ 0 & -\tau_\alpha(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix},$$

where

$$\kappa_\alpha(s) = \|D_\alpha(T)(s)\|,$$

and

$$\tau_\alpha(s) = \langle D_\alpha(N)(s), B(s) \rangle.$$

2.7. Contact Magnetic Fields and Normal Magnetic Curves

Consider an almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$, where φ functions as an endomorphism on the tangent bundle, ξ denotes a vector field, and η is a 1-form. The fundamental 2-form Ω is determined by $\Omega(X, Y) = g(\varphi X, Y)$. When M^{2n+1} is recognized as a cosymplectic manifold, Ω remains closed, enabling the development of F expressed as

$$F_q(X, Y) = q\Omega(X, Y).$$

This F_q is termed the *contact magnetic force of strength q*.

$\nabla \varphi = 0$, F_q is uniform. Note that using the term "contact" for the magnetic force does not lead to confusion, since this framework pertains to a cosymplectic manifold, which drastically differs from a contact manifold. The expression for the Lorentz force can be derived by merging the relations $\Omega(X, Y) = g(\varphi X, Y)$ and $F(X, Y) = g(\Phi X, Y)$, with $X, Y \in X(M)$, specifically $\Phi q = q\varphi$, leading to

$$\nabla_{\gamma'} \gamma' = q\varphi\gamma', \tag{2}$$

where $\gamma : I \subseteq R \to M^{2n+1}$. The solution of (2) is a normal $\mathcal{MC}$ for the case of a strength contact magnetic force F_q [24].

2.8. Pseudo-Hermitian Magnetic Curves

Consider a Riemannian n-manifold $(M^n, \varphi, \xi, \eta, g)$, and let $\gamma : I \to M$ be parameterized by arc length. Assume that there exist $\{T = \dot{\alpha}, v_1, \ldots, v_{r-1}\}$ on γ such that

$$\begin{aligned}
&\nabla_T T = \kappa_1 v_1,\\
&\nabla_T v_1 = -\kappa_1 T + \kappa_2 v_2,\\
&\nabla_T v_j = -\kappa_j v_{j-1} + \kappa_{j+1} v_{j+1} \text{ for } j = 2, \ldots, r-2,\\
&\nabla_T v_{r-1} = -\kappa_{r-1} v_{r-2}.
\end{aligned}$$

Then γ is a Frenet curve with respect to $\widehat{\nabla}$, and the osculating order of γ is r, where $1 \leq r \leq n$. The functions $\widehat{\kappa}_1, \ldots, \widehat{\kappa}_{r-1}$ are known as the pseudo-Hermitian curvature of γ and are strictly positive on the interval I.

When $r = 1$, γ is a geodesic for $\widehat{\nabla}$, referred to as a pseudo-Hermitian geodesic.

If $r = 2$ and $\widehat{\kappa}_1$ is constant, then we call γ a pseudo-Hermitian circle.

For $r \geq 3$, if all $\widehat{\kappa}_1, \ldots, \widehat{\kappa}_{r-1}$ are positive constants, then γ is a pseudo-Hermitian helix of order r. By convention, the term "pseudo-Hermitian helix" usually refers to the case when $r = 3$ [43].

2.9. Biharmonic, f-Harmonic, and f-Biharmonic Maps

We briefly review the fundamental definitions and variational characterizations of harmonic, biharmonic, f-harmonic, and f-biharmonic maps (see [44] for general references).

Suppose that $\psi : (M, g) \to (\bar{M}, \bar{g})$ is a map between Riemannian manifolds. Then the energy functional associated with ψ is specified by

$$\mathfrak{E}_1(\psi) = \frac{1}{2} \int_M |d\psi|^2 \, v_g,$$

where v_g is the volume form on M. ψ is called harmonic [45] if the tension field $\tau(\psi)$ is null, i.e.,

$$\tau(\psi) := \operatorname{trace} \nabla d\psi = 0.$$

Note that biharmonic maps arise as the critical points of the bienergy functional and are described as

$$\mathfrak{E}_2(\psi) = \frac{1}{2} \int_M |\tau(\psi)|^2 \, v_g.$$

It is well-known that a map ψ is biharmonic [44,46] if its bitension field satisfies

$$\tau_2(\psi) = \operatorname{trace}\Big(\nabla^\psi \nabla^\psi - \nabla^\psi_{\nabla}\Big)\tau(\psi) - \operatorname{trace}\Big(R^{\bar{M}}(d\psi, \tau(\psi))d\psi\Big) = 0,$$

where $R^{\bar{M}}$ denotes the Riemann curvature tensor of $\bar{M}$ and ∇^ψ is the pullback connection. $R^{\bar{M}}$ is stated as

$$R^{\bar{M}}(\bar{X}, \bar{Y})\bar{Z} = \nabla^{\bar{M}}_{\bar{X}} \nabla^{\bar{M}}_{\bar{Y}} \bar{Z} - \nabla^{\bar{M}}_{\bar{Y}} \nabla^{\bar{M}}_{\bar{X}} \bar{Z} - \nabla^{\bar{M}}_{[\bar{X}, \bar{Y}]} \bar{Z}, \quad \forall \bar{X}, \bar{Y}, \bar{Z} \in \Gamma(T\bar{M}).$$

Notice that while every harmonic map is automatically biharmonic, the converse is not generally true (see, e.g., [44,46,47]).

An extension of harmonic maps, known as f-harmonic maps, is defined via the f-energy functional given by

$$\mathfrak{E}_f(\psi) = \frac{1}{2} \int_M f |d\psi|^2 \, v_g, \quad f \in C^\infty(M, \mathbb{R}).$$

ψ is called f-harmonic if we have

$$\tau_f(\psi) = f\tau(\psi) + d\psi(\operatorname{grad} f) = 0,$$

where $\tau_f(\psi)$ represents the f-tension field.

Similarly, f-biharmonic maps are defined as the critical points of the f-bienergy functional:

$$\mathfrak{E}_{2,f}(\psi) = \frac{1}{2}\int_M f|\tau(\psi)|^2 v_g.$$

A map ψ is f-biharmonic if

$$\tau_{2,f}(\psi) = f\tau_2(\psi) + (\Delta f)\tau(\psi) + 2\nabla^{\psi}_{\operatorname{grad} f}\tau(\psi) = 0,$$

where $\tau_{2,f}(\psi)$ is known as the f-bitension field.

Definition 2 ([22]). *A unit-speed magnetic curve $\gamma(t)$ in a quasi-Sasakian 3-manifold is considered biharmonic when it meets*

$$\nabla^{3\prime}_{\gamma}\gamma' + R(\nabla'_{\gamma}\gamma', \gamma')\gamma' = 0.$$

2.10. Hopf Magnetic Curves in Anti-De Sitter 3-Space $\mathbb{H}^3_1$

Consider

$$\gamma(s) = (x_0(s), x_1(s), x_2(s), x_3(s)) \subset \mathbb{H}^3_1$$

to be a unit-speed curve in $\mathbb{H}^3_1$. For a $\mathcal{KVF}$ ξ on $\mathbb{H}^3_1$, there exists a magnetic curve γ in $\mathbb{H}^3_1$ associated with ξ given by a solution of

$$\nabla'_{\gamma}\gamma' = q\xi \times \gamma', \tag{3}$$

where $q \neq 0$. Such a magnetic curve γ is called a *Hopf magnetic curve* in $\mathbb{H}^3_1$, as discussed in [48].

Consider a frame $\{\xi, U, V\}$ on $\mathbb{H}^3_1$ such that ξ is time-like and U and V are space-like. Then, there are smooth functions T_1, T_2, and T_3 dependent on s such that $\gamma' = T_1\xi + T_2U + T_3V$. Note that T_1 is a constant real value. Indeed, we have

$$\begin{array}{lll} \nabla_{\xi}\xi = 0, & \nabla_U\xi = V, & \nabla_V\xi = -U, \\ \nabla_{\xi}U = -V, & \nabla_U U = 0, & \nabla_V U = -\xi, \\ \nabla_{\xi}V = U, & \nabla_U V = \xi, & \nabla_V V = 0. \end{array} \tag{4}$$

Using (3) and (4), we have

$$T_1 = \frac{d}{dt}g(\gamma', \xi) = g(q\xi \times \gamma', \xi) + g(\gamma', \nabla_{\gamma'}\xi) = 0.$$

Be aware that $g(\gamma', \gamma')$ remains constant along γ. As a result, the unconstrained nature of γ is maintained at every point and relies solely on the starting value of γ'. γ is light-like iff we have $0 = \|\gamma'\|^2 = -T_1^2 + T_2^2 + T_3^2$ (see [48]).

2.11. A Historic Remark on Slant Curves by Inoguchi and Munteanu

Consider a unit-speed curve $\gamma(s)$ within a Sasakian manifold $(M, \varphi, \xi, \eta, g)$. The contact angle $\theta(s)$ of $\gamma(s)$ is specified by $\cos\theta(s) = g(T(s)\xi)$, where $T(s)$ represents the unit tangent vector of $\gamma(s)$. γ in M is referred to as a *slant curve* if its $\theta(s)$ remains constant.

The history of "slant curves" was written by Inoguchi and Munteanu, published [49]. The following is a shortened version of the history of slant curves in [49].

Chen proposed, in 1990, the idea of slant submanifolds in almost Hermitian manifolds [50] (see also [51,52]). Later, in 1996, Lotta extended Chen's idea to define a slant submanifold in an almost contact metric manifold [53] (see also [54]).

Suppose that $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold. A submanifold N of M is called *slant* if for any nonzero vector $X \in T_xN$, linearly independent of ξ_x, the angle between φX and T_xM is a constant $\theta \in [0, \frac{\pi}{2}]$, and it is also independent of $x \in N$, which is called the *slant angle* of N. A slant submanifold that is neither invariant nor anti-invariant is called a *proper slant submanifold*. Lotta's definition omits one-dimensional submanifolds, i.e., slant curves. If the goal is to examine curves in an almost contact metric manifold with the slant property, it is necessary to adjust Lotta's definition. For this, we revisit the initial inspiration for slant submanifold geometry. A key motivation in Chen's research is to propose a new category of submanifolds that includes both complex submanifolds and totally real submanifolds as extreme instances.

In the context of curves within an almost contact metric manifold, the term "invariant one-dimensional curve" refers specifically to a characteristic flow. More precisely, a curve is considered invariant iff it serves as an *integral curve* of the characteristic vector field $\pm\xi$. It is important to observe that every regular curve in M is actually an anti-invariant curve.

Conversely, in the realm of almost contact metric geometry, *C-totally real submanifolds*, which are submanifolds perpendicular to ξ, have been examined as analogous to totally real submanifolds. Specifically, C-totally real curves are termed *almost contact curves*. Within three-dimensional contact geometry, these almost contact curves are commonly referred to as *Legendre curves*. To extend the concept of slant submanifolds to curves in an almost contact metric manifold, Cho, Inoguchi, and Lee introduced an alternative notion known as slant curves (slant curve) in [55]. After [55], slant curves in almost contact metric manifolds were explored intensively (see [49,56] for details).

3. Magnetic Curves in Riemannian Manifolds

Let F be a magnetic field in a Riemannian manifold (M, g) and Φ the *Lorentz force* associated to F so that we have $F(X, Y) = g(\Phi X, Y)$ (cf., e.g., [8,37,38]).

Consider a Killing vector field V and define the 2-form

$$F_V = \iota_V dv_g,$$

where ι_V denotes the interior (contraction) operator. The 2-form F_V is the magnetic field associated with V. The Lorentz force corresponding to F_V is the $(1,1)$-tensor Φ_V determined by

$$F_V(X, Y) = g(\Phi_V X, Y), \quad \forall\, X, Y \in TM.$$

In dimension three, the Lorentz force can be written using the cross product as

$$\Phi_V(X) = V \times X.$$

A magnetic curve γ associated with F_V satisfies

$$\nabla_{\gamma'}\gamma' = \Phi_V(\gamma').$$

Such a curve is called a *Killing magnetic curve* (see, e.g., [4,13]).

In the realm of physics, F_V denotes the magnetic 2-form produced by the flow of the Killing field V. On the other hand, Φ_V refers to the Lorentz force exerted on a charged particle as it moves through the associated electric and magnetic fields.

3.1. Magnetic Curves in Three-Dimensional Space Forms

In [13], Bozkurt et al. investigated the trajectories of a Killing magnetic field F_V, namely, N- and B-$\mathcal{MC}$s. They also provided a variational method to study $\mathcal{MC}$s related to a Killing magnetic field defined in an oriented Riemannian manifold (M^3, g).

Proposition 1 ([13]). *If γ is a unit-speed N-$\mathcal{MC}$ in an oriented Riemannian 3-manifold with Frenet frame (T, N, B, κ, τ), then the Frenet frame is expressed by*

$$\begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

And the Lorentz force in the Frenet frame takes the form

$$\begin{bmatrix} \Phi(T) \\ \Phi(N) \\ \Phi(B) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & \Omega_1 \\ -\kappa & 0 & \tau \\ -\Omega_1 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where Ω_1 is a function.

Theorem 1 ([13]). *Consider a $\mathcal{KVF}$ V defined on a 3-dimensional simply connected space form $\left(M^3(c), g\right)$ of constant curvature c. Then, a unit-speed N-magnetic trajectory is a curve that satisfies*

$$\left[\frac{1}{\tau}\Omega_1''\right]' - \Omega_1\tau + \frac{\kappa^2}{\tau}\Omega_1'' + ck = 0, \quad \text{and} \quad V = \tau T - \Omega_1 N + \kappa B,$$

where

$$\Omega_1 = -\frac{\tau'}{\kappa}.$$

Theorem 2 ([13]). *Under the hypothesis of Theorem 1, κ and τ of every unit-speed B-$\mathcal{MC}$ of V satisfy*

$$\tau^2(\kappa - a) + 2C\kappa + \kappa'' + \kappa^2 k' = 0, \tau = \text{ constant},$$

where Ω_2 is a certain function satisfying

$$V = \tau T + \Omega_2 B,$$

or

$$V = \tau T + \left(\frac{\kappa}{2} + a\right) B.$$

3.1.1. Magnetic Curves in Euclidean 3-Space $\mathbb{E}^3$

The notions of T-, N_1-, and N_2-$\mathcal{MC}$s in $\mathbb{E}^3$ associated with the Bishop frame were introduced by Kazan and Karadag in [15].

Proposition 2 ([15]). *In a Euclidean 3-space $\mathbb{E}^3$, if γ is a unit-speed T-, N_1-, or N_2-$\mathcal{MC}$, then the corresponding Lorentz force, as specified by the Bishop frame, is given by*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N_1) \\ \Phi(N_2) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \rho \\ -\kappa_2 & -\Omega_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

$$\begin{bmatrix} \Phi(T) \\ \Phi(N_1) \\ \Phi(N_2) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \Omega_3 \\ -\kappa_1 & 0 & 0 \\ -\Omega_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

or

$$\begin{bmatrix} \Phi(T) \\ \Phi(N_1) \\ \Phi(N_2) \end{bmatrix} = \begin{bmatrix} 0 & \Omega_4 & \kappa_2 \\ -\Omega_4 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

respectively, with $\Omega_2 = g(\Phi N_1, N_2)$, $\Omega_3 = g(\Phi T, N_2)$, *and* $\Omega_4 = g(\Phi T, N_1)$.

In [12], Druță-Romaniuc and Munteanu studied magnetic fields in $\mathbb{E}^3$ and their corresponding magnetic trajectories determined by $\mathcal{KVF}$s. In particular, they identified all the $\mathcal{MC}$s related to a Killing magnetic field F_V in a Euclidean 3-space $\mathbb{E}^3$. They obtained the following.

Theorem 3 ([12]). *The Killing* $\mathcal{MC}$*s of a Killing magnetic field* F_V *in* $\mathbb{E}^3$ *consist of the following curves:*

(a) *Planar curves located within a vertical strip.*
(b) *Circular helices.*
(c) *Curves defined by the following parametric equations:*

$$x(t) = \rho(t)\cos\phi(t), \quad y(t) = \rho(t)\sin\phi(t), \quad z(t) = -\frac{1}{2}\int^t \rho^2(\zeta)d\zeta,$$

where ρ *denotes the radial distance from the z-axis in the xy-plane,* ϕ *denotes the angular position in the xy-plane, and* ζ *is a dummy variable for the integral. The functions* ρ *and* ϕ *fulfill*

$$\left(\frac{d\rho^2}{dt}\right)^2 + P\left(\rho^2(t)\right) = 0, \quad \rho^2(t)\phi'(t) = \text{ constant },$$

where P is a polynomial of degree 3.

Remark 1. *In* [57], *Munteanu and Nistor analyzed magnetic fields on the 3-torus derived from two distinct contact forms on* $\mathbb{E}^3$. *They investigated the conditions under which their associated normal* $\mathcal{MC}$*s are closed. Their findings demonstrated that the periodicity conditions are linked to a set of rational numbers. Furthermore, they illustrated that the spectrum of closed geodesics on the torus is comparable to the quantization of certain energy levels in atomic models.*

3.1.2. Magnetic Curves in Hyperbolic 3-Space $\mathbb{H}^3$

In [58], Killing $\mathcal{MC}$s in the hyperbolic 3-space $\mathbb{H}^3$ are explored, and homogeneous $\mathcal{MC}$s and Killing $\mathcal{MC}$s in $\mathbb{H}^3$ are classified. Moreover, the Lorentz equations for the basic $\mathcal{KVF}$s of $\mathbb{H}^3$ are exhibited, and some particular solutions to those Lorentz equations are given.

In [59] Kelekçi et al. computed the $\mathcal{MC}$s induced by $\mathcal{KVF}$s of $\mathbb{H}^3$, which are magnetic vector fields. To study the Lorentz equation and to find solutions, Kelekçi et al. applied a perturbation method up to the first order (for some constants), and in such a way, they could control their strength.

3.2. Magnetic Curves According to Bishop Frame

In [15], Kazan and Karadağ studied the magnetic vector field V considering an $\mathcal{MC}$ that is a T-, N_1-, or N_2-$\mathcal{MC}$ of V with respect to the Bishop frame and also a ξ_1-, ξ_2-, or B-$\mathcal{MC}$ of V with respect to Bishop frame and type-2 Bishop frame.

Theorem 4 ([15]). *Assume that γ is a unit-speed T-$\mathcal{MC}$ in the context of the Bishop frame within a space form $M^3(c)$ having sectional curvature c, and $V \in (M^3(c), g)$. If γ serves as one of the T-magnetic curves in $M^3(c)$, then the following two conditions hold:*

(a) $\Omega_2 = constant$.

(b) *The harmonic curvature function of γ concerning the Bishop frame is*

$$H = \frac{\Omega_2\kappa_2' + \kappa_1''}{\kappa_2'' - \rho\kappa_1'},$$

or

$$H = \frac{-\kappa_1\kappa_1'\kappa_2 + \Omega_2\kappa_1'' - \kappa_2''' - \kappa_2^2\kappa_2' + c\Omega_2\kappa_1 - c\kappa_2' - \frac{\kappa_1\kappa_1' + \kappa_2\kappa_2'}{\kappa_1^2+\kappa_2^2}\left(\Omega_2\kappa_1' - \kappa_2'' - c\kappa_2\right)}{\kappa_1\kappa_2\kappa_2' + \Omega_2\kappa_2'' + \kappa_1''' + \kappa_1^2\kappa_1' + c\Omega_2\kappa_2 + c\kappa_1' - \frac{\kappa_1\kappa_1' + \kappa_2\kappa_2'}{\kappa_1^2+\kappa_2^2}\left(\Omega_2\kappa_2' + \kappa_1'' + c\kappa_1\right)}.$$

Theorem 5 ([15]). *Let γ be a unit-speed N_1-$\mathcal{MC}$ specified by the Bishop frame in $M^3(c)$, and let $V \in (M^3(c), g)$. If γ is an N_1-$\mathcal{MC}$ of $(M^3(c), g, V)$, then we have*

(a) $\Omega_3 = \kappa_2$.

(b) *H of γ conforming to the Bishop frame is*

$$H = \frac{\kappa_1''}{\kappa_2''},$$

or

$$H = \frac{-\kappa_1\kappa_1'\kappa_2 - \kappa_2''' - \kappa_2^2\kappa_2' - c\kappa_2' + \frac{\kappa_1\kappa_1' + \kappa_2\kappa_2'}{\kappa_1 2\kappa_2{}^2}\left(\kappa_2'' + c\kappa_2\right)}{\kappa_1\kappa_2\kappa_2' + \kappa_1'''{}^2\kappa_1' + c\kappa_1' - \frac{\kappa_1\kappa_1' + \kappa_2\kappa_2'}{\kappa_1^2+\kappa_2^2}\left(\kappa_1'' + c\kappa_1\right)}.$$

Proposition 3 ([15]). *If γ is a unit-speed N_2-$\mathcal{MC}$ with respect to the Bishop frame in $\mathbb{E}^3$, then γ is an N_2-$\mathcal{MC}$ of a Killing magnetic vector field V iff, along γ, V is given by*

$$V = -\kappa_2 N_1 + \Omega_4 N_2.$$

Theorem 6 ([15]). *Suppose that γ is a unit-speed N_2-$\mathcal{MC}$ according to the Bishop frame in $M^3(c)$ and $V \in (M^3(c), g)$. If γ is an N_2-$\mathcal{MC}$ of $(M^3(c), g, V)$, then we have*

(a) $\Omega_4 = \kappa_1$.

(b) *H of γ as per the Bishop frame is*

$$H = \frac{\kappa_1''}{\kappa_2''},$$

or

$$H = \frac{-\kappa_1\kappa_1'\kappa_2 - \kappa_2''' - \kappa_2^2\kappa_2' - c\kappa_2' + \frac{\kappa_1\kappa_1' + \kappa_2\kappa_2'}{\kappa_1^2+\kappa_2^2}\left(\kappa_2'' + c\kappa_2\right)}{\kappa_1\kappa_2\kappa_2' + \kappa_1''' + \kappa_1^2\kappa_1' + c\kappa_1' - \frac{\kappa_1\kappa_1' + \kappa_2\kappa_2'}{\kappa_1^2+\kappa_2^2}\left(\kappa_1'' + c\kappa_1\right)}.$$

Remark 2. *ξ_1-, ξ_2-, and B-$\mathcal{MC}$s for type-2 Bishop frame are also examined in* [15].

Theorem 7 ([15]). *Consider γ to be parameterized by arc length, a ξ_1-integral curve as per the type-2 Bishop frame in $M^3(c)$, and $V \in (M^3(c), g)$. If γ is an ξ_1-integral curve of $(M^3(c), g, V)$, then*

$$\tan\theta = \frac{\varepsilon_1'}{\Omega_5'},$$

$$\tan\theta = \frac{-\Omega_5''\theta' + \Omega_5\varepsilon_1^2\theta' + \varepsilon_1^2\varepsilon_2\theta' + 2\Omega_5'\varepsilon_1\varepsilon_2 + 2\varepsilon_1'\varepsilon_2^2 + \Omega_5\varepsilon_1'\varepsilon_2 + \varepsilon_1\varepsilon_2\varepsilon_2' - c\Omega_5\theta'}{\varepsilon_1''\theta' - \Omega_5\varepsilon_1\varepsilon_2\theta' - \varepsilon_1\varepsilon_2^2\theta' + 2\Omega_5'\varepsilon_1^2 + 2\varepsilon_1'\varepsilon_1\varepsilon_2 + \Omega_5\varepsilon_1'\varepsilon_1 + \varepsilon_1^2\varepsilon_2' + c\varepsilon_1\theta'},$$

and

$$\theta'\left\{3\Omega_5''\varepsilon_1+3\Omega_5'\varepsilon_1'+3\varepsilon_1''\varepsilon_2+3\varepsilon_1'\varepsilon_2'+\Omega_5\varepsilon_1''+\varepsilon_1\varepsilon_2''-\Omega_5\varepsilon_1^3-\Omega_5\varepsilon_1\varepsilon_2^2-\varepsilon_1^3\varepsilon_2-\varepsilon_1\varepsilon_2^3\right\}$$
$$-\theta''\left\{2\Omega_5'\varepsilon_1+2\varepsilon_1'\varepsilon_2+\Omega_5\varepsilon_1'+\varepsilon_1\varepsilon_2'\right\}+\theta'^3\left\{\Omega_5\varepsilon_1+\varepsilon_1\varepsilon_2\right\}+c\theta'\left\{\Omega_5\varepsilon_1+\varepsilon_1\varepsilon_2\right\}=0,$$

where $\Omega_5=g(\Phi\xi_2,B)$.

Theorem 8 ([15]). *Let γ be a unit-speed ξ_2-integral curve concerning the type-2 Bishop frame in $M^3(c)$ and $V\in(M^3(c),g)$. If γ is an ξ_2-integral curve of $(M^3(c),g,V)$, then*

$$\tan\theta=\frac{\Omega_6'}{\varepsilon_2'},$$
$$\tan\theta=\frac{\varepsilon_2''\theta'-\varepsilon_1^2\varepsilon_2\theta'-\Omega_6\varepsilon_1\varepsilon_2\theta'-2\varepsilon_1\varepsilon_2\varepsilon_2'-2\Omega_6'\varepsilon_2^2-\varepsilon_1'\varepsilon_2^2-\Omega_6\varepsilon_2\varepsilon_2'+c\varepsilon_2\theta'}{-\Omega_6''\theta'+\varepsilon_1\varepsilon_2^2\theta'+\Omega_6\varepsilon_2^2\theta'-2\varepsilon_1^2\varepsilon_2'-2\Omega_6'\varepsilon_1\varepsilon_2-\varepsilon_1'\varepsilon_1\varepsilon_2-\Omega_6\varepsilon_1\varepsilon_2'-c\Omega_6\theta'},$$

and

$$\theta'\left\{3\varepsilon_1\varepsilon_2''-\varepsilon_1^3\varepsilon_2-\Omega_6\varepsilon_1^2\varepsilon_2+3\Omega_6''\varepsilon_2-\varepsilon_1\varepsilon_2^3-\Omega_6\varepsilon_2^3+3\varepsilon_1'\varepsilon_2'+3\Omega_6'\varepsilon_2'+\varepsilon_1''\varepsilon_2+\Omega_6\varepsilon_2''\right\}$$
$$-\theta''\left\{2\varepsilon_1\varepsilon_2'+2\Omega_6'\varepsilon_2+\varepsilon_1'\varepsilon_2+\Omega_6\varepsilon_2'\right\}+\theta'^3\left\{\varepsilon_1\varepsilon_2+\Omega_6\varepsilon_2\right\}+c\theta'\left\{\varepsilon_1\varepsilon_2+\Omega_6\varepsilon_2\right\}=0,$$

where $\Omega_6=g(\Phi\xi_1,B)$.

Theorem 9 ([15]). *Let γ be parameterized by an arc-length B-integral curve depending on the type-2 Bishop frame in M with c and $V\in(M^3(c),g)$. If γ is a B-integral curve of $(M^3(c),g,V)$, then*

$$\tan\theta=\frac{\varepsilon_1'+\Omega_7\varepsilon_2}{-\varepsilon_2'+\Omega_7\varepsilon_1}\times$$
$$\tan\theta=\frac{\begin{array}{c}\left\{\varepsilon_2''\theta'-2\Omega_7'\varepsilon_1\theta'-\Omega_7\varepsilon_1'\theta'-\varepsilon_1\varepsilon_2\varepsilon_2'+\varepsilon_1'\varepsilon_2^2+\Omega_7\varepsilon_1^2\varepsilon_2+\Omega_7\varepsilon_2^3\right\}\\-\Omega_7'''\varepsilon_2+2\theta'^2\varepsilon_1'+2\theta'^2\Omega_7\varepsilon_2+c\varepsilon_2\theta'-c\Omega_7\varepsilon_2\end{array}}{\begin{array}{c}\varepsilon_1''\theta'+2\Omega_7'\varepsilon_2\theta'+\Omega_7\varepsilon_2'\theta'-\varepsilon_1^2\varepsilon_2'+\varepsilon_1\varepsilon_1'\varepsilon_2+\Omega_7\varepsilon_1^3+\Omega_7\varepsilon_1\varepsilon_2^2\\-\Omega_7'''\varepsilon_1-2\theta'^2\varepsilon_2'+2\theta'^2\Omega_7\varepsilon_1+c\varepsilon_1\theta'-c\Omega_7\varepsilon_1\end{array}}$$

and

$$\theta'\left\{2\varepsilon_1\varepsilon_2''-2\varepsilon_1''\varepsilon_2-3\Omega_7'\varepsilon_1^2-3\Omega_7'\varepsilon_2^2-3\Omega_7\varepsilon_1\varepsilon_1'-3\Omega_7\varepsilon_2\varepsilon_2'+\Omega_7'''\right\}$$
$$-\theta''\left\{\varepsilon_1\varepsilon_2'-\varepsilon_1'\varepsilon_2-\Omega_7\varepsilon_1^2-\Omega_7\varepsilon_2^2+\Omega_7''\right\}+\theta'^3\Omega_7'+c\theta'\Omega_7'-c\theta''\Omega_7=0,$$

where $\Omega_7=g(\phi\xi_1,\xi_2)$.

3.3. Fractional T-, N-, and B-Magnetic Curves in Riemannian 3-Manifolds

In [42], Has and Yılmaz conducted a study on the trajectories of magnetic force lines, characterized as *T*-, *N*-, or *B*-integral curves, within the framework of fractional calculus. They analyzed the impact of fractional derivatives and integrals on the integral curve, utilizing the conformable fractional derivative due to its compatibility with the algebraic structure of differential geometry.

Proposition 4 ([42]). *If γ is a unit-speed fractional T-integral curve in (M^3, g) and if F is a magnetic field on M with Frenet frame $(T, N, B, \kappa_\alpha, \tau_\alpha)$, then the Lorentz force, as per the conformable frame, is represented by*

$$\begin{pmatrix} \Phi(t) \\ \Phi(n) \\ \Phi(b) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\alpha(s) & 0 \\ -\kappa_\alpha(s) & 0 & \varrho_1(s) \\ 0 & -\varrho_1(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix},$$

where ϱ_1 is a certain function.

Proposition 5 ([42]). *A unit-speed curve γ in (M^3, g, V) is a fractional T-integral curve of the magnetic field V iff, along γ, V can be expressed as*

$$V = \varrho_1 t + \kappa_\alpha b.$$

Theorem 10 ([42]). *Suppose that γ is a unit-speed fractional T-integral curve and $V \in (M^3, g)$. Then*

$$\varrho_1 = c, \quad c \in R, \quad \left(\kappa_\alpha^2 \left[\frac{1}{2}\varrho_1 - \tau_\alpha\right]\right)' = 0,$$

and

$$\left[\frac{1}{\kappa_\alpha}\left(\varrho_1 \kappa_\alpha \tau_\alpha - \kappa_\alpha \tau_\alpha^2 + (1-\alpha)s^{1-2\alpha}\kappa_\alpha' + s^{2-2\alpha}\kappa_\alpha'' + C\kappa_\alpha\right)\right]' + \kappa_\alpha \kappa_\alpha' = 0.$$

Proposition 6 ([42]). *A unit-speed curve γ in (M^3, g, V) is a fractional N-integral curve of V iff V, along γ, is described as*

$$V = \tau_\alpha t - \varrho_2 n + \kappa_\alpha b.$$

Theorem 11 ([42]). *If γ is a unit-speed fractional N-integral curve and V is a $\mathcal{KVF}$ on (M^3, g), then we have*

$$\begin{aligned} &s^{1-\alpha}\tau_\alpha' + \varrho_2 \kappa_\alpha = 0, \\ &(\alpha-1)s^{1-2\alpha}\varrho_2' - s^{2-2\alpha}\varrho_2'' - s^{1-\alpha}\kappa_\alpha'\tau_\alpha + \varrho_2\tau_\alpha^2 = C\varrho_2, \\ &\left(\frac{1}{\kappa_\alpha}\left[-s^{1-\alpha}\varrho_2'\tau_\alpha + s^{2-2\alpha}\kappa_\alpha'' - s^{1-\alpha}\{\tau_\alpha\varrho_2\}' + \{1-\alpha\}s^{1-2\alpha}\kappa_\alpha'\right]\right)' \\ &\qquad + \kappa_\alpha\left(\kappa_\alpha' - s^{\alpha-1}\tau_\alpha\varrho_2\right) = 0. \end{aligned}$$

Definition 3 ([42]). *Suppose that $\gamma : I \subset R \to M^3$ is a conformable curve in (M^3, g). Provided that the vector area of the tangent trajectory for the conformable frame adheres to the Lorentz force equation, then γ is termed a fractional B-integral curve if it fulfills*

$$\frac{D_\alpha B}{s^{1-\alpha}} = \phi(b) = V \times B.$$

Proposition 7 ([42]). *A unit-speed curve γ in (M^3, g, V) is a fractional B-integral curve of V iff V, along γ, is indicated as*

$$V = \tau_\alpha t + \varrho_3 b.$$

Theorem 12 ([42]). *If γ is a unit-speed fractional B-integral curve in (M^3, g, V), then we have*

$$s^{1-\alpha}\tau_\alpha' = 0,$$
$$\kappa_\alpha'\tau_\alpha - 2\tau_\alpha\varrho_3' = 0,$$
$$\left(\frac{1}{\kappa_\alpha}\left[\kappa_\alpha\tau_\alpha^2 - \varrho_3\tau_\alpha^2 + \{1-\alpha\}s^{1-2\alpha}\varrho_3' + s^{2-2\alpha}\varrho_3''' + C\kappa_\alpha\right]\right)' + s^{1-\alpha}\varrho_3'\kappa_\alpha = 0.$$

3.4. Magnetic Curves in Riemannian 3-Manifolds

In [60], Korpinar and Baş introduced a special kind of $\mathcal{MC}$ related to a given magnetic field F in a Riemannian 3-manifold (M^3, g). More precisely, they obtained the following.

Proposition 8 ([60]). *If Frenet frame $\{T, N, B\}$ of a unit-speed curve γ in (M^3, g) has the binormal indicatrix given by $\gamma_{\mathfrak{b}} = B$, i.e., $\{T_{\mathfrak{b}}, N_{\mathfrak{b}}, B_{\mathfrak{b}}\}$, then we have*

$$\begin{bmatrix} T_{\mathfrak{b}}' \\ N_{\mathfrak{b}}' \\ B_{\mathfrak{b}}' \end{bmatrix} = \begin{bmatrix} 0 & \sigma_{\mathfrak{b}} & 0 \\ -\sigma_{\mathfrak{b}} & 0 & \tau_{\mathfrak{b}} \\ 0 & -\tau_{\mathfrak{b}} & 0 \end{bmatrix} \begin{bmatrix} T_{\mathfrak{b}} \\ N_{\mathfrak{b}} \\ B_{\mathfrak{b}} \end{bmatrix},$$

where

$$T_{\mathfrak{b}} = -N, \quad N_{\mathfrak{b}} = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}T - \frac{\tau}{\sqrt{\kappa^2+\tau^2}}B, \quad B_{\mathfrak{b}} = \frac{-\kappa}{\sqrt{\kappa^2+\tau^2}}T + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}B,$$

and

$$\sigma_{\mathfrak{b}} = \frac{\sqrt{\kappa^2+\tau^2}}{\tau}, \quad \tau_{\mathfrak{b}} = \frac{\kappa\tau' - \kappa'\tau}{\tau(\kappa^2+\tau^2)}.$$

Definition 4 ([60]). *Consider a unit-speed magnetic curve $\gamma_{\mathfrak{b}}$ in (M^3, g), and let $V_{\mathfrak{b}}$ be a magnetic force on M^3. Then $\gamma_{\mathfrak{b}}$ is called a $B_{\mathfrak{b}}$-$\mathcal{MC}$ if $B_{\mathfrak{b}}$ satisfies*

$$\nabla_{\gamma_{\mathfrak{b}}'} B_{\mathfrak{b}} = \Phi(B_{\mathfrak{b}}) = V_{\mathfrak{b}} \times B_{\mathfrak{b}}.$$

Theorem 13 ([60]). *Let $\frac{\partial\gamma}{\partial t}$ be an inextensible flow of a unit-speed curve γ in (M^3, g). Then,*

$$\frac{\partial}{\partial t}\Phi(T_{\mathfrak{b}}) = \left[\frac{\partial}{\partial t}(\kappa\lambda\delta) - \tau^2\lambda\delta\mathfrak{g} + \tau\lambda\delta\frac{\partial\mathfrak{h}}{\partial s}\right]T + \left[\kappa^2\lambda\delta\mathfrak{f} + \kappa\lambda\delta\mathfrak{h}\tau + \kappa\lambda\delta\frac{\partial\mathfrak{g}}{\partial s} + \tau\lambda\delta\chi\right]N + \left[-\kappa\lambda\delta\mathfrak{g}\tau + \kappa\lambda\delta\frac{\partial\mathfrak{h}}{\partial s} - \frac{\partial}{\partial t}(\tau\lambda\delta)\right]B,$$

$$\frac{\partial}{\partial t}\Phi(N_{\mathfrak{b}}) = \left[-\mathfrak{f}\kappa\lambda - \mathfrak{h}\lambda\tau - \lambda\frac{\partial\mathfrak{g}}{\partial s} + \frac{\partial\Gamma}{\partial t} + \Gamma\mathfrak{g}\kappa - \frac{\Gamma\kappa}{\tau}\frac{\partial\mathfrak{h}}{\partial s}\right]T + \left[\frac{\partial\lambda}{\partial t} + \Gamma\left(\mathfrak{f}\kappa + \mathfrak{h}\tau + \frac{\partial\mathfrak{g}}{\partial s} - \frac{\chi\kappa}{\tau}\right)\right]N + \left[\lambda\chi + \Gamma\left(\frac{\partial\mathfrak{h}}{\partial s} - g\tau\right) + \frac{\partial}{\partial t}\left(\frac{\Gamma\kappa}{\tau}\right)\right]B,$$

$$\frac{\partial}{\partial t}\Phi(B_{\mathfrak{b}}) = \left[-\frac{\partial}{\partial t}\left(\frac{\kappa\Gamma}{\tau}\right) + \Gamma\left(\mathfrak{g}\tau - \frac{\partial\mathfrak{h}}{\partial s}\right)\right]T - \left[\left(\frac{\kappa\Gamma}{\tau}\right)\left(\mathfrak{f}\kappa + \mathfrak{h}\tau + \frac{\partial\mathfrak{g}}{\partial s}\right) + \Gamma\chi\right]N + \left[-\left(\frac{\kappa\Gamma}{\tau}\right)\left(\frac{\partial\mathfrak{h}}{\partial s} - g\tau\right) + \frac{\partial\Gamma}{\partial t}\right]B,$$

where $\delta = \frac{1}{\sqrt{\kappa^2+\tau^2}}$, $\Gamma = \frac{\kappa\tau'-\kappa'\tau}{\tau(\kappa^2+\tau^2)^{5/2}}$, $\chi = \left\langle \frac{\partial N}{\partial t}, B \right\rangle$, *and* $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ *are smooth functions of time and arc length.*

Theorem 14 ([60]). *Let* $\frac{\partial \gamma}{\partial t}$ *be an inextensible flow of a unit-speed curve* γ *in* $\left(M^3, g\right)$. *Then the subsequent is valid:*

(a)

$$\begin{aligned}
&\kappa^2\tau\lambda\delta\mathfrak{f} + \kappa\tau^2\lambda\delta\mathfrak{h} + \kappa\tau\lambda\delta\frac{\partial \mathfrak{g}}{\partial s} + \tau^2\lambda\delta\chi - \frac{\partial}{\partial s}(\kappa\lambda\delta\mathfrak{g}\tau) + \frac{\partial}{\partial s}\left(\kappa\lambda\delta\frac{\partial \mathfrak{h}}{\partial s}\right) - \frac{\partial^2}{\partial s\partial t}(\tau\lambda\delta)\\
&\quad = \left(-\mathfrak{g}\tau + \frac{\partial \mathfrak{h}}{\partial s}\right)\frac{\partial}{\partial s}(\kappa\lambda\delta) + \kappa^2\lambda\delta\chi + \tau^2\lambda\delta\chi - \frac{\partial^2}{\partial t\partial s}(\tau\lambda\delta)\\
&\left(-\kappa\lambda + \frac{\partial \Gamma}{\partial s}\right)\left(-\mathfrak{g}\tau + \frac{\partial \mathfrak{h}}{\partial s}\right) + \chi\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial t}(\lambda\tau) + \frac{\partial^2}{\partial t\partial s}\left(\frac{\Gamma\kappa}{\tau}\right)\\
&\quad = \tau\frac{\partial \lambda}{\partial t} + \Gamma\tau\left(\mathfrak{f}\kappa + \mathfrak{h}\tau + \frac{\partial \mathfrak{g}}{\partial s} - \frac{\chi\kappa}{\tau}\right) + \frac{\partial}{\partial s}\left(\lambda\chi + \Gamma\left(-\mathfrak{g}\tau + \frac{\partial \mathfrak{h}}{\partial s}\right) + \frac{\partial}{\partial t}\left(\frac{\Gamma\kappa}{\tau}\right)\right.\\
&\quad \left. - \frac{\partial^2}{\partial s\partial t}\left(\frac{\Gamma\kappa}{\tau}\right) + \Gamma\frac{\partial}{\partial s}(\mathfrak{g}\tau) - \Gamma\frac{\partial^2 \mathfrak{h}}{\partial s^2} = (\tau\Gamma)\left(\mathfrak{f}\kappa + \mathfrak{h}\tau + \frac{\partial \mathfrak{g}}{\partial s}\right) - \frac{\partial^2}{\partial t\partial s}\left(\frac{\Gamma\kappa}{\tau}\right)\right),
\end{aligned}$$

where

$$\delta = \frac{1}{\sqrt{\kappa^2+\tau^2}}, \quad \Gamma = \frac{\kappa\tau'-\kappa'\tau}{\tau(\kappa^2+\tau^2)^{5/2}} \quad \text{and} \quad \chi = \left\langle \frac{\partial N}{\partial t}, B \right\rangle.$$

(b)

$$\begin{aligned}
&\kappa\frac{\partial}{\partial t}(\kappa\lambda\delta) - \kappa\tau^2\lambda\delta\mathfrak{g} + \kappa\tau\lambda\delta\frac{\partial \mathfrak{h}}{\partial s} + \frac{\partial}{\partial s}\left(\kappa^2\lambda\delta\mathfrak{f} + \kappa\lambda\delta\mathfrak{h}\tau + \kappa\lambda\delta\frac{\partial \mathfrak{g}}{\partial s} + \tau\lambda\delta\chi\right)\\
&+ \kappa\tau^2\lambda\delta\mathfrak{g} - \kappa\tau\lambda\delta\frac{\partial \mathfrak{h}}{\partial s} + \tau\frac{\partial}{\partial t}(\tau\lambda\delta)\\
&= \left(\mathfrak{f}\kappa + \mathfrak{h}\tau + \frac{\partial \mathfrak{g}}{\partial s}\right)\frac{\partial}{\partial s}(\kappa\lambda\delta) + \frac{\partial}{\partial t}\left(\kappa^2\lambda\delta + \tau^2\lambda\delta\right) + \chi\frac{\partial}{\partial s}(\tau\lambda\delta).
\end{aligned}$$

(c)

$$\begin{aligned}
&\frac{\partial}{\partial s}\left(f\kappa\lambda - \mathfrak{h}\tau\lambda - \lambda\frac{\partial \mathfrak{g}}{\partial s} + \frac{\partial \Gamma}{\partial t} + \Gamma\mathfrak{g}\kappa - \frac{\Gamma\kappa}{\tau}\frac{\partial \mathfrak{h}}{\partial s}\right) - \kappa\frac{\partial \lambda}{\partial t} - \kappa\Gamma\left(\mathfrak{f}\kappa + \mathfrak{h}\tau + \frac{\partial \mathfrak{g}}{\partial s} - \frac{\chi\kappa}{\tau}\right)\\
&= -\frac{\partial}{\partial t}(\kappa\lambda) + \frac{\partial^2 \Gamma}{\partial t\partial s} - \left(\frac{\partial \lambda}{\partial s}\right)\left(\mathfrak{f}\kappa + \mathfrak{h}\tau + \frac{\partial \mathfrak{g}}{\partial s}\right)\\
&+ \left(\lambda\tau + \frac{\partial}{\partial s}\left(\frac{\Gamma\kappa}{\tau}\right)\right)\left(\mathfrak{g}\tau - \frac{\partial \mathfrak{h}}{\partial s}\right).
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{\partial \mathcal{V}_{\mathfrak{b}}}{\partial t} &= \left[\frac{\partial}{\partial t}(\tau\lambda\delta) + \Gamma_1\left(f\kappa + \mathfrak{h}\tau + \frac{\partial \mathfrak{g}}{\partial s}\right) + (\kappa\lambda\delta)\left(\mathfrak{g}\tau - \frac{\partial \mathfrak{h}}{\partial s}\right)\right]T\\
&\quad + \left[(\tau\lambda\delta)\left(f\kappa + \mathfrak{h}\tau + \frac{\partial \mathfrak{g}}{\partial s}\right) - \frac{\partial}{\partial t}\Gamma_1 - \kappa\lambda\delta\chi\right]N\\
&\quad + \left[(\tau\lambda\delta)\left(-\mathfrak{g}\tau + \frac{\partial \mathfrak{h}}{\partial s}\right) - \Gamma_1\chi + \frac{\partial}{\partial t}(\kappa\lambda\delta)\right]B,
\end{aligned}$$

where

$$\delta = \frac{1}{\sqrt{\kappa^2+\tau^2}}, \quad \Gamma_1 = \frac{\kappa\tau'-\kappa'\tau}{\tau(\kappa^2+\tau^2)^2}, \quad \chi = \left\langle \frac{\partial N}{\partial t}, B \right\rangle.$$

(e)

$$\begin{aligned}
&\frac{\partial}{\partial s}((\tau\lambda\delta)+\Gamma_1\kappa)\left(-\mathfrak{g}\tau+\frac{\partial \mathfrak{h}}{\partial s}\right)-\chi\frac{\partial\Gamma_1}{\partial s}+\frac{\partial^2}{\partial t\partial s}(\kappa\lambda\delta)-\frac{\partial}{\partial t}(\Gamma_1\tau)\\
&=\left[\frac{\partial}{\partial s}\left((\tau\lambda\delta)\left(-\mathfrak{g}\tau+\frac{\partial \mathfrak{h}}{\partial s}\right)-\Gamma_1\chi+\frac{\partial}{\partial t}(\kappa\lambda\delta)\right)+\left(\tau^2\lambda\delta\right)\left(\mathfrak{f}\kappa+\mathfrak{h}\tau+\frac{\partial \mathfrak{g}}{\partial s}\right)\right.\\
&-\tau\frac{\partial\Gamma_1}{\partial t}-\kappa\tau\lambda\delta\chi,\kappa\frac{\partial}{\partial t}(\tau\lambda\delta)+\Gamma_1\kappa\left(f\kappa+\mathfrak{h}\tau+\frac{\partial \mathfrak{g}}{\partial s}\right)+\left(\kappa^2\lambda\delta\right)\left(\mathfrak{g}\tau-\frac{\partial \mathfrak{h}}{\partial s}\right)\\
&+\frac{\partial}{\partial s}\left(\left((\tau\lambda\delta)\left(f\kappa+\mathfrak{h}\tau+\frac{\partial \mathfrak{g}}{\partial s}\right)\right)-\frac{\partial^2\Gamma_1}{\partial s\partial t}-\frac{\partial}{\partial s}(\kappa\lambda\delta\chi)\right)\\
&\left.-\left(\tau^2\lambda\delta\right)\left(-\mathfrak{g}\tau+\frac{\partial \mathfrak{h}}{\partial s}\right)+\Gamma_1\tau\chi-\tau\frac{\partial}{\partial t}(\kappa\lambda\delta)\right]\\
&=\frac{\partial}{\partial s}((\tau\lambda\delta)+\Gamma_1\kappa)\left(\mathfrak{f}\kappa+\mathfrak{h}\tau+\frac{\partial \mathfrak{g}}{\partial s}\right)-\frac{\partial^2}{\partial t\partial s}\Gamma_1-\chi\left(\frac{\partial}{\partial s}(\kappa\lambda\delta)-\Gamma_1\tau\right),
\end{aligned}$$

where

$$\delta=\frac{1}{\sqrt{\kappa^2+\tau^2}},\quad \Gamma_1=\frac{\kappa\tau'-\kappa'\tau}{\tau(\kappa^2+\tau^2)^2},\quad \chi=\left\langle\frac{\partial N}{\partial t},B\right\rangle.$$

3.5. Gravitational Magnetic Curves in Riemannian 3-Manifolds

Körpınar and Demirkol investigated in [61] a special type of $\mathcal{MC}$s assuming a charged particle in motion that is thought to be subject to a gravitational force G in a Riemannian 3-manifold (M^3,g). They proved the following results for such curves.

Definition 5 ([61]). *Let γ be a unit-speed integral curve and $V\in\left(M^3,g\right)$. Then γ is called a gravitational integral curve if the gravitational force G of γ satisfies*

$$\nabla_{\gamma'}G=\Phi(G)=V\times G.$$

Here, in this three-dimensional model, the Lorentz force is explicitly given by

$$\Phi(X)=V\times X,\quad \forall X\in TM.$$

This agrees with the identification of the magnetic 2-form with a vector via the metric and volume form in dimension three.

Theorem 15 ([61]). *A unit-speed curve γ is a gravitational $\mathcal{MC}$ of V iff, along γ, we have $V=\tau\mathbf{e}_{(0)}+\kappa\mathbf{e}_{(2)}$.*

The energy on the gravitational $\mathcal{MC}$ is also investigated in [61] as follows.

Theorem 16 ([61]). *If Γ represents a charged particle in motion corresponding to a unit-speed gravitational magnetic curve γ within V on $\left(M^3,g\right)$, then the particle's energy in the region V is*

$$\mathfrak{E}(G_V)=\frac{1}{2}\int_0^s\left(1+\left(\kappa'\right)^2+\left(\tau'\right)^2\right)ds.$$

Theorem 17 ([61]). *A gravitational magnetic curve obeys uniformly accelerated motion iff it is a plane curve with constant κ.*

4. Magnetic Curves in Semi-Riemannian Manifolds

4.1. Magnetic Curves in Semi-Riemannian 3-Manifolds

In their research on T-$\mathcal{MC}$s in an oriented semi-Riemannian 3-manifold, Özdemir et al. established the following results in [14].

Proposition 9 ([14]). *The relation for the Lorentz force in the Frenet frame for a non-null, unit-speed curve γ in a semi-Riemannian manifold (M, g) is denoted by*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N) \\ \Phi(B) \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2\kappa & 0 \\ -\epsilon_1\kappa & 0 & \epsilon_3\Omega \\ 0 & -\epsilon_2\Omega & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $\Omega = g(\Phi(N), B)$ and ϵ_1, ϵ_2, and ϵ_3 are described by $g(T, T), g(N, N)$, and $g(B, B)$, respectively.

Proposition 10 ([14]). *Assume that γ is a unit-speed non-null curve in (M, g). Then γ is a T-$\mathcal{MC}$ of V iff , along γ, V can be represented as*

$$V = \epsilon_3(\Omega T + \kappa B).$$

Theorem 18 ([14]). *Let γ be a unit-speed T-$\mathcal{MC}$ of $(M(c), g, V)$. Thus*

$$\kappa^2(\frac{\Omega}{2} + \tau) + \epsilon_1 A = 0,$$
$$\epsilon_3\kappa'' - \epsilon_2 k\tau(\Omega + \tau) - \epsilon_3 ck + \frac{\kappa^3}{3} - B\kappa = 0,$$

where A and B are constants.

4.2. Magnetic Curves in Minkowski 3-Space $\mathbb{R}^3_1$

In their work, Kazan and Karadag [62] introduced and explored various types of $\mathcal{MC}$s in $\mathbb{R}^3_1$.

Definition 6. *A curve in a Minkowski space is called* pseudo-null *if its tangent and binormal vectors are space-like, but its principal normal vector is null.*

Definition 7 ([62]). *Consider a pseudo-null curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3_1$ and a magnetic force F_V in $\mathbb{R}^3_1$. When γ' meets the Lorentz force equation, then γ is referred to as a* T-magnetic pseudo-null curve. *In this case the Lorentz force equation is stated as $\nabla_{\gamma'}\gamma' = \Phi(\gamma') = V \times \gamma'$.*

Proposition 11 ([62]). *Consider γ to be a unit-speed T-magnetic pseudo-null curve in $\mathbb{R}^3_1$. Then the Frenet frame yields the Lorentz force as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N) \\ \Phi(B) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\Omega & 0 \\ 1 & 0 & \Omega \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $\Omega = g(\Phi N, B)$.

Proposition 12 ([62]). *If γ is a unit-speed T-magnetic pseudo-null curve in $\mathbb{R}^3_1$, then γ is a T-$\mathcal{MC}$ of a Killing magnetic vector field V iff, along γ, we have $V = -\Omega T - N$.*

Corollary 1 ([62]). *If γ is a unit-speed T-magnetic pseudo-null curve in $\mathbb{R}^3_1$, then the space-like vector corresponding to the Killing magnetic vector field V describes a T-$\mathcal{MC}$ that is the pseudo-null curve γ.*

Proposition 13 ([62]). *Let γ be a unit-speed N-magnetic pseudo-null curve in $\mathbb{R}^3_1$. Then, the Lorentz force concerning the Frenet frame is described as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N) \\ \Phi(B) \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_1 & 0 \\ 0 & -\tau & 0 \\ -\Omega & 0 & \tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $\Omega_1 = g(\Phi T, B)$.

Corollary 2 ([62]). *If γ is a unit-speed B-magnetic pseudo-null curve in $\mathbb{R}^3_1$, then the Killing magnetic vector field V, whose B-$\mathcal{MC}$ is the pseudo-null curve γ, is among the subsequent:*

(a) *A space-like curve, if $\tau^2 > 2\Omega_2$;*
(b) *A time-like curve, if $\tau^2 < 2\Omega_2$ and $\Omega_2 > 0$;*
(c) *A null curve, if $\tau^2 = 2\Omega_2$ and $\Omega_2 \geq 0$.*

Theorem 19 ([62]). *Consider γ to be a null curve in $\mathbb{R}^3_1$. In this case, there is no nonzero V along γ such that γ serves as a ξ-magnetic, N-magnetic, or W-magnetic null trajectory associated with V.*

Remark 3. *Let V be a magnetic vector field on $\mathbb{R}^3_1$. In* [63]*, the concepts of T-, N_1-, and N_2-magnetic time-like and space-like curves in $\mathbb{R}^3_1$ are defined. They find V in the case where a space-like or time-like curve is either a T-, N_1-, or N_2-$\mathcal{MC}$ of V. Finally, they also provide examples for such $\mathcal{MC}$s.*

Remark 4. *Wang et al. studied $\mathcal{MC}$s based on $\mathcal{KVF}$s in $\mathbb{R}^3_1$* [64]*. According to the properties of $\mathcal{KVF}$s and the special measurement of $\mathbb{R}^3_1$, Wang et al. then computed the six $\mathcal{KVF}$s corresponding to rotation and translation in $\mathbb{R}^3_1$. After that, by using the Lorentz force equation, they classified the magnetic force curves associated with different $\mathcal{KVF}$s. Finally, the derived five types of $\mathcal{MC}$s were represented by Jacobian elliptic integrals or parametric equations.*

4.3. Magnetic Curves in Anti-De Sitter 3-Space $\mathbb{H}^3_1$

In [48], the authors studied $\mathcal{MC}$s in $\mathbb{H}^3_1$. They proved the following results.

Theorem 20. *Let $\bar{\gamma}$ be the image of a light-like Hopf magnetic curve γ in $\mathbb{H}^3_1$ via the hyperbolic Hopf map. Then $\bar{\gamma}$ maintains a constant curvature within $\mathbb{H}^2(-\frac{1}{2})$.*

Theorem 21. *In $\mathbb{H}^3_1$, a light-like magnetic curve γ serves as a geodesic within its associated hyperbolic Hopf tube.*

5. Magnetic Curves in Contact Metric Manifolds

5.1. Magnetic Curves in Sasakian Manifolds

In [18], Druţă-Romaniuc et al. investigated the $\mathcal{MC}$s of a contact magnetic field in a Sasakian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$. They calculated the curvatures of these trajectories by using the Lorentz equation and the Frenet–Serret formulas. They proved the subsequent classification results.

Theorem 22 ([18]). *Consider a Sasakian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ equipped with a contact magnetic force F_q of strength $q \neq 0$. Then a curve γ in M^{2n+1} is a normal $\mathcal{MC}$ related to F_q iff γ is among the curves given below:*

(a) *A geodesic that is an integral curve of ξ;*
(b) *φ-Circles that are not geodesic, having $\kappa_1 = \sqrt{q^2 - 1}$ for $|q| > 1$, with a constant $\theta = \arccos \frac{1}{q}$;*

(c) *A Legendre φ-curve in M^{2n+1} characterized by $\kappa_1 = |q|$ and $\kappa_2 = 1$, essentially defining an integral curve of the contact distribution;*
(d) *A φ-helix of $r = 3$ oriented around ξ, with $\kappa_1 = |q| \sin\theta$, $\kappa_2 = |q\cos\theta - 1|$, and $\theta \neq \frac{\pi}{2}$.*

Additionally, they proved the following results.

Proposition 14 ([18]). *In a Sasakian manifold $\left(M^{2n+1}, \varphi, \xi, \eta, g\right)$, if a curve γ is a non-geodesic Legendre φ-curve with $r = 3$, then it holds that $v_2 = \pm\xi$ and $\kappa_2 = 1$.*

Theorem 23 ([18]). *For a φ-helix γ with $r \leq 3$ in a Sasakian manifold $\left(M^{2n+1}, \varphi, \xi, \eta, g\right)$, the subsequent is observed:*

(a) *When $\cos\theta = \pm 1$, γ acts as an integral curve aligned with ξ, categorizing it as a normal $\mathcal{MC}$ linked with the contact magnetic field F_q of any q.*
(b) *When $\cos\theta = 0$ and $\kappa_1 \neq 0$, γ is a non-geodesic integral curve that corresponds to an $\mathcal{MC}$ associated with $F_{\pm\kappa_1}$.*
(c) *When $\cos\theta = \frac{\varepsilon}{\sqrt{\kappa_1^2+1}}$, γ acts as an $\mathcal{MC}$ for $F_{\varepsilon\sqrt{\kappa_1^2+1}}$, where $\varepsilon = \operatorname{sgn}(\tau_{01})$, and in this scenario, γ is a φ-circle, meaning that $\kappa_2 = 0$.*
(d) *If $\cos\theta = \frac{(\varepsilon\pm\kappa_2)}{\sqrt{\kappa_1^2+(\varepsilon\pm\kappa_2)^2}}$, then γ is configured as an $\mathcal{MC}$ for $F_{\varepsilon\sqrt{\kappa_1^2+(\varepsilon\pm\kappa\kappa_2)^2}}$, where $\varepsilon = \operatorname{sgn}(\tau_{01})$ and $\pm$ aligns to the sign of $\eta(E_3)$.*
(e) *Outside these conditions, γ does not qualify as an $\mathcal{MC}$ in the context of F_q.*

Proposition 15 ([18]). *In a Sasakian manifold $\left(M^{2n+1}, \varphi, \xi, \eta, g\right)$, there do not exist non-geodesic circles that serve as an $\mathcal{MC}$ corresponding to the contact magnetic force F_q of strength q with $0 < |q| \leq 1$ on M^{2n+1}.*

Theorem 24 ([28]). *Let $\left(M^{2n+1}, \varphi, \xi, \eta, g\right)$ be a Sasakian manifold equipped with a contact magnetic field F_q of $q \neq 0$. Then γ is a normal $\mathcal{MC}$ associated with F_q in M^{2n+1} iff it belongs to the subsequent criteria:*

(a) *A geodesic, arising as integral curve of ξ;*
(b) *A non-geodesic φ-circle with $\kappa_1 = \sqrt{q^2-1}$, applicable when $|q| > 1$ and maintaining a fixed angle $\theta = \arccos\frac{1}{q}$;*
(c) *A Legendre φ-curve within M^{2n+1} characterized by $\kappa_1 = |q|$ and $\kappa_2 = 1$, representing one-dimensional integral submanifolds of the contact distribution;*
(d) *A φ-helix of $r = 3$ along the axis defined by ξ, characterized by $\kappa_1 = |q|\sin\theta$ and $\kappa_2 = |q\cos\theta - 1|$, provided that $\theta \neq \frac{\pi}{2}$.*

Remark 5. *In* [65], *Zhang et al. studied $\mathcal{MC}$s and Killing vectors in Lorentzian α-Sasakian 3-manifolds. By using the Killing equation, they derived the expression of $\mathcal{KVF}$s. At the same time, they obtained explicit formulas for the Killing $\mathcal{MC}$s related to Bott's connection. In addition, they provided an example for each type of Killing $\mathcal{MC}$.*

5.2. Magnetic Curves in Sasakian Spheres

In [66], Inoguchi and Munteanu proved the following.

Theorem 25 ([66]). *The collection of all periodic $\mathcal{MC}$s with any given intensity on the Sasakian space form $M^3(c)$ is quantizable within the set of rational numbers.*

They also proved the following theorem.

Theorem 26 ([66]). *Consider γ to be a normal $\mathcal{MC}$ in a Sasakian sphere $\mathbb{S}^3$. The path γ is periodic iff*

$$\frac{q}{\sqrt{q^2-4q\cos\theta+4}} \in \mathbb{Q}.$$

In [18], Druţă-Romaniuc et al. also proved the following.

Theorem 27 ([18]). *Suppose that γ represents a normal $\mathcal{MC}$ in the Sasakian manifold $\left(\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g\right)$ related to F_q. Then, γ constitutes a normal $\mathcal{MC}$ on $\mathbb{S}^3(1)$, which is integrated as a totally geodesic, Sasakian submanifold of $\mathbb{S}^{2n+1}(1)$.*

In [67], Munteanu and Nistor established the following result for normal $\mathcal{MC}$s in a Sasakian manifold $(S^{2n+1}, \varphi, \xi, \eta, g)$.

Theorem 28 ([67]). *Let γ represent a normal $\mathcal{MC}$ in a Sasakian manifold $(S^{2n+1}, \varphi, \xi, \eta, g)$, associated with a contact magnetic force F_q of strength q. Then, γ is identified as a normal $\mathcal{MC}$ in a 3-sphere $S^3(1)$, which is integrated as a totally geodesic, Sasakian submanifold of $S^{2n+1}(1)$.*

Pseudo-Hermitian $\mathcal{MC}$s in a Sasakian manifold $\left(M^{2n+1}, \varphi, \xi, \eta, g\right)$ equipped with a Tanaka–Webster connection $\hat{\nabla}$ were studied in [68] as follows.

Theorem 29 ([68]). *A pseudo-Hermitian $\mathcal{MC}$ on the Sasakian $\mathbb{R}^{2n+1}(-3)$ equipped with $\hat{\nabla}$ admits the parametric equation*

$$\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^{2n+1}(-3), \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_{n+1}, \ldots, \gamma_{2n}, \gamma_{2n+1}),$$

where $\gamma_i (i = 1, \ldots, 2n+1)$ fulfills either of the following:

(a)

$$\gamma_i = h_i + \frac{c_i}{\lambda}\sin(\lambda t + d_i),$$

$$\gamma_{n+i} = h_{n+i} + \frac{-c_i}{\lambda}\cos(\lambda t + d_i),$$

$$\gamma_{2n+1} = h_{2n+1} + 2\cos\theta t + \sum_{i=1}^{n}\left\{\frac{-c_i^2}{4\lambda^2}[2(\lambda t + d_i) + \sin(2(\lambda t + d_i))] + \frac{c_i h_{n+i}}{\lambda}\sin(\lambda t + d_i)\right\},$$

for constants $c_i, d_i\,(i = 1, \ldots, n)$, and $h_i\,(i = 1, \ldots, 2n+1)$, where $\lambda = q - 2\cos\theta \neq 0$ and $c_i\,(i = 1, \ldots, n)$ satisfies $\sum_{i=1}^{n} c_i^2 = 4\sin^2\theta$;

(b)

$$\gamma_i = d_i +_i t,$$

$$\gamma_{2n+1} = 2t\cos\theta + c_{2n+1} + \sum_{i=1}^{n} c_i\left(\frac{c_{n+i}}{2}t^2 + d_{n+i}t\right),$$

where $q = 2\cos\theta$, $c_i (i = 1, 2, \ldots, 2n+1)$, and $d_i (i = 1, 2, \ldots, 2n)$ satisfy $\sum_{i=1}^{2n} c_i^2 = 4 - q^2$.

5.3. Magnetic Curves in Quasi-Sasakian Manifolds

The next result was proved by Inoguchi et al. in [21].

Proposition 16. *For a quasi-Sasakian 3-manifold, one obtains*

(a) *A Legendre $\mathcal{MC}$ that fulfills the Lorentz equation's counterpart when an Okumura-type connection $\bar{\bar{\nabla}}$ when any arbitrary function a is applied.*

(b) *A magnetic curve γ that adheres to the Lorentz equation's counterpart when it pertains to $\bar{\bar{\nabla}}$*

$$\bar{\bar{\nabla}}_X Y = \nabla_X Y + \alpha(\eta(Y)\varphi X - \eta(X)\varphi Y + V(X,Y)\xi).$$

Sarkar et al. [22] explored normal $\mathcal{MC}$s in a quasi-Sasakian 3-manifold (M^3, φ, ξ, g) and established the subsequent.

Theorem 30 ([22]). *A quasi-Sasakian 3-manifold that admits an $\mathcal{MC}$ is a constant structure function.*

Sarkar et al. also investigated biharmonic $\mathcal{MC}$s on the same ambient space.

Theorem 31 ([22]). *The scalar κ of a quasi-Sasakian manifold (M^3, φ, ξ, g) admitting a biharmonic $\mathcal{MC}$ is constant.*

Theorem 32 ([22]). *A quasi-Sasakian manifold (M^3, φ, ξ, g) admitting a biharmonic $\mathcal{MC}$ is locally φ-symmetric.*

Sarkar et al. [23] provided a positive response to support the hypothesis of the order of an $\mathcal{MC}$ in a quasi-Sasakian manifold. Specifically, they demonstrated that the $\mathcal{MC}$ in a quasi-Sasakian manifold, which is the product of a Kähler and a Sasakian manifold, has a maximum order of 5.

Proposition 17 ([23]). *Suppose that γ is an $\mathcal{MC}$ in the quasi-Sasakian manifold $M = M^{2p+1} \times B^{2k}$. Then, the two projection curves γ_M and γ_B are $\mathcal{MC}$s on N and B, respectively.*

Theorem 33 ([23]). *If $M = M \times B$ is a quasi-Sasakian manifold of product type, then the $\mathcal{MC}$ corresponding to F stated by the fundamental 2-form of M has maximum order of* 5.

Proposition 18 ([23]). *Suppose that γ is a normal contact $\mathcal{MC}$ in $M \times B$. Then, γ is a slant curve.*

Proposition 19 ([23]). *Assume that γ is a normal contact $\mathcal{MC}$ in $M \times B$. Then, the first curvature κ_1 of γ is described by*

$$\kappa_1^2 = q^2(\cos^2\alpha \sin^2\theta + \sin^2\alpha).$$

5.4. Magnetic Curves in Trans-Sasakian Manifolds

Consider $(M^{2n+1}, \varphi, \xi, \eta, g)$ to be a contact metric manifold with φ as a $(1,1)$-tensor field, ξ as a unit vector field, and η as the 1-form dual to ξ fulfilling

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0,$$
$$g(\varphi X, \varphi Y) = g(X, Y - \eta(X)\eta(Y).$$

If there are functions α and β on M^{2n+1} fulfilling

$$(\nabla_X \varphi)(Y) = \alpha\big(g(X,Y)\xi - \eta(Y)X\big) + \beta\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big), \tag{5}$$

where

$$(\nabla\varphi)(X,Y) = \nabla_X(\varphi Y) - \varphi(\nabla_X Y),$$

then M^{2n+1} is said to be a *trans-Sasakian manifold of type* (α, β) (see [69]). Accordingly,

$$\nabla_X \xi = -\alpha\varphi(X) + \beta(X - \eta(X\xi), \quad X \in \mathfrak{X}(M). \tag{6}$$

A trans-Sasakian manifold is called *proper* if $\alpha, \beta \neq 0$.

It is known from [70] that there do not exist proper trans-Sasakian manifolds of dimension $n \geq 5$. Hence, our focus is restricted to trans-Sasakian 3-manifolds.

Within the trans-Sasakian manifold, Bozdağ [20] examined f-harmonic, biharmonic, and f-biharmonic $\mathcal{MC}$s. Bozdağ obtained adequate and essential criteria to guarantee that $\mathcal{MC}$s and Legendre $\mathcal{MC}$s are f-harmonic, f-biharmonic, and biharmonic. More precisely, Bozdağ obtained the following.

Theorem 34 ([20]). *Suppose that $\gamma : I \to M^3$ is an $\mathcal{MC}$ in a trans-Sasakian 3-manifold M^3 satisfying $\omega_1, \omega_2 =$ constant, where ω_1 and ω_2 are smooth functions. Then, γ is a biharmonic $\mathcal{MC}$ iff the subsequent is fulfilled:*

$$\begin{cases} \omega_1\omega_2\eta(\gamma')^2 + 2\omega_2\eta(\gamma') + \omega_1\omega_2 = 0, \\ \omega_2^2\eta(\gamma')^2 - 2\omega_1\eta(\gamma') - \omega_1^2 - 1 + \frac{\rho}{2} - 2(\omega_1^2 - \omega_2^2) - \eta(\gamma')^2\left(\frac{\rho}{2} - 3(\omega_1^2 - \omega_2^2)\right) = 0, \\ \omega_2\left(\omega_1\eta(\gamma') + \eta(\gamma')^2\right) = 0, \end{cases}$$

where ρ denotes the scalar curvature of M^3.

Corollary 3 ([20]). *If $\gamma : I \to M$ is an $\mathcal{MC}$, then γ is biharmonic iff*

$$\rho = \frac{-6\omega_1^2\eta(\gamma')^2 + 6\omega_1^2 + 4\omega_1\eta(\gamma') + 2}{1 - \eta(\gamma')^2},$$

where $\eta(\gamma')^2 \neq 1$.

Corollary 4 ([20]). *There is no non-Legendre biharmonic $\mathcal{MC}$ in an ω_2-Kenmotsu 3-manifold.*

Corollary 5 ([20]). *Consider an $\mathcal{MC}$ $\gamma : I \to M^3$. Then, γ is a proper f-biharmonic slant $\mathcal{MC}$ iff*

$$\rho = \frac{6\omega_1^6 - 14\omega_1^4 + 8\omega_1^2 - \omega_1}{\omega_1^4 - 2\omega_1^2 + 1}, \quad f(s) = e^{\frac{s}{2\omega_1^3 - 2\omega_1}},$$

and $\eta(\gamma') = -\frac{1}{\omega_1}$.

5.5. Magnetic Curves in Para-Sasakian Manifolds

The concept of para-Sasakian manifolds was defined by Sato in [71]. In [72], Bejan et al. studied the case of a para-Sasakian geometry in a para-Sasakian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ for which the endomorphism φ is exactly provided by the paracontact structure. In particular, they classified normal $\mathcal{MC}$s in the para-Sasakian structure on $\mathbb{R}^{2n+1}$.

5.6. Magnetic Curves in Cosymplectic Manifolds

In [24], $\mathcal{MC}$s in a cosymplectic manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ are explored, and the following result is obtained.

Theorem 35 ([24]). *If γ is a φ-helix of $r \leq 3$ in a cosymplectic manifold $\left(M^{2n+1}, \varphi, \xi, \eta, g\right)$, then the following apply:*

(a) *If $\cos\theta = \pm 1$, γ is an integral curve of ξ. Thus, γ is a normal $\mathcal{MC}$ for a contact magnetic force F_q of strength q.*

(b) *If $\cos\theta = 0$, γ is a magnetic circle generated by $F_{\pm\widehat{\kappa}_1}$.*

(c) *If $\cos\theta = \pm\frac{\widehat{\kappa}_2}{\sqrt{\widehat{\kappa}_1^2 + \widehat{\kappa}_2^2}}$, γ is an $\mathcal{MC}$ for $F_{\pm\sqrt{\widehat{\kappa}_1^2 + \widehat{\kappa}_2^2}}$, where the selection of each sign is independent.*

Druță-Romaniuc et al. [24] studied $\mathcal{MC}$s in $\bar{M}^{2n}(k) \times \mathbb{R}$ and $\mathbb{M}^2 \times \mathbb{R}$. They obtained the following.

Theorem 36 ([24]). *If $\gamma \in (\mathbb{M}^2 \times \mathbb{R}(t), g)$ is a unit-speed curve and F_V is a $\mathcal{KVF}$ $V = \frac{\partial}{\partial t}$, then γ is a normal Killing $\mathcal{MC}$ in $\mathbb{M}^2 \times \mathbb{R}(t)$ iff γ is one of the subsequent:*

(a) *A geodesic line $(p, t_0 \pm s)$, where $p \in \mathbb{M}^2$ and t_0 is a constant;*
(b) *A circle $\alpha_0 \times \{t_0\}$, where α_0 has constant $\bar{\kappa} = \pm 1$ in $\mathbb{M}^2$;*
(c) *A helix on $\beta_\theta \times \mathbb{R}$ with $\kappa = \sin\theta$ and $\tau = \cos\theta$, where β_θ has $\bar{\kappa} = \pm\csc\theta$ in $\mathbb{M}^2$ and $\theta \in (0, \pi)$.*

5.7. Magnetic Curves in Normal Almost Paracontact Metric 3-Manifolds

Consider $(M^{2n+1}, \varphi, \xi, \eta, g)$ to be a $(2n+1)$-dimensional manifold. If the semi-Riemannian metric g is with the signature $(n+1, n)$ on M^{2n+1}, then the tuple (φ, ξ, η, g) is referred to as an *almost paracontact metric structure* on M^{2n+1}, provided that the subsequent conditions are met:

(1) $\eta(\xi) = 1, \ \varphi(\xi) = 0, \ \eta \circ \varphi = 0, \ \varphi^2 = I - \eta \otimes \xi$;
(2) $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \ \forall X, Y \in TM$;
(3) φ induces an almost para-complex structure P on $\mathcal{D} = \ker(\eta)$, where $(P^2 = I)$. The eigen-subbundles T^+ and T^- related to the eigenvalues 1 and -1 of P, respectively, both have dimension n.

(φ, ξ, η, g) on M^{2n+1} is called *normal* if the Nijenhuis tensor N_φ fulfills $N_\varphi = 2d\eta \otimes \xi$. Additionally, note that when M^{2n+1} is three-dimensional, g is Minkowski–Lorentzian with a signature of $(2, 1)$.

In [19], the authors have explored normal $\mathcal{MC}$s in normal almost paracontact 3-manifolds. In [19], they computed their curvature, torsion, Lancret invariant, and mean curvature vector field. Further, Călin and Crasmareanu discovered two one-parameter families of $\mathcal{MC}$s (where the first one is space-like and the second one time-like) in quasi-para-Sasakian manifolds, which are not para-Sasakian, as follows.

Proposition 20 ([19]). *For a normal almost paracontact 3-manifold $(M^3, \varphi, \xi, \eta, g)$, a unit-speed magnetic curve γ is a ψ-slant curve with $\eta(N) = 0$ with ψ restricted to $\epsilon_2(\psi^2 - \epsilon_1) > 0$, where $\epsilon_1 = g(T, T)$ and $\epsilon_2 = g(N, N)$.*

Proposition 21 ([19]). *κ and τ of a ψ-$\mathcal{MC}$ in a normal almost paracontact 3-manifold $(M^3, \varphi, \xi, \eta, g)$ are given by*

$$\kappa = \sqrt{\epsilon_2(c^2 - \epsilon_1)} = constant, \quad \tau = \epsilon_2(\epsilon_1\beta - \psi)sgn(\eta(B)).$$

The corresponding Lancret invariant is

$$Lancret(\gamma) = \frac{\epsilon_1\beta - \epsilon_2 sgn(\eta(B))\tau}{k}.$$

It adheres to a new restriction:

$$\epsilon_2(\epsilon_1\beta - \psi)sgn(\eta(\beta)) \geq 0,$$

where $\beta = \frac{1}{2}trace(\varphi\nabla\xi)$.

In [73], Bozdağ and Erdoğan investigated various types of $\mathcal{MC}$s in a normal almost paracontact metric 3-manifold $(M^3, \phi, \xi, \eta, g)$, and they obtained the following results.

Theorem 37 ([73]). *$(M^3, \phi, \xi, \eta, g)$ does not admit an f-harmonic non-null $\mathcal{MC}$ with constant α and β.*

Theorem 38 ([73]). *Consider $(M^3, \phi, \xi, \eta, g)$ with $\alpha, \beta =$ constant. If $\gamma : I \subseteq \mathbb{R} \to M^3$ is a unit-speed non-null $\mathcal{MC}$, then γ is an f-biharmonic curve iff we have*

$$\begin{aligned}
&(1-\beta\eta(\gamma'))\big(f'-\alpha\eta(\gamma')f\big)=0,\\
&\Big(1-\frac{r}{2}\Big((\eta(\gamma'))^2-\varepsilon_1\Big)-2\beta\eta(\gamma')+\Big(\alpha^2+3\beta^2\Big)\Big(\varepsilon_1-(\eta(\gamma'))^2\Big)\Big)f\\
&\qquad\qquad -2\alpha\eta(\gamma')f'+f''=0,\\
&\Big(3\alpha(\eta(\gamma'))^2-2\varepsilon_1\alpha\beta\eta(\gamma')-\varepsilon_1\alpha\Big)f+2\big(\varepsilon_1\beta-\eta(\gamma')\big)f'=0,
\end{aligned}$$

where $\varepsilon_1 = g(\gamma', \gamma')$.

Theorem 39 ([73]). *In a normal almost paracontact metric 3-manifold $(M^3, \phi, \xi, \eta, g)$ with constant α and β, there are no f-biharmonic non-null magnetic Legendre curves.*

For a β-para-Sasakian manifold M, Bozdağ and Erdoğan obtained the following.

Corollary 6 ([73]). *Let $\gamma : I \subseteq \mathbb{R} \to M^3$ be a unit-speed non-null slant $\mathcal{MC}$. Then γ is an f-biharmonic curve iff f and k satisfy $f \neq$ constant and*

$$k=\frac{2\beta^2f''-8\beta^2f+6\beta^4\varepsilon_1 f}{(1-\varepsilon_1\beta^2)f},$$

with $\beta\eta(\gamma') = 1$.

For an f-biharmonic non-null $\mathcal{MC}$, Bozdağ and Erdoğan [73] proved the following.

Theorem 40 ([73]). *Consider $(M^3, \phi, \xi, \eta, g)$ with $\alpha, \beta =$ constant, and let $\gamma : I \subseteq \mathbb{R} \to M^3$ be a non-null $\mathcal{MC}$. Then γ is bi-f-harmonic iff we have*

$$\begin{aligned}
&(ff'')'+(1-\beta\eta(\gamma'))(4ff'-2f^2\alpha\eta(\gamma'))=0,\\
&3ff''+2(f')^2-4ff'\alpha\eta(\gamma')+f^2\Big[1-2\beta\eta(\gamma')+\frac{r}{2}((\eta(\gamma'))^2-\varepsilon_1)\\
&\qquad+(\eta(V))^2(5\alpha^2+3\beta^2)-\varepsilon_1(\beta^2+3\alpha^2)\Big]=0,\\
&4ff'(\varepsilon_1\beta-\eta(\gamma'))+f^2(3\alpha(\eta(\gamma'))^2-2\varepsilon_1\alpha\beta\eta(\gamma')-\varepsilon_1\alpha)=0.
\end{aligned}$$

For an f-biminimal non-null $\mathcal{MC}$, Bozdağ and Erdoğan also obtained the following results.

Theorem 41 ([73]). *Consider $(M^3, \phi, \xi, \eta, g)$ with constant α, β. Let $\gamma : I \subseteq \mathbb{R} \to M^3$ be a unit-speed non-null $\mathcal{MC}$. Then γ is f-biminimal iff we have*

$$\begin{aligned}
&\Big(1-\frac{r}{2}((\eta(\gamma'))^2-\varepsilon_1)-2\beta\eta(\gamma')+(\alpha^2+3\beta^2)(\varepsilon_1-(\eta(\gamma'))^2)-\lambda\Big)f\\
&\qquad -2\alpha\eta(\gamma')f'+f''=0,\\
&(3\alpha(\eta(\gamma'))^2-2\varepsilon_1\alpha\beta\eta(\gamma')-\varepsilon_1\alpha)f+2(\varepsilon_1\beta-\eta(\gamma'))f'=0.
\end{aligned}$$

Corollary 7 ([73]). *Consider $(M^3, \phi, \xi, \eta, g)$ with α, β. Let $\gamma : I \subseteq \mathbb{R} \to M^3$ be a unit-speed non-null $\mathcal{MC}$. Then γ is f-biminimal iff f and k satisfy*

$$f(s) = e^{\left(\alpha\eta(\gamma') + \frac{\alpha\left((\eta(\gamma'))^2 - \varepsilon_1\right)}{2(\eta(\gamma') - \varepsilon_1\beta)}\right)s + c},$$

and

$$k = \frac{2\left(\left(\frac{\alpha\left((\eta(\gamma'))^2 - \varepsilon_1\right)}{2(\eta(\gamma') - \varepsilon_1\beta)}\right)^2 - (\eta(\gamma'))^2(3\beta^2 + 2\alpha^2) + \varepsilon_1(3\beta^2 + \alpha^2) - 2\beta\eta(\gamma') + 1 - \lambda\right)}{(\eta(\gamma'))^2 - \varepsilon_1},$$

where $s \in I$ and $\lambda \in \mathbb{R}$ is constant.

Remark 6. *In* [74], *Perrone studied $\mathcal{MC}$s in almost paracontact 3-manifolds associated with the Reeb vector field. She established an equivalence between the normality of an almost paracontact structure and the property of a magnetic field to be Killing. As a consequence, she studied $\mathcal{MC}$s in paracontact 3-manifolds. Then she described a non-geodesic $\mathcal{MC}$ associated with a (divergence-free) Reeb vector field of the 3-manifold.*

Remark 7. *Consider M^3 to be a normal almost contact metric 3-manifold. Lee demonstrated in* [75] *that a pseudo-Hermitian $\mathcal{MC}$ within M^3 gains a canonical affine connection, resulting in it being a slant helix. Additionally, they calculated the pseudo-Hermitian curvature and torsion within M^3. Consequently, it is evident that every pseudo-Hermitian $\mathcal{MC}$ in M^3, except quasi-Sasakian 3-manifolds, is classified as a slant helix. It is important to note that a pseudo-Hermitian $\mathcal{MC}$ in a quasi-Sasakian 3-manifold becomes a slant curve but not a helix.*

5.8. Magnetic Curves in Kenmotsu Manifolds

Pandey and Mohammad [27] investigated $\mathcal{MC}$s for contact magnetic force in Kenmotsu manifolds. Moreover, they also investigated magnetic slant curves in Kenmotsu manifolds. Consequently, they obtained the following.

Theorem 42 ([27]). *Assume that $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ is a Kenmotsu manifold and F_q a contact magnetic field of strength q with $q \neq 0$ on M^{2m+1}. Then γ is a normal $\mathcal{MC}$ related to F_q if γ is among the subsequent curves:*

(a) *A geodesic which is an integral curve of ξ.*

(b) *A non-geodesic φ-circle with $\kappa_1 = \sqrt{q^2 - \sin^2\theta}$, with $|q| > 1$ and $\theta = \operatorname{arccot}\frac{1}{|q|}$.*

(c) *A Legendre φ-curve with $\kappa_1 = |q|$ and $\kappa_2 = 1$.*

(d) *A φ-helix of order 3 with ξ as its axis and with curvatures being given by*

$$\kappa_1 = |q|, \quad \kappa_2 = |\operatorname{sgn}(q)\sin\theta + q\cos\theta|, \quad \theta \neq \frac{\pi}{2}.$$

Remark 8. *The condition $|q| > 1$ in part (b) of this theorem ensures that the curvature $\kappa_1 = \sqrt{q^2 - \sin^2\theta}$ is real and positive, and it is necessary for the existence of non-geodesic φ-circles in a Kenmotsu manifold (cf.* [27], *which described a non-geodesic $\mathcal{MC}$ associated with a (divergence-free) Reeb vector field of the 3-manifold).*

5.9. Magnetic Curves in S-Manifolds

A Riemannian manifold is referred to as a *metric f-manifold* if it includes an f-structure f on M. This f-structure is described as a tensor field f of type (1, 1) that fulfills the condition $f^3 + f = 0$ [76]. Additionally, it involves s global vector fields $\xi_1, \ldots, \xi_s$, known

as *structure vector fields*, such that if $\eta_1, \ldots, \eta_s$ represent the corresponding dual 1-forms of $\xi_1, \ldots, \xi_s$, then

$$f\xi_\alpha = 0, \;\; \eta_\alpha \circ f = 0, \;\; f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha,$$
$$g(X,Y) = g(fX,fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y),$$

for any vector fields X and Y tangent to M and $\alpha = 1, \ldots, s$. In the above conditions,

$$g(X,fY) = -g(fX,Y), \tag{7}$$

so it is possible to consider Ω given by $\Omega(X,Y) = g(X,fY)$. If $\Omega = d\eta_\alpha$ for each $\alpha = 1, \ldots, s$, M is referred to as a *metric f-contact manifold*. If $s = 1$, these metric f-contact manifolds coincide with metric contact manifolds.

A metric f-manifold is classified as a *K-manifold* [77] if it possesses normality and satisfies $d\Omega = 0$. In a K-manifold, the structure vector fields are $\mathcal{KVF}$s. Moreover, a K-manifold becomes a Kähler manifold when $s = 0$, and it is identified as a quasi-Sasakian K-manifold when $s = 1$.

In [78], Güvenç studied quarter-symmetric $\mathcal{MC}$s in S-manifolds. He proved that quarter-symmetric $\mathcal{MC}$s are θ_α-slant curves of osculating order $r \leq 3$ with constant quarter-symmetric curvature functions. Furthermore, he classified curves in S-manifolds which represent quarter-symmetric $\mathcal{MC}$s.

In [79], Güvenç and Özgür defined the notion of a homothetic s-th Sasakian manifold M as an expansion of an S-manifold. Güvenç and Özgür proved that a curve γ is a normal $\mathcal{MC}$ in M iff the osculating order r of γ fulfills $r \leq 3$ and γ is part of a group of θ_i-slant helices. Furthermore, applying generalized D-homothetic transformations, Güvenç and Özgür constructed a homothetic s-th Sasakian manifold and found the parametric equations of normal $\mathcal{MC}$s in such a manifold.

5.10. Magnetic Curves in C-Manifolds

A K-manifold is called an *S-manifold* if we have $F = d\eta_\alpha$, for any $\alpha = 1, \ldots, s$. Güvenç studied in [26] normal $\mathcal{MC}$s in a C-manifold, and he established the following.

Theorem 43 ([26]). *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a C-manifold. If γ is a unit-speed curve in M, then γ is a normal $\mathcal{MC}$ for a contact magnetic force F_q of strength q with $q \neq 0$ iff γ is one of the subsequent:*

(a) *A geodesic θ_α-slant curve which is an integral curve of $\pm\sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_a$ that satisfies $\sum_{\alpha=1}^{s} \cos^2\theta_\alpha = 1$.*
(b) *A Legendre circle with first curvature $\kappa_1 = |q|$ with Frenet frame $\{T, -\mathrm{sgn}(q)\varphi(T)\}$.*
(c) *A non-Legendre θ_a-slant helix with curvatures*

$$\kappa_1 = |q|\sqrt{1 - \sum_{\alpha=1}^{s} \cos^2\theta_\alpha}, \quad \kappa_2 = |q|\sqrt{\sum_{\alpha=1}^{s} \cos^2\theta_\alpha},$$

such that $\sum_{\alpha=1}^{s} \cos^2\theta_\alpha < 1$.

Remark 9. *Güvenç also provided, in* [26], *parametrizations of normal $\mathcal{MC}$s in the C-manifold $\left(\mathbb{R}^{2n+s}, \xi_\alpha, \eta^\alpha, g\right)$.*

5.11. A Generalization of Magnetic Curves in Contact Metric Geometry

In [80], Bejan and Druta-Romaniuc studied a triple (M, G, ∇) containing a $(1,1)$-tensor field G and a linear connection ∇ on M. They stated that a unit-speed curve γ is to be called *G-geodesic* if it satisfies

$$\nabla_{\gamma'}\gamma' = G\gamma'.$$

Using an alternating parameter, this G-geodesic condition becomes

$$\nabla_{\gamma'(s)}\gamma'(s) = a(s)\varphi\gamma'(s) + b(s)F\gamma'(s),$$

for some functions $a(s)$ and $b(s)$.

In [81] Alegre et al. studied unit-speed trajectories γ in an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ that satisfy

$$\nabla_{\gamma'}\gamma' = q\varphi\gamma' + p(\xi - \eta(\gamma')\gamma'),$$

where p and q are constants. They called such curves (p,q)*-curves*. Alegre et al. extended geodesics and $\mathcal{MC}$s and considered F-geodesics with $FX = q\varphi X + p(\xi - \eta(X)X)$. They stated that T associated with M is a p,q*-curve* if it satisfies

$$\nabla_T T = q\varphi T + p(\xi - \eta(T)),$$

for some constant p and with strength q.

An interesting motivation for these curves is the following: If we make a D-conformal deformation of a Sasakian manifold, we obtain a trans-Sasakian manifold. The magnetic properties of a curve are not retained under such a deformation. Alegre et al. showed that a D-conformal deformation, under additional assumptions, changes an $\mathcal{MC}$ into a p,q-curve. The Heisenberg 3-dimensional space is rigid for p,q-curves. But in the hyperbolic 3-space, Alegre et al. obtained explicit parametrization for non-geodesic p,q-curves. They also consider β-Kenmotsu manifolds, as well as cosymplectic manifolds of arbitrary dimension, as ambient spaces.

6. Magnetic Curves in Some Special Spaces

6.1. Magnetic Curves in Galilean 3-Space

A Galilean n-space $\mathbb{G}_n$ is a semi-Euclidean n-space of nullity 1. Such a space can be regarded as the limit case of a semi-Euclidean space in which the isotropic cone degenerates to a plane.

Aydın investigated in [4] $\mathcal{MC}$s in the Galilean 3-space $\mathbb{G}_3$. He obtained the subsequent.

Theorem 44. *Let γ be a unit-speed normal $\mathcal{MC}$ related to $V = v_1\partial_x + v_2\partial_y + v_3\partial_z$ in the Galilean 3-space satisfying $y(0) = y_0, y'(0) = Y_0, z(0) = z_0$, and $z'(0) = Z_0$. Then one of the subsequent conditions applies to γ:*

(a) *When V is isotropic, γ is given by*

$$\gamma(s) = \left(s, \frac{v_3}{2}s^2 + Y_0 s + y_0, -\frac{v_2}{2}s^2 + Z_0 s + z_0\right).$$

(b) *γ is a cylindrical helix on $S^1_{\mathbb{G}}(r) \times l$, where $S^1_{\mathbb{G}}(r)$ is a Euclidean circle in $\mathbb{G}_3$ with radius $r = \sqrt{\left(\frac{z_0}{v_1} - \frac{v_3}{v_1^2}\right)^2 + \left(\frac{Y_0}{v_1} - \frac{v_2}{v_1^2}\right)^2}$ and l is given by*

$$\left(s, \frac{v_2}{v_1}s + \left(y_0 - \frac{Z_0}{v_1} + \frac{v_3}{v_1^2}\right), \frac{v_3}{v_1}s + \left(z_0 + \frac{Y_0}{v_1} - \frac{v_2}{v_1^2}\right)\right),$$

and is parameterized by

$$\gamma(s) = \left(s, \frac{\left(Z_0 - \frac{v_3}{v_1}\right)}{v_1}\cos(v_1 s) + \frac{\left(Y_0 - \frac{v_2}{v_1}\right)}{v_1}\sin(v_1 s) + \frac{v_2}{v_1}s + \left(y_0 - \frac{\left(Z_0 - \frac{v_3}{v_1}\right)}{v_1}\right), \right.$$
$$\left. \frac{\left(Z_0 - \frac{v_3}{v_1}\right)}{v_1}\sin(v_1 s) - \frac{\left(Y_0 - \frac{v_2}{v_1}\right)}{v_1}\cos(v_1 s) + \frac{v_3}{v_1}s + \left(z_0 + \frac{\left(Y_0 - \frac{v_2}{v_1}\right)}{v_1}\right)\right).$$

Aydin obtained in [4] the following theorem for N-$\mathcal{MC}$s in Galilean 3-space.

Theorem 45. *Suppose that γ is a unit-speed normal N-$\mathcal{MC}$ in $\mathbb{G}_3$ having constant κ_0 connected to $V = v_1\partial_x + v_2\partial_y + v_3\partial_z$ and fulfilling the subsequent:*

$$y(0) = y_0, \quad y'(0) = Y_0, \quad y''(0) = T_0, \quad z(0) = z_0, \quad z'(0) = Z_0, \quad z''(0) = U_0.$$

Then γ is of one of the following types:

(a) *If $v_1 = v_2 = v_3 = 0$ ($V = 0$, trivial $\mathcal{KVF}$),*

$$\gamma(s) = \left(s, \frac{T_0}{2}s^2 + Y_0 s + y_0, \frac{U_0}{2}s^2 + Z_0 s + z_0\right).$$

(b) *If $v_1 = v_2 = 0$ and $v_3 \neq 0$,*

$$\gamma(s) = \left(s, Y_0 s + y_0, \frac{U_0}{2}s^2 + Z_0 s + z_0\right).$$

(c) *If $v_1 = v_3 = 0$ and $v_2 \neq 0$,*

$$\gamma(s) = \left(s, \frac{T_0}{2}s^2 + Y_0 s + y_0, Z_0 s + z_0\right).$$

(d) *If $v_1 = 0, v_2 \neq 0$, and $v_3 \neq 0$,*

$$\gamma(s) = \left(s, \frac{U_0}{T_0}v_2 s^2 + Y_0 s + y_0, \frac{U_0}{T_0}v_3 s^2 + Z_0 s + z_0\right).$$

(e) *If V is non-isotropic, then γ is the cylindrical helix on $S^1_{\mathbb{G}}(r) \times l$, where $S^1_{\mathbb{G}}(r)$ is a planar circle in $\mathbb{G}_3$ with radius $r = \sqrt{\left(\frac{T_0}{v_1^2}\right)^2 + \left(\frac{U_0}{v_1^2}\right)^2}$ and l is given by*

$$\left(s, \left(Y_0 - \frac{U_0}{v_1}\right)s + \left(y_0 + \frac{T_0}{v_1^2}\right), \left(Z_0 + \frac{T_0}{v_1}\right)s + \left(z_0 + \frac{U_0}{v_1^2}\right)\right),$$

and is parameterized by

$$\gamma(s) = \left(s, \frac{-T_0}{v_1^2}\cos(v_1 s) + \frac{U_0}{v_1^2}\sin(v_1 s) + \left(Y_0 - \frac{U_0}{v_1}\right)s + \left(y_0 + \frac{T_0}{v_1^2}\right) \right.$$
$$\left. \frac{-U_0}{v_1^2}\cos(v_1 s) - \frac{T_0}{v_1^2}\sin(v_1 s) + \left(Z_0 + \frac{T_0}{v_1}\right)s + \left(z_0 + \frac{U_0}{v_1^2}\right) \right).$$

6.2. Magnetic Curves in a Flat Para-Kähler Manifold

An almost para-Hermitian manifold is a triple $(M, \mathcal{P}, g)$ consisting of an almost product structure $\mathcal{P}$ and a semi-Riemannian metric g on M such that

$$\mathcal{P}^2 = I \ \ (\mathcal{P} \neq \pm I), \ \ g(\mathcal{P}X, \mathcal{P}Y) = -g(X, Y), \ \ \forall X, Y \in T(M).$$

The almost para-Hermitian manifold $(M, \mathcal{P}, g)$ is called *para-Kähler* if $\nabla\mathcal{P} = 0$ holds identically.

Jleli and Munteanu [16] established that all para-Kähler $\mathcal{MC}$s in the flat para-Kähler space $\mathbb{E}_n^{2n}$ are either space-like or time-like.

Theorem 46 ([16]). *Consider $\gamma : I \longrightarrow \mathbb{E}_n^{2n}$ to be an $\mathcal{MC}$ related to the standard flat para-Kähler structure on $\mathbb{E}_n^{2n}$. Then, up to Lorentzian transformations in $\mathbb{E}_n^{2n}$, γ appears in the subsequent criteria:*

(a1) $\gamma(s) = \frac{1}{c}(e^{cs}w; e^{cs}w)$;

(b1) $\gamma(s) = \frac{1}{c}(-e^{-cs}w; e^{-cs}w)$;

(a2) $\gamma(s) = \frac{1}{c}(\coth(cs), 0, \ldots, 0; \sinh(cs), 0, \ldots, 0)$;

(b2) $\gamma(s) = \frac{1}{c}(\sinh(cs), 0, \ldots, 0; \cosh(qcs), 0, \ldots, 0)$;

(c2) $\gamma(s) = \frac{1}{c}(\sinh(cs), \cosh(cs), 0, \ldots, 0; \cosh(cs), \sinh(cs), 0, \ldots, 0)$.

Case (c2) *occurs only when $n \geq 2$, c is a nonzero constant, and $0 \neq w \in \mathbb{R}^n$.*

Definition 8 ([17]). *A unit-speed curve $\gamma(s)$ in (M, J, g) is regarded as a* Kähler $\mathcal{MC}$ *if it satisfies $\nabla_{\gamma'}\gamma' = q\varphi\gamma'$, where strength q is a constant.*

Remark 10. *Kelekçi* [82] *investigated $\mathcal{MC}$s in a Kähler manifold conformally equivalent to a Euclidean Schwarzschild space. By solving the Lorentz equation, analytical expressions for the $\mathcal{MC}$s were derived, consistent with the almost complex structure of the Kähler manifold. In addition, the energy of the $\mathcal{MC}$s were also computed in* [82].

6.3. Magnetic Curves in Heisenberg Group $\mathbb{H}_3$

The Heisenberg group $\mathbb{H}_3$ is the group of matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

under matrix multiplication. The Heisenberg group $\mathbb{H}_3$ admits an almost contact metric structure $(\mathbb{H}_3, \varphi, \xi, \eta, g)$ (see, e.g., [83]).

In [29], Ozgur explored normal $\mathcal{MC}$s in $\mathbb{H}_3$. He derived the parametric equations for normal slant $\mathcal{MC}$s in $\mathbb{H}_3$ as follows.

Theorem 47 ([29]). *A normal slant $\mathcal{MC}$ in $\mathbb{H}_3$ is described by one of the subsequent cases:*

(a)

$$x(s) = \frac{1}{b}\sin\theta\sin(ks+a) + d_1,$$
$$y(s) = -\frac{1}{b}\sin\theta\cos(ks+a) + d_2,$$
$$z(s) = \left(\cos\theta + \frac{\Lambda}{2b}\sin^2\theta\right)s - \frac{\Lambda}{2b}d_1\sin\theta\cos(bs+c)$$
$$-\frac{\Lambda}{2b}d_2\sin\theta\sin(bs+a) + d_3,$$

where $b = \Lambda\cos\theta + q \neq 0$; a, d_1, d_2, *and* d_3 *are real numbers; and* $\theta \in (0, \pi)$.

(b)

$$x(s) = (\sin\theta\cos c)s + d_4,$$
$$y(s) = (\sin\theta\sin c)s + d_5,$$

and

$$z(s) = \left(-\frac{q}{\Lambda} + \frac{\Lambda}{2}\sin\theta(d_4\sin c - d_5\cos c)\right)s + d_6,$$

where c, d_4, d_5, *and* d_6 *are real numbers;* $\theta = \arccos\left(-\frac{q}{\Lambda}\right)$; *and* $-\frac{q}{\Lambda} \in [-1, 1]$.

Theorem 48 ([29]). *Normal slant* $\mathcal{MC}$*s on* $\mathbb{H}_3$ *have one of the following parametric equations:*

(a)

$$x(s) = \frac{1}{b}\sin\theta\sin(bs+c) + d_1,$$
$$y(s) = -\frac{1}{b}\sin\theta\cos(bs+c) + d_2,$$
$$z(s) = \left(\cos\theta + \frac{\Lambda}{2b}\sin^2\theta\right)s - \frac{\Lambda}{2b}d_1\sin\theta\cos(bs+c)$$
$$-\frac{\Lambda}{2b}d_2\sin\theta\sin(bs+c) + d_3,$$

where $b = \Lambda\cos\alpha + q \neq 0$; c, d_1, d_2, *and* d_3 *are real numbers; and* $\theta \in (0, \pi)$.

(b)

$$x(s) = (\sin\theta\cos a)s + d_4,$$
$$y(s) = (\sin\theta\sin a)s + d_5,$$

and

$$z(s) = \left(\frac{\Lambda}{2}\sin\theta(d_4\sin a - d_5\cos a) - \frac{q}{\Lambda}\right)s + d_6,$$

where a, d_4, d_5, *and* d_6 *are real numbers;* $\theta = \arccos\left(-\frac{q}{\Lambda}\right)$; *and* $-\frac{q}{\Lambda} \in [-1, 1]$.

Remark 11. *Druţă-Romaniucn et al. found the equations of the* $\mathcal{MC}$*s in* [84] *associated with the magnetic fields given by the* $\mathcal{KVF}$*s on* $\mathbb{H}_3$.

Remark 12. *In* [84], *Derkaoui and Hathout presented the geometry of* $\mathbb{H}_3$ *and its geodesic curves. By examining Killing and geodesic Killing* $\mathcal{MC}$*s, Derkaoui and Hathout presented the precise expressions for these curves.*

Remark 13. *In* [85], *Druţă-Romaniuc et al. found the equations of* $\mathcal{MC}$*s pertaining to the magnetic fields described by* $\mathcal{KVF}$*s on* $\mathbb{H}_3$.

Remark 14. *In* [86], *Gonzales and Chinea investigated homogeneous structures on the generalized Heisenberg group* $\mathbb{H}(n, 1)$ *endowed with the natural left-invariant metric. They also introduced a*

class of quasi-Sasakian structures in [86], *that is, normal almost contact metric structures with a closed fundamental 2-form on* $\mathbb{H}(n,1)$.

Remark 15. *$\mathcal{MC}$s in an* $\mathbb{H}(n,1)$ *equipped with a quasi-Sasakian structure were studied in* [87]. *It was shown that these trajectories are Frenet curves of a maximum order of 5, and their complete classification was provided.*

6.4. Magnetic Curves in $SL(2,\mathbb{R})$ *and* $\widetilde{SL}(2,\mathbb{R})$

Normal homogeneous spaces of positive curvature were classified in [88,89]. It is known from [88] that $(\mathrm{SU}(2)\times\mathbb{R})/H_r$ is diffeomorphic to the 3-sphere $\mathbb{S}^3$. This space is currently referred to as the *Berger 3-sphere*. The Berger 3-sphere carries a Sasakian structure with constant ϕ-sectional curvature c and serves as the principal circle bundle over $S^2(c+3)$ with curvature $(c+3)$.

The special linear group $SL(2,\mathbb{R})$, considered the hyperbolic analogue of the Berger 3-sphere, admits the structure of a principal circle bundle over the hyperbolic 2-space $\mathbb{H}^2(c+3)$ with $(c+3)<0$. Moreover, $SL(2,\mathbb{R})$ carries a canonical left-invariant Sasakian structure with constant ϕ-sectional curvature c. Using this relationship between $SL(2,\mathbb{R})$ and the Sasakian manifold mentioned above, Inoguchi and Munteanu [28] investigated contact $\mathcal{MC}$s and contact homogeneous $\mathcal{MC}$s in $SL(2,\mathbb{R})$ as follows.

Proposition 22 ([28]). *Consider the hyperbolic Hopf fibering*

$$SL(2,\mathbb{R})\to SL(2,\mathbb{R})/SO(2)=\mathbb{H}^2(-4).$$

The projected curve $\beta(s)=\pi(\gamma(s))$ *of a contact* $\mathcal{MC}$ γ *is a Kähler* $\mathcal{MC}$ *within* $\mathbb{H}^2(-4)$. *Specifically, the curve* $\beta:\mathbb{R}\to\mathbb{H}^2(-4)$ *adheres to* $\bar{\nabla}_{\beta'}\beta'=(q-2\cos\sigma)J\beta'$, *where* J *is the complex structure. Consequently, contact magnetic trajectories in* $SL(2,\mathbb{R})$ *are geodesics of the Hopf tubes over the projection curve.*

Remark 16. *In* [90], *Bosak et al. investigated Killing $\mathcal{MC}$s in* $\widetilde{SL}(2,\mathbb{R})$. *In particular, they established several classification results for such Killing $\mathcal{MC}$s.*

6.5. Magnetic Curves in Sol_3

In mathematics, the space Sol_3 is a type of homogeneous 3-dimensional manifold which is a geometric space modeled on R^3 with the metric given by

$$g=(e^z dx)^2+(e_z dy)^2+(dz)^2,$$

and with a special group operation.

In [25], Erjavec and Inoguchi studied $\mathcal{MC}$s and the almost cosymplectic structure of the Sol_3 space, obtaining curvature properties for these curves.

Theorem 49 ([25]). *$\mathcal{MC}$s in the* Sol_3 *space concerning* $F=dx\wedge dy$ *are given by*

(a) *Geodesic lines* $\gamma(s)=(0,0,as)$;

(b) *Curves* γ *defined by* $(b\sin s\pm b\cos s\mp b,\pm b\sin s-b\cos s+b,0)$;

(c) *Curves γ parameterized by*

$$x(s) = \int_0^a e^{-2(r)}\sqrt{(a^2+2(ab-c)+1)e^{-2x(\tau)}-1-f(\tau)}d\tau,$$
$$y(s) = \int_0^2 e^{2(\tau)}\sqrt{(b^2-2(ab-c)+1)e^{2x(r)}-1+g(\tau)}d\tau,$$
$$ds = \frac{dz}{\pm\sqrt{\delta}-(a^2+2(ab-c)+1)e^{-2x}-(b^2-2(ab-c)+1)e^{2x}-2\int_0^n(f(\tau)+g(\tau))x^2(\tau)d\tau},$$

where δ, a, b, and c are real numbers, and f and g are functions.

Proposition 23 ([25]). *A unit-speed $\mathcal{MC}$ in* Sol_3 *regarding $F = dx \wedge dy$ is a slant curve iff its first curvature is constant.*

Proposition 24 ([25]). *Consider $\gamma(s)$ a unit-speed $\mathcal{MC}$ in* Sol_3 *and $\theta \neq 0, \pi$. Then we have*

$$\begin{aligned}
T(s) &= e^{z(s)}x'(s)e_1 + e^{-z(s)}y'(s)e_2,\\
N(s) &= (\csc\theta)\Phi T(s) = (\csc\theta)\left(-e^{-z(s)}y'(s)e_1 + e^{z(s)}x'(s)e_2\right),\\
B(s) &= -(\cot\theta)T(s) + (\csc\theta)\xi\\
&= -(\csc\theta)\left(\cos\theta\left(e^{z(s)}x'(s)e_1 + e^{-z(s)}y'(s)e_2\right) - e_3\right),\\
\kappa_1 &= \sin\theta \ \text{ and } \ \kappa_2 = \frac{2x'y' + \cos\theta\sin^2\theta}{\sin^2\theta}.
\end{aligned}$$

$\mathcal{MC}$s in Sol_3 were also investigated in [91], and the following result was proved.

Theorem 50 ([91]). *The normal $\mathcal{MC}$ of* Sol_3 *with respect to $F = 2d\eta$ with $q \neq 0$ is provided by*

$$\begin{aligned}
x(s) &= \left(b - \frac{q}{\sqrt{2}}\right)\int_0^s e^{-2z(\tau)}d\tau + \frac{q}{\sqrt{2}}\int_0^s e^{-z(\tau)}d\tau,\\
y(s) &= \left(c - \frac{q}{\sqrt{2}}\right)\int_0^s e^{2z(\tau)}d\tau + \frac{q}{\sqrt{2}}\int_0^{z(\tau)} e^{z(\tau)}d\tau,\\
ds &= \frac{dz}{\pm\sqrt{1-\left[\left(b-\frac{q}{\sqrt{2}}\right)e^{-z}+\frac{q}{\sqrt{2}}\right]^2-\left[\left(c-\frac{q}{\sqrt{2}}\right)e^{z}+\frac{q}{\sqrt{2}}\right]^2}},
\end{aligned}$$

where a, b, and c are real constants satisfying $a^2 + b^2 + c^2 = 1$.

Remark 17. *There are three $\mathcal{KVF}$s in Sol_3 (see* [92]*). In* [93]*, Erjavec and Inoguchi also studied Killing $\mathcal{MC}$s for such vector fields.*

6.6. Magnetic Curves in 3-Symmetric Space F^4

A strictly almost Kähler manifold is an almost Kähler manifold that is not Kähler. The space F^4 is the homogeneous Riemannian 4-space $SA(2)/SO(2)$, where $SA(2)$ denotes the group of orientation-preserving equiaffine transformations of the equiaffine plane. It is known that F^4 admits both a Kähler structure and a strictly almost Kähler structure [94].

In [17], a symplectic pair of Kähler forms was defined on F^4, giving rise to both an invariant strictly almost Kähler structure and an invariant Kähler structure on the space. Typical submanifolds of F^4 were then investigated, along with general properties of $\mathcal{MC}$s in an almost Kähler 4-manifold. Additionally, Kähler $\mathcal{MC}$s were characterized for the symplectic pair of Kähler forms, and homogeneous geodesics as well as homogeneous $\mathcal{MC}$s in F^4 were studied.

Remark 18. *The generic features of $\mathcal{MC}$s and Kähler $\mathcal{MC}$s in F^4 were also investigated by Erjavec and Inoguchi in* [17].

6.7. Magnetic Curves in Berger Sphere

In [66], Inoguchi and Munteanu extended the notion of Berger's 3-sphere as a simply-connected, complete Sasakian 3-manifold $\mathcal{B}^3(c)$ whose holomorphic sectional curvature is a constant $c \neq 1$ with $c > -3$. Such a Berger sphere can be considered a naturally reductive homogeneous space $\mathcal{B}^3(c) = U(2)/U(1)$. Inoguchi and Munteanu investigated in [66] the homogeneity of contact $\mathcal{MC}$s in the Berger sphere. Furthermore, they proved that each contact magnetic trajectory in $\mathcal{B}^3(c)$ is the product of a homogeneous geodesic and a charged Reeb flow.

6.8. Magnetic Curves in Walker Manifolds

A Walker manifold is a semi-Riemannian n-manifold (W^n, g) equipped with a light-like distribution $\mathcal{D}$ such that $\mathcal{D}$ is a parallel distribution regarding ∇ (see, e.g., [95–98]).

Bejan and Druţă-Romaniuc investigated in [99] $\mathcal{MC}$s in (W^3, g). After imposing certain conditions for the existence of $\mathcal{KVF}$s on W^3, they obtained several classification results. In particular, they demonstrated the subsequent.

Theorem 51 ([99])**.** *Assume that W^3 is a Walker 3-manifold equipped with a unitary (time-like or space-like) $\mathcal{KVF}$ ξ. Then every time-like or space-like integral curve of $\xi^{\perp}$ is a normal $\mathcal{MC}$ of ξ.*

Theorem 52 ([99])**.** *Suppose that W^3 admits a unitary space-like $\mathcal{KVF}$ ξ. If there is a light-like curve γ in W^3 that is normal to ξ, then γ is a geodesic.*

6.9. Magnetic Curves in Tangent Sphere Bundles

Let UM^n denote the tangent sphere bundle $(T^{(1)}M, \bar{g})$ of a Riemannian manifold (M^n, g). The tangent sphere bundle of a real space form $M^n(c)$ admits a canonical Sasaki metric induced from the original metric g (see [100]).

In 1975, Klingenberg and Sasaki [101] proved that every geodesic in US^2 is a unit vector field along a circle in the 2-sphere S^2, which makes a constant angle with the circle. After that, Sasaki [100] studied $UM^n(c)$ of a $M^n(c)$.

Sasaki proved that all geodesics on $UM^n(c)$ can be divided into horizontal, vertical, and slant types. Every horizontal-type geodesic is a parallel vector field along a geodesic on $M^n(c)$, and every vertical-type geodesic is a great circle of a fiber. Sasaki also showed that slant-type geodesics can be divided into three families, namely,

(1) The family of slant-type geodesics over geodesics in $M^n(c)$;
(2) The family of geodesics over curves whose first curvature is constant and the second curvature is zero;
(3) The family of geodesics over curves whose first curvature is constant, the second curvature is a nonzero constant, and the third curvature is zero.

In [30], Inoguchi and Munteanu studied curves of the form $\bar{\gamma}(s) = (\gamma(s), V(s))$ on UM^n that consists of a base curve $\gamma(s)$ in M^n and a unit vector field along the base curve. First, they derived the characterization of contact $\mathcal{MC}$s on UM^n. Next, they studied slant contact $\mathcal{MC}$s in UM^n when M^n is a real space form $M^n(c)$. For $c = 1$, they showed that every contact normal $\mathcal{MC}$ is slant. For $c \neq 1$, they derived an equivalence condition for a contact normal $\mathcal{MC}$ to be slant. Furthermore, they also characterized slant contact normal $\mathcal{MC}$s in $U\mathbb{H}^n$, $U\mathbb{S}^2$, and the universal cover of $U\mathbb{S}^2$. In addition, Inoguchi and Munteanu also verified the following two results.

Theorem 53 ([30]). *Every contact normal $\mathcal{MC}$ in the tangent bundle $T\mathbb{S}^n$ is slant.*

Theorem 54 ([30]). *Let $\gamma(s)$ be a unit-speed curve in $\mathbb{S}^2$. Then its Gauss map is a contact $\mathcal{MC}$ with $q \neq 0$ iff $\gamma(s)$ fulfills*

$$\kappa(s) = -\frac{q(s-s_0)}{\sqrt{1-q^2(s-s_0)^2}},$$

for a certain constant s_0.

Remark 19. *In* [102], *Inoguchi and Munteanu studied contact $\mathcal{MC}$s in $U\mathbb{E}^2$. In particular, they showed that all contact $\mathcal{MC}$s in $U\mathbb{E}^2$ are slant.*

Remark 20. *Bejan and Druţă-Romaniuc* [103] *studied the cotangent bundle T^*M^n of an M^n equipped with a symmetric linear connection. Their main results are about geodesics of $\tilde{g}$ and $\mathcal{MC}$a, both on T^*M^n and on a certain family of level hypersurfaces of T^*M^n.*

7. Killing Magnetic Curves in Special Spaces

7.1. Killing Magnetic Curve in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$

In their study [31], Munteanu and Nistor explored a comprehensive description of the local characteristics of $\mathcal{MC}$s linked to V within $\mathbb{S}^2 \times \mathbb{R}$ space, and they obtained the ensuing classification result.

Theorem 55 ([31]). *Suppose that $\gamma : I \to \mathbb{S}^2 \times \mathbb{R}$ is a curve in $\mathbb{S}^2 \times \mathbb{R}$ that satisfies the initial conditions*

$$(x,y,z,t)(0) = (x_0,y_0,z_0,t_0), \text{ and } (x',y',z',t')(0) = (u_0,v_0,w_0,\mu_0),$$

with $x_0^2+y_0^2+z_0^2+\mu_0^2 = u_0^2+v_0^2+z_0^2+\mu_0^2 = 1$, $x_0u_0+y_0v_0+z_0w_0 = 0$, and a $\mathcal{KVF}$ given by $V = \frac{\partial}{\partial_t}$. Then, γ is a normal Killing $\mathcal{MC}$ produced by F associated with V iff γ is one of the following:

(a) *A geodesic defined by $\gamma(s) = (x_0,y_0,z_0,t_0 \pm s)$;*
(b) *The curve $\mathrm{S}^1 1/\sqrt{2}) \times \{t_0\}$ defined by*

$$\gamma(s) = \frac{1}{\sqrt{2}}\Big(u_0\sin(\sqrt{2}s) - \frac{a_0}{\sqrt{2}}\cos(\sqrt{2}s) + \frac{a_0}{\sqrt{2}} + \sqrt{2}x_0,\ v_0\sin(\sqrt{2}s) - \frac{b_0}{\sqrt{2}}\cos(\sqrt{2}s) + \frac{b_0}{\sqrt{2}} + \sqrt{2}y_0,\ w_0\sin(\sqrt{2}s) - \frac{c_0}{\sqrt{2}}\cos(\sqrt{2}s) + \frac{c_0}{\sqrt{2}} + \sqrt{2}z_0,\ \sqrt{2}t_0\Big),$$

(c) *A non-degenerate cylindrical helix on $\mathbb{S}^1(r) \times \mathbb{R}$ with $r = \frac{\sqrt{\mu_0^2-1}}{\mu_0}$ which is parameterized by*

$$\gamma(s) = \Big(\frac{u_0}{\mu_0}\sin(\mu_0 s) - \frac{a_0}{\mu_0^2}\cos(\mu_0 s) + \frac{a_0}{\mu_0^2} + x_0,\ \frac{v_0}{\mu_0}\sin(\mu_0 s) - \frac{b_0}{\mu_0^2}\cos(\mu_0 s) + \frac{b_0}{\mu_0^2} + y_0,\ \frac{w_0}{\mu_0}\sin(\mu_0 s) - \frac{c_0}{\mu_0^2}\cos(\mu_0 s) + \frac{c_0}{\mu_0^2} + z_0,\ t_0 + \zeta_0 s\Big),$$

where $\mu_0 = \sqrt{2-\zeta_0^2}$; a_0, b_0, c_0, and μ_0 are real numbers, with $\mu \geq 1$; and $\zeta_0 = t_0'$.

In addition, Munteanu and Nistor also classified, in [31], $\mathcal{MC}$s associated to the $\mathcal{KVF}$ $V = y\frac{\partial}{\partial_x} + x\frac{\partial}{\partial_y}$.

Remark 21. *Erjavec and Inoguchi* [104] *investigated magnetic trajectories in* $\mathbb{H}^3 \times \mathbb{R}$ *associated with a strictly almost Kähler structure. They discovered three families of* $\mathcal{MC}$*s associated with the almost complex structure that is compatible with the product metric in* $\mathbb{H}^3 \times \mathbb{R}$.

7.2. Killing Magnetic Curves in Minkowski 3-Space

Druță-Romaniuc and Munteanu [32] classified $\mathcal{MC}$s in Minkowski space in accordance with the Killing vector field $V = v_1\frac{\partial}{\partial_x} + v_2\frac{\partial}{\partial_y} + v_3\frac{\partial}{\partial_z}$, with $v_1, v_2, v_3 \in \mathbb{R}$. Druță-Romaniuc and Munteanu obtained the corresponding results:

Theorem 56 ([32]). *Assume* γ *that is an* $\mathcal{MC}$ *according to the* $\mathcal{KVF}$ $V = a\frac{\partial}{\partial_x} + b\frac{\partial}{\partial_y} + c\frac{\partial}{\partial_z}$ *in* $\mathbb{E}_1^3$. *Then,* γ *is among the following curves:*

(a) *If* V *is time-like, then* γ *is the cylindrical helix*

$$\gamma(t) = (1-\cos t)V \boxtimes \gamma'(0) + \gamma(0) - \langle V, \gamma'(0)\rangle tV + \sin t\big(\gamma'(0) + \langle V, \gamma'(0)\rangle V\big);$$

(b) *If* V *is space-like, then* γ *is the hyperbolic helix*

$$\gamma(t) = \sinh t\big(\gamma'(0) - \langle V, \gamma'(0)\rangle V\big) + \gamma(0) + \langle V, \gamma'(0)\rangle tV + (\cosh t - 1)V \boxtimes \gamma'(0);$$

(c) *If* V *is light-like, then* γ *is parameterized by*

$$\gamma(t) = t\gamma'(0) + \gamma(0) + \frac{t^2}{2}V \boxtimes \gamma'(0) - \langle V, \gamma'(0)\rangle\frac{t^3}{6}V.$$

7.3. Killing Magnetic Curves in Almost Paracontact 3-Manifolds

Calvaruso et al. studied in [33] the magnetic field corresponding to ξ of a quasi-para-Sasakian 3-manifold M^3. They classified the $\mathcal{MC}$s and subsequently obtained the following four results.

Theorem 57 ([33]). *If* γ *is a light-like curve that is not a geodesic and is parametrized by pseudo-arc length within a quasi-para-Sasakian manifold* M^3 *where* $\beta \neq 0$, *then we have the following:*

(a) *If* $\gamma(s)$ *is an* $\mathcal{MC}$ *associated with* ξ *and* q, *then* ξ *is given by* $\xi = \frac{1}{2a_0}T + a_0 B$ *with* $a_0 = \eta(\gamma')$ *and* $a_0^2 q^2 = 1$. *Additionally, under these conditions,* γ *becomes a helix iff* β *is constant.*

(b) *Conversely, when* $\xi = \omega(s)T + a(s)B$, *with* a *and* ω *being smooth along* γ, *then both* $a(s)$ *and* $\omega(s)$ *are constants, leading to* γ *being an* $\mathcal{MC}$ *corresponding to* ξ.

Theorem 58 ([33]). *If* γ *is a unit-speed, non-geodesic, time-like curve in a quasi-para-Sasakian manifold* M^3 *with* $\beta \neq 0$, *then the following apply:*

(a) *If* γ *is an* $\mathcal{MC}$ *with* q *related to* ξ, *then* ξ *is given by* $\xi = -a_0 T + \frac{\kappa}{q}B$, *where* $a_0 = \eta(\gamma')$ *and the curvature* κ *satisfies* $\kappa^2 = \big(1 + a_0^2\big)q^2$. *Furthermore,* γ *is a helix if and only if* M^3 *is a* β*-para-Sasakian, i.e.,* β *is a constant.*

(b) *Conversely, if* $\xi = \frac{\kappa(s)}{q}B - a(s)T$ *with strength* $q \neq 0$, *then* a *and* κ *are both constant and* γ *is an* $\mathcal{MC}$ *corresponding to* ξ *with* $\pm q$ *as its strength.*

Theorem 59 ([33]). *If* γ *is a unit-speed, non-geodesic space-like curve with space-like acceleration in a quasi-para-Sasakian manifold* M^3 *and if* $\beta \neq 0$, *then the following apply:*

(a) *If* γ *is an* $\mathcal{MC}$ *of strength* q *associated with the* $\mathcal{KVF}$ ξ, *then* ξ *is* $\xi = a_0 T - \frac{\kappa}{q}B$, *where* $a_0 = \eta(\gamma')$ *and* $\kappa^2 = \big(1 - a_0^2\big)q^2$. *Furthermore,* γ *is a helix iff* β *is a constant. Thus* M^3 *is a* β*-para-Sasakian.*

(b) *On the other hand, if* $\xi = a(s)T - \frac{\kappa(s)}{q}B$ *with strength* $q \neq 0$, *then both* a *and* κ *are constant, and* γ *is an* $\mathcal{MC}$ *with* $\pm q$ *associated with* ξ.

Theorem 60 ([33]). *If γ is a unit-speed, non-geodesic, space-like curve with time-like acceleration in a quasi-para-Sasakian manifold M^3 with $\beta \neq 0$, then we have the following:*

(a) *When γ is an $\mathcal{MC}$ with q associated with ξ, then ξ is $a_0 T - \frac{\kappa}{q} B$, where $a_0 = \eta(\gamma')$ and the curvature satisfies $\kappa^2 = (1 - a_0^2)q^2$. In this scenario, γ is a helix iff β is constant.*

(b) *On the other hand, provided that $\xi = a(s)T + \frac{\kappa(s)}{q} B$ with $q \neq 0$, where a is a smooth function along γ, then a and κ are constant, and γ acts as an $\mathcal{MC}$ with strength $\pm q$ associated with ξ.*

Theorem 61 ([33]). *If γ is a unit-speed, space-like, non-geodesic curve with light-like acceleration in a quasi-para-Sasakian manifold M^3 with $\beta, \neq 0$, then we have the following:*

(a) *When γ is an $\mathcal{MC}$ with strength q associated with ξ, then ξ is $\varepsilon_0 T \pm \frac{1}{q} N$ with $\varepsilon_0 = \eta(\gamma') = \pm 1$. In this scenario, γ is a helix iff β is constant.*

(b) *On the other hand, provided that $\xi = a(s)T + \frac{\omega(s)}{q} N$, with $q \neq 0$, and a and ω are smooth functions along γ that obey $\omega' + \omega\tau + aq = -\beta$, then γ is an $\mathcal{MC}$ with strength q associated with ξ.*

In [34], Erjavec studied Killing $\mathcal{MC}$s in the real special linear group $SL(2, \mathbb{R})$. First, he derived the system of differential equations that characterizes the Killing $\mathcal{MC}$ associated with $V = \frac{\partial}{\partial_x}$ in $SL(2, \mathbb{R})$ (see Equation (18), p. 337 in [34]). Then, he obtained the solutions of this system of differential equations to establish the following results.

Theorem 62 ([34]). *All solutions of this system of differential equations that characterize Killing $\mathcal{MC}$s in $SL(2, \mathbb{R})$ associated with $V = \frac{\partial}{\partial_x}$ are given as follows:*

(a) *Space curves defined by*

$$\begin{aligned} x(t) &= c_1 + \frac{c}{4a\sqrt{a}} e^{-\sqrt{a}(t+b)} - \frac{2c}{\sqrt{a}} \left(1 + \frac{8c^2}{a}\right) e^{\sqrt{a}(t+b)} + \frac{1}{2}\left(1 + \frac{8c^2}{a}\right) t, \\ y(t) &= \left(1 + \frac{8c^2}{a}\right) e^{\sqrt{a}(t+b)} + \frac{1}{8a} e^{-\sqrt{a}(t+b)} - \frac{2c}{a}, \\ \theta(t) &= 2ct + \sqrt{2} \arctan\left(\frac{2\sqrt{2}c}{\sqrt{a}} - 2\sqrt{2a}\left(1 + \frac{8c^2}{a}\right) e^{\sqrt{a}(s+b)}\right) + c_2; \end{aligned}$$

(b) *Space curves defined by*

$$\begin{aligned} x(t) &= c_3 - \frac{c}{4a\sqrt{a}} e^{\sqrt{a}(t+b)} + \frac{2c}{\sqrt{a}} \left(1 + \frac{8c^2}{a}\right) e^{-\sqrt{a}(t+b)} + \frac{1}{2}\left(1 + \frac{8c^2}{a}\right) t, \\ y(t) &= \frac{1}{8a} e^{\sqrt{a}(t+b)} + \left(1 + \frac{8c^2}{a}\right) e^{-\sqrt{a}(t+b)} - \frac{2c}{a}, \\ \theta(t) &= c_4 + 2ct - \sqrt{2} \arctan\left(\frac{e^{\sqrt{a}(t+b)} - 8c}{2\sqrt{2a}}\right); \end{aligned}$$

(c) *Lines lie in the plane $y = y_0$ defined by*

$$\gamma(t) = \left(x_0 + kt,\, y_0,\, \theta_0 - \frac{2k^2}{y_0(2k+1)} t\right),$$

where $b, c, x_0, \theta_0, c_1, c_2, c_3$, and c_4 are real numbers; a and y_0 are positive numbers; and $k \in \mathbb{R} \backslash \left\{-\frac{1}{2}\right\}$.

Similarly, he derived the system of differential equations that characterizes Killing $\mathcal{MC}$s associated with $V = \frac{\partial}{\partial_\theta}$ in $SL(2, \mathbb{R})$ (see Equation (29), p. 341 in [34]).

Theorem 63 ([34]). *All solutions of this system of differential equations that characterize Killing $\mathcal{MC}$s in $SL(2,\mathbb{R})$ associated with $V = \frac{\partial}{\partial_\theta}$ are as follows:*

(a) *Space curves specified by*

$$\gamma(t) = \left(c_1 + e^{at}\frac{(1-2c)y_0}{a},\, y_0 e^{at},\, \theta_0 + 2\left(c - \frac{1}{4}\right)t\right) \text{ and;}$$

(b) *Lines lie in the plane $y = y_0$ provided by*

$$\gamma(t) = \left(x_0 + (1-2c)y_0 t,\, y_0,\, \theta_0 + 2\left(c - \frac{1}{4}\right)t\right).$$

8. Slant Curves in Some Special Spaces

8.1. Slant Curves in Quasi-Sasakian Manifolds

For a given unit-speed curve γ in a Sasakian manifold $(M, \varphi, \xi, \eta, g)$, its $\theta(s)$ is given by $\cos\theta(s) = g(T(s)\xi)$, where $T(s)$ is the unit tangent vector field. Recall that γ in M is termed a slant curve if its $\theta(s)$ is constant.

The subsequent results were presented by Inoguchi et al. in [21] by investigating $\mathcal{MC}$s that correspond to a contact magnetic force F_q of strength q within a quasi-Sasakian 3-manifold.

Proposition 25 ([21]). *A non-geodesic Frenet curve γ in a quasi-Sasakian 3-manifold is a slant curve iff it fulfills $\eta(N) = 0$.*

Proposition 26 ([21]). *Each normal contact $\mathcal{MC}$ in a quasi-Sasakian 3-manifold is a slant curve.*

8.2. Slant Curves in S-Manifolds

Consider an S-manifold $\left(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g\right)$. A unit-speed curve $\gamma : I \to M^{2m+s}$ is called a Legendre curve [105] if it satisfies $\eta^\alpha(T) = 0$ for $\alpha = 1, \ldots, s$. More broadly, if θ is a constant angle satisfying

$$\eta^\alpha(T) = \cos\theta,$$

for each $\alpha = 1, \ldots, s$, γ is called a slant curve. In this case, θ is said to be the contact angle fulfilling $|\cos\theta| \leq 1/\sqrt{n}$ (cf. [106]).

The work in [35] explores slant normal $\mathcal{MC}$s in S-manifolds, demonstrating that γ qualifies as a slant normal $\mathcal{MC}$ precisely when it is part of a set of slant φ-curves that adhere to certain specific curvature relationships.

Theorem 64 ([35]). *Suppose that $\left(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g\right)$ is an S-manifold, and let F_q be a contact magnetic force of strength $q \neq 0$. Then γ is a slant normal $\mathcal{MC}$ of F_q iff γ is one of the following:*

(a) *A geodesic which is an integral curve of $\pm\frac{1}{\sqrt{s}}\sum_{i=1}^{s}\xi_i$;*

(b) *A non-geodesic slant circle with $\kappa_1 = \sqrt{q^2 - s}$ $\left(|q| > \sqrt{s}\right)$, contact angle $\theta = \arccos\left(\frac{1}{q}\right)$, and $\left\{T, \frac{-\operatorname{sgn}(q)\varphi T}{\sqrt{1 - s\cos^2\theta}}\right\}$ as the Frenet frame;*

(c) *A Legendre helix with $\kappa_1 = |q|$, $\kappa_2 = \sqrt{s}$, and with the Frenet frame field given by*

$$\left\{T, -\operatorname{sgn}(q)\varphi T, \frac{-\operatorname{sgn}(q)}{\sqrt{s}}\sum_{i=1}^{s}\xi_i\right\};$$

(d) *A slant helix with* $\kappa_1 = |q|\sqrt{1 - s\cos^2\theta}$, $\left(\theta \neq \frac{\pi}{2}, |\cos\theta| < \frac{1}{\sqrt{s}}\right)$, $\kappa_2 = \sqrt{s}|1 - q\cos\theta|$, *and*

$$\left\{T, \frac{-\operatorname{sgn}(q)\varphi T}{\sqrt{1 - s\cos^2\theta}}, \frac{\varepsilon \operatorname{sgn}(q)}{\sqrt{s}\sqrt{1 - s\cos^2\theta}}\left(s\cos\theta T - \sum_{i=1}^{s}\xi_i\right)\right\}$$

as the Frenet frame, where $\varepsilon = \operatorname{sgn}(1 - q\cos\theta)$.

Theorem 65 ([35]). *Consider a slant* φ*-helix* γ *of order* $r \leq 3$ *in* $\left(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g\right)$ *with contact angle* θ*. Then, the following apply:*

(a) *For* $\cos\theta = \pm\frac{1}{\sqrt{s}}$, γ *is an integral curve of* $\pm\frac{1}{\sqrt{s}}\sum_{i=1}^{s}\xi_i$*, and consequently, it serves as a normal* $\mathcal{MC}$ *for a contact magnetic force* F_q *of strength* q*.*

(b) *For* $\cos\theta = 0$ *and* $\kappa_1 \neq 0$, γ *is an* $\mathcal{MC}$ *for* $F_{\pm\kappa_1}$*.*

(c) *For* $\cos\theta = \frac{\varepsilon}{\sqrt{\kappa_1^2 + s}}$, γ *stands as an* $\mathcal{MC}$ *for* $F_{\varepsilon\sqrt{\kappa_1^2+s}}$ *with* $\varepsilon = -\operatorname{sgn}(g(\varphi T, v_2))$*. Consequently,* γ *represents a slant* φ*-circle.*

(d) *For* $\cos\theta = \frac{\varepsilon\sqrt{s}\pm\kappa_2}{\sqrt{s}\sqrt{\kappa_1^2+\left(\varepsilon\sqrt{s}\pm\kappa_2\right)^2}}$, γ *becomes an* $\mathcal{MC}$ *for* $F_{\varepsilon\sqrt{\kappa_1^2+\left(\varepsilon\sqrt{s}\pm\kappa_2\right)^2}}$*, where* $\varepsilon = -\operatorname{sgn}(g(\varphi T, v_2))$ *and* $\pm$ *aligns with the sign of* $\eta^\alpha(v_3)$*.*

(e) *Apart from these situations,* γ *cannot be an* $\mathcal{MC}$ *for any* F_q*.*

In [35], Güvenç and Özgür also constructed examples of slant normal $\mathcal{MC}$s in $\mathbb{R}^{2n+s}(-s)$. Additionally, they provided parametric equations for these curves.

8.3. Slant Curves in Lorentzian Sasakian 3-Manifolds

A curve in a contact Lorentzian 3-manifold $\left(M^3, \varphi, \xi, \eta, g\right)$ is termed slant if its tangent vector field forms a constant angle with the Reeb vector field. For γ on $\left(M^3, \varphi, \xi, \eta, g\right)$, the quantity $F_{\xi,q}$ was described in [36] as

$$F_{\xi,q}(X, Y) = -qd\eta(X, Y), \quad \forall X, Y \in TM^3,$$

where q is a nonzero constant. He calls $F_{\xi,q}$ the *contact magnetic field with strength* q.

In [36], Lee established the following results.

Theorem 66 ([36]). *Assume that* γ *is a non-geodesic, Frenet curve in a Sasakian Lorentzian 3-manifold* $\left(M^3, \varphi, \xi, \eta, g\right)$*. If* γ *is a contact* $\mathcal{MC}$*, then* γ *is a slant pseudo-helix with geodesic curvature* $\kappa = \sqrt{(\epsilon_1 + a^2)\epsilon_2}$ *and geodesic torsion* $\tau = 1 + \epsilon_1 aq$*, where* $\epsilon_1 = g(T, T) = \pm 1$ *is the causal character of* γ*,* $\epsilon_2 = g(N, N)$*, and* $a = \eta(T), T = \gamma'$*. In particular, it follows that the ratio* $\kappa : (\tau - 1)$ *is constant.*

Theorem 67 ([36]). *Under the hypothesis of Theorem 66, if* γ *is a contact* $\mathcal{MC}$*, then it is either a time-like curve or a space-like curve with a space-like normal vector field.*

Theorem 68 ([36]). *A non-geodesic Frenet slant curve in the Sasakian Lorentzian* $\left(M^3, \varphi, \xi, \eta, g\right)$ *has a constant ratio of* $\kappa : (\tau - 1)$*.*

Theorem 69 ([36]). *If* γ *is a contact* $\mathcal{MC}$ *within a contact Lorentzian 3-manifold* M^3*, then* γ *is a slant curve iff* M^3 *is a Sasakian manifold.*

Remark 22. *For additional results regarding slant curves, see* [49].

9. Killing Submersions and Magnetic Curves

A Riemannian submersion $\pi : (M, g) \to (B, \bar{g})$ between Riemannian manifolds is called a *Killing submersion* if it equips a complete vertical unit Killing vector field ξ on M.

In [107], Inoguchi and Munteanu investigated $\mathcal{MC}$s within Killing submersions. They proved that the bundle curvature of $\pi : (M, g) \to (B, \bar{g})$ is constant along every $\mathcal{MC}$ concerning a $\mathcal{KVF}$ if every vertical tube via these $\mathcal{MC}$s has constant mean curvature. Then Inoguchi and Munteanu proved the following.

Theorem 70 ([107]). *If $\pi : (M, g) \to (B, \bar{g})$ is a Killing submersion with a unit $\mathcal{KVF}$ ξ, then, the subsequent statements are equivalent:*

(1) *The bundle curvature is constant along all $\mathcal{MC}$s regarding ξ.*
(2) *All the vertical tubes derived from $\mathcal{MC}$s for ξ have constant mean curvature.*
(3) *All $\mathcal{MC}$s concerning ξ have constant second curvature, so they are helices.*

Further, they examined magnetic Jacobi fields along horizontal Killing $\mathcal{MC}$s in 3-dimensional Sasakian space forms. It is known that the total space M of a Killing submersion $\pi : (M, g) \to (B, \bar{g})$ of constant bundle curvature 1 is Sasakian. They proved the following results.

Theorem 71 ([107]). *Let $\pi : (M, g) \to (B, \bar{g})$ be a Killing submersion of bundle curvature 1. Then the Reeb vector field is a magnetic Jacobi field along any normal Killing magnetic curve γ in a Sasakian 3-manifold.*

Theorem 72 ([107]). *Let γ be a normal Killing $\mathcal{MC}$ in the total space of a Killing submersion of bundle curvature 1. Then $\varphi\gamma'$ is a magnetic Jacobi field along γ if and only if either it is an integral curve of the $\mathcal{KVF}$ ξ or the horizontal curvature of M is 1.*

Remark 23. *In addition, Inoguchi and Munteanu generalized several results of O'Neill from horizontal geodesics to horizontal $\mathcal{MC}$s. They also investigated magnetic Jacobi fields along horizontal Killing magnetic trajectories in Sasakian space forms of dimension 3.*

10. Epilogue

The concept of magnetic lines of force, or $\mathcal{MC}$s, was introduced in the 1830s by Michael Faraday; his work provided the foundation for the modern understanding of magnetic fields. A decade before Faraday, Hans C. Ørsted observed that an electric current creates a magnetic field, which indicates that electricity and magnetism are closely linked. Building on Faraday's work, James C. Maxwell provided a rigorous mathematical framework to the concept of electromagnetic fields in the 1860s. The famous Maxwell equations unified the theories of electricity and magnetism, confirming Faraday's ideas.

In differential geometry, a magnetic field in a Riemannian manifold (M, g) is a closed 2-form F, and the Lorentz force of a magnetic field F is a $(1, 1)$-tensor field satisfying the relation $g(\Phi(X), Y) = F(X, Y)$ for vector fields X, Y in M. The $\mathcal{MC}$s of F are $\mathcal{MC}$ γ obeying the Lorentz equation $\nabla_{\gamma'}\gamma' = \Phi(\gamma')$.

In recent decades, many notable findings regarding $\mathcal{MC}$s in Riemannian and semi-Riemannian manifolds have been achieved by many mathematicians. The authors intend that this review paper on $\mathcal{MC}$s will become a useful reference for graduate students and beginning researchers who want to work on this important subject, as well as for researchers who have already worked in the field.

Author Contributions: Conceptualization, B.-Y.C., F.A., and M.A.C.; methodology, B.-Y.C., F.A., and M.A.C.; software, F.A., M.A.K., and M.A.C.; validation, F.A., M.A.K., and M.A.C.; formal analysis, B.-Y.C., F.A., and M.A.C.; investigation, F.A., M.A.K., and M.A.C.; resources, F.A.; data curation, F.A., M.A.K., and M.A.C.; methodology, B.-Y.C., F.A., and M.A.C.; software and writing—original draft preparation, F.A., M.A.K., and M.A.C.; writing—review and editing, B.-Y.C.; visualization, F.A. and M.A.C.; supervision, B.-Y.C., F.A., and M.A.C.; project administration, F.A. and M.A.C.; funding acquisition, F.A. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2502).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used throughout this paper:

$\mathcal{MC}$	Magnetic curve
$\mathcal{KVF}$	Killing vector field
iff	if and only if

References

1. Faraday, M. Experimental researches in electricity. In *Abstracts of the Papers Communicated to the Royal Society of London*; Philosophical Transactions of the Royal Society of London: London, UK, 1839; pp. 125–162.
2. Fahie, J.J. *A History of Electric Telegraphy to the Year 1837*; E. & F. N. Spon: London, UK, 1884.
3. Nahin, P.J. Maxwell's grand unification. *IEEE Spectr.* **1992**, *29*, 45. [CrossRef]
4. Aydın, M.E. Magnetic curves associated to Killing vector fields in a Galilean space. *Math. Sci. Appl. E-Notes* **2016**, *4*, 144–150. [CrossRef]
5. Arnol'd, V.I. First steps in symplectic topology. *Russ. Math. Surv.* **1986**, *41*, 1. [CrossRef]
6. Arnol'd, V.I. Some remarks on flows of line elements and frames. In *Collected Works: Representations of Functions, Celestial Mechanics and KAM Theory, 1957–1965*; Springer: Berlin/Heidelberg, Germany, 2009; pp. 230–232.
7. Ginzburg, V.L. A charge in a magnetic field: Arnold's problems 1981–9, 1982–24, 1984–4, 1994–14, 1994–35, 1996–17, 1996–18. In *Arnold's Problems*; Springer: Berlin/Heidelberg, Germany, 2004; pp. 395–401.
8. Comtet, A. On the Landau levels on the hyperbolic plane. *Ann. Phys.* **1987**, *173*, 185–209. [CrossRef]
9. Sunada, T. Magnetic flows on a Riemannian surface. *Proc. KAIST Math. Workshop* **1993**, *8*, 93–108.
10. Adachi, T. Kähler magnetic flows for a manifold of constant holomorphic sectional curvature. *Tokyo J. Math.* **1995**, *18*, 473–483. [CrossRef]
11. Adachi, T. Kähler magnetic fields on a complex projective space, Proceedings of the Japan Academy. *Ser. A Math. Sci.* **1994**, *70*, 12–13.
12. Druță-Romaniuc, S.L.; Munteanu, M.I. Magnetic curves corresponding to Killing magnetic fields in $\mathbb{E}^3$. *J. Math. Phys.* **2011**, *52*, 113506. [CrossRef]
13. Bozkurt, Z.; Gök, I.; Yaylı, I.; Ekmekci, F.N. A new approach for magnetic curves in 3D Riemannian manifolds. *J. Math. Phys.* **2014**, *55*, 053501. [CrossRef]
14. Özdemir, Z.; Gök, I.; . Yayli, Y.; Ekmekci, F.N. Notes on magnetic curves in 3D semi-Riemannian manifolds. *Turk. J. Math.* **2015**, *39*, 412–426. [CrossRef]
15. Kazan, A.; Karadag, H.B. Magnetic curves according to Bishop frame and type-2 Bishop frame in Euclidean 3-space. *Br. J. Math. Comput. Sci.* **2017**, *22*, 1–18. [CrossRef]
16. Jleli, M.; Munteanu, M.I. Magnetic curves on flat para-Kähler manifolds. *Turk. J. Math.* **2015**, *39*, 963–969. [CrossRef]
17. Erjavec, Z.; Inoguchi, J. Geodesics and magnetic curves in the 4-dim almost Kähler model space F^4. *Complex Manifolds* **2024**, *11*, 20240001. [CrossRef]
18. Druță-Romaniuc, S.L.; Inoguchi, J.; . Munteanu, M.I.; Nistor, A.I. Magnetic curves in Sasakian manifolds. *J. Nonlinear Math. Phys.* **2015**, *22*, 428–447. [CrossRef]
19. Călin, C.; Crasmareanu, T.M. Magnetic curves in three-dimensional quasi-para-Sasakian geometry. *Mediterr. J. Math.* **2016**, *13*, 2087–2097. [CrossRef]

20. Bozdağ, Ş.N. A study on magnetic curves in trans-Sasakian manifolds, Analele Științifice ale Universității "Ovidius" Constanța. *Ser. Mat.* **2023**, *31*, 47–60.
21. Inoguchi, J.; Munteanu, M.I.; Nistor, A.I. Magnetic curves in quasi-Sasakian 3-manifolds. *Anal. Math. Phys.* **2019**, *9*, 43–61. [CrossRef]
22. Sarkar, A.; Sil, A.; Paul, A.K. On three-dimensional quasi-Sasakian manifolds and Magnetic curves. *Appl. Math. E-Notes* **2019**, *19*, 55–64.
23. Munteanu, M.I.; Nistor, A.I. Magnetic curves in quasi-Sasakian manifolds of product type. In *New Horizons in Differential Geometry and Its Related Fields*; World Scientific Publishing: Singapore, 2022; pp. 1–22.
24. Druță-Romaniuc, S.-L.; Inoguchi, J.; Munteanu, M.I.; Nistor, A.I. Magnetic curves in cosymplectic manifolds. *Rep. Math. Phys.* **2016**, *78*, 33–48. [CrossRef]
25. Erjavec, Z.; Inoguchi, J. On magnetic curves in almost cosymplectic *Sol* space. *Results Math.* **2020**, *75*, 113. [CrossRef]
26. Güvenç, S. Magnetic curves in C-manifolds. *Int. Electron. J. Geom.* **2022**, *15*, 145–152. [CrossRef]
27. Pandey, P.K.; Mohammad, S. Magnetic and slant curves in Kenmotsu manifolds. *Surv. Math. Its Appl.* **2020**, *15*, 139–151.
28. Inoguchi, J.; Munteanu, M.I. Magnetic curves in the real special linear group. *Adv. Theor. Math. Phys.* **2019**, *23*, 2161–2205. [CrossRef]
29. Özgür, C. On magnetic curves in the 3-dimensional Heisenberg group. *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerbaijan* **2017**, *43*, 278–286.
30. Inoguchi, J.; Munteanu, M.I. Magnetic curves on tangent sphere bundles, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. *Ser. A Mat.* **2019**, *113*, 2087–2112.
31. Munteanu, M.I.; Nistor, A.I. The classification of Killing magnetic curves in $S^2 \times \mathbb{R}$. *J. Geom. Phys.* **2012**, 62, 170–182. [CrossRef]
32. Druță-Romaniuc, S.L.; Munteanu, M.I. Killing magnetic curves in a Minkowski 3-space, *Nonlinear Anal. Real World Appl.* **2013**, *14*, 383–396. [CrossRef]
33. Calvaruso, G.; Munteanu, M.I.; Perrone, A. Killing magnetic curves in three-dimensional almost paracontact manifolds. *J. Math. Anal. Appl.* **2015**, *426*, 423–439. [CrossRef]
34. Erjavec, Z. On killing magnetic curves in $SL(2,\mathbb{R})$ geometry. *Rep. Math. Phys.* **2019**, *84*, 333–350. [CrossRef]
35. Güvenç, S.; Özgür, C. On slant magnetic curves in S-manifolds. *J. Nonlinear Math. Phys.* **2019**, *26*, 536–554. [CrossRef]
36. Lee, J. Slant curves and contact magnetic curves in Sasakian Lorentzian 3-manifolds. *Symmetry* **2019**, *11*, 784. [CrossRef]
37. Adachi, T. Curvature bound and trajectories for magnetic fields on a Hadamard surface. *Tsukuba J. Math.* **1996**, *20*, 225–230. [CrossRef]
38. Barros, M.; Romero, A.; Cabrerizo, J.L.; Fernández, M. The Gauss-Landau-Hall problem on Riemannian surfaces. *J. Math. Phys.* **2005**, *46*, 112905. [CrossRef]
39. do Carmo, M.P. *Differential Geometry of Curves and Surfaces*; Prentice Hall: Englewood Cliffs, NJ, USA, 1976.
40. Bishop, R.L. There is more than one way to frame a curve. *Am. Math. Mon.* **1975**, *82*, 246–251. [CrossRef]
41. Gözütok, U.; Çoban, H.; Sağiroğlu, Y. Frenet frame with respect to conformable derivative. *Filomat* **2019**, *33*, 1541–1550. [CrossRef]
42. Has, A.; Yilmaz, B. Effect of fractional analysis on magnetic curves. *Rev. Mex. Física* **2022**, *68*, 041401. [CrossRef]
43. Cho, J.T.; Lee, J.-E. Slant curves in contact pseudo-Hermitian 3-manifolds. *Bull. Aust. Math. Soc.* **2008**, *78*, 383–396. [CrossRef]
44. Ou, Y.L.; Chen, B.Y. *Biharmonic Submanifolds and Biharmonic Maps in Riemannian Geometry*; World Scientific Publishing: Hackensack, NJ, USA, 2020.
45. Eells, J.; Sampson, J.H. Harmonic mappings of Riemannian manifolds. *Am. J. Math.* **1964**, *86*, 109–160. [CrossRef]
46. Jiang, G.Y. 2-harmonic maps and their first and second variational formulas. *Chin. Ann. Math. Ser. A* **1986**, *7*, 389–402.
47. Yüksel Perktas, S.; Blaga, A.M.; Acet, B.E.; Erdogan, F. Magnetic biharmonic curves on 3-dimensional normal almost paracontact metric manifolds. *AIP Conf. Proc.* **2018**, *1991*, 020004. [CrossRef]
48. Calvaruso, G.; Munteanu, M.I. Hopf magnetic curves in the anti-de Sitter space $\mathbb{H}^3_1$. *J. Nonlinear Math. Phys.* **2018**, *25*, 462–484. [CrossRef]
49. Inoguchi, J.; Munteanu, M.I. Slant curves and magnetic curves. In *Contact Geometry of Slant Submanifolds*; Springer: Singapore, 2022; pp. 199–259.
50. Chen, B.Y. Slant immersions. *Bull. Aust. Math. Soc.* **1990**, *41*, 135–147. [CrossRef]
51. Chen, B.Y. *Geometry of Slant Submanifolds*; Katholieke Universiteit Leuven: Leuven, Belgium, 1990.
52. Chen, B.Y.; Tazawa, Y. Slant submanifolds in complex Euclidean spaces. *Tokyo J. Math.* **1991**, *14*, 101–120. [CrossRef]
53. Lotta, A. Slant submanifolds in contact geometry, Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie. *Nouv. Ser.* **1996**, *39*, 183–198.
54. Cabrerizo, J.L.; Carriazo, A.; Fernández, L.M.; Fernández, M. Slant submanifolds in Sasakian manifolds. *Glasg. Math. J.* **2000**, *42*, 125–138. [CrossRef]
55. Cho, J.T.; Inoguchi, J.; Lee, J.E. On slant curves in Sasakian 3-manifolds. *Bull. Aust. Math. Soc.* **2006**, *74*, 359–367. [CrossRef]

56. Inoguchi, J.; Lee, J.E. Slant curves in 3-dimensional almost contact metric geometry. *Int. Electron. J. Geom.* **2015**, *8*, 106–146. [CrossRef]
57. Munteanu, M.I.; Nistor, A.I. On some closed magnetic curves on a 3-torus. *Math. Phys. Anal. Geom.* **2017**, *20*, 8. [CrossRef]
58. Erjavec, Z.; Inoguchi, J. Killing magnetic curves in $\mathbb{H}^3$. *Int. Electron. J. Geom.* **2023**, *16*, 181–195. [CrossRef]
59. Kelekçi, Ö.; Dündar, F.S.; Ayar, G. Classification of Killing magnetic curves in $\mathbb{H}^3$. *Int. J. Geom. Methods Mod. Phys.* **2023**, *20*, 2450006. [CrossRef]
60. Korpinar, T.; Baş, S. A new approach for inextensible flows of binormal spherical indicatrices of magnetic curves. *Int. J. Geom. Methods Mod. Phys.* **2019**, *16*, 1950020. [CrossRef]
61. Körpınar, T.; Demirkol, R.C. Gravitational magnetic curves on 3D Riemannian manifolds. *Int. J. Geom. Methods Mod. Phys.* **2018**, *15*, 1850184. [CrossRef]
62. Kazan, A.; Karadag, H.B. Magnetic pseudo null and magnetic null curves in Minkowski 3-space. *Int. Math. Forum* **2017**, *12*, 119–132. [CrossRef]
63. Kazan, A.; Karadağ, H.B. Magnetic non-null curves according to parallel transport frame in Minkowski 3-space, Communications. Faculty of Sciences. University of Ankara. Series A1. *Math. Stat.* **2018**, *67*, 147–160.
64. Wang, Y.; An, P.; Zhao, C. The magnetic curves in 3-dimensional Minkowski space according to killing vector. *Acta Math. Appl. Sin.* **2023**, *46*, 835–844.
65. Han Zhang, H.; Liu, H.; Chen, X. Killing vectors and magnetic curves in Lorentzian α-Sasakian 3-manifolds. *Mod. Phys. Lett. A* **2024**, *39*, 2450163. [CrossRef]
66. Inoguchi, J.; Munteanu, M.I. Periodic magnetic curves in Berger spheres. *Tohoku Math. J. Second. Sers.* **2017**, *69*, 113–128. [CrossRef]
67. Munteanu, M.I.; Nistor, A.I. A note on magnetic curves on S^{2n+1}. *C. R. Math.* **2014**, *352*, 447–449. [CrossRef]
68. Güvenç, S.; Özgur, C. On pseudo-Hermitian magnetic curves in Sasakian manifolds. *arXiv* **2021**, arXiv:2002.05214.
69. Deshmukh, S. Trans-Sasakian manifolds homothetic to Sasakian manifolds. *Mediterr. J. Math.* **2016**, *13*, 2951–2958. [CrossRef]
70. Marrero, J.C. The local structure of trans-Sasakian manifolds. *Ann. Mat. Pura Appl.* **1992**, *162*, 77–86. [CrossRef]
71. Sato, I. On a structure similar to the almost contact structure. *Tensor* **1976**, *30*, 219–224.
72. Bejan, C.L.; Binh, T.Q.; Druţă-Romaniuc, S.L. Magnetic Frenet curves on para-Sasakian manifolds. *Filomat* **2023**, *37*, 1479–1496. [CrossRef]
73. Bozdağ, Ş.N.; Erdoğan, F.E. f-biharmonic and bi-f-harmonic magnetic curves in three-dimensional normal almost paracontact metric manifolds. *Int. Electron. J. Geom.* **2021**, *14*, 331–347. [CrossRef]
74. Perrone, A. Magnetic curves of the Reeb vector field of a normal almost paracontact three-manifold, Rendiconti del Seminario Matematico. *Univ. Politec. Torino* **2015**, *73*, 171–182.
75. Lee, J.-E. Pseudo-Hermitian magnetic curves in normal almost contact metric 3-manifolds. *Commun. Korean Math. Soc.* **2020**, *35*, 1269–1281.
76. Yano, K. On a structure defined by a tensor field f of type $(1,1)$ satisfying $f^3 + f = 0$. *Tensor* **1963**, *14*, 99–109.
77. Blair, D.E. Geometry of manifolds with structural group $\mathcal{U}(n) \times \mathcal{O}(s)$. *J. Differ. Geom.* **1970**, *4*, 155–167. [CrossRef]
78. Güvenç, Ş. The complete classification of quarter-symmetric magnetic curves in S-manifolds. *J. Mahani Math. Res.* **2023**, *12*, 151–160.
79. Güvenç, Ş.; Özgür, C. Magnetic curves in homothetic s-th Sasakian manifolds. *Mathematics* **2025**, *13*, 159. [CrossRef]
80. Bejan, C.L.; Druta-Romaniuc, S.L. F-geodesic on manifolds. *Filomat* **2015**, *29*, 2367–2379 [CrossRef]
81. Alegre, P.; Barrera, J.; Carriazo, A.; Garcia-Cano, D. A generalization of magnetic curves in contact metric geometry. *Results Math.* **2024**, *79*, 141. [CrossRef]
82. Kelekçi, Ö. Kähler magnetic curves in conformally Euclidean Schwarzschild space. *Cumhur. Sci. J.* **2024**, *45*, 147–152. [CrossRef]
83. Blair, D.E. *Riemannian Geometry of Contact and Symplectic Manifolds*; Progress in Mathematics; Birkhäuser: Boston, MA, USA, 2010; xii+260p.
84. Derkaoui, K.; Hathout, F. Explicit formulas for Killing magnetic curves in three-dimensional Heisenberg group. *Int. J. Geom. Methods Mod. Phys.* **2021**, *18*, 2150135. [CrossRef]
85. Druţă-Romaniuc, S.L.; Inoguchi, J.; Munteanu, M.I. Killing slant magnetic curves in the 3-dimensional Heisenberg group Nil_3. *Int. J. Geom. Methods Mod. Phys.* **2023**, *20*, 2350094. [CrossRef]
86. Gonzalez, J.C.; Chinea, D. Quasi-sasakian homogeneous structures on the generalized Heisenberg group $\mathbb{H}(p,1)$. *Proc. Am. Math. Soc.* **1989**, *105*, 173–184.
87. Munteanu, M.I.; Nistor, A.I. Magnetic curves in the generalized Heisenberg group. *Nonlinear Anal.* **2022**, *214*, 112571. [CrossRef]
88. Berger, M. Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive. *Ann. Sc. Norm. Super. Pisa* **1961**, *15*, 179–246.
89. Wilking, B. The normal homogeneous space $(\mathsf{SU}(3) \times \mathsf{SO}(3))/\mathsf{U}^\bullet(2)$ has positive sectional curvature. *Proc. Am. Math. Soc.* **1999**, *127*, 1191–1194. [CrossRef]

90. Bosak, M.; Erjavec, Z.; Klemenčić, D. On killing magnetic curves in the hyperboloid model of $\widetilde{SL}(2,\mathbb{R})$ geometry. *Sarajevo J. Math.* **2024**, *20*, 291–307.
91. Erjavec, Z.; Inoguchi, J. Magnetic curves in Sol_3. *J. Nonlinear Math. Phys.* **2018**, *25*, 198–210. [CrossRef]
92. Troyanov, M. L'horizon de SOL. *Expo. Math.* **1998**, *16*, 441–479.
93. Erjavec, Z.; Inoguchi, J. Killing magnetic curves in *Sol* space. Mathematical Physics. *Anal. Geom.* **2018**, *21*, 15. [CrossRef]
94. Kowalski, O. *Generalized Symmetric Spaces, Lecture Notes in Mathematics*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1980.
95. Brozos-Vázquez, M.; García-Río, E.; Gilkey, P.; S Nikčević, S.; Vázquez-Lorenzo, R. The geometry of Walker manifolds. In *Synthesis Lectures on Mathematics and Statistics*; Krantz, S.G., Ed.; Washington University: St. Louis, MO, USA, 2009.
96. Walker, A.G. Canonical form for a Riemannian space with parallel field of null planes. *Q. J. Math.* **1950**, *1*, 69–79. [CrossRef]
97. Chen, B.Y. Differential geometry of identity maps: A survey. *Mathematics* **2020**, *8*, 1264. [CrossRef]
98. Chen, B.Y.; Nagano, T. Harmonic metrics, harmonic tensors, and Gauss maps. *J. Math. Soc. Jpn.* **1984**, *36*, 295–313. [CrossRef]
99. Bejan, C.L.; Druţă-Romaniuc, S. Walker manifolds and Killing magnetic curves. *Differ. Geom. Its Appl.* **2014**, *35*, 106–116. [CrossRef]
100. Sasaki, S. Geodesics on the tangent sphere bundles over space forms. *J. Reine Angew. Math.* **1976**, *288*, 106–120.
101. Klingenberg, W.; Sasaki, S. On the tangent sphere bundle of a 2-sphere. *Tohoku Math. J.* **1975**, *27*, 49–56. [CrossRef]
102. Inoguchi, J.; Munteanu, M.I. Magnetic curves in tangent sphere bundles II. *J. Math. Anal. Appl.* **2018**, *466*, 1570–1581. [CrossRef]
103. Bejan, C.L.; Druţă-Romaniuc, S.L. Magnetic curves on cotangent bundles endowed with the Riemann extension. *Colloq. Math.* **2022**, *168*, 47–58. [CrossRef]
104. Erjavec, Z.; Inoguchi, J. Magnetic curves in $\mathbb{H}^3 \times \mathbb{R}$. *J. Korean Math. Soc.* **2021**, *58*, 1501–1511.
105. Özgür, C.; Güvenç, S. On biharmonic Legendre curves in S-space forms. *Turk. J. Math.* **2014**, *38*, 454–461. [CrossRef]
106. Guvenc, S.; Ozgur, C. On slant curves in *S*-manifolds. *Commun. Korean Math. Soc.* **2018**, *33*, 293–303.
107. Inoguchi, J.; Munteanu, M.I. Killing submersions and magnetic curves. *J. Math. Anal. Appl.* **2023**, *520*, 126889. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.