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The Compactness of Right Inverse of Imaginary Part of Reggeon Field Theory Hamiltonian on Bargmann Space

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Abstract

The Hamiltonian of Reggeon field theory is defined by $H_{\mu,\lambda} = \mu A^* A + i\lambda A^*(A + A^*)A$, where A and A^* are the annihilation and creation operators satisfying $[A, A^*] = I$ and μ, λ are real parameters, and $i^2 = -1$. This operator acts on Bargmann space $\mathbb{B}$ where $\mathbb{B}$ is a Hilbert space of holomorphic square integrable functions with respect to the Gaussian-weighted Lebesgue measure. In this work, we consider the operator $H_\lambda = i\lambda A^*(A + A^*)A$ with maximum domain $D(H_\lambda) = \{\varphi \in \mathbb{B}; H_\lambda \varphi \in \mathbb{B}\}$. If we limit the domain to polynomials and take the closure of the obtained operator, we denote it by H_λ^{min} , of which H_λ is obviously an extension. Contrary to what happens for $\mu \neq 0$, it is well known that these two operators are different. The main purpose of the present work is to show that H_λ admits a right-inverse K_λ , i.e., $H_\lambda K_\lambda = I$ on negative imaginary axis and that K_λ is compact.

Keywords: non-self-adjoint operators; cubic Heun's operator; Bargmann space; reggeon field theory

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1. Introduction

We recall that the Hamiltonian of Reggeon Field Theory is defined by

$$H_{\mu,\lambda} = \mu A^* A + i\lambda A^*(A + A^*)A \quad (1)$$

where A and A^* are the annihilation and creation operators, μ and λ are real parameters, and $i^2 = -1$.

This operator is considered to act on Bargmann space [1]:

$$\mathbb{B} = \{\varphi : \mathbb{C} \longrightarrow \mathbb{C} \text{ entire; } \int_{\mathbb{C}} |\varphi(z)|^2 e^{-|z|^2} dx dy < \infty\} \quad (2)$$

where its usual basis is given by

$$e_n(z) = \frac{z^n}{\sqrt{n!}}; n \in \mathbb{N} \quad (3)$$

or

$$\tilde{e}_n(z) = \frac{(iz)^n}{\sqrt{n!}}; n \in \mathbb{N} \quad (4)$$

and the annihilation operator A and creation operator A^* are defined by

$$A\varphi := \frac{d\varphi}{dz} \quad \varphi \in \mathbb{B} \quad (5)$$

$$A^*\varphi = z\varphi \quad \varphi \in \mathbb{B} \quad (6)$$

Action of A and A^* on usual basis $(e_n(z))_{n \in \mathbb{N}}$ are given, respectively, by

$$Ae_n = \sqrt{n}e_{n-1}, \quad n \geq 1 \quad \text{and} \quad Ae_0 = 0 \quad (7)$$

$$A^*e_n = \sqrt{n+1}e_{n+1}, \quad n \geq 0 \quad (8)$$

Remark 1 (Some Specific Properties on Bargmann's Space).

- In [1] it is well known that $L^2(\mathbb{R})$ is related to $\mathbb{B}$ by an unitary transform (called Bargmann transform) of $L^2(\mathbb{R})$ onto $\mathbb{B}$, given by the following transform

$$\mathcal{B}(f) := \varphi(z) = c \int_{\mathbb{R}} e^{-\frac{1}{2}z^2 - \frac{1}{2}q^2 + \sqrt{2}zq} f(q); \quad f \in L^2(\mathbb{R}) \quad \text{and} \quad c > 0$$

- In [2], I established several properties on Bargmann space:

(a) $\mathbb{B}$ is related to $\mathbb{B}_s$ by an unitary transform of $\mathbb{B}$ onto $\mathbb{B}_s$, given by the following transform:

$$\mathcal{I} : \mathbb{B} \longrightarrow \mathbb{B}_s; \quad \mathcal{I}(\varphi) = \left(\frac{1}{n!} \varphi^{(n)}(0) \right)_0^\infty := (a_n)_0^\infty$$

where

$$\mathbb{B}_s = \left\{ (a_n)_0^\infty \in \mathbb{C} \text{ such that } \sum_{n=0}^{\infty} n! |a_n|^2 < \infty \right\}.$$

- (b) Lemmas 1 and 2 and in particular Lemma 3 link a function of the Bargmann space with its derivative. This lemma plays a fundamental role in the study of the Hamiltonian of reggeon field theory:

Lemma 1 (Lemma 0.3 [2]).

$$\text{In Bargmann space, we have } \varphi \in \mathbb{B} \iff z \longrightarrow \frac{\varphi'(z) - \varphi'(0)}{z} \in \mathbb{B}.$$

The expressions of $H_{\mu,\lambda}$ and H_λ are given, respectively, by

$$H_{\mu,\lambda}\varphi = \mu z \frac{d\varphi}{dz} + i\lambda \left(z \frac{d^2\varphi}{dz^2} + z^2 \frac{d\varphi}{dz} \right) \quad \varphi \in \mathbb{B} \quad (9)$$

$$H_\lambda\varphi = i\lambda \left(z \frac{d^2\varphi}{dz^2} + z^2 \frac{d\varphi}{dz} \right) \quad \varphi \in \mathbb{B} \quad (10)$$

- In [2], we have given a complete spectral analysis of the following $\mathbb{C}$ -symmetric matrices which play an important role in Reggeon field theory

$$\mathbb{H}_n^{\mu,\lambda} = \begin{pmatrix} \mu & i\lambda\sqrt{2} & 0 & \cdot & \cdot & \cdot & \cdot \\ i\lambda\sqrt{2} & 2\mu & i\lambda 2\sqrt{3} & \ddots & \cdot & \cdot & \cdot \\ 0 & i\lambda 2\sqrt{3} & 3\mu & i\lambda 3\sqrt{4} & \ddots & \cdot & \cdot \\ \cdot & \ddots & \ddots & \ddots & \ddots & \ddots & \cdot \\ \cdot & \cdot & \ddots & \ddots & \ddots & \ddots & 0 \\ \cdot & \cdot & \cdot & \ddots & \ddots & \ddots & i\lambda(n-1)\sqrt{n} \\ \cdot & \cdot & \cdot & \cdot & 0 & i\lambda(n-1)\sqrt{n} & n\mu \end{pmatrix}. \quad (11)$$

where μ, λ are real parameters and $i^2 = -1$.

In his consideration of the classical Carathéodory–Féjer problem in function theory, Takagi [3] observed the relevance of the anti-linear eigenvalue problem $TX = \lambda\bar{X}$, where T is an $n \times n$ symmetric complex matrix and X denotes complex conjugation of a vector X in $\mathbb{C}^n$. He noted that this equation implies that $T^*TX = |\lambda|^2X$ and hence that $|\lambda|$ is an eigenvalue of $|T| = \sqrt{T^*T}$.

According this observation, Garcia and Putnar give in [4] the following definition:

Definition 1 (Complex-Symmetric Operators). *A bounded linear operator T on a complex Hilbert space $\mathcal{H}$ is called complex symmetric if $T = CT^*C$, where C is a conjugation (an isometric, anti-linear involution) of $\mathcal{H}$.*

Remark 2.

- (i) $\mathbb{H}_n^{\mu,\lambda} \neq {}^t\overline{\mathbb{H}_n^{\mu,\lambda}}$ (i.e., $\mathbb{H}_n^{\mu,\lambda}$ is non-self adjoint).
- (ii) These $\mathbb{C}$ -symmetric matrices $\mathbb{H}_n^{\mu,\lambda}$ approximate our unbounded operator $H_{\mu,\lambda}$ in the sense that $[\mathbb{H}_n^{\mu,\lambda}]^{-1}$ approximate $H_{\mu,\lambda}^{-1}$ (see [2], Theorem 2.1 page 345).
- (iii) It is well known that for $\mu \neq 0$, $H_{\mu,\lambda}$ is invertible and has compact resolvent because its maximal domain $D(H_{\mu,\lambda})$ is compactly embedded in Bargmann space $\mathbb{B}$ (see [2]). Now for $\mu = 0$, $H_\lambda := H_{0,\lambda}$ is not invertible because its spectrum $\sigma(H_\lambda) = \mathbb{C}$ [5]. Hence it is natural to ask whether H_λ admits a left inverse or a right inverse, and if either of them exists, to see whether it is unbounded or bounded, and even better, whether it is compact.
- It is well known for several years that the eigenvalues of this operator $H_{\mu,\lambda}$; $\mu \neq 0$ are real and that, recently in 2023, we have shown the completeness of its generalized eigenvectors (for this topic see [6]).

Let

$$\mathbb{B}_0 = \{\varphi \in \mathbb{B}; \varphi(0) = 0\} \quad (12)$$

Then, it is well known that on $\mathbb{B}_0$, for $\psi \in \mathbb{B}_0$, an explicit inverse of $H_{\mu,\lambda}$ restricted on imaginary axis; $y \in [0, +\infty[$ is given (see [7], Proposition 9) by

$$K_{\mu,\lambda} := H_{\mu,\lambda}^{-1}\psi(-iy) = \int_0^\infty \mathcal{N}_{\mu,\lambda}(y,s)\psi(-is)ds \quad (13)$$

where

$$\mathcal{N}_{\mu,\lambda}(y,s) = \frac{1}{\lambda s} e^{-\frac{s^2}{2} - \frac{\mu}{\lambda}s} \int_0^{\min(y,s)} e^{\frac{u^2}{2} + \frac{\mu}{\lambda}u} du \quad (14)$$

By taking $\mu = 0$ and $\lambda \neq 0$, we obtain the following:

$$K_\lambda := H_\lambda^{-1}\psi(-iy) = \int_0^\infty \mathcal{N}_\lambda(y,s)\psi(-is)ds \quad (15)$$

where

$$\mathcal{N}_\lambda(y, s) = \frac{1}{\lambda s} e^{-\frac{s^2}{2}} \int_0^{\min(y, s)} e^{\frac{u^2}{2}} du \quad (16)$$

or by taking $\lambda = 1$ and $\mu \neq 0$ (see Lemme 4 in [8])

$$K_\mu := H_{\mu,1}^{-1} \psi(-iy) = \int_0^\infty \mathcal{N}_\mu(y, s) \psi(-is) ds \quad (17)$$

where

$$\mathcal{N}_\mu(y, s) = \frac{1}{s} e^{-\frac{s^2}{2} - \mu s} \int_0^{\min(y, s)} e^{\frac{u^2}{2} + \mu u} du \quad (18)$$

The kernel of operator $H_{\mu,1}^{-1}$ is analytic with respect to μ . The integral operator defined by this kernel extends into a compact operator on a space L_2 with weight, including for negative values of μ .

In particular, we have the following.

Proposition 1 (See Proposition 9 in [7,8] or Lemmas 4 and 8 in [8]).

Let $L_2([0, \infty[, e^{-x^2 - 2\frac{\mu}{\lambda}x} dx)$ with $\lambda \neq 0$ be a space of square integrable functions with respect to the measure $e^{-x^2 - 2\frac{\mu}{\lambda}x} dx$, then we have:

- (i) For all $\mu > 0$, $H_{\mu,\lambda}^{-1}$ can be extended to a Hilbert–Schmidt operator of $L_2([0, \infty[, e^{-x^2 - 2\frac{\mu}{\lambda}x} dx)$ to itself.
- (ii) The map $\mu \rightarrow K_{\mu,\lambda}$ is analytic on $[0, +\infty[$ in Hilbert–Schmidt norm operators on $L_2([0, \infty[, e^{-x^2 - 2\frac{\mu}{\lambda}x} dx)$.
- (iii) For $\mu > 0$, the smallest eigenvalue $\sigma_0(\mu)$ of $H_{\mu,\lambda}$ which is simple can be extended to a real positive analytical function and is increasing with respect to μ on entire real axis.

Remark 3. $H_{\mu,\lambda}$ is non-normal operator; nevertheless, it has several properties analogous to those of the self-adjoint operators.

- ₁ In 1987, we have given in [7] (see also [2]) many non-trivial spectral properties of $H_{\mu,\lambda}$ for $\mu > 0$:
 - (i) $\rho(H_{\mu,\lambda}) \neq \emptyset$ where $\rho(H_{\mu,\lambda})$ is the resolvent set of $H_{\mu,\lambda}$.
 - (ii) $D_{\min}(H_{\mu,\lambda}) = D_{\max}(H_{\mu,\lambda})$ where $D_{\max}(H_{\mu,\lambda}) = \{\varphi \in \mathbb{B}; H_{\mu,\lambda} \varphi \in \mathbb{B}\}$.
 - (iii) $D_{\min}(H_{\mu,\lambda}) = \{\varphi \in \mathbb{B}; \exists p_n \in \mathcal{P}, \exists \psi \in \mathbb{B}; \lim_{n \rightarrow +\infty} p_n = \varphi \text{ and } \lim_{n \rightarrow +\infty} H_{\mu,\lambda} p_n = \psi\}$.
 - (iv) For $\mu > 0$ $\|H_{\mu,\lambda} \varphi\| \geq \mu \|A \varphi\| \quad \forall \varphi \in D_{\max}(H_{\mu,\lambda})$.
 - (v) The injection of $D_{\max}(H_{\mu,\lambda})$ in $D(A) = \{\varphi \in \mathbb{B}; \frac{d}{dz} \varphi \in \mathbb{B}\}$ is continuous.
 - (vi) The injections of $D(A)$ and of $D_{\max}(H_{\mu,\lambda})$ in $\mathbb{B}$ are compact.
 - (vii) The existence of the smallest eigenvalue $\sigma_0 \neq 0$ and asymptotic expansion of semigroup $e^{-tH_{\mu,\lambda}}$ and the elements of the diffusion matrix as $t \rightarrow +\infty$.

More precisely, for $\mu > 0$, σ_0 and σ_1 , denote the smallest and the second eigenvalue of $H_{\mu,\lambda}$, respectively, then we have the following:

$$\|e^{-tH_{\mu,\lambda}}\| = e^{-t\sigma_0} + e^{-\epsilon t}, \forall \epsilon < \sigma_1 \quad (19)$$

$$\langle e^{-tH_{\mu,\lambda}} \varphi, \psi \rangle = \langle \varphi, \phi_0^* \rangle \langle \phi_0, \psi \rangle e^{-\sigma_0 t} + O(e^{-\sigma_1 t}), \forall \varphi, \psi \in \mathbb{B} \quad (20)$$

where $H_{\mu,\lambda} \phi_0 = \sigma_0 \phi_0$ and ϕ_0^* is the eigenfunction of $H_{\mu,\lambda}^*$ associated to σ_0 such that $\langle \phi_0, \phi_0^* \rangle = 1$.

Remark 4.

- (i) The operator $H_\lambda = i\lambda A^*(A + A^*)A$ with maximal domain $D(H_\lambda) = \{\varphi \in \mathcal{H}; H_\lambda \varphi \in \mathcal{H}\}$ is formally anti-adjoint, i.e., it is equal to the opposite of its adjoint.
- (ii) The maximal domain of H_λ is not equal to its minimal domain.
- (iii) For $\mu = 0$ in $H_{\mu,\lambda}$, the $\mathbb{C}$ -symmetric matrices:

$$\mathbb{H}_n^\lambda = \begin{pmatrix} 0 & i\lambda\sqrt{2} & 0 & \cdot & \cdot & \cdot & \cdot \\ i\lambda\sqrt{2} & 0 & i\lambda 2\sqrt{3} & \ddots & \cdot & \cdot & \cdot \\ 0 & i\lambda 2\sqrt{3} & 0 & i\lambda 3\sqrt{4} & \ddots & \cdot & \cdot \\ \cdot & \ddots & \ddots & \ddots & \ddots & \ddots & \cdot \\ \cdot & \cdot & \ddots & \ddots & \ddots & \ddots & 0 \\ \cdot & \cdot & \cdot & \ddots & \ddots & \ddots & i\lambda(n-1)\sqrt{n} \\ \cdot & \cdot & \cdot & \cdot & 0 & i\lambda(n-1)\sqrt{n} & 0 \end{pmatrix}. \quad (21)$$

approximate H_λ :

$$H_\lambda = i\lambda A^*(A + A^*)A = i\lambda \left[z \frac{d^2}{dz^2} + z^2 \frac{d}{dz} \right] \quad (22)$$

where λ is real parameter and $i^2 = -1$.

- (iv) $\mathbb{H}_n^\lambda$ is equivalent to

$$\mathbb{H}_n^\lambda = \begin{pmatrix} 0 & -2\lambda^2 & 0 & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & -2^2.3\lambda^2 & \ddots & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & -3^2.4\lambda^2 & \ddots & \cdot & \cdot \\ \cdot & \ddots & \ddots & \ddots & \ddots & \ddots & \cdot \\ \cdot & \cdot & \ddots & \ddots & \ddots & \ddots & 0 \\ \cdot & \cdot & \cdot & \ddots & \ddots & \ddots & -n(n-1)^2\lambda^2 \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 \end{pmatrix}. \quad (23)$$

then $\mathbb{H}_n^\lambda$ is not symmetric but is $\mathbb{C}$ -symmetric.

Remark 5. Let $\alpha_n = 1, n = 1, 2, \dots, \beta_n = 0; n = 1, 2, \dots$ and $|\gamma_n| = n(n-1)^2\lambda^2; n = 2, 3, \dots$, then we have

- (i) $\lim_{n \rightarrow \infty} |\gamma_n| = \infty$ and $|\gamma_n| \sim n^3$
- (ii) $\sum_{n=1}^{\infty} \frac{1}{|\gamma_n|} < \infty$

In the sequel, we will consider the following integral operator:

$$K_\lambda \psi(-iy) = \int_0^\infty \mathcal{N}_\lambda(y, s) \psi(-is) ds \quad (24)$$

where

$$\mathcal{N}_\lambda(y, s) = \frac{1}{\lambda s} e^{-\frac{s^2}{2}} \int_0^{\min(y, s)} e^{\frac{u^2}{2}} du \quad (25)$$

The cubic term H_λ parameterized by λ is analogous to the Lindblad operators for describing non-unitarity in open quantum systems.

- In 1998, we have given in [9] the boundary conditions at infinity for a description of all maximal dissipative extensions in Bargmann space of the minimal Heun's

operator $H_I = z \frac{d^2}{dz^2} + z^2 \frac{d}{dz}$. The characteristic functions of the dissipative extensions were computed and some completeness theorems were obtained for the system of generalized eigenvectors. It is well known that the restriction H_I^{min} of the closure of H_I on the polynomials set $\mathcal{P}$ is symmetric. But the minimal domain $D(H_I^{min}) = \{\varphi \in \mathbb{B}, \exists p_n \in \mathcal{P}, p_n \rightarrow \varphi, \exists \psi \in \mathbb{B}; H_I p_n \rightarrow \psi\}$ of H_I is different of its maximal domain $D(H_I) = \{\varphi \in \mathbb{B}; H_I \varphi \in \mathbb{B}\}$.

- ₃ It is also well known that H_I is chaotic operator in Devaney's sense [10] (see reference [5]). In particular its spectrum is $\sigma(H_I) = \mathbb{C}$. It follows that its action on the standard orthonormal basis $\{e_n\}_{n \geq 0}$ in the Bargmann $\mathbb{B}$ is given by

$$H_I^{min} e_n = b_n e_{n+1} + a_n e_n + b_{n-1} e_{n-1}; a_n = 0, b_n = n\sqrt{n+1} > 0, n \geq 1 \quad (26)$$

Then H_I^{min} can be represented in $\mathbb{B}_0 = \{\varphi \in \mathbb{B}; \varphi(0) = 0\}$ by an infinite tridiagonal matrix

$$\mathcal{H}_I = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & \dots & \dots \\ \sqrt{2} & 0 & 2\sqrt{3} & 0 & \dots & \dots \\ 0 & 2\sqrt{3} & 0 & 3\sqrt{4} & \dots & \dots \\ 0 & 0 & 3\sqrt{4} & 0 & \ddots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \quad (27)$$

known as the Jacobi–Gribov matrix.

- ₄ In 2025, we studied in [11] the deficiency numbers of the generalized operator $H^{p,m} = A^{*p}(A^m + A^{*m})A^p; p, m = 1, 2, \dots$ acting on Bargmann space where ($H^{1,1} = H_I$) and we gave some conditions on the parameters p and m such that $H^{p,m}$ must be completely indeterminate. It follows from these conditions that $H^{p,m}$ is entirely of minimal type and is connected to chaotic operator.

We now describe briefly the contents of this paper, section by section.

In Section 2, according to Askey and Wilson in [12] on some hypergeometric orthogonal polynomials, we give some spectral properties associated with H_I ; in Section 3, we give some properties of action of H_λ on the Bargmann basis $\tilde{e}_n(z) = \frac{(iz)^n}{\sqrt{n!}}$ where $z = -iy$ with $y > 0$ and in Section 4, we show that H_λ has a right-inverse K_λ on negative imaginary axis and that K_λ is compact.

2. Some Spectral Properties Associated to H_I in the Askey–Wilson Sense

Let $H_I = A^*(A + A^*)A$ with domain $D(H_I) = \{\varphi \in \mathbb{B}_0; H_I \varphi \in \mathbb{B}_0\}$.

Now, we denote $H_{I|_{\mathcal{P}}}$ the above operator if we limit its domain to polynomials $\mathcal{P}_0$ where $\mathcal{P}_0 = \{p \in \mathcal{P}; p(0) = 0\}$ and we denote the closure of this restriction by H_I^{min} .

Hence H_I is obviously an extension to this closure.

It is well known that the minimal domain of H_I is different of its maximal domain contrary to minimal and maximal domains of $H_{\mu,\lambda}$ which coincide for $\mu \neq 0$.

According to Askey and Wilson [12] on some hypergeometric orthogonal polynomials, we deduce that the above Jacobi matrix associated with H_I is related to the set of polynomials $P_n(x)$ of degree n satisfying the recurrent relation,

$$b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x) = x P_n(x), n \geq 2 \quad (28)$$

with the following initial conditions:

$$P_0(x) = 0 \quad P_1(x) = 1 \quad (29)$$

where $a_n = 0$, and $b_n = n\sqrt{n+1}$.

Definition 2. The polynomials $P_n(x)$, which solve to above recurrence relation and subject to the initial conditions $P_1(x) = 1$ and $P_0(x) = 0$, are called polynomials of the first kind.

The polynomials $Q_n(x)$, which satisfy the above recurrence relation and are subject to the initial conditions $Q_1(x) = 1$ $Q_2(x) = \frac{1}{\sqrt{2}}$, are called polynomials of the second kind.

Remark 6 (discrete Wronskian).

Let $b_n u_{n+1} + b_{n-1} u_{n-1} = \sigma u_n$, $n \geq 2$ the spectral equation for H_I . Then for every two solutions $u = (u_n)$ and $v = (v_n)$ of above equation with the same parameter σ , the discrete Wronskian is defined by

$$W(u, v) = b_n(u_n v_{n+1} - u_{n+1} v_n) \quad (30)$$

which is independent of n .

For the polynomials $(P_n(x))$ and $(Q_n(x))$, the wronskian is equal to one for every x . It follows that

$$P_n Q_{n+1} - P_{n+1} Q_n = \frac{1}{b_n} \quad (31)$$

Definition 3 (Borzov [13]).

A polynomial set $\{\psi_n(x)\}_{n=1}^{\infty}$ is called a canonical polynomial system if it is defined by the following recurrence relations:

$$c_{n-1} \psi_{n-1}(x) + c_n \psi_{n+1}(x) = x \psi_n(x) \quad n \geq 1, \quad c_0 = 0 \quad (32)$$

$$\psi_1(x) = 1 \quad (33)$$

where the positive sequence $\{c_n\}_{n=1}^{\infty}$ is given.

Lemma 2.

- (i) The polynomial set $\{P_n(x)\}_{n=1}^{\infty}$ is a canonical polynomial system.
- (ii) The polynomials $P_n(x)$ have real coefficients and fulfill the following parity conditions

$$P_n(-x) = (-1)^{n-1} P_n(x) \quad (34)$$

(iii)

$$b_{n-1} b_{n+1} \leq b_n^2 \quad (35)$$

(iv)

$$\sum_{n=1}^{\infty} \frac{1}{b_n} < \infty \quad (36)$$

(v) The operator H_I has the deficiency indices $(1, 1)$.

Proof.

- (i) As the sequence b_n is positive, then the proof of this property is trivial by taking $c_n = b_n$.
- (ii) As the coefficients of $P_1(x) = 1$ and the coefficients of $P_2(x) = \frac{x}{b_1}$ are real, then we deduce by recurrence that the polynomials $P_n(x)$ have real coefficients. Now as $P_2(-x) = -\frac{x}{b_1} = (-1)P_2(x)$, then if we suppose that $P_n(-x) = (-1)^{n-1}P_n(x)$ and $P_{n-1}(-x) = (-1)^{n-2}P_{n-1}(x)$, then from the recurrence relation

$$b_n P_{n+1}(-x) = -x P_n(-x) - b_{n-1} P_{n-1}(-x)$$

we deduce that

$$b_n P_{n+1}(-x) = -x(-1)^{n-1} P_n(x) - b_{n-1}(-1)^{n-2} P_{n-1}(x)$$

and

$$b_n P_{n+1}(-x) = (-1)^n [x P_n(x) - b_{n-1} P_{n-1}(x)] = (-1)^n b_n P_{n+1}(x)$$

It follows that

$$P_{n+1}(-x) = (-1)^n P_{n+1}(x).$$

- (iii) Let $b_n = n\sqrt{n+1}$ then $b_{n-1} = (n-1)\sqrt{n}$ and $b_{n+1} = (n+1)\sqrt{n+2}$. It follows that $b_{n-1}b_{n+1} = (n-1)(n+1)\sqrt{n(n+2)}$ and $b_n^2 = n^2(n+1)$. This implies that

$$b_{n-1}b_{n+1} \leq b_n^2 \iff (n-1)^2 \leq n^3 \iff 3n \geq 2.$$

As $3n \geq 2$ holds for all $n \geq 1$, we deduce that $b_{n-1}b_{n+1} \leq b_n^2$.

- (iv) It is well known that the following series $\zeta(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges for $\alpha > 1$, where α

is real, then as $\sum_{n=1}^{\infty} \frac{1}{b_n} = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}} \equiv \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, it follows that $\sum_{n=1}^{\infty} \frac{1}{b_n} < \infty$.

- (v) By using theorem 1.5, Ch.VII [14].

For any operator T , we denote by $\sigma(T)$, $\sigma_{\text{ess}}(T)$ and $\sigma_p(T)$ the spectrum, the essential spectrum and the point spectrum of T , respectively. $\square$

Proposition 2. Let $H_I = z \frac{d^2}{dz^2} + z^2 \frac{d}{dz}$ acting on Bargmann space $\mathbb{B}_0$. For all $\varphi(z) = \sum_{n=1}^{\infty} \varphi_n \frac{z^n}{\sqrt{n!}}$, H_I can be defined by

$$\left\{ \begin{array}{l} [H_I \varphi](z) = \sum_{n=1}^{\infty} (H_I)_n \frac{z^n}{\sqrt{n!}} \\ \text{where} \\ (H_I \varphi)_1 = \sqrt{2} \varphi_2 \\ (H_I \varphi)_n = (n-1)\sqrt{n} \varphi_{n-1} + n\sqrt{n+1} \varphi_{n+1} \end{array} \right. \quad (37)$$

Then the point spectrum of H_I is $\mathbb{C}$.

Direct Proof. This direct proof is based on the following classical proposition: $\square$

Proposition 3. The Bertrand series $\sum_{n>1} u_n$ with general term $u_n = \frac{1}{n \ln(n)^\beta}$ converges if and only if $\beta > 1$.

Now, let us consider the sequence $(u_n(\xi)), n = 1, 2, \dots; \xi \in \mathbb{C}$; defined by the recurrence relation:

$$\left\{ \begin{array}{l} u_1(\xi) = 1 \\ u_2(\xi) = \frac{\xi}{\sqrt{2}} \\ (n-1)\sqrt{n} u_{n-1}(\xi) + n\sqrt{n+1} u_{n+1}(\xi) = \xi u_n(\xi) \quad n \geq 2 \end{array} \right. \quad (38)$$

and let $\varphi_{\xi}(z) = \sum_{n=1}^{\infty} u_n(\xi) \frac{z^n}{\sqrt{n!}}$, where $u_n(\xi)$ is defined by (38).

It is clear that $H_I \varphi_{\xi} = \xi \varphi_{\xi}$.

φ_{ξ} belongs to $\mathbb{B}_0$ we must prove that for all $\xi \in \mathbb{C}$ we have $(u_n(\xi)) \in \ell_2(\mathbb{N})$. For this, we will show that

$$\exists n_0 \text{ such that } \forall n \geq n_0 \text{ we have } |u_n(\xi)| \leq \frac{M}{\sqrt{n} \ln(n)} \quad (39)$$

where M is a constant independent of n .

To determine n_0 , we use the below relation when $n \rightarrow \infty$

$$\frac{1}{2n^{\frac{3}{2}} \ln(n)} \sim \frac{1}{\sqrt{n+1} \ln(n+1)} - \frac{\sqrt{n-1}}{\sqrt{n(n-1)} \ln(n-1)} - \frac{|\xi|}{n \sqrt{n(n+1)} \ln(n)} \quad (40)$$

which proves that the second part of the above equality is positive for $n \geq n_0$.

Now, we set $M = \max\{|u_{n_0}| \ln(n_0), |u_{n_0-1}| \ln(n_0-1)\}$ and we deduce (39) by recurrence. We use the following inequalities

$$|u_{n+1}| \leq |\xi| \frac{|u_n|}{n \sqrt{n+1}} + \frac{(n-1) \sqrt{n} |u_{n-1}|}{n \sqrt{n+1}}$$

and

$$\frac{(n-1) \sqrt{n}}{n \sqrt{(n+1)(n-1)} \ln(n-1)} + \frac{|\xi|}{n \sqrt{n(n+1)} \ln(n)} \leq \frac{1}{\sqrt{(n+1)} \ln(n+1)}$$

It follows that

$$|u_n(\xi)|^2 \leq \frac{M^2}{n \ln^2(n)} \text{ and } \sum_{n=1}^{\infty} |u_n(\xi)|^2 \leq M^2 \sum_{n=1}^{\infty} \frac{1}{n \ln^2(n)} < \infty$$

This implies that $(u_n(\xi)) \in \ell_2(\mathbb{N})$ for all $\xi \in \mathbb{C}$.

Second Proof. The second proof is based on the following classical lemma: $\square$

Lemma 3 (Raabe-Duhamel test).

We suppose $\forall n > 0 \ a_n > 0$.

If

$$\exists \alpha \in \mathbb{R}, \frac{a_{n+1}}{a_n} = 1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right),$$

then

- (i) $\alpha > 1 \implies \sum a_n$ converges
- (ii) $\alpha < 1 \implies \sum a_n$ diverges

Same conclusions if

$$\exists \alpha \in \mathbb{R}, \frac{a_{n+1}}{a_n} = 1 - \frac{1}{n} - \frac{\alpha}{n \ln(n)} + o\left(\frac{1}{n \ln(n)}\right),$$

and we apply Theorem 2.3 of the reference [11].

3. Some Properties of Integral Operator K_{λ} Action on the Bargmann Basis

$\tilde{e}_n(z) = \frac{(iz)^n}{\sqrt{n!}}$ Where $z = -iy$ with $y > 0$

We recall that on $\mathbb{B}_0 = \{\varphi \in \mathbb{B}; \varphi(0) = 0\}$, it was well known that an explicit inverse of

$$H_{\mu,\lambda} = i\lambda z \frac{d^2}{dz^2} + (i\lambda z^2 + \mu z) \frac{d}{dz} \quad (41)$$

restricted on imaginary axis, $y \in [0, +\infty[$ is given by

$$K_{\mu,\lambda}\psi(-iy) := H_{\mu,\lambda}^{-1}\psi(-iy) = \int_0^\infty \mathcal{N}_{\mu,\lambda}(y,s)\psi(-is)ds$$

where

$$\mathcal{N}_{\mu,\lambda}(y,s) = \frac{1}{\lambda s} e^{-\frac{s^2}{2} - \frac{\mu}{\lambda}s} \int_0^{\min(y,s)} e^{\frac{u^2}{2} + \frac{\mu}{\lambda}u} du.$$

It follows that

$$K_{\mu,\lambda}\psi(-iy) := H_{\mu,\lambda}^{-1}\psi(-iy) = \int_0^y e^{\frac{u^2}{2} + \frac{\mu}{\lambda}u} du \int_u^\infty \frac{1}{\lambda s} e^{-\frac{s^2}{2} - \frac{\mu}{\lambda}s} ds \quad (42)$$

and for $\mu > 0$, let $\sigma(\mu)$ be the smallest eigenvalue of the operator $H_{\mu,\lambda}$, then it is well known in [8] that $\sigma(\mu)$ extends to a positive, increasing, analytic function on the whole real line and that the limit value $\sigma(0)$ is an eigenvalue of $H_\lambda = i\lambda A^*(A + A^*)A$ in particular, $\sigma(0) \neq 0$.

Despite the difficulty of the absence of any relation between the domains of the self-adjoint and anti-adjoint parts of $H_{\mu,\lambda}$, this operator has a fine spectral property in the Bargmann representation: if we restrict to an imaginary semi-axis, its inverse is an integral operator with a positive kernel, which allows us to apply the Krein–Rutman theorem [15] and the Jentzsch theorem [16].

For $\mu = 0$, let

$$H_\lambda\varphi(z) = i\lambda z \left(\frac{d}{dz} + z \right) \frac{d}{dz} \varphi(z) \quad (43)$$

then for $z = -iy; y \in [0, \infty[$ and $\varphi(-iy) = u(y)$, we have the following:

$$H_\lambda u(y) = \lambda [-yu''(y) + y^2 u'(y)] \quad (44)$$

where $u'(y)$ denotes the first derivative of $u(y)$, and $u''(y)$ is its second derivative. And

$$\begin{aligned} K_\lambda\psi(-iy) &= \frac{1}{\lambda} \int_0^y e^{\frac{u^2}{2}} du \int_u^\infty \frac{1}{s} e^{-\frac{s^2}{2}} \psi(-is) ds = \frac{1}{\lambda} \int_0^y du \int_u^\infty \frac{1}{s} e^{-\frac{1}{2}(s^2 - u^2)} \psi(-is) ds \\ &= \frac{1}{\lambda} \left[- \int_0^y du \int_0^u \frac{1}{s} e^{-\frac{1}{2}(s^2 - u^2)} \psi(-is) ds + \int_0^y du \int_0^\infty \frac{1}{s} e^{-\frac{1}{2}(s^2 - u^2)} \psi(-is) ds \right] \end{aligned}$$

Remark 7.

- ₁ It is well known that $K_{\mu,\lambda}$ is compact if $\mu > 0$ see [8] or [7]. It belongs same to the Carleman-class $\ell_{1+\epsilon} \forall \epsilon > 0$, for this topic see the reference [17]. A compact operator K in the Hilbert space $\mathcal{H}$ is said to belong to the Carleman-class $\ell_p, p > 0$ if $\sum_{n=1}^\infty [s_n(\sqrt{K^*K})]^p$ converges where $s_n(\sqrt{K^*K})$ are the eigenvalues of compact operator $\sqrt{K^*K}$.
- ₂ It is well known that for $\mu = 0$, H_λ satisfies some properties which are to far form the properties of $H_{\mu,\lambda}; \mu \neq 0$:
 - (i) $\rho(H_\lambda) = \emptyset$ where $\rho(H_\lambda)$ is the resolvent set of H_λ .
 - (ii) $D_{\min}(H_\lambda) \neq D_{\max}(H_\lambda)$.
 - (iii) The spectrum of H_λ is $\mathbb{C}$.
- ₃ If $e_n(z) = \frac{z^n}{\sqrt{n!}}; z = x + iy$ is the usual basis of Bargmann space then $\tilde{e}_n(z) = \frac{(iz)^n}{\sqrt{n!}}; z = x + iy$ is also an orthonormal basis of Bargmann space.

Now, if we restrict to an imaginary semi-axis, for example, $z = -iy, y \in [0, +\infty[$, we deduce that $e_n(-iy) = \frac{(-iy)^n}{\sqrt{n!}}$ and $\tilde{e}_n(-iy) = \frac{(i(-iy))^n}{\sqrt{n!}} = \frac{y^n}{\sqrt{n!}} := u_n(y)$.

Lemma 4 (Action of K_λ on the basis $\tilde{e}_n(-iy)$). Let $\tilde{e}_n(-iy) = u_n(y) = \frac{y^n}{\sqrt{n!}}$ then

$$K_\lambda u_1 := v_1(y) = \frac{1}{\lambda} \int_0^y e^{\frac{u^2}{2}} du \int_u^\infty e^{-\frac{s^2}{2}} ds \quad (45)$$

and

$$K_\lambda u_{n+1} = \frac{n-1}{\sqrt{n(n+1)}} K_\lambda u_{n-1} + \frac{u_n}{\lambda n \sqrt{n+1}}, n \geq 1 \quad (46)$$

i.e.,

$$v_{n+1} = \frac{n-1}{\sqrt{n(n+1)}} v_{n-1} + \frac{u_n}{\lambda n \sqrt{n+1}} \quad (47)$$

Proof. Let $v_{n+1}(y) = K_\lambda u_{n+1} = \frac{1}{\lambda} \int_0^y e^{\frac{u^2}{2}} du \int_u^\infty \frac{1}{s} e^{-\frac{s^2}{2}} \frac{s^{n+1}}{\sqrt{(n+1)!}} ds = \frac{1}{\lambda} \int_0^y e^{\frac{u^2}{2}} du \int_u^\infty e^{-\frac{s^2}{2}} \frac{s^n}{\sqrt{(n+1)!}} ds$
 $\frac{s^n}{\sqrt{(n+1)!}} ds = \frac{1}{\lambda} \int_0^y e^{\frac{u^2}{2}} du \int_u^\infty s e^{-\frac{s^2}{2}} \frac{s^{n-1}}{\sqrt{(n+1)!}} ds.$

Setting $v'(s) = s e^{-\frac{s^2}{2}}$ and $u(s) = s^{n-1}$, this implies that $v(s) = -e^{-\frac{s^2}{2}}$ and $u'(s) = (n-1)s^{n-2}$.

Now we use an integration by part to obtain:

$$\begin{aligned} \int_u^\infty e^{-\frac{s^2}{2}} s^n ds &= [-e^{-\frac{s^2}{2}} s^{n-1}]_u^\infty + (n-1) \int_u^\infty e^{-\frac{s^2}{2}} s^{n-2} ds. \\ &= e^{-\frac{u^2}{2}} u^{n-1} + (n-1) \int_u^\infty e^{-\frac{s^2}{2}} s^{n-2} ds. \end{aligned}$$

$$\begin{aligned} \text{It follows that } v_{n+1}(y) &= \frac{1}{\lambda \sqrt{(n+1)!}} [\int_0^y e^{\frac{u^2}{2}} e^{-\frac{u^2}{2}} u^{n-1} du + (n-1) \int_0^y e^{\frac{u^2}{2}} \int_u^\infty e^{-\frac{s^2}{2}} s^{n-2} ds] \\ &= \frac{1}{\lambda \sqrt{(n+1)!}} \frac{y^n}{n} + \frac{n-1}{\lambda \sqrt{(n+1)!}} \int_u^\infty e^{-\frac{s^2}{2}} s^{n-2} ds. \\ &= \frac{1}{\lambda \sqrt{(n+1)!}} \frac{\sqrt{n!} u_n}{n} + \frac{n-1}{\sqrt{n(n+1)}} \frac{1}{\lambda \sqrt{(n-1)!}} \int_u^\infty e^{-\frac{s^2}{2}} s^{n-2} ds. \\ &= \frac{u_n}{\lambda n \sqrt{n+1}} + \frac{n-1}{\sqrt{n(n+1)}} v_{n-1}, \text{ i.e.,} \end{aligned}$$

$$K_\lambda u_{n+1}(y) = \frac{u_n}{\lambda n \sqrt{n+1}} + \frac{n-1}{\sqrt{n(n+1)}} K_\lambda u_{n-1}. \quad (48)$$

□

Definition 4 (Right (or Left) invertibility of unbounded operators).

Let $\mathcal{H}$ be a Hilbert space and let T be a linear operator with domain $D(T) \subset \mathcal{H}$.

We say that T is right invertible if there exists an everywhere defined $R \in \mathcal{B}(\mathcal{H})$ (the space of bounded operators on $\mathcal{H}$) such that $TR = I$; and we say that T is left invertible if there is an everywhere defined $L \in \mathcal{B}(\mathcal{H})$ such that $LT \subset I$.

Remark 8 (Examples).

- (i) Let $\mathcal{P}(\mathbb{C})$ be the set of polynomials, and $A = \frac{d}{dz}$. We would like to undo differentiation, so we integrate the following:

$$\mathbb{J}(p) = \int_0^z p(\xi) d\xi.$$

The fundamental theorem of calculus says that the derivative of this integral is p ; that is, $A\mathbb{J} = I_{\mathcal{P}(\mathbb{C})}$. So $\mathbb{J}$ is a right inverse of A ; it provides a solution, not the only one of the differential equation:

$$\frac{d}{dz}q(z) = p(z)$$

If we try things in the other direction, there is a problem:

$$\mathbb{J}A(p) = \int_0^z p'(\xi)d\xi = p(z) - p(0).$$

That is, $\mathbb{J}A$ sends p to $p - p(0)$, which is not the same as p . So $\mathbb{J}$ is not a left inverse to A ; since A has a nonzero null space, we will see that no left inverse can exist.

- (ii) Consider the space E of real sequences, the linear mapping T that maps a sequence $(a_0, a_1, \dots)$ to the sequence $(0, a_0, a_1, \dots)$ and the linear mapping S that maps a sequence $(a_0, a_1, a_2, \dots)$ to the sequence $(a_1, a_2, \dots)$. It is clear that $ST = I$. Now consider the sequence $a = (1, 0, 0, 0, \dots)$. We have $S(a) = 0$ where 0 is the sequence that vanishes identically and also $TS(a) = 0$ hence $TS \neq I$.
- (iii) Consider the usual (unilateral) shift S on ℓ_2 . Then $S^*S = I$, $SS^* \neq I$ and S is hyponormal. Hence
 - (1) S is left invertible without being invertible.
 - (2) Also, $\sigma(S^*S) = \{1\} \neq \sigma(SS^*) = 0, 1$. where $\sigma(T)$ denotes the spectrum of an operator T .

Remark 9. $v_1(y)$ is a primitive of function defined by

$$\phi(u) = e^{\frac{u^2}{2}} \int_u^\infty e^{-\frac{s^2}{2}} ds \quad (49)$$

The function $\phi(u)$ has interesting properties (see [18]).

- ₁ For every non negative integer n , there exists an unique couple (P_n, Q_n) of polynomials that satisfy

$$\forall u \in \mathbb{R}, \quad \phi^{(n)}(u) = P_n(u)\phi(u) - Q_n(u) \quad (50)$$

Moreover, these polynomials are defined, starting from $(P_0, Q_0) = (1, 0)$, by the recurrence relations

$$\forall n \in \mathbb{N}, \quad P_{n+1} = xP_n + P'_n \quad (51)$$

$$\forall n \in \mathbb{N}, \quad Q_{n+1} = P_n + Q'_n \quad (52)$$

and by Lemma 1 and Proposition 3 of Kouba in [19], the sequence $(P_n(x))_{n \in \mathbb{N}}$ satisfies:

$$\forall n \in \mathbb{N}, P_n(x) = e^{-\frac{x^2}{2}} \frac{d^n}{dx^n} e^{\frac{x^2}{2}} \quad (53)$$

The polynomials $P_n(x)$ are linked to the well-known physicist's Hermite polynomials by the following relation

$$P_n(x) = \left(-\frac{i}{\sqrt{2}}\right) H_n\left(\frac{i}{\sqrt{2}}x\right) \quad (54)$$

In [18], some similar properties of the generating function $\phi(u) = e^{u^2} \int_u^\infty e^{-s^2} ds$ are given to study the “Plasma dispersion function widely used in the field of plasma physics”.

In particular

- ₂ The n -th derivative of $\phi(u) = e^{u^2} \int_u^\infty e^{-s^2} ds$ is given by:

$$\phi^{(n)}(u) = P_n(u)\phi(u) - Q_n(u) \quad (55)$$

where P_n and Q_n are given by the following recurrence relations:

$$\begin{cases} P_0(u) = 1, & P_1(u) = 2x \\ Q_0(u) = 0, & Q_1(u) = 1 \\ P_{n+1}(u) = 2xP_n(u) + 2nP_{n-1}(u) \\ Q_{n+1}(u) = 2xQ_n(u) + 2nQ_{n-1}(u) \end{cases} \quad (56)$$

Furthermore we have the following identity

$$Q_{n+1}(u)P_n(u) - P_{n+1}(u)Q_n(u) = (-2)^n n! \quad (57)$$

and the following expression for $P_n(u)$:

$$P_n(u) = e^{-u^2} \frac{d^n}{du^n} e^{u^2} \quad (58)$$

In following lemma, we give an explicit expression of $v_1(y)$ and we show that it belongs to Bargmann space.

Lemma 5 (An explicit expression of $v_1(y)$).

$$v_1(y) = \sum_{n=0}^{\infty} (-1)^n a_n u_{2n+1} - \sum_{n=0}^{\infty} b_n u_{2(n+1)}(y) \quad (59)$$

$$\text{where } a_n = \frac{\sqrt{(2n)! \frac{\pi}{2}}}{2^n n! \sqrt{2n+1}} \sim \frac{1}{2} n^{-\frac{3}{4}} \pi^{\frac{1}{4}} \text{ and } b_n = \frac{2^n n!}{\sqrt{2(n+1)(2n+1)!}} \sim \frac{1}{2} n^{-\frac{3}{4}} \pi^{\frac{1}{4}}$$

Proof. We begin by writing the following:

$$\begin{aligned} v_1(y) &:= K_\lambda u_1(y) = K_\lambda y = \frac{1}{\lambda} \int_0^y e^{\frac{u^2}{2}} du \int_u^\infty e^{-\frac{s^2}{2}} ds \\ &= \frac{1}{\lambda} \left\{ \int_0^y e^{\frac{u^2}{2}} du \left[\int_0^\infty e^{-\frac{s^2}{2}} ds - \int_0^u e^{-\frac{s^2}{2}} ds \right] \right\} \\ &= -\frac{1}{\lambda} \int_0^y du \int_0^u e^{-\frac{1}{2}(s^2-u^2)} ds + \frac{1}{\lambda} \int_0^y e^{\frac{u^2}{2}} du \int_0^\infty e^{-\frac{s^2}{2}} ds \\ &= f(y) + g(y) \end{aligned}$$

$$\text{where } f(y) = -\frac{1}{\lambda} \int_0^y du \int_0^u e^{-\frac{1}{2}(s^2-u^2)} ds \text{ and } g(y) = \frac{1}{\lambda} \int_0^y e^{\frac{u^2}{2}} du \int_0^\infty e^{-\frac{s^2}{2}} ds.$$

$$\text{Let } f(y) = -\frac{1}{\lambda} \int_0^y du \int_0^u e^{-\frac{1}{2}(s^2-u^2)} ds.$$

Setting $t = \frac{s}{u}$ then we have

$$\begin{aligned} \int_0^u e^{-\frac{1}{2}(s^2-u^2)} ds &= \int_0^1 e^{-\frac{1}{2}(t^2-1)u^2} u dt \\ &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})^n (t^2-1)^n u^{2n+1}}{n!} dt \\ &= \sum_{n=0}^{\infty} \frac{u^{2n+1}}{2^n n!} \int_0^1 (1-t^2)^n dt \\ &= \sum_{n=0}^{\infty} \frac{u^{2n+1}}{2^n n!} W_{n+1} \end{aligned}$$

where $W_{n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}$ is Wallis formula).

It follows that

$$\begin{aligned} \int_0^u e^{-\frac{1}{2}(s^2-u^2)} ds &= \sum_{n=0}^{\infty} \frac{u^{2n+1}}{2^n n!} \frac{2^{2n}(n!)^2}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{2^n n! \sqrt{(2n+1)!} u_{2n+1}(u)}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{2^n n!}{\sqrt{(2n+1)!}} u_{2n+1}(u). \end{aligned}$$

Now as $\int_0^y u_{2n+1}(u) du = \frac{1}{\sqrt{2(n+1)}} u_{2(n+1)}(y)$ then we deduce that

$$f(y) = -\frac{1}{\lambda} \int_0^y du \int_0^u e^{-\frac{1}{2}(s^2-u^2)} ds = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{2^n n!}{\sqrt{2(n+1)}(2n+1)!} u_{2(n+1)}(y)$$

i.e.,

$$f(y) = -\frac{1}{\lambda} \sum_{n=0}^{\infty} a_n u_{2(n+1)} \text{ where } a_n = \frac{2^n n!}{\sqrt{2(n+1)}(2n+1)!} \quad (60)$$

Now, we consider $g(y) = \frac{1}{\lambda} \int_0^y e^{\frac{u^2}{2}} du \int_0^{\infty} e^{-\frac{s^2}{2}} ds$, as $\int_0^{\infty} e^{-\frac{s^2}{2}} ds = \sqrt{2\pi}$, then we deduce that

$$g(y) = \frac{\sqrt{2\pi}}{\lambda} \int_0^y e^{\frac{u^2}{2}} du = \frac{\sqrt{2\pi}}{\lambda} \int_0^y \sum_{n=0}^{\infty} \frac{u^{2n}}{2^n n!} = \frac{\sqrt{2\pi}}{\lambda} \sum_{n=0}^{\infty} \frac{y^{2n+1}}{2^n (2n+1)n!}$$

As $y^n = \sqrt{n!} u_n(y)$, then it follows that

$$\begin{aligned} g(y) &= \frac{\sqrt{2\pi}}{\lambda} \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)!} u_{2n+1}(y)}{2^n (2n+1)n!} \\ &= \frac{\sqrt{2\pi}}{\lambda} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!} u_{2n+1}(y)}{2^n n! \sqrt{2n+1}} \end{aligned}$$

i.e.,

$$g(y) = \frac{\sqrt{2\pi}}{\lambda} \sum_{n=0}^{\infty} b_n u_{2n+1}(y) \text{ where } b_n = \frac{\sqrt{(2n)!}}{2^n n! \sqrt{2n+1}} \quad (61)$$

□

Corollary 1.

(α) The function $v_1(y)$ belongs to Bargmann space, i.e., $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ and $\sum_{n=0}^{\infty} |b_n|^2 < \infty$

(β) $f(y) = g(y) - v_1$

Remark 10.

$$\text{Let } a_n^2 = \frac{2^{2n}(n!)^2}{2(n+1)(2n+1)!}, b_n^2 = \frac{(2n)!}{2^n(n!)^2(2n+1)}.$$

Then by observing that $(2n+1)! = (2n)!(2n+1)$ and by setting $c_n = \frac{(2n)!}{2^n(n!)^2}$, we deduce that $b_n^2 = \frac{c_n}{2n+1}$, $a_n^2 = \frac{1}{2(n+1)(2n+1)c_n}$ and

$$a_n^2 b_n^2 = \frac{1}{2(n+1)(2n+1)^2} \sim \frac{1}{n^3} \quad (62)$$

In particular,

$$a_n b_n \sim \frac{1}{n^{\frac{3}{2}}} \quad (63)$$

Below we show that $b_n \sim \frac{1}{n^{\frac{3}{2}}}$ to deduce the convergence of $\sum_{n=0}^{\infty} |b_n|^2$ and of $\sum_{n=0}^{\infty} |a_n|^2$.

The convergence of the series $\sum_{n=1}^{\infty} |b_n|^2$ requires Stirling's approximation, $\lim_{n \rightarrow +\infty} \frac{1}{n!} \left(\frac{n}{e}\right)^n \sqrt{2\pi n} = 1$ and $\lim_{n \rightarrow +\infty} \frac{1}{(n!)^2} \left(\frac{n}{e}\right)^{2n} 2\pi n = 1^2$.

We give two methods to prove this convergence: the first method uses Stirling's approximation by establishing the convergence with limits, and for the second method, we use an upper bound of $n!$ using the trapezoidal method and a lower bound of $n!$ by the median point method.

(i) First method: Establishing convergence with limits

Let $a_n^2 = \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}$ and $c_n^2 = \frac{1}{n^{\frac{3}{2}}}$.

Then,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{a_n^2}{c_n^2} &= \lim_{n \rightarrow +\infty} \frac{\frac{(2n)!}{2^{2n}(n!)^2(2n+1)}}{\frac{1}{n^{\frac{3}{2}}}} \\ &= \lim_{n \rightarrow +\infty} \frac{(2n)! n^{3/2}}{2^{2n} (n!)^2 n} \quad (\text{by using } 2n+1 \sim 2n) \\ &= \lim_{n \rightarrow +\infty} \frac{(2n)! \sqrt{n}}{2^{2n} (n!)^2} \\ &= \lim_{n \rightarrow +\infty} \frac{(2n)! \sqrt{n}}{2^{2n} (n!)^2} \frac{1}{1^2} \\ &= \lim_{n \rightarrow +\infty} \frac{(2n)! \sqrt{n}}{2^{2n} (n!)^2} \frac{\frac{1}{(2n)!} \left(\frac{2n}{e}\right)^{2n} \sqrt{4\pi n}}{\left(\frac{1}{n!} \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^2} \\ &= \lim_{n \rightarrow +\infty} \frac{\sqrt{n} \left(\frac{2n}{e}\right)^{2n} \sqrt{4\pi n}}{2^{2n} \left(\frac{n}{e}\right)^{2n} (2\pi n)} \\ &= \lim_{n \rightarrow +\infty} \frac{\sqrt{n} 2^{2n} \sqrt{4\pi n}}{2^{2n} (2\pi n)} \\ &= \lim_{n \rightarrow +\infty} \frac{\sqrt{n} \sqrt{4\pi n}}{2\pi n} \\ &= \frac{1}{\sqrt{\pi}}. \end{aligned}$$

It follows that $a_n^2 \sim \frac{1}{n^{\frac{3}{2}}}$.

(ii) Second method: lower bound and upper bound of $n!$

With the upper bound of $n!$, by using the trapezoidal method, we get

$$n! \leq e n^n e^{-n} \sqrt{n} \quad (64)$$

This implies that

$$(2n)! \leq \alpha (2n)^{2n} e^{-2n} \sqrt{2n} \quad (65)$$

where α is a positive constant.

It follows that

$$\sqrt{(2n)!} \leq \sqrt{\alpha} (2n)^n e^{-n} (2n)^{\frac{1}{4}} \quad (66)$$

With the lower bound of $n!$, by using the median point method, we get

$$n! \geq cn^n e^{-n} \sqrt{n} \quad (67)$$

where c is a positive constant.

This implies that

$$(2n)! \geq \beta(2n)^{2n} e^{-2n} \sqrt{2n} \quad (68)$$

where β is a positive constant. And

$$\sqrt{(2n)!} \geq \sqrt{\beta}(2n)^n e^{-n} (2n)^{\frac{1}{4}} \quad (69)$$

From (64) and (67) we deduce the following:

$$cn^n e^{-n} \sqrt{n} \leq n! \leq en^n e^{-n} \sqrt{n} \quad (70)$$

From (66) and (69) we deduce the following:

$$\sqrt{\beta}(2n)^n e^{-n} (2n)^{\frac{1}{4}} \leq \sqrt{(2n)!} \leq \sqrt{\alpha}(2n)^n e^{-n} (2n)^{\frac{1}{4}} \quad (71)$$

Now from (70) and (71) we deduce the following:

$$\frac{\sqrt{\beta}(2n)^n e^{-n} (2n)^{\frac{1}{4}}}{en^n e^{-n} \sqrt{n}} \leq \frac{\sqrt{(2n)!}}{n!} \leq \frac{\sqrt{\alpha}(2n)^n e^{-n} (2n)^{\frac{1}{4}}}{cn^n e^{-n} \sqrt{n}} \quad (72)$$

(Note in above denominators that $en^n e^{-n} \sqrt{n}$ is upper bound of $n!$ and $cn^n e^{-n} \sqrt{n}$ is lower bound of $n!$). It follows that

$$\frac{\sqrt{\beta}2^{n+\frac{1}{4}}}{en^{\frac{1}{4}}} \leq \frac{\sqrt{(2n)!}}{n!} \leq \frac{\sqrt{\alpha}2^{n+\frac{1}{4}}}{cn^{\frac{1}{4}}} \quad (73)$$

Now as $b_n = \frac{\sqrt{(2n)!}}{2^n n! \sqrt{2n+1}}$, then we deduce that

(i)

$$\frac{\sqrt{\beta}2^{\frac{1}{4}}}{en^{\frac{1}{4}} \sqrt{2n+1}} \leq b_n \leq \frac{\sqrt{\alpha}2^{\frac{1}{4}}}{cn^{\frac{1}{4}} \sqrt{2n+1}} \quad (74)$$

(ii)

$$\sum_{n=1}^{\infty} b_n^2 \leq \frac{\alpha\sqrt{2}}{c^2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(2n+1)} < \infty \quad (75)$$

As $b_n \sim \frac{1}{n^{\frac{3}{4}}} \in \ell_2$ and $a_n b_n \sim \frac{1}{n^{\frac{3}{2}}}$ then $a_n \sim \frac{1}{n^{\frac{3}{4}}} \in \ell_2$ and $v_1 \in \mathbb{B}_0$.

Remark 11.

(α) $b_n = O\left(\frac{1}{n^{\frac{3}{4}}}\right)$, $b_n^2 = O\left(\frac{1}{n^{\frac{3}{2}}}\right)$, $a_n = O\left(\frac{1}{n^{\frac{3}{4}}}\right)$ and $a_n^2 = O\left(\frac{1}{n^{\frac{3}{2}}}\right)$.

(β) As $v_1 = f + g = \frac{1}{\lambda} \int_0^y e^{\frac{u^2}{2}} du \int_u^\infty e^{-\frac{s^2}{2}} ds$ then $f = g - v_1$ where $g(y) = \frac{\sqrt{2\pi}}{\lambda} \sum_{n=0}^{\infty} b_n u_{2n+1}(y)$ such that $b_n = \frac{\sqrt{(2n)!}}{2^n n! \sqrt{2n+1}} = O\left(\frac{1}{n^{\frac{3}{4}}}\right)$.

Corollary 2. v_n belongs to Bargmann space for all $n = 1, 2, \dots$

Proof. As v_n satisfies the recurrence relation:

$$v_{n+1} = \frac{n-1}{\sqrt{n(n+1)}}v_{n-1} + \frac{u_n}{\lambda n\sqrt{n+1}} \quad \text{for all } n = 1, 2, \dots$$

where u_n belongs to Bargmann space.

Then we deduce that $v_2 = \frac{u_1}{\lambda\sqrt{2}}$, this implies that $v_2 \in \mathbb{B}_0$, and by recurrence, we deduce that $v_n \in \mathbb{B}_0$ for all $n = 1, 2, \dots$ $\square$

Proposition 4 (Determination explicit of v_n with respect to u_n and v_1).

Let

$$\left\{ \begin{array}{l} v_{n+1} = A_n u_n + B_n v_{n-1} \\ \text{where} \\ A_n = \frac{1}{\lambda n \sqrt{n+1}} \\ \text{and} \\ B_n = \frac{n-1}{\sqrt{n(n+1)}} \end{array} \right.$$

Then, we have

$$v_n = P_{n-1} + \alpha_n v_1 \quad (76)$$

with

$$\left\{ \begin{array}{l} \alpha_n = 0 \quad \text{if } n = 2p; p \in \mathbb{N} \\ \alpha_{2n+1} = \prod_{j=1}^p B_{2j} \quad \text{if } n = 2p+1; p \in \mathbb{N} \\ P_0 = 0 \\ P_1 = \frac{u_1}{\lambda\sqrt{2}} \\ P_n = \frac{n-1}{\sqrt{n(n+1)}}P_{n-2} + \frac{u_n}{\lambda n\sqrt{n+1}} \\ \text{or} \\ P_n = B_n P_{n-2} + A_n u_n \end{array} \right.$$

Proof.

$$v_2 = A_1 u_1 + P_1 + \alpha_2 v_1; P_1 = A_1 u_1 \text{ and } \alpha_2 = 0$$

$$v_3 = A_2 u_2 + B_2 v_1 = P_2 + \alpha_3 v_1; P_2 = A_2 u_2 \text{ and } \alpha_3 = B_2$$

$$v_4 = A_3 u_3 + B_3 v_2 = A_3 u_3 + B_3 A_1 u_1 = P_3 + \alpha_4 v_1; P_3 = A_3 u_3 + B_3 A_1 u_1 = B_3 P_1 + A_3 u_3 \text{ and } \alpha_4 = 0$$

$$v_5 = A_4 u_4 + B_4 v_3 = A_4 u_4 + B_4 [A_2 u_2 + B_2 v_1] = A_4 u_4 + B_4 A_2 u_2 + B_4 B_2 v_1 = P_4 + \alpha_5 v_1;$$

$$P_4 = A_4 u_4 + B_4 A_2 u_2 = B_4 P_2 + A_4 u_4 \text{ and } \alpha_5 = B_4 B_2 = \prod_{j=1}^2 B_{2j}$$

$$v_6 = A_5 u_5 + B_5 v_4 = A_5 u_5 + B_5 [A_3 u_3 + B_3 A_1 u_1] = A_5 u_5 + B_5 A_3 u_3 + B_5 B_3 A_1 u_1 = P_5 + \alpha_6 v_1; P_5 = A_5 u_5 + B_5 A_3 u_3 + B_5 B_3 A_1 u_1 = B_5 P_3 + A_5 u_5 \text{ and } \alpha_6 = 0$$

$$v_7 = A_6 u_6 + B_6 v_5 = A_6 u_6 + B_6 [A_4 u_4 + B_4 A_2 u_2 + B_4 B_2 v_1] = A_6 u_6 + B_6 A_4 u_4 + B_6 B_4 A_2 u_2 + B_6 B_4 B_2 v_1 = P_6 + \alpha_7 v_1; P_6 = B_6 P_4 + A_6 u_6 \text{ and } \alpha_7 = \prod_{j=1}^3 B_{2j}.$$

By recurrence with respect to p , we suppose that

$$v_{2p} = P_{2p-1} + \alpha_{2p} v_1 \text{ such that } P_{2p} = B_{2p} P_{2p-2} + A_{2p} u_{2p} \text{ and } \alpha_{2p} = 0.$$

$v_{2p+1} = P_{2p} + \alpha_{2p+1}v_1$ such that $P_{2p+1} = B_{2p+1}P_{2p-1} + A_{2p+1}u_{2p+1}$ and $\alpha_{2p+1} = \prod_j^p B_{2j}$.

Then we deduce that

• $_{2(p+1)}$

$$\begin{aligned} v_{2(p+1)} &= A_{2p+1}u_{2p+1} + B_{2p+1}v_{2p} \\ &= A_{2p+1}u_{2p+1} + B_{2p+1}[P_{2p-1} + \alpha_{2p}v_1] \\ &= A_{2p+1}u_{2p+1} + B_{2p+1}P_{2p-1} + \alpha_{2p}B_{2p+1}v_1 \\ &= B_{2p+1}P_{2p-1} + A_{2p+1}u_{2p+1} \text{ (because } \alpha_{2p} = 0) \\ &= B_{2p+1}P_{2p-1} + A_{2p+1}u_{2p+1} + \alpha_{2(p+1)}v_1. \end{aligned}$$

This implies

$$\alpha_{2(p+1)} = 0$$

and

$$P_{2p+1} = B_{2p+1}P_{(2p+1)-2} + A_{2p+1}u_{2p+1}.$$

i.e.,

$$v_{2(p+1)} = P_{2p+1} + \alpha_{2(p+1)}v_1$$

• $_{2(p+1)+1}$

$$\begin{aligned} v_{2(p+1)+1} &= A_{2(p+1)}u_{2(p+1)} + B_{2(p+1)}v_{2p+1} \\ &= A_{2(p+1)}u_{2(p+1)} + B_{2(p+1)}[P_{2p} + \alpha_{2p+1}v_1] \\ &= A_{2(p+1)}u_{2(p+1)} + B_{2(p+1)}P_{2p} + B_{2(p+1)}\alpha_{2p+1}v_1 \\ &= A_{2(p+1)}u_{2(p+1)} + B_{2(p+1)}P_{2p} + B_{2(p+1)}\prod_{j=1}^p B_{2j}v_1 \\ &= B_{2(p+1)}P_{2p} + A_{2(p+1)}u_{2(p+1)} + \prod_{j=1}^{p+1} B_{2j}v_1. \end{aligned}$$

Then we deduce that

$$\alpha_{2(p+1)+1} = \prod_{j=1}^{p+1} B_{2j}v_1$$

and

$$P_{2(p+1)} = B_{2(p+1)}P_{2p} + A_{2(p+1)}u_{2(p+1)}$$

i.e.,

$$v_{2(p+1)+1} = P_{2(p+1)} + \alpha_{2(p+1)+1}v_1.$$

□

Lemma 6. Let $u_n(y) = \frac{y^n}{\sqrt{n!}}$ with norm $\|u_n\| = 1$ and $P_n(y)$ is a polynomial which satisfies:

$$P_n(y) = B_n P_{n-2}(y) + A_n u_n(y)$$

where $B_n = \frac{n-1}{\sqrt{n(n+1)}}$ and $A_n = \frac{1}{\lambda n \sqrt{n+1}}$

Then

- (i) P_{n-2} is orthogonal to u_n .
- (ii) Let $p_n = \|P_n\|^2$ then $p_n = B_n^2 p_{n-2} + A_n^2$.

$$(iii) \quad p_n = O\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

Proof.

- (i) As the degree of P_{n-2} is $n-2$ then $\langle P_{n-2}, u_n \rangle = 0$.
 (ii) We have

$$\begin{aligned} p_n &= \|P_n\|^2 = \langle B_n P_{n-2}(y) + A_n u_n(y), B_n P_{n-2}(y) + A_n u_n(y) \rangle \\ &= \langle B_n P_{n-2}(y), B_n P_{n-2}(y) \rangle + \langle B_n P_{n-2}(y), A_n u_n(y) \rangle \\ &\quad + \langle A_n u_n(y), B_n P_{n-2}(y) \rangle + \langle A_n u_n(y), A_n u_n(y) \rangle \end{aligned}$$

By using (i) and $\|u_n\| = 1$, we deduce that $p_n = B_n^2 p_{n-2} + A_n^2$.

- (iii) is deduced from (ii) and Lemma 5.

□

4. On Right-Inverse K_λ of H_λ and Its Compactness

Proposition 5. Let $\mathcal{H}$ be a Hilbert space and let T be a linear operator with domain $D(T) \subset \mathcal{H}$. Let $N(T)$ be the kernel of T and $R(T)$ its range.

- (i) T is right invertible $\iff R(T) = \mathcal{H}$.
 (ii) The inverse of an unbounded operator could be compact, or bounded, or unbounded, as one can easily see by considering multiplication operators on L^2 spaces.
 (iii) The left inverse of a compact operator on an infinite dimensional Hilbert space, if it exists, must be unbounded.

Proof.

- (i) • For $\boxed{\Leftarrow}$: assume T is surjective. Then, for all $\psi \in \mathcal{H}$, there exists $\varphi_\psi \in \mathcal{H}$ such that $T(\varphi_\psi) = \psi$. Define $R: \mathcal{H} \rightarrow \mathcal{H}$ to be the function which maps each $\psi \in \mathcal{H}$ to such $\varphi \in \mathcal{H}$ (if there is more than one φ , then the function R maps ψ to one of them chosen in an arbitrary way). This excludes the possibility that R maps ψ to two distinct values, in which case it would not be a function). It follows that

$$\forall \psi \in \mathcal{H}, \quad (T \circ R)(\psi) = T(R(\psi)) = T(\varphi_\psi) = \psi$$

and $T \circ R = id_{\mathcal{H}}$.

- For $\boxed{\Rightarrow}$, assume $T: D(T) \subset \mathcal{H} \rightarrow \mathcal{H}, R: \mathcal{H} \rightarrow \mathcal{H}$ are such that $T \circ R = id_D(T)$. Then, for each $\psi \in \mathcal{H}$, $\varphi_\psi \stackrel{\text{def}}{=} R(\psi) \in D(T)$ is a pre-image of ψ by T , as $T(\varphi_\psi) = T \circ R(\psi) = id_D(T)(\psi) = \psi$. Hence, T is surjective.
 • K_λ and H_λ are the examples satisfying (ii) and (iii) (see the following propositions).

□

Proposition 6.

- (i) K_λ is a right inverse of H_λ , i.e., $H_\lambda K_\lambda = I$.
 (ii) H_λ is surjective.
 (iii) H_λ is non-injective.

Proof.

- (i) As $u_n = \frac{y^n}{\sqrt{n!}}$ then we deduce that:

$$H_\lambda u_n = \lambda \left[-\frac{n(n-1)y^{n-1}}{\sqrt{n!}} + \frac{ny^{n+1}}{\sqrt{n!}} \right]$$

and

$$\frac{H_\lambda u_n}{n\sqrt{n+1}} = \lambda \left[-\frac{(n-1)u_{n-1}}{\sqrt{n(n+1)}} + u_{n+1} \right]$$

Now, from the following recurrence relation:

$$v_{n+1} = \frac{u_n}{\lambda n\sqrt{n+1}} + \frac{n-1}{\sqrt{n(n+1)}} v_{n-1}. \quad (77)$$

we deduce that

$$K_\lambda u_{n+1} = \frac{u_n}{\lambda n\sqrt{n+1}} + \frac{n-1}{\sqrt{n(n+1)}} K_\lambda u_{n-1} \quad (78)$$

and

$$\begin{aligned} H_\lambda K_\lambda u_{n+1} &= \frac{H_\lambda u_n}{\lambda n\sqrt{n+1}} + \frac{n-1}{\sqrt{n(n+1)}} H_\lambda K_\lambda u_{n-1} \\ &= -\frac{(n-1)}{\sqrt{n(n+1)}} u_{n-1} + u_{n+1} + \frac{n-1}{\sqrt{n(n+1)}} H_\lambda K_\lambda u_{n-1}. \end{aligned}$$

Now as $K_\lambda u_2(y) = \frac{u_1(y)}{\lambda\sqrt{2}}$ then $H_\lambda K_\lambda u_2(y) = \frac{y^2}{\sqrt{2}} = u_2(y)$.

And by recurrence we deduce that:

$$H_\lambda K_\lambda u_{n+1} = -\frac{(n-1)}{\sqrt{n(n+1)}} u_{n-1} + u_{n+1} + \frac{n-1}{\sqrt{n(n+1)}} u_{n-1} = u_{n+1} \quad \forall n.$$

By taking the range of K_λ as domain of H_λ , It follows that:

$$H_\lambda K_\lambda = I \quad (79)$$

(ii) It is well known that if an operator T has a right inverse, then T is surjective (see above proposition). Then H_λ is surjective. Conversely, if T is surjective, then T has a right inverse (see above proposition or [20]).

(iii) H_λ is non-injective since it cancels out the function $g(y) = \frac{\sqrt{2\pi}}{\lambda} \int_0^y e^{\frac{u^2}{2}} du$. In fact, as

$$\begin{aligned} H_\lambda u(y) &= \lambda[-yu''(y) + y^2 u'(y)] \text{ and as } g'(y) = \frac{\sqrt{2\pi}}{\lambda} e^{\frac{y^2}{2}} \text{ and } g''(y) = \frac{\sqrt{2\pi}}{\lambda} y e^{\frac{y^2}{2}} \text{ then} \\ H_\lambda(g(y)) &= \sqrt{2\pi}[-y^2 e^{\frac{y^2}{2}} + y^2 e^{\frac{y^2}{2}}] = 0. \end{aligned}$$

□

Proposition 7 (Compactness of K_λ).

Let $u = \sum_{n=1}^{\infty} c_n u_n$, $K_\lambda u = \sum_{n=1}^{\infty} c_n v_n = \sum_{n=1}^{\infty} c_n P_{n-1} + c_1 v_1$ where $c_1 = \sqrt{2} \langle \varphi, g - v_1 \rangle$ and let $K_{\lambda,m} u = \sum_{n=1}^m c_n P_{n-1} + \sqrt{2} \langle \varphi, g - v_1 \rangle v_1$. where $g(y) = \frac{\sqrt{2\pi}}{\lambda} \sum_{n=0}^{\infty} b_n u_{2n+1}(y)$ such that $b_n = \frac{\sqrt{(2n)!}}{2^n n! \sqrt{2n+1}}$ (see Formula (59)).

Then,

- (i) $K_{\lambda,m} \longrightarrow K_\lambda$ as $m \longrightarrow \infty$ (in operator norm).
- (ii) K_λ is compact.

Proof.

- (i) Let $u = \sum_{n=1}^{\infty} c_n u_n$ then $K_\lambda u = \sum_{n=1}^{\infty} c_n P_{n-1} + \sqrt{2} \langle \varphi, g - v_1 \rangle v_1$ and $K_{\lambda,m} u = \sum_{n=1}^m c_n P_{n-1} + \sqrt{2} \langle \varphi, g - v_1 \rangle v_1$.

Let us estimate the remainder of the series using Cauchy–Schwarz: $\|\sum_{n=m+1}^{\infty} c_n P_{n-1}\| \leq$

$$\begin{aligned} \sum_{n=m+1}^{\infty} |c_n| \cdot \|P_{n-1}\| &\leq \left(\sum_{n=m+1}^{\infty} |c_n|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{n=m+1}^{\infty} \|P_{n-1}\|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=m+1}^{\infty} |\langle u, u_n \rangle|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{n=m}^{\infty} \|p_n\|^2 \right)^{\frac{1}{2}} \leq \|u\| \cdot \left(\sum_{n=m}^{\infty} \|p_n\|^2 \right)^{\frac{1}{2}} \end{aligned}$$

Then, $K_{\lambda,m} \rightarrow K_{\lambda}$ as $m \rightarrow \infty$ (in operator norm).

(ii) As K_{λ} is an operator norm limit of finite rank operators, K_{λ} is a compact operator.

□

Remark 12.

(i) We can take the range of K_{λ} as domain of H_{λ} . This is possible because $R(K_{\lambda})$ is dense in $\mathbb{B}_0$. To do this, let us take a function $\varphi \in \mathbb{B}_0$ orthogonal to $R(K_{\lambda})$; in particular, it is orthogonal to all v_n . Starting with v_1 from $v_n = P_{k-1} + \alpha_n v_1$ where $\alpha_{2n} = 0$ and $\alpha_{2n+1} = \sqrt{2}b_n$ we deduce the following:

$$\langle P_k, \varphi \rangle = \langle v_{n+1}, \varphi \rangle = 0. \quad (80)$$

Since P_n is of degree n exactly, the set of P_n is an algebraic basis for the space of polynomials that are zero at the origin, so it is the total in $\mathbb{B}_0$ then $\varphi = 0$.

(ii) H_{λ} is an imaginary part of $H_{\mu,\lambda}$, we observe that it is also the imaginary part of $H_{\lambda',\mu,\lambda} = \lambda' A^{*2} A^2 + H_{\mu,\lambda}$. In [21], we have presented an interesting “non-linear” factorization of this operator $H_{\lambda',\mu,\lambda}$; $\lambda' \neq 0$.

5. Conclusions

Reggeon field theory (RFT) is an attempt to predict the high-energy behavior of soft processes; the RFT can be derived from the assumed softness of hadronic interactions at low transverse momenta, which seems to be well established experimentally in hadron–hadron and hadron–nucleus interactions.

Originally, RFT was formulated as a field theory (or as quantum mechanics in zero transverse dimensions) of pomerons. The basic degrees of freedom in this formulation are the Gribov fields ψ and ψ^+ that create and annihilate the pomeron. The action defining the theory with triple pomeron couplings only (MRFT) is defined in the following way:

$$S = \int dy \{ \psi^+ \partial_y \psi - \mu \psi^+ \psi + i\lambda \psi^+ (\psi^+ + \psi) \psi \} \quad (81)$$

The Hamiltonian of this theory is given by $H_{\mu,\lambda} = \mu A^* A + i\lambda A^* (A + A^*) A$ acting on Bargmann space, where $\mu \neq 0$ is intercept coupling, λ is the triple Pomeron coupling, and A and A^* are the annihilation and creation operators satisfying $[A, A^*] = I$. Many spectral properties of $H_{\mu,\lambda}$ have been studied by the author and recalled in this work. The mathematical difficulties of this problem come of course from the non-self-adjointness of $H_{\mu,\lambda}$. Notice that this non-self-adjointness is a rather wild one; the word “wild” meaning here that the domains of the adjoint and anti-adjoint parts are not included in one another, nor is the domain of their commutator.

The main purpose of the present work (Section 4) is to give some new spectral properties of anti-adjoint part $H_{\lambda} = i\lambda A^* (A + A^*) A$; $i^2 = -1$ of $H_{\mu,\lambda}$, in particular, to show that H_{λ} admits a right-inverse K_{λ} , i.e., $H_{\lambda} K_{\lambda} = I$ on negative imaginary axis and that K_{λ} is compact. This gives us an original example of an operator that admits a right inverse but does not admit a left inverse.

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