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Adaptive Consensus Control of Multiple Underactuated Marine Surface Vessels with Input Saturation and Severe Uncertainties

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Abstract

This paper is devoted to the consensus control of a networked system constituted by multiple underactuated marine surface vessels (MSVs) with input saturation. Compared with the related works on this topic, two remarkable features are involved in the system under investigation: (1) Input saturation of each follower MSV is considered in the paper but ignored in most of the related works. (2) More coarse information is allowed about the network since more severer uncertainties (external disturbance joint with unknown system parameters) are involved in each follower MSV, and moreover, the output of the leader MSV is not necessarily second-order continuously differentiable while its time derivatives are not necessarily available for feedback. The above two aspects lead to the incapability of the traditional control schemes on this topic. To solve the control problem, a novel adaptive control scheme is proposed by adaptive dynamic compensation technique combining with certain methods for the handling of saturation input and under-actuation. Specifically, a smooth function is introduced to approximate the saturation input, by which and a couple of state transformations, a new system is obtained with a skillful injection of an auxiliary input for the handling of under-actuation. Then, an iterative procedure is given to derive an adaptive controller which ensures that all the signals of the closed-loop system are bounded while the output of the follower MSVs practically tracks that of the leader. Finally, simulation results are provided to validate the effectiveness of the proposed theoretical results.

Keywords: marine surface vessels; underactuated; consensus; input saturation; adaptive

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1. Introduction

Cooperative control (such as consensus or formation) of multiple underactuated MSVs has attracted extensive attention due to its notable advantages compared with the control of single MSV on the performing of various marine tasks, such as ocean transportation, exploration and rescue, etc. In this field, considerable results are obtained but constrained on the uncertainties of the followers and the prior knowledge of the leader. For detail, the constraints of the existing results are specified by the following two aspects:

① *The uncertainties (such as unknown parameters and external disturbance) of system are seriously restricted*. Remark that uncertainties are always existing in MSVs due to the presence of modelling error or the influence from the external disturbance. However, the existing results require severe assumptions on the uncertainties. Specifically, in [1–4],

unknown external disturbance is allowed but all the system parameters must be accurately known, for such a case control method based on disturbance observer (DS) is proposed. In [5], partial system parameters are allowed to be unknown but the mass of each follower MSV must be accurately known while free of external disturbance, for such case neural network (NN) control method is used. In [6–15], external disturbance is allowed but some system parameters, such as the mass of the follower MSVs, required to be known. For such systems, adaptive method is used in [6,9], H_∞ method is used in [7], control based on extended state observer (ESO) and NN are respectively used in [8,11,14] and [10,13,15]. In [16–22], some bounded unmodeled dynamics and external disturbance are allowed but the system parameters must be known. For such case, sliding model control (SMC) and adaptive control are respectively used in [20] and [21], while control based on extended state observer (ESO) and NN are respectively used in [19] and [16–18,22]. Once the uncertainties become more severer, for example all the system parameters are unknown while suffered from external disturbance, the control methods in the above literature become ineffective.

② *The generality and measurability of leader's output are seriously restricted.* Specifically, in [1,2,16–19], the leader's output must be at least twice continuously differentiable, which excludes those without so much smoothness (such as those which are only first-order continuously differentiable) and hence restricts the generality of the leader's output. On the other hand, in [1,2,19–21], the second-order derivative of the leader's output require to be available for feedback. This implies that the acceleration of the leader MSV should be measured but is difficult (or even impossible) to be implemented in practice. Noting that the constraints on the generality and measurability of the leader's output limit the applicability of the proposed control methods, for some leaders whose prior knowledge is difficult to be obtained (such as those which are not twice continuously differentiable or their time derivatives are not available), the traditional control methods will be inapplicable.

In engineering, input saturation is inevitable due to the physical constraint of actuator. For example, the rudder angle of a ship's steering gear is usually constrained by a maximum deflection limit and hence satisfies a saturation constraint. On one hand, the presence of the input saturation damages the system performance or even leads to instability. On the other hand, input saturation implies an intermittent control to the controlled system while brings extra nonlinearities and uncertainties, which lead to certain technique obstacles in control design and hence essentially challenge the control problems. In the past decade, input saturation is considered for the tracking control of single MSV [23–32], but seldom for the cooperative control of multiple MSVs. As some tentative works, refs. [8,9,14–16] consider the formation control of multiple underactuated MSVs but constrained by the uncertainties since the mass of the follower MSVs in [8,9,14,15] in known while all the system parameters must be known in [16], just as mentioned before. Recognizing the constraints of the existing results mentioned above, the traditional control methods in the literature would be incapable for the cooperative control of the saturated underactuated MSVs with severe uncertainties (such as all the system parameters are unknown while suffered from the external disturbance) or/and only coarse information about the leader's output (such as the cases that the time derivative of the leader's output is not available for feedback). Then, an interesting and nontrivial control problem arises, i.e., *for a networked system constituted by multiple saturated underactuated MSVs, how to design a distributed cooperative controller under weak conditions on the uncertainties about followers and the prior knowledge about the leader?*

Towards the control problem raised above, this paper devotes the consensus of multiple underactuated MSVs with input saturation and coarse information about the follower and leader MSVs. Remarkably, severe uncertainties are involved in the dynamics of follower MSVs since all the system parameters and the saturation level parameters are unknown while the external disturbances are allowed. Moreover, the leader's output is

only first-order continuously differentiable while its time derivatives is not necessarily available for feedback. The above two aspects reflect coarse information about the follower and leader MSVs which result into the ineffectiveness of the traditional methods on this topic (a detailed comparison with the related literature can be found in Table 1). To solve the control problem, an adaptive control scheme is proposed in this paper. Specifically, a smooth function is firstly introduced to approximate the saturation function and helps the handling of the saturation input. Then, a couple of state transformations are introduced to change the original system into a new one, in which an auxiliary function is injected for the handling of the under-actuation. For the new system, an adaptive distributed controller is designed join with the smart choice of the auxiliary function and the updating law of one pivotal dynamic gain for the compensation of system uncertainties. Finally, a recursive procedure is given to show that all the signals of the closed-loop system are bounded while the output of each follower MSVs practically tracks that of the leader MSVs.

Table 1. Comparison with the related works for consensus control of underactuated MSVs.

	Control Methods	Uncertainties (All the Parameters Are Unknown with Disturbance)	Twice Continuity of Leader's Output	Input Saturation
[1–4]	DS	×	✓	×
[5,10,12,13,22]	NN	×	×	×
[15]	NN	×	×	✓
[16]	NN	×	✓	✓
[17,18]	NN	×	✓	×
[6,21]	Adaptive	×	✓	×
[9]	Adaptive	×	×	✓
[8]	ESO	×	✓	✓
[14]	ESO	×	×	✓
[11,19]	ESO	×	✓	×
[7]	H_∞	×	×	×
[20]	SMC	×	×	×
This paper	Adaptive	✓	×	✓

The remainder of the paper is organized as follows. Section 2 presents the problem formulation which includes some preliminary knowledge of the graph theory and the system under investigation. Section 3 gives the procedure of control design. Section 4 brings the performance analysis of the resulting closed-loop system. Section 5 provides the simulation results to validate the effectiveness of the proposed theoretical results. Section 6 gives some concluding remarks.

2. Problem Formulation

2.1. Graph Theory

In this paper, a direct graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ represents the communication of N followers where $\mathcal{V} = \{n_1, \cdots, n_N\}$ is the node set, $\mathcal{E} = \{(n_i, n_j) \in \mathcal{V} \times \mathcal{V}\}$ is the edge set and $\mathcal{A} = [a_{ij}]_{N\times N}$ is an adjacent matrix with $a_{ij} > 0$ if the i-th follower can obtain information from the j-th follower and $a_{ij} = 0$ otherwise. Self-edges are not allowed, i.e., $a_{ii} = 0, i = 1, \cdots, N$. $B = \mathbf{diag}(b_1, \ldots, b_N)$ represents the communication weight from the leader to followers, where $b_i > 0$ if the leader can directly send information to the

i-th follower while $b_i = 0$ otherwise. $L = D - \mathcal{A}$ denotes the Laplacian matrix of $\mathcal{G}$ with $D = \mathbf{diag}(\sum_{j=1}^{N} a_{1j}, \cdots, \sum_{j=1}^{N} a_{Nj})$. Let $H = L + B$.

2.2. System Formulation and Control Objective

A networked system consisted by N MSVs is investigated under a direct graph (an example for the communication topology can be found in Figure 1 below), in which the dynamics of the i-th follower is described by the following equations:

$$\begin{cases} \begin{cases} \dot{x}_i = u_i \cos\psi_i - v_i \sin\psi_i, \\ \dot{y}_i = u_i \sin\psi_i + v_i \cos\psi_i, \\ \dot{\psi}_i = r_i, \end{cases} \\ \begin{cases} \dot{u}_i = f_{u_i}(\nu_i) + \frac{1}{m_{u_i}}(\tau_{u_i} + \omega_{u_i}), \\ \dot{v}_i = f_{v_i}(\nu_i) + \frac{1}{m_{v_i}}\omega_{v_i}, \\ \dot{r}_i = f_{r_i}(\nu_i) + \frac{1}{m_{r_i}}(\tau_{r_i} + \omega_{r_i}), \end{cases} \end{cases} \tag{1}$$

with

$$\begin{cases} f_{u_i}(\nu_i) = \frac{1}{m_{u_i}}\Big(m_{v_i} v_i r_i - d_{u_i} u_i - d^*_{u_i}|u_i|u_i\Big), \\ f_{v_i}(\nu_i) = \frac{1}{m_{v_i}}\Big(m_{u_i} u_i r_i - d_{v_i} v_i - d^*_{v_i}|v_i|v_i\Big), \\ f_{r_i}(\nu_i) = \frac{1}{m_{r_i}}\Big((m_{u_i} - m_{v_i}) u_i v_i - d_{r_i} r_i - d^*_{r_i}|r_i|r_i\Big), \end{cases} \tag{2}$$

where (x_i, y_i) and ψ_i respectively denote the position and heading angel of the i-th MSV in the earth-fixed frame, $\nu_i = (u_i, v_i, r_i)^{\mathrm{T}}$ denotes the velocity of the i-th MSV's with u_i, v_i and r_i respectively denoting surge velocity, sway velocity and yaw velocity in the body-fixed frame, ω_{γ_i} $(\gamma_i = u_i, v_i, r_i)$ is the external disturbance caused by the ocean waves, wind, currents, etc., $\tau_{\gamma_i^*}(\gamma_i^* = u_i, r_i)$ is control input of the i-th MSV which is subject to the following saturation constraint:

$$\tau_{\gamma_i^*} = \begin{cases} \tau_{\gamma^*_{i,\max}}, & \text{if } \tau_{\gamma^*_{i,c}} > \tau_{\gamma^*_{i,\max}} \\ \tau_{\gamma^*_{i,c}}, & \text{if } \tau_{\gamma^*_{i,\min}} < \tau_{\gamma^*_{i,c}} < \tau_{\gamma^*_{i,\max}} \\ \tau_{\gamma^*_{i,\min}}, & \text{if } \tau_{\gamma^*_{i,c}} < \tau_{\gamma^*_{i,\min}} \end{cases} \tag{3}$$

$\tau_{\gamma^*_{i,c}}$ being the control input to be designed, $\tau_{\gamma^*_{i,\max}}$ and $\tau_{\gamma^*_{i,\min}}$ being *unknown* constants, $m_{\gamma_i} > 0$, d_{γ_i}, $d^*_{\gamma_i}$, $\gamma_i = u_i, v_i, r_i$ are some *unknown* constants in certain practical sense.

The control objective of the paper is to design a distributed cooperative tracking controller such that all the states of the closed-loop system are bounded while the position $p_i = (x_i, y_i)^{\mathrm{T}}$ and the heading angel ψ_i of the i-th follower enter and then stay at the given neighborhoods of those of the leader's output, i.e., for any $\varepsilon > 0$, there exists a finite time $T > 0$ such that

$$|p_i - p_d| < \varepsilon, \ |\psi_i - \psi_d| < \varepsilon, \ i = 1, \ldots, N.$$

with $p_d = (x_d, y_d)^{\mathrm{T}}$ and ψ_d being the position and heading angel of the virtual leader, respectively.

The following gives an important property of the MSV which can be easily verified from the nonlinear terms (2) and will be used in the later control design.

Property 1. *There exists an unknown constant σ such that $|f_{\gamma_i}(\nu_i)| \le \sigma(1 + |\nu_i|^2)$, $\gamma_i = u_i, v_i, r_i$.*

The following gives three assumptions about disturbance, the output of the leader and the graph for the networked system.

Assumption 1. *There exists an unknown constant* $\bar{\omega}$ *such that* $|\omega_{\gamma_i}| < \bar{\omega}$, $\gamma_i = u_i, v_i, r_i$.

Assumption 2. *There exist unknown constants* $\bar{p}$ *and* $\bar{\psi}$ *such that* $|p_d| + |\dot{p}_d| < \bar{p}$, $|\psi_d| + |\dot{\psi}_d| < \bar{\psi}$.

Assumption 3. *The leader is globally reachable in* $\overline{\mathcal{G}}$.

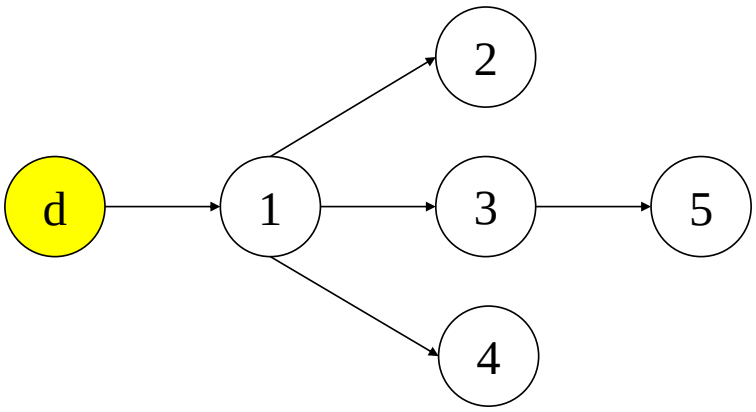


Figure 1. The communication topology graph.

3. Controller Design

First, in order to handle the saturation constraint of control input, the following smooth function is introduced [25,33,34]:

$$\rho_{\gamma_i^*}\left(\tau_{\gamma_{i,c}^*}\right) = \Gamma \mathrm{erf}\left(\frac{\sqrt{\pi}\tau_{\gamma_{i,c}^*}}{2\Gamma}\right), \tag{4}$$

with $\Gamma = \left(\frac{\tau_{\gamma_{i,\max}^*} + \tau_{\gamma_{i,\min}^*}}{2} + \frac{\tau_{\gamma_{i,\max}^*} - \tau_{\gamma_{i,\min}^*}}{2}\mathrm{sign}\left(\tau_{\gamma_{i,c}^*}\right)\right)$ and $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}x$ being a Gaussian error function. By the above function, the following approximation error is defined:

$$e_{\gamma_i^*} = \tau_{\gamma_i^*} - \rho_{\gamma_i^*}\left(\tau_{\gamma_{i,c}^*}\right), \tag{5}$$

which is clearly bounded by using the definition given by (3) and (6), i.e., there exists an unknown constant $\bar{e}_\gamma$ such that $\left|e_{\gamma_i^*}\right| \leq \bar{e}_\gamma$. Then, by using the mean value theorem, $\rho_{\gamma_i^*}$ is rewritten as follows:

$$\rho_{\gamma_i^*}\left(\tau_{\gamma_{i,c}^*}\right) = \rho_{\gamma_i^*}(0) + W_{\gamma_i^*}\left(\tau_{\gamma_{i,c}^*} - 0\right), \tag{6}$$

with $W_{\gamma_i^*} = e^{-\left(\frac{\sqrt{\pi}\iota\gamma_{i,c}^*}{2\Gamma}\right)^2}$, $\iota \in (0,1)$. Note that, by the definition of $W_{\gamma_i^*}$, there exists an unknown constant W_0 such that $W_{\gamma_i^*} > W_0 > 0$. Thus, combining (5) with (6) while noting $\rho_{\gamma_i^*}(0) = 0$, we obtain that

$$\tau_{\gamma_i^*} = W_{\gamma_i^*}\tau_{\gamma_{i,c}^*} + e_{\gamma_i^*}, \tag{7}$$

with $\gamma_i^* = u_i, r_i$, $\gamma_{i,c}^* = u_{i,c}, r_{i,c}$.

Second, define the following state transformations:

$$\begin{cases} \eta_i = \sum_{j=1}^{N} a_{ij}\left(p_i - p_j\right) + b_i(p_i - p_d), \\ \xi_i = \sum_{j=1}^{N} a_{ij}(\psi_i - \psi_j) + b_i(\psi_i - \psi_d), \\ u_{e_i} = u_i - \beta_{1i}, \\ v_{e_i} = v_i - \beta_{2i} - \tanh\zeta_i, \\ r_{e_i} = r_i - \beta_{3i}, \end{cases} \tag{8}$$

with β_{1i}, β_{2i} and β_{3i} being virtual controls which will be designed later, $\tanh\zeta_i$ being some additional term for controller whose updating law will be specified later. Then, by (1) and (7), a new system is obtained from (8):

$$\begin{cases} \dot{\eta}_i = d_i R_i \bar{U}_i - \sum_{j=1}^{N} a_{ij} R_j \bar{U}_j - b_i \dot{p}_d, \\ \dot{\xi}_i = d_i(r_{e_i} + \beta_{3i}) - \sum_{j=1}^{N} a_{ij}(r_{e_j} + \beta_{3j}) - b_i\dot{\psi}_d, \\ \dot{u}_{e_i} = f_{u_i}(\nu_i) + \frac{1}{m_{u_i}}\left(W_{u_i}\tau_{u_{i,c}} + e_{u_i} + \omega_{u_i}\right) - \dot{\beta}_{1i}, \\ \dot{v}_{e_i} = f_{v_i}(\nu_i) + \frac{1}{m_{v_i}}\omega_{v_i} - \dot{\beta}_{2i} - \frac{\dot{\zeta}_i}{\cosh^2\zeta_i}, \\ \dot{r}_{e_i} = f_{r_i}(\nu_i) + \frac{1}{m_{r_i}}\left(W_{r_i}\tau_{r_{i,c}} + e_{r_i} + \omega_{r_i}\right) - \dot{\beta}_{3i}, \end{cases} \tag{9}$$

with $d_i = \sum_{j=1}^{N} a_{ij} + b_i$ and

$$\bar{U}_i = \begin{pmatrix} u_{e_i} + \beta_{1i} \\ v_{e_i} + \beta_{2i} + \tanh\zeta_i \end{pmatrix}, R_i = \begin{pmatrix} \cos\psi_i & -\sin\psi_i \\ \sin\psi_i & \cos\psi_i \end{pmatrix}.$$

Now, we are ready for the presentation of the controller for the i-th follower which is given as follows:

$$\begin{cases} \tau_{u_{i,c}} = -c_{3i}u_{e_i} - c_{3i}u_{e_i}v_{e_i}^2 - \frac{1}{4}\kappa u_{e_i}\left(\left(1 + |\nu_i|^2\right)^2 + 1 + \left(\frac{\partial\beta_{1i}}{\partial\psi_i}r_i\right)^2 \right. \\ \qquad \left. + \left(\left(\frac{\partial\beta_{1i}}{\partial p}\right)^{\mathrm{T}} RU\right)^2 + \left|\frac{\partial\beta_{1i}}{\partial p_d}\right|^2 + \left(\frac{\partial\beta_{1i}}{\partial\kappa}\left(|\eta|^2 + |\xi|^2\right)\right)^2\right), \\ \tau_{r_{i,c}} = -c_{5i}r_{e_i} - c_{5i}v_{e_i}^2 r_{e_i} - \frac{1}{4}\kappa r_{e_i}\left(\left(1 + |\nu_i|^2\right)^2 + 1 + \left(\left(\frac{\partial\beta_{3i}}{\partial\psi}\right)^{\mathrm{T}} r\right)^2 \right. \\ \qquad \left. + \left(\frac{\partial\beta_{3i}}{\partial\psi_d}\right)^2 + \left(\frac{\partial\beta_{1i}}{\partial\kappa}\left(|\eta|^2 + |\xi|^2\right)\right)^2\right), \end{cases} \tag{10}$$

where $\eta = (\eta_1^{\mathrm{T}}, \cdots, \eta_N^{\mathrm{T}})^{\mathrm{T}}$, $\xi = (\xi_1, \cdots, \xi_N)^{\mathrm{T}}$, $p = (p_1^{\mathrm{T}}, \cdots, p_N^{\mathrm{T}})^{\mathrm{T}}$, $\psi = (\psi_1, \cdots, \psi_N)^{\mathrm{T}}$, $R = \mathbf{diag}(R_1, \cdots, R_N)$, $U = (U_1^{\mathrm{T}}, \cdots, U_N^{\mathrm{T}})^{\mathrm{T}}$, $U_i = (u_i, v_i)^{\mathrm{T}}$, $r = (r_1, \cdots, r_N)^{\mathrm{T}}$, κ is tuned by the following updating law:

$$\dot{\kappa} = \begin{cases} |\eta|^2 + |\xi|^2 - \frac{\varepsilon^2}{2\|H^{-1}\|^2}, & |\eta|^2 + |\xi|^2 \geq \frac{\varepsilon^2}{\|H^{-1}\|^2}, \\ 0, & |\eta|^2 + |\xi|^2 < \frac{\varepsilon^2}{\|H^{-1}\|^2}, \end{cases} \tag{11}$$

with $\kappa(0) \geq 1$, β_{1i}, β_{2i} β_{3i} and ζ_i being chosen as follows:

$$\begin{cases} \beta^{12} &= -c_{1i}R^{-1}(H^{-1}\otimes I_2)\eta - \frac{1}{2}\kappa R^{-1}(H^{-1}\otimes I_2)\mathbf{diag}(d_1,\cdots,d_N)\eta, \\ \beta_{31} &= -c_{2i}H^{-1}\xi - \frac{1}{4}\kappa H^{-1}B\xi, \\ \dot{\zeta}_i &= \cosh^2\zeta_i\cdot\Big(-c_{3i}u_{e_i}^2v_{e_i} - c_{5i}v_{e_i}r_{e_i}^2 + c_4v_{e_i} + \frac{1}{4}\kappa v_{e_i}\Big(\Big(1+|\nu_i|^2\Big)^2+1 \\ &\quad + \Big(\frac{\partial\beta_{2i}}{\partial\psi_i}r_i\Big)^2 + \Big(\Big(\frac{\partial\beta_{2i}}{\partial p}\Big)^{\mathrm{T}}RU\Big)^2 + \Big|\frac{\partial\beta_{2i}}{\partial p_d}\Big|^2 + \Big(\frac{\partial\beta_{2i}}{\partial\kappa}\big(|\eta|^2+|\xi|^2\big)\Big)^2\Big)\Big), \end{cases} \tag{12}$$

$\beta^{12} = ((\beta_{11},\beta_{21})^{\mathrm{T}},\cdots,(\beta_{1N},\beta_{2N})^{\mathrm{T}})^{\mathrm{T}}$, $\beta_3 = (\beta_{31},\cdots,\beta_{3N})^{\mathrm{T}}$, $H = L+B$, $c_{ki} > \frac{1}{2}d_i + \frac{1}{2}\sum_{j=1}^{N}a_{ij} + \frac{1}{2}$, $k = 1,\cdots,5$.

Remark 1. *Compared with other adaptive consensus methods in related works (such as [5,10,17,18]), the proposed adaptive controller has lower complexity in controller implementation and parameter tuning induced by the iterative adaptive design. Specifically, adaptive laws should be designed for each follower for the estimation of neural network weights in [5,17,18] or the estimation of unknown system parameters in [10], which clearly expand the dimension of the closed-loop system and hence enhance the complexity in controller implementation and parameter tuning. Differently, as shown by (11), all the followers share one dynamic gain for the compensation of system uncertainties. Then, such a feature decreases the complexity in controller implementation and parameter tuning.*

Proposition 1. *For the MSVs with Assumptions 1–3 and the followers' dynamics satisfying (1), the designed controller (10) and (11) guarantees that the Lyapunov function $V = \sum_{i=1}^{N}V_i$ with*

$$V_i = \frac{1}{2}\eta_i^{\mathrm{T}}\eta_i + \frac{1}{2}\xi_i^2 + \frac{1}{2}\frac{m_{u_i}}{W_0}u_{e_i}^2 + \frac{1}{2}v_{e_i}^2 + \frac{1}{2}\frac{m_{r_i}}{W_0}r_{e_i}^2,$$

satisfies that

$$\dot{V} \leq -cV + \frac{M^*}{\kappa} \tag{13}$$

with c and M^* being some positive constant.

Proof. Such proposition is proved by the following recursive steps.

Step 1: Define $V_{i1} = \frac{1}{2}\eta_i^{\mathrm{T}}\eta_i + \frac{1}{2}\xi_i^2$. Then, computing the time derivative of V_{i1} along the solutions of system (9) leads to the following:

$$\begin{aligned} \dot{V}_{i1} &= \eta_i^{\mathrm{T}}d_iR_i\begin{pmatrix}u_{e_i}\\ v_{e_i}\end{pmatrix} - \eta_i^{\mathrm{T}}\sum_{j=1}^{N}a_{ij}R_j\begin{pmatrix}u_{e_j}\\ v_{e_j}\end{pmatrix} + \xi_id_ir_{e_i} - \xi_i\sum_{j=1}^{N}a_{ij}r_{e_j} \\ &\quad + \eta_i^{\mathrm{T}}d_iR_i\begin{pmatrix}\beta_{1i}\\ \beta_{2i}\end{pmatrix} - \eta_i^{\mathrm{T}}\sum_{j=1}^{N}a_{ij}R_j\begin{pmatrix}\beta_{1j}\\ \beta_{2j}\end{pmatrix} + \xi_id_i\beta_{3i} - \xi_i\sum_{j=1}^{N}a_{ij}\beta_{3j} \\ &\quad + \eta_i^{\mathrm{T}}d_iR_i\begin{pmatrix}0\\ \tanh\zeta_i\end{pmatrix} - \eta_i^{\mathrm{T}}\sum_{j=1}^{N}a_{ij}R_j\begin{pmatrix}0\\ \tanh\zeta_j\end{pmatrix} - \eta_i^{\mathrm{T}}b_i\dot{p}_d - \xi_ib_i\dot{\psi}_d. \end{aligned} \tag{14}$$

Some terms on the right-hand side of (14) should be estimated for later control design. First, by using Young's inequality, the first four items on the right-hand side of (14) are estimated as follows:

$$\begin{cases} \eta_i^{\mathrm{T}}d_iR_i\begin{pmatrix}u_{e_i}\\ v_{e_i}\end{pmatrix} \leq \frac{d_i}{2}|\eta_i|^2 + \frac{d_i}{2}\Big(u_{e_i}^2+v_{e_i}^2\Big), \\ \eta_i^{\mathrm{T}}\sum_{j=1}^{N}a_{i_j}R_j\begin{pmatrix}u_{e_j}\\ v_{e_j}\end{pmatrix} \leq \frac{1}{2}\sum_{j=1}^{N}a_{ij}|\eta_i|^2 + \frac{1}{2}\sum_{j=1}^{N}a_{ij}\Big(u_{e_j}^2+v_{e_j}^2\Big), \\ \xi_id_ir_{e_i} \leq \frac{d_i}{2}\xi_i^2 + \frac{d_i}{2}r_{e_i}^2, \\ -\xi_i\sum_{j=1}^{N}a_{ij}r_{e_j} \leq \frac{1}{2}\sum_{j=1}^{N}a_{ij}\xi_i^2 + \frac{1}{2}\sum_{j=1}^{N}a_{ij}r_{e_j}^2. \end{cases} \tag{15}$$

Second, by Assumption 3 and Young's inequality while noting the fact $|\tanh\zeta_i| < 1$, the estimations of the last four terms on the right-hand side of (14) are given as follows:

$$\begin{cases} \eta_i^{\mathrm{T}} d_i R_i \begin{pmatrix} 0 \\ \tanh\zeta_i \end{pmatrix} \le \frac{1}{4} d_i \kappa |\eta_i|^2 + \frac{1}{\kappa} d_i, \\ -\eta_i^{\mathrm{T}} \sum_{j=1}^{N} a_{ij} R_j \begin{pmatrix} 0 \\ \tanh\zeta_j \end{pmatrix} \le \frac{1}{4} \sum_{j=1}^{N} a_{ij} \kappa |\eta_i|^2 + \frac{1}{\kappa} \sum_{j=1}^{N} a_{ij}, \\ -\eta_i^{\mathrm{T}} b_i \dot{p}_d \le \frac{1}{4} b_i \kappa |\eta_i|^2 + \frac{1}{\kappa} b_i \bar{p}^2, \\ -\xi_i b_i \dot{\psi}_d \le \frac{1}{4} b_i \kappa \xi_i^2 + \frac{1}{\kappa} b_i \bar{\psi}^2, \end{cases} \tag{16}$$

where κ is the dynamic gain whose updating law has been given in (11). Substituting (15) and (16) into (14) gives that

$$\begin{aligned} \dot{V}_{i1} \le & \left(\frac{1}{2} d_i + \frac{1}{2} \sum_{j=1}^{N} a_{ij} \right) \eta_i^{\mathrm{T}} \eta_i + \left(\frac{1}{2} d_i + \frac{1}{2} \sum_{j=1}^{N} a_{ij} \right) \xi_i^2 \\ & + \frac{1}{2} d_i u_{e_i}^2 + \frac{1}{2} \sum_{j=1}^{N} a_{ij} u_{e_j}^2 + \frac{1}{2} d_i v_{e_i}^2 + \frac{1}{2} \sum_{j=1}^{N} a_{ij} v_{e_j}^2 + \frac{1}{2} d_i r_{e_i}^2 + \frac{1}{2} \sum_{j=1}^{N} a_{ij} r_{e_j}^2 \\ & + \eta_i^{\mathrm{T}} d_i R_i \begin{pmatrix} \beta_{1i} \\ \beta_{2i} \end{pmatrix} - \eta_i^{\mathrm{T}} \sum_{j=1}^{N} a_{ij} R_j \begin{pmatrix} \beta_{1j} \\ \beta_{2j} \end{pmatrix} + \xi_i d_i \beta_{3i} - \xi_i \sum_{j=1}^{N} a_{ij} \beta_{3j} \\ & + \frac{1}{2} d_i \kappa \eta_i^{\mathrm{T}} \eta_i + \frac{1}{4} b_i \kappa \xi_i^2 + \frac{M_{1i}}{\kappa}, \end{aligned} \tag{17}$$

with $M_{1i} = b_i(\bar{p}^2 + \bar{\psi}^2) + d_i + \sum_{j=1}^{N} a_{ij}$.

Moreover, by the choice of virtual controls β_{1i}, β_{2i} and β_{3i} given by (12), we obtain

$$\begin{cases} d_i R_i \begin{pmatrix} \beta_{1i} \\ \beta_{2i} \end{pmatrix} - \sum_{j=1}^{N} a_{ij} R_j \begin{pmatrix} \beta_{1j} \\ \beta_{2j} \end{pmatrix} + \frac{1}{2} d_i \kappa \eta_i = -c_{1i} \eta_i, \\ d_i \beta_{3i} - \sum_{j=1}^{N} a_{ij} \beta_{3j} + \frac{1}{4} b_i \kappa \xi_i = -c_{2i} \xi_i, \end{cases}$$

substituting which into (17) brings

$$\begin{aligned} \dot{V}_{i1} \le & - \left(c_{1i} - \frac{1}{2} d_i - \frac{1}{2} \sum_{j=1}^{N} a_{ij} \right) \eta_i^{\mathrm{T}} \eta_i - \left(c_{2i} - \frac{1}{2} d_i - \frac{1}{2} \sum_{j=1}^{N} a_{ij} \right) \xi_i^2 + \frac{M_{1i}}{\kappa} \\ & + \frac{1}{2} d_i u_{e_i}^2 + \frac{1}{2} \sum_{j=1}^{N} a_{ij} u_{e_j}^2 + \frac{1}{2} d_i v_{e_i}^2 + \frac{1}{2} \sum_{j=1}^{N} a_{ij} v_{e_j}^2 + \frac{1}{2} d_i r_{e_i}^2 + \frac{1}{2} \sum_{j=1}^{N} a_{ij} r_{e_j}^2. \end{aligned} \tag{18}$$

Step 2: For the Lyapunov function $V_i = V_{i1} + \frac{1}{2} \frac{m_{u_i}}{W_0} u_{e_i}^2 + \frac{1}{2} v_{e_i}^2 + \frac{1}{2} \frac{m_{r_i}}{W_0} r_{e_i}^2$, its time derivative along the solutions of system (9) gives

$$\begin{aligned} \dot{V}_i = & \dot{V}_{i1} + \frac{u_{e_i}}{W_0} \left(W_{u_i} \tau_{u_{i,c}} + e_{u_i} + m_{u_i} f_{u_i} + \omega_{u_i} - m_{u_i} \dot{\beta}_{1i} \right) \\ & + v_{e_i} \left(- \frac{\dot{\zeta}_i}{\cosh^2 \zeta_i} + f_{v_i} + \frac{1}{m_{v_i}} \omega_{v_i} - \dot{\beta}_{2i} \right) \\ & + \frac{r_{e_i}}{W_0} \left(W_{r_i} \tau_{r_{i,c}} + e_{r_i} + m_{r_i} f_{r_i} + \omega_{r_i} - m_{r_i} \dot{\beta}_{3i} \right). \end{aligned} \tag{19}$$

Some terms on the right-hand side of (19) should be further estimated. First, by using Assumptions 1–3 and Young's inequality, the following estimations are obtained:

$$\begin{cases} u_{e_i} m_{u_i} f_{u_i} \le |u_{e_i}| m_{u_i} \sigma \left(1 + |\nu_i|^2\right) \le \frac{1}{4} W_0 \kappa u_{e_i}^2 \left(1 + |\nu_i|^2\right)^2 + \frac{m_{u_i}^2 \sigma^2}{W_0 \kappa} \\ \qquad \le \frac{1}{4} W_{u_i} \kappa u_{e_i}^2 \left(1 + |\nu_i|^2\right)^2 + \frac{m_{u_i}^2 \sigma^2}{W_0 \kappa}, \\ u_{e_i}(e_{u_i} + \omega_{u_i}) \le \frac{1}{4} W_{u_i} \kappa u_{e_i}^2 + \frac{e_m^2 + \bar{\omega}^2}{W_0 \kappa}, \\ v_{e_i} f_{v_i} \le \frac{1}{4} \kappa v_{e_i}^2 \left(1 + |\nu_i|^2\right)^2 + \frac{\sigma^2}{\kappa}, \\ \frac{1}{m_{v_i}} v_{e_i} \omega_{v_i} \le \frac{1}{4} \kappa v_{e_i}^2 + \frac{\bar{\omega}^2}{m_{v_i}^2 \kappa}, \\ r_{e_i} m_{r_i} f_{r_i} \le \frac{1}{4} W_{r_i} \kappa r_{e_i}^2 \left(1 + |\nu_i|^2\right)^2 + \frac{m_{r_i}^2 \sigma^2}{W_0 \kappa}, \\ r_{e_i}(e_{r_i} + \omega_{r_i}) \le \frac{1}{4} W_{r_i} \kappa r_{e_i}^2 + \frac{e_m^2 + \bar{\omega}^2}{W_0 \kappa}. \end{cases} \tag{20}$$

Second, by (12), $\dot{\beta}_{1i}$, $\dot{\beta}_{2i}$ and $\dot{\beta}_{3i}$ are given as follows:

$$\begin{cases} \dot{\beta}_{1i} = \frac{\partial \beta_{1i}}{\partial \psi_i} \dot{\psi}_i + \left(\frac{\partial \beta_{1i}}{\partial p}\right)^{\mathrm{T}} \dot{p} + \left(\frac{\partial \beta_{1i}}{\partial p_d}\right)^{\mathrm{T}} \dot{p}_d + \frac{\partial \beta_{1i}}{\partial \kappa} \dot{\kappa}, \\ \dot{\beta}_{2i} = \frac{\partial \beta_{2i}}{\partial \psi_i} \dot{\psi}_i + \left(\frac{\partial \beta_{2i}}{\partial p}\right)^{\mathrm{T}} \dot{p} + \left(\frac{\partial \beta_{2i}}{\partial p_d}\right)^{\mathrm{T}} \dot{p}_d + \frac{\partial \beta_{2i}}{\partial \kappa} \dot{\kappa}, \\ \dot{\beta}_{3i} = \left(\frac{\partial \beta_{3i}}{\partial \psi}\right)^{\mathrm{T}} \dot{\psi} + \frac{\partial \beta_{3i}}{\partial \psi_d} \dot{\psi}_d + \frac{\partial \beta_{3i}}{\partial \kappa} \dot{\kappa}, \end{cases}$$

Then, by using Young's inequality, the definition of $\dot{\kappa}$ given by (11) while noting $\dot{p} = RU$, $\dot{\psi} = r$, the following estimations are obtained:

$$\begin{cases} -u_{e_i} m_{u_i} \dot{\beta}_{1i} \le \frac{1}{4} W_{u_i} \kappa u_{e_i}^2 \left(\left(\frac{\partial \beta_{1i}}{\partial \psi_i} r_i\right)^2 + \left(\left(\frac{\partial \beta_{1i}}{\partial p}\right)^{\mathrm{T}} RU \right)^2 + \left| \frac{\partial \beta_{1i}}{\partial p_d} \right|^2 \right. \\ \qquad \left. + \left(\frac{\partial \beta_{1i}}{\partial \kappa} \left(|\eta|^2 + |\xi|^2 \right) \right)^2 \right) + \frac{m_{u_i}^2 (3 + \bar{p}^2)}{W_0 \kappa}, \\ -v_{e_i} \dot{\beta}_{2i} \le \frac{1}{4} \kappa v_{e_i}^2 \left(\left(\frac{\partial \beta_{2i}}{\partial \psi_i} r_i\right)^2 + \left(\left(\frac{\partial \beta_{2i}}{\partial p}\right)^{\mathrm{T}} RU \right)^2 + \left| \frac{\partial \beta_{2i}}{\partial p_d} \right|^2 \right. \\ \qquad \left. + \left(\frac{\partial \beta_{2i}}{\partial \kappa} \left(|\eta|^2 + |\xi|^2 \right) \right)^2 \right) + \frac{3 + \bar{p}^2}{\kappa}, \\ -r_{e_i} m_{r_i} \dot{\beta}_{3i} \le \frac{1}{4} W_{r_i} \kappa r_{e_i}^2 \left(\left(\left(\frac{\partial \beta_{3i}}{\partial \psi}\right)^{\mathrm{T}} r \right)^2 + \left(\frac{\partial \beta_{3i}}{\partial \psi_d}\right)^2 + \left(\frac{\partial \beta_{3i}}{\partial \kappa} \left(|\eta|^2 + |\xi|^2 \right) \right)^2 \right) \\ \qquad + \frac{m_{r_i}^2 (2 + \bar{\psi}^2)}{W_0 \kappa}. \end{cases} \tag{21}$$

Substituting (20) and (21) into (19) arrives at

$$\dot{V}_i \le \dot{V}_{i1} + \frac{u_{e_i} W_{u_i}}{W_0} \left(\tau_{u_{i,c}} + \frac{1}{4} \kappa u_{e_i} \left(\left(1 + |\nu_i|^2\right)^2 + 1 + \left(\frac{\partial \beta_{1i}}{\partial \psi_i} r_i\right)^2 + \left(\left(\frac{\partial \beta_{1i}}{\partial p}\right)^{\mathrm{T}} RU \right)^2 \right.$$

$$+\left|\frac{\partial\beta_{1i}}{\partial p_d}\right|^2+\left(\frac{\partial\beta_{1i}}{\partial\kappa}\left(|\eta|^2+|\xi|^2\right)\right)^2\Bigg)\Bigg)+v_{e_i}\Bigg(-\frac{\dot{\zeta}_i}{\cosh^2\zeta_i}+\frac{1}{4}\kappa v_{e_i}\Bigg(\left(1+|\nu_i|^2\right)^2$$
$$+1+\left(\frac{\partial\beta_{2i}}{\partial\psi_i}r_i\right)^2+\left(\left(\frac{\partial\beta_{2i}}{\partial p}\right)^{\mathrm{T}}RU\right)^2+\left|\frac{\partial\beta_{2i}}{\partial p_d}\right|^2+\left(\frac{\partial\beta_{2i}}{\partial\kappa}\left(|\eta|^2+|\xi|^2\right)\right)^2\Bigg)\Bigg)$$
$$+\frac{r_{e_i}W_{r_i}}{W_0}\Bigg(\tau_{r_{i,c}}+\frac{1}{4}\kappa r_{e_i}\Bigg(\left(1+|\nu_i|^2\right)^2+1+\left(\left(\frac{\partial\beta_{3i}}{\partial\psi}\right)^{\mathrm{T}}r\right)^2+\left(\frac{\partial\beta_{3i}}{\partial\psi_d}\right)^2$$
$$+\left(\frac{\partial\beta_{3i}}{\partial\kappa}\left(|\eta|^2+|\xi|^2\right)\right)^2\Bigg)\Bigg)+\frac{M_{2i}}{\kappa}, \tag{22}$$

where $M_{2i}=\frac{m_{u_i}^2(\sigma^2+3+\bar{p}^2)+m_{r_i}^2(\sigma^2+2+\bar{\psi}^2)+2(e_m^2+\bar{\omega}^2)+\left(\sigma^2+\frac{\bar{\omega}^2}{m_{v_i}^2}+3+\bar{p}^2\right)W_0}{W_0}$. Thus, the choice of controller $\tau_{u_{i,c}}$, $\tau_{r_{i,c}}$ and additional control $\dot{\zeta}_i$ given by (10) and (12) while noting $W_{\gamma_i^*}>W_0>0$, $\gamma_i^*=u_i,r_i$, gives

$$\begin{aligned}\dot{V}_i &\leq \dot{V}_{i1}-c_{3i}\frac{W_{u_i}}{W_0}u_{e_i}^2-c_{3i}\left(\frac{W_{u_i}}{W_0}-1\right)u_{e_i}^2v_{e_i}^2-c_4v_{e_i}^2-c_5\frac{W_{r_i}}{W_0}r_{e_i}^2-c_{5i}\left(\frac{W_{r_i}}{W_0}-1\right)v_{e_i}^2r_{e_i}^2+\frac{M_{2i}}{\kappa}\\ &\leq -\left(c_{1i}-\frac{1}{2}d_i-\frac{1}{2}\sum_{j=1}^N a_{ij}\right)\eta_i^{\mathrm{T}}\eta_i-\left(c_{2i}-\frac{1}{2}d_i-\frac{1}{2}\sum_{j=1}^N a_{ij}\right)\xi_i^2\\ &\quad-\left(c_{3i}u_{e_i}^2-\frac{1}{2}d_iu_{e_i}^2-\frac{1}{2}\sum_{j=1}^N a_{ij}u_{e_j}^2\right)-\left(c_{4i}v_{e_i}^2-\frac{1}{2}d_iv_{e_i}^2-\frac{1}{2}\sum_{j=1}^N a_{ij}v_{e_j}^2\right)\\ &\quad-\left(c_{5i}r_{e_i}^2-\frac{1}{2}d_ir_{e_i}^2-\frac{1}{2}\sum_{j=1}^N a_{ij}r_{e_j}^2\right)+\frac{M_{1i}+M_{2i}}{\kappa}.\end{aligned} \tag{23}$$

Step 3: For the Lyapunov function $V=\sum_{i=1}^N V_{i2}$, its time derivative is given as follows by (23):

$$\begin{aligned}\dot{V}\leq&-\sum_{i=1}^N\left(c_{1i}-\frac{1}{2}d_i-\frac{1}{2}\sum_{j=1}^N a_{ij}\right)\eta_i^{\mathrm{T}}\eta_i-\sum_{i=1}^N\left(c_{2i}-\frac{1}{2}d_i-\frac{1}{2}\sum_{j=1}^N a_{ij}\right)\xi_i^2\\ &-\sum_{i=1}^N\left(c_{3i}-\frac{1}{2}d_i-\frac{1}{2}\sum_{j=1}^N a_{ji}\right)u_{e_i}^2-\sum_{i=1}^N\left(c_{4i}-\frac{1}{2}d_i-\frac{1}{2}\sum_{j=1}^N a_{ji}\right)v_{e_i}^2\\ &-\sum_{i=1}^N\left(c_{5i}-\frac{1}{2}d_i-\frac{1}{2}\sum_{j=1}^N a_{ji}\right)r_{e_i}^2+\frac{M^*}{\kappa}.\end{aligned}$$

Then, by the choice of c_{ki} given above, we arrive at

$$\begin{aligned}\dot{V}&\leq-\frac{1}{2}\eta_i^{\mathrm{T}}\eta_i-\frac{1}{2}\xi_i^2-\frac{1}{2}u_{e_i}^2-\frac{1}{2}v_{e_i}^2-\frac{1}{2}r_{e_i}^2+\frac{M^*}{\kappa}\\ &\leq-cV+\frac{M^*}{\kappa},\end{aligned}$$

with $M^*=\sum_{i=1}^N(M_{1i}+M_{2i})$, $c=\min_{1\leq i\leq N}\left\{1,\frac{W_0}{m_{u_i}},\frac{W_0}{m_{r_i}}\right\}$, which gives (13). This completes the proof. □

4. Performance Analysis

As preparation of the performance analysis of the resulting closed-loop system, the boundedness of the dynamic gain κ is first given and summarized in the following proposition.

Proposition 2. *The time-varying gain κ updated by (11) is bounded on $[0,+\infty)$.*

Proof. The proposition is proved by contradiction. Suppose that κ is unbounded on $[0,+\infty)$. Then, for some positive constant $\frac{4M^*\|H^{-1}\|^2}{\varepsilon^2 c}$, there exists $t_1 > 0$ such that $\kappa(t_1) > \frac{4M^*\|H^{-1}\|^2}{\varepsilon^2 c}$. Since κ is monotone increasing by noting that $\dot{\kappa}(t) > 0$ from (11), there holds that $\kappa(t) \geq \kappa(t_1) > \frac{4M^*\|H^{-1}\|^2}{\varepsilon^2 c}, \forall t > t_1$.

Integrating both sides of (13) on $[t_1, t]$ while noting $\kappa > \frac{4M^*\|H^{-1}\|^2}{\varepsilon^2 c}$ leads to

$$V(t) \leq V(t_1)e^{c(t_1-t)} + \frac{\varepsilon^2}{4\|H^{-1}\|^2}, t > t_1. \tag{24}$$

Since $\lim_{t\to+\infty} V(t_1)e^{c(t_1-t)} = 0$, there exists a $t_2 > t_1$ such that $V(t_1)e^{c(t_1-t)} < \frac{\varepsilon^2}{4\|H^{-1}\|^2}$, $t > t_2$. Then, there holds that

$$\frac{1}{2}\eta^{\mathrm{T}}\eta + \frac{1}{2}\xi^{\mathrm{T}}\xi \leq V(t) < \frac{\varepsilon^2}{2\|H^{-1}\|^2}, \ t > t_2, \tag{25}$$

which implies that $|\eta|^2 + |\xi|^2 < \frac{\varepsilon^2}{\|H^{-1}\|^2}$. This leads to that $\dot{\kappa}(t) \equiv 0$ by using (11), and hence κ remains a constant after t_2. Noting that κ is continuous and increasing with $\kappa(0) \geq 1$, κ is bounded on $[0, t_2)$, and hence κ is bounded on $[0,+\infty)$. This contradicts with the induce assumption. □

Now, it is a position to present the main results of the paper which are summarized in the following theorem.

Theorem 1. *For the MSVs with Assumptions 1–3 and the followers' dynamics satisfying (1), the designed controller given by (10) and (11) guarantees the following properties:*

(1) all the signals of the closed-loop system are bounded on $[0,+\infty)$.

(2) the position p_i *and heading angel* ψ_i *of the i-th followers practically track those of the leader, i.e., for any* $\varepsilon > 0$, *there exists a finite time* $T > 0$ *such that*

$$|p_i - p_d| < \varepsilon, \ |\psi_i - \psi_d| < \varepsilon, \ i = 1, \ldots, N.$$

Proof. We first prove Claim (1). Noting $\kappa(0) \geq 1$ and $\dot{\kappa} \geq 0$, (13) gives

$$\dot{V} \leq -cV + M^*. \tag{26}$$

Integrating both sides of (26) over $[0, t]$, we obtain

$$V(t) \leq V(0)e^{-ct} + \frac{M^*}{c}(1 - e^{-ct}) < V(0) + \frac{M^*}{c}, \tag{27}$$

which implies that $V(t)$, and hence ξ_i, η_i, u_{e_i}, v_{e_i} and r_{e_i}, are bounded. Then, (8) gives the boundedness of p_i and ψ_i, and hence those of p and ψ. The boundedness of p, p_d, ψ, ψ_d and κ give that β_{1i}, β_{2i} and β_{3i} are bounded. Moreover, transfomation (8) implies that u_i, v_i and r_i are bounded. Thus, the boundedness of $\tau_{u_{i,c}}$ and $\tau_{r_{i,c}}$ are obtained by (10).

We next prove Claim (2). First, $\dot{p}_i$ and $\dot{\psi}_i$ are bounded by the boundedness of u_i, v_i and r_i. This, together with (9), gives that $\dot{\eta}_i$ and $\dot{\xi}_i$ are bounded. So are $\dot{\eta}$ and $\dot{\xi}$, and hence, $|\eta|^2$ and $|\xi|^2$ are uniformly continuous on $[0,+\infty)$. Moreover, the boundedness of κ gives that

$$\int_0^{+\infty} \dot{\kappa}(t)\mathrm{d}t = \kappa(+\infty) - \kappa(0) < +\infty,$$

which implies that $\dot{\kappa}(t)$ is integrable. Thus, by Babalat's lemma, we obtain that $\lim_{t\to+\infty}\dot{\kappa}(t)=0$. Therefore, for any $\varepsilon>0$, there exists a finite time $T>0$, such that

$$|\dot{\kappa}(t)|<\frac{\varepsilon^2}{2\|H^{-1}\|^2},\ \forall t>T.$$

Then, by (11), we obtain

$$|\eta|^2+|\xi|^2<\frac{\varepsilon^2}{\|H^{-1}\|^2},\ \forall t>T.$$

Noting $\eta=(\eta_1^{\mathrm{T}},\cdots,\eta_N^{\mathrm{T}})^{\mathrm{T}}$, $\xi=(\xi_1,\cdots,\xi_N)^{\mathrm{T}}$ and the definitions of η_i and ξ_i in (8), we have Noting the definition of η_i and ξ_i in (8), we have $\eta=(H\otimes I_2)(p-\mathbf{1}_N\otimes p_d)$, and $\xi=H(\psi-\mathbf{1}_N\otimes\psi_d)$, which brings

$$\begin{cases}|p-\mathbf{1}_N\otimes p_d|^2\le\left\|H^{-1}\otimes I_2\right\|^2|\eta|^2,\\ |\psi-\mathbf{1}_N\otimes\psi_d|^2\le\left\|H^{-1}\right\|^2|\xi|^2.\end{cases}\tag{28}$$

Since $|\eta|^2+|\xi|^2<\frac{\varepsilon^2}{\|H^{-1}\|^2}$, we have $|p-\mathbf{1}_N\otimes p_d|<\varepsilon$ and $|\psi-\mathbf{1}_N\otimes\psi_d|<\varepsilon$, and hence $|p_i-p_d|<\varepsilon$ and $|\psi_i-\psi_d|<\varepsilon$, $t>T$. □

5. Simulation Results

In this section, a networked system consisted by one leader and five follower is considered to verify the effectiveness of the proposed theoretical results. The communication topology of the network is shown in Figure 1 with node d denoting the leader while nodes 1–5 denoting the followers. Moreover, seen from Figure 1 that, matrices L and B are defined as follows:

$$L=\begin{pmatrix}0&0&0&0&0\\-1&1&0&0&0\\-1&0&1&0&0\\-1&0&0&1&0\\0&0&-1&0&1\end{pmatrix},\quad B=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}.$$

For each MSV (either the leader or the follower), the dynamics are given as (1) with initial conditions being chosen as $q_1(0)=(2.5,-4)^{\mathrm{T}}$, $q_2(0)=(2.8,3.8)^{\mathrm{T}}$, $q_3(0)=(2,-2.2)^{\mathrm{T}}$, $q_4(0)=(1.8,-2)^{\mathrm{T}}$, $q_5(0)=(1,-3.5)^{\mathrm{T}}$, $\psi_1(0)=0.15$, $\psi_2(0)=-0.2$, $\psi_3(0)=-0.35$, $\psi_4(0)=0.4$, $\psi_5(0)=0.5$, $\nu_1(0)=(0.1,0.3,0.2)^{\mathrm{T}}$, $\nu_2(0)=(0.4,0.2,0.5)^{\mathrm{T}}$, $\nu_3(0)=(0.3,-0.2,0.2)^{\mathrm{T}}$, $\nu_4(0)=(-0.1,0.3,0.1)^{\mathrm{T}}$, $\nu_5(0)=(0.2,0.3,0.4)^{\mathrm{T}}$. Suppose that the actual values of the unknown parameters in the system are $m_{u_i}=23$, $m_{v_i}=34$, $m_{r_i}=1.1$, $d_{u_i}=1.5$, $d_{v_i}=2.8$, $d_{r_i}=3.8$, $d^*_{u_i}=12$, $d^*_{v_i}=10.5$, $d^*_{r_i}=1.8$, $\tau_{u_{i,max}}=85$, $\tau_{u_{i,min}}=-80$, $\tau_{v_{i,max}}=85$, $\tau_{v_{i,min}}=-75$, the disturbance ω_{γ_i} $(\gamma_i=u_i,v_i,r_i)$ are supposed to be as follows:

$$\begin{cases}\omega_{u_i}&=1.2+0.8\sin 2t+2\cos 1.5t,\\ \omega_{v_i}&=-1.5-0.2\cos 2t+0.8\sin 0.5t,\\ \omega_{v_i}&=2.5-0.3\cos 2.2t+0.6\sin 0.6t.\end{cases}$$

Suppose that the output of the leader is given as

$$p_d=(3\cos 0.02t+2\sin 0.01t,2.5\sin 0.02t-3.2\cos 0.01t)^{\mathrm{T}},\psi_d=0.1\sin 0.01t.$$

Then, for the tracking accuracy parameter $\varepsilon = 0.01$, we implement controller (10), (11) with $\kappa(0) = 1$, $c_{k1} = 20$, $c_{k2} = 10$, $c_{k3} = 20$, $c_{k4} = 10$, $c_{k5} = 20$, $k = 1, \cdots, 5$. Consequently, four simulation figures are obtained (see Figures 2–5 for detail). Specifically, Figure 2 shows that the errors for the position and heading angel between the leader and the follower (i.e., $x_i - x_d$, $y_i - y_d$, $\psi_i - \psi_d$) enter and then stay at the given neighborhood of the origin after some time. Figure 3 demonstrates that the system states u_i, v_i, r_i of each follower are bounded. Figure 4 shows that the control input $\tau_{u_{i,c}}$, $\tau_{v_{i,c}}$ are bounded and satisfy the saturation constraints. Figure 5 shows that the dynamic gain $\hat{\kappa}$ is bounded and converges to a constant ultimately.

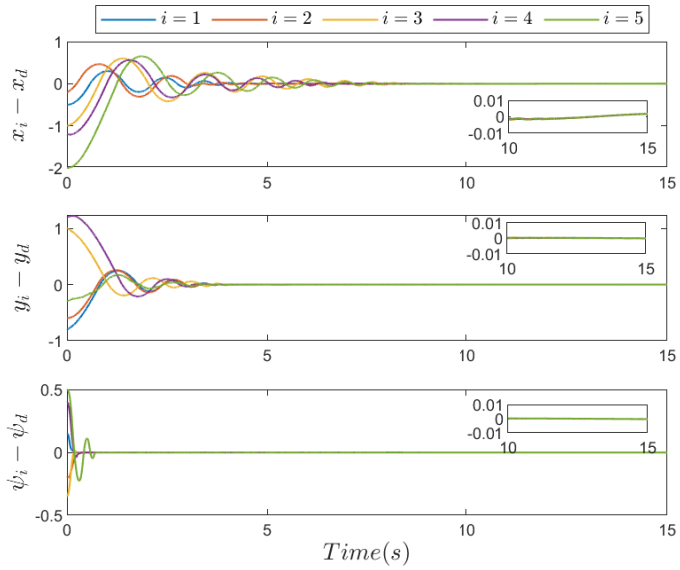


Figure 2. Trajectories of tracking error $x_i - x_d$, $y_i - y_d$, $\psi_i - \psi_d$.

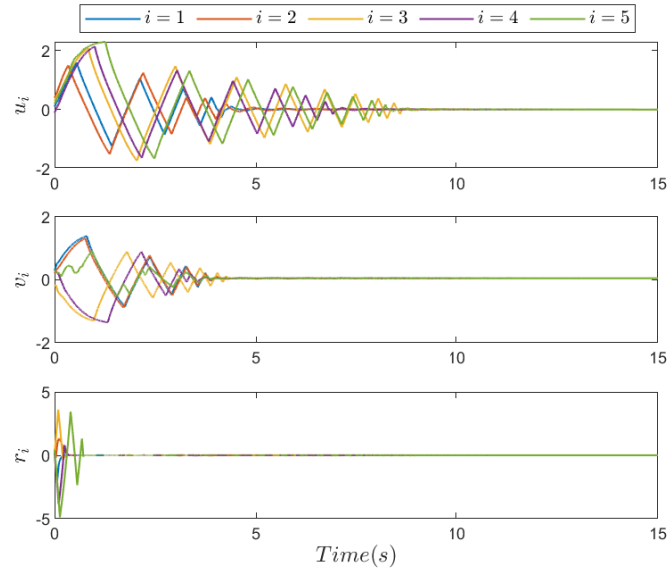


Figure 3. Trajectories of states u_i, v_i, r_i.

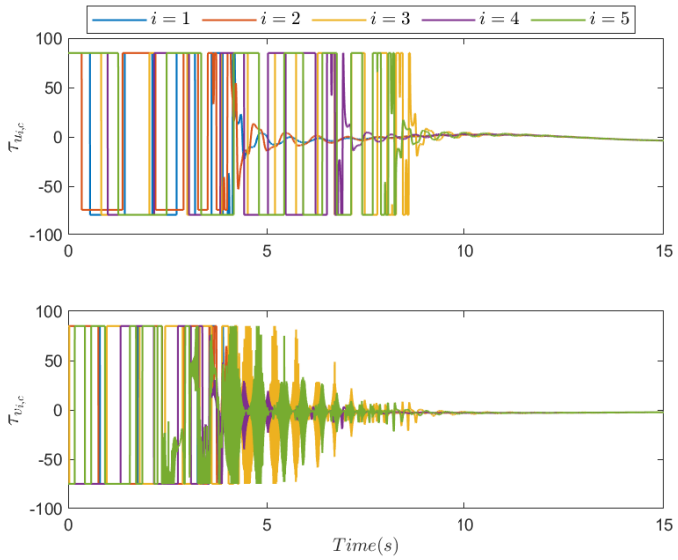


Figure 4. Trajectories of control inputs $\tau_{u_{i,c}}$, $\tau_{v_{i,c}}$.

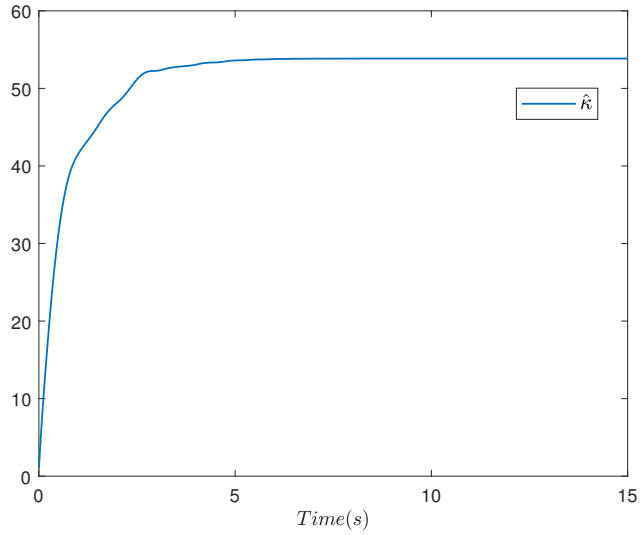


Figure 5. Trajectory of dynamic gain $\hat{\kappa}$.

6. Concluding Remarks

In this paper, consensus control is solved for a multiple underactuated MSV system with input saturation. Mainly due to the two key features, i.e., the consideration of input saturation and the presence of coarse information about the network which is reflected from the severe uncertainties of the followers and the generality of the leader's output, the traditional control methods in the related literature are inapplicable to the control problem. To achieve the goal of the paper, a novel adaptive control scheme is proposed which is capable of the coarse information about the MSVs which are reflected from the severe uncertainties and the generality as well as the measurability of the output of the leader's output. Consequently, a distributed adaptive state-feedback controller is explicitly designed which ensures that all the states of the closed-loop system are bounded while the output of the follower MSVs practically track that of the leader MSVs. The simulation results show the effectiveness of the proposed theoretical results for an investigated system. It is worthy pointing out that the proposed method is applicable for the consensus control of other systems, such as the networked system constituting by multiple manipulators.

The future research directions are the following two aspects: ① Consensus control via output feedback. Noting that all the states of the system are required to be available for

feedback in the paper which is difficult to implement in practice, then one of the interesting research focuses on the consensus control via output feedback for the investigated system. For such control problem, a state observer is required to reconstruct the system states, which is rather difficult due to the presence of uncertainties and hence greatly challenges the control problem. ② Consensus control for the networked system with other topologies. In this paper, a networked system with directed graph is considered. However, in some practical cases, a networked system may have other compunction topologies, such as the switching or undirected topology, for which the state transformations used in the paper is inapplicable and hence the proposed control method is incapable. Thus, it is interesting to investigate the consensus control of the investigated system under other topologies.

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