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On Solving the MHD Problem for Several Classes of Three-Dimensional Domains Within the Framework of Discrete Potential Theory

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Abstract

The MHD (magnetic hydrodynamics) boundary problem in three-dimensional domains of certain types is considered within the framework of discrete potential theory. The discrete character of the information obtained from remote sensing of the Earth and planets of the Solar System can be taken into account when using the basic principles of this theory. This approach makes it possible to reconstruct the spatial distribution of magnetic fields and the velocity field with relatively high accuracy using the heterogeneous data in some network points. In order to restore the magnetic image of a planet with a so-called dynamo, the subsequent approximations approach is implemented. The unknown physical field is represented as a sum of terms of different magnitudes. Such an approach allows us to simplify the nonlinear partial differential equation system of magnetic hydrodynamics and extend it to discrete magnetic field and velocity vectors. The solution of the simplified MHD equation system is constructed for some classes of bounded domains in Cartesian coordinates in three-dimensional space.

Keywords: uniqueness; magnetic hydrodynamics; discrete potential; m-tuply connected domains

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1. Introduction

Remote sensing of the Earth and other objects in the Solar System has caused a series of challenges for researchers [1–5]. First of all, the problem of adequate interpretation of large and very large time-dependent datasets arises. Second, the problem of data storage and manipulation is of great importance. The magnetic fields of planets with a so-called dynamo can be described within the framework of magnetic hydrodynamics (MHD). The most complicated problem is finding the solutions to MHD equations when not all necessary initial and boundary functions are given.

Information obtained from satellite missions should be carefully analyzed and checked. Some of this data should be "thrown out"; the remaining part is crucial for the interpretation process. Analytical modelling of the magnetic fields of planets implies the use of up-to-date information about the measured values of these fields as well as the details of processes taking place in the interior of the planet. Many parameters of the motion of charged fluid within a planet's core are unknown. Solutions to nonlinear inverse MHD equation systems

can only be determined uniquely if very strong constraints are imposed on the domains under investigation. We should note that the question of the existence of a solution to MHD systems of nonlinear partial differential equations is not considered in this work. This could be a topic of future studies.

An approximation approach to solving linear and nonlinear ill-posed problems in geophysics [6] allows us to reduce almost all settings to linear algebraic equation systems (LAESs). The features of such LAESs depend on the differential operator (or its finite-differenced analogue) describing the direct problem. For inverse equations, especially ill-posed ones, the eigenvalue problem comes to the forefront. More exactly, we are to find the resolvent of the direct problem operator.

The discrete character of the data should be considered when processing digital signals. For this reason, the basic principles of the discrete potential theory elaborated by V.N. Strakhov [6] at the beginning of the 1990s could serve, in our opinion, as an effective tool in imaging, restoring the sources of physical fields, and so on.

A cube in Cartesian three-dimensional space has a rotational symmetry group consisting of 24 elements. Permutations are fulfilled with different edges and planes of such a cube. For example, cyclic permutations correspond to the permutations of four diagonals of maximal length. Eigenvalues and eigenfunctions of the Laplace operator are also saved. If take some small cubes from the large one, then those cubes will not remain stable, but their size as well as the character of their eigenfunctions will be the same. Given the eigenfunctions and eigenvalues of a boundary problem, the direct and inverse settings can be reduced to finding the scalar product of two finite-dimensional vectors. Therefore, there is no need to solve sparse and badly conditioned linear algebraic systems many times. The settings for the boundary layer between the magnetosphere and solar wind are not discussed in this paper. However, we can give some recommendations on finding the solutions of MHD equations in the case of viscosity and magnetic viscosity: the eigenfunctions and eigenvalues depend on these parameters nonlinearly.

2. Problem Statement and Methods for Solving It

Inverse problems in geophysics can be solved with the help of the linear integral representation method [6]. Key parameters of the regional version of this method are the radii of spheres, which are considered as supports of simple and double layers and as the sources of magnetic fields. Additionally, the depths at which those spheres are located are important when constructing effective tools for solving linear and nonlinear ill-posed problems. The analytical approximations of the magnetic field of planets with the so-called dynamo (such the Earth and Mercury) proposed by the authors in [7] can be considered as zero approximations to the solutions of the nonlinear MHD equation system as well as being useful in further investigating the processes in the interior of planets. We elaborated a regularized algorithm for finding stable approximate solutions to linear inverse problems. The satellite data obtained from the Messenger mission [1] were interpreted within the framework of the linear integral representation method mentioned above. The theory of kinematic dynamo [8] gives a description of the motion of an incompressible fluid in a magnetic field. In this work, a bounded domain in three-dimensional space is considered. The boundary conditions are not fixed and can be different (depending on the setting). In [7], we have proposed an effective algorithm for analytically modelling the magnetic field of Mercury from the satellite data. The solution to the inverse problem which can be set within the process of this modelling approach could serve as an initial approximation of the vector field satisfying the MHD equation system.

Let us write the full MHD equation system (in dimensionless form) in a three-dimensional multiply connected domain. In this paper, we “work” with two types of

such domains: rectangular parallelepipeds with several gaps and the direct product of a rectangle with gaps and infinite lines (in some directions). First, let us consider the following vector functions in a cube K_l with L representing the number of removed small cubes.

$$SK_m = \left\{ x_1,\ x_2,\ x_3 : \left| x_\alpha - x_{\alpha 0}^{(m)} \right| \le l, \alpha = 1,2,3 \right\}, m = 1, \ldots, L,$$

(we will denote this spatial domain as $D_{S,L}$):

$$\begin{cases} \frac{\partial \bar{B}}{\partial t} = -\{\bar{v}, \bar{B}\} + \eta \Delta \bar{B}, \\ div \bar{B} = 0,\ div \bar{v} = 0, \\ \frac{\partial \bar{v}}{\partial t} = -(\bar{v},\ \bar{\nabla})\bar{v} + \mathrm{curl}\bar{B} \times \bar{B} + \chi \Delta \bar{v} - \nabla p, \\ \bar{B}_n|_{\partial D_{S,L}} = \bar{\phi}_1(x_1, x_2, x_3, t);\ \bar{B}(x_1, x_2, x_3, 0) = \bar{\phi}_0(x_1, x_2, x_3); \\ \bar{v}_n|_{\partial D_{S,L}} = \bar{\psi}_1(x_1, x_2, x_3, t);\ \bar{v}(x_1, x_2, x_3, 0) = \bar{\psi}_0(x_1, x_2, x_3); \end{cases} \quad (1)$$

where $\bar{B}(\bar{r}, t)$ is the vector of the magnetic field, $\bar{v}(\bar{r}, t)$ corresponds to the velocity field of the charged fluid, χ is the viscosity coefficient, and η is a small dimensionless parameter that could be connected with so-called magnetic viscosity, $\bar{B}, \bar{v} \in C^2(D_{S,L} \times (0,T)), \bar{r} = (x_1, x_2, x_3) \in D_{S,L}$.

Here, $\{\bar{v}, \bar{B}\}$ is the Poisson bracket of two solenoidal vectors:

$$\{\bar{v}, \bar{B}\}_i = \sum_{k=1}^{3} \left\{ v_k \frac{\partial B_i}{\partial x^k} - B_k \frac{\partial v_i}{\partial x^k} \right\}.$$

∇p is the pressure gradient, and $\Delta \bar{B}$ is the notation of the Laplacian of $\bar{B}(\bar{r}, t)$.

In Equation (1), $C^2(D_{S,L} \times (0,T))$ denotes the space of continuous twice-differentiable functions given in the considered domain; $\bar{\psi}_i, \bar{\phi}_i,\ i = 0, 1$, are differentiable vector functions in the corresponding space domains. As is stressed in [8], the vector fields to be found from the MHD equation system (1) should "touch" the boundary of the three-dimensional domain in which these fields are considered. Thus, we have to operate using mathematical settings with different boundary conditions for each component of velocity and magnetic induction. If we study the behaviour of these vector fields either in a cube or in the domain that can be represented as the direct product of a rectangle and an infinite line, then the constraint on boundary conditions is as follows:

$$B_n|_{\partial D_{S,L}} = 0;\ v_n|_{\partial D_{S,L}} = 0.$$

Here, B_n and v_n are the normal components of the magnetic induction and velocity vectors with respect to the boundary $D_{S,L}$.

In this connection, the boundary conditions will take the form

$$B_n|_{\partial D_{S,L}} = 0;\ v_n|_{\partial D_{S,L}} = 0;\ \bar{B}_\tau|_{\partial D_{S,L}} = \bar{\phi}_1(x_1, x_2, x_3, t);\ \bar{v}_\tau|_{\partial D_{S,L}} = \bar{\psi}_1(x_1, x_2, x_3, t).$$

We denote here the tangential components of the vector fields via $\bar{B}_\tau$ and $\bar{v}_\tau$. Both the magnetic induction and velocity vectors can be written as the sum

$$B_n|_{\partial D_{S,L}} = 0;\ v_n|_{\partial D_{S,L}} = 0;\ \bar{B}_\tau|_{\partial D_{S,L}} = \bar{\phi}_1(x_1, x_2, x_3, t);\ \bar{v}_\tau|_{\partial D_{S,L}} = \bar{\psi}_1(x_1, x_2, x_3, t)$$

$\bar{B} = B_n \bar{n} + \bar{B}_\tau; v = v_n \bar{n} + \bar{v}_\tau$. Here, $\bar{n}$ is the unit vector of the external normal to the boundary of $D_{S,L}$.

Each component of the magnetic induction and velocity vectors can be represented as a sum of terms of different magnitudes. We can suppose that the zeroth approximations to the magnetic induction and velocity vectors are known or have previously been analytically

expressed, as in our previous work [7]. We would like to serve the notations for the field components of the first order of smallness as $\bar{v}$ and $\bar{B}$.

In this case, then we can assume that $\{\bar{v}, \bar{B}\} = \bar{W}$ is known, as well as curl$\bar{B} = \bar{v}$ and $\chi\Delta\bar{v} = 0$. It was stressed in [8] that ∇p can be uniquely determined from the condition div$\frac{\partial \bar{v}}{\partial t} = 0$.

We will assume that ∇p is defined from this condition.

Moreover, we can consider $\frac{\partial \bar{v}}{\partial t} + (\bar{v}, \bar{\nabla})\bar{v}$ as the full derivative, with respect to time, of the vector function $\bar{v}$: $\frac{\partial \bar{v}}{\partial t} + (\bar{v}, \bar{\nabla})\bar{v} = \frac{d\bar{v}}{dt}$.

If we consider the setting with a homogenous boundary value, then the eigenfunctions of the corresponding spatial boundary problem can be used to find the solution of the time-dependent problem. To accomplish this, let us represent the vector field $\bar{B}(\bar{r}, t)$ as the sum

$$\bar{B}(\bar{r}, t) = \bar{B}_0(\bar{r}, t) + \bar{B}_1(\bar{r}, t)$$

where $\bar{B}_1(\bar{r}, t)$ satisfies the following boundary conditions:

$$B_{1n}|_{\partial D_{S,L}} = 0, \ \bar{B}_{1\tau}|_{\partial D_{S,L}} = \bar{\phi}_1(x_1, x_2, x_3, t).$$

Then, for $\bar{B}_0(\bar{r}, t)$, it holds that

$$\bar{B}_0(x_1, x_2, x_3, 0) = \bar{\phi}_0(x_1, x_2, x_3) - \bar{B}_1(x_1, x_2, x_3, 0)$$

So, the direct problem can be formulated as follows: let the functions $\bar{\phi}_i, \bar{\psi}_i, i = 0, 1$, $\nabla p(x_1, x_2, x_3, t)$ and the viscosity $\chi(x_1, x_2, x_3)$ and magnetic viscosity $\eta(x_1, x_2, x_3)$ coefficients be given in the corresponding domains. Let us also assume that $\{\bar{v}, \bar{B}\} = \bar{W}$, curl$\bar{B} = \bar{v}$ and $\chi\Delta\bar{v} = 0$. In reality, these physical entities are not vanishing, but if they are known we can reduce system (1) to a more simple form that would allow us to compute the eigenfunctions of the corresponding boundary problem. In addition, the condition $\{\bar{v}, \bar{B}\} = 0$ holds if the magnetic field can be characterized as "frozen" into the liquid flow.

Namely, we should find the solution of the following system:

$$\begin{cases} \frac{\partial \bar{B}_0}{\partial t} = \eta\Delta\bar{B}_0 + \eta\Delta\bar{B}_1 - \frac{\partial \bar{B}_1}{\partial t} + \bar{W}, \\ div\bar{B} = 0, \ div\bar{v} = 0, \\ \frac{\partial \bar{v}}{\partial t} = -(\bar{v}, \bar{\nabla})\bar{v} - \nabla p + \bar{v} \times \bar{B}, \\ \bar{B}_0\ (x_1, x_2, x_3, 0) = \bar{\phi}_0(x_1, x_2, x_3) - \bar{B}_1\ (x_1, x_2, x_3, 0); \\ \bar{v}(x_1, x_2, x_3, 0) = \bar{\psi}_0(x_1, x_2, x_3); \bar{v}_\tau|_{\partial D_{S,L}} = \bar{\psi}_1(x_1, x_2, x_3, t) \\ B_n|_{\partial D_{S,L}} = 0; \ v_n|_{\partial D_{S,L}} = 0; \bar{B}_\tau|_{\partial D_{S,L}} = \bar{\phi}_1(x_1, x_2, x_3, t)\ . \end{cases} \tag{2}$$

Setting Equation (2) is considered, as mentioned above, in a specific domain $D_{S,L} \times (0, T)$. Without loss of generality, let us assume that all boundary conditions are homogeneous. Then, we have to use the equation system without a right-hand side, which is equal to $\eta\Delta\bar{B}_1 - \frac{\partial \bar{B}_1}{\partial t}$. It is well known that a system with a non-zero right-hand side can be transformed into a homogeneous one by representing the solution as the sum of two functions:

$$\bar{B}_0 = \bar{u}(x_1, x_2, x_3, t) + \bar{w}(x_1, x_2, x_3, t)$$

where one satisfies the homogeneous equation and modified initial value.

In this paper, we will concentrate our efforts on finding approximate solutions to system (2) within the framework of discrete potential theory [9]. To accomplish this, it is important to remember the basic moments of the eigenvalue problem setting for the Laplace operator, in both the continuous and finite-differenced cases. The continuous case will serve us as a guide when constructing grid solutions to finite-differenced analogues of system (2).

We should also note that the first equation in (2) and homogeneous boundary conditions for the magnetic induction only allow us to leverage the variable separation method when finding the magnetic field. The velocity field can be determined from the third equation in (2) provided that the whole magnetic field $\bar{B}$ is found and the initial values of the velocity vector components $\bar{v}(x_1, x_2, x_3, 0) = \bar{\psi}_0(x_1, x_2, x_3)$ are known. We also suppose that an approximation to the velocity is given. This means that we can introduce a vector field $\bar{X} = \frac{\partial}{\partial t} + (\bar{v}_0, \bar{\nabla}) \equiv \frac{d}{dt}$. The full derivative with respect to the time will be considered in terms of this vector field. If we want to describe stationary motion of a charged fluid, then the partial derivative over time of the velocity is equal to zero. We obtain the velocity field depending on the point of trajectory only. Integrating the velocity field over time, we can determine the path of the charged particle associated with the fluid. Let us describe a method of obtaining the magnetic field distribution in the case of discrete potentials provided that the Poisson bracket $\{\bar{v}, \bar{B}\} = \bar{W}$ is a known function.

2.1. Estimates of the Continual Laplace Operator Eigenvalues for the Dirichlet and Neumann Problems in a Three-Dimensional Domain Represented as a Cube with Several Cutouts

Consider a cube in R^3 (three-dimensional Cartesian space): $K_l = \{x_1, x_2, x_3 : |x_\alpha| \leq l, \alpha = 1, 2, 3\}$. Let L subdomains $SK_m, m = 1, \ldots, L$ be given that are cubes with edges of lengths $2l_m, m = 1, \ldots, L$.

$SK_m = \left\{x_1, x_2, x_3 : \left|x_\alpha - x_{\alpha 0}^{(m)}\right| \leq l_m, \alpha = 1, 2, 3\right\}, m = 1, \ldots, L$ denotes the subdomains of K_l.

Then, the eigenvalue problem with the Dirichlet boundary conditions in the "large cube with small cutouts" is as follows:

$$\begin{gathered} \Delta u(x_1, x_2, x_3) + \lambda u(x_1, x_2, x_3) = 0,\ x \in K_l, x = (x_1, x_2, x_3); \\ u|_{\Gamma K_l \cup \bigcup_{m=1}^{L} \Gamma SK_m} = 0. \end{gathered} \tag{3}$$

Function $u(x_1, x_2, x_3) \in C^2(K_l) \cap C^1(\bar{K}_l)$ in Equation (3) belongs to the intersection of two functional spaces: one that is twice continuously differentiable in the whole cube and once continuously differentiable in the closure of this cube.

Here $\Gamma K_l \cup \bigcup_{m=1}^{L} \Gamma SK_m$ means the union of the whole cube's boundary and the boundaries of the subdomains.

Note 1. We could consider rectangular parallelepipeds, but there is no difference in principle between the eigenvalue problem settings and the solutions to those problems in both cases (cube or parallelepiped).

Eigenfunctions of the Laplace operator in K_l have the following form [10,11]:

$$\begin{gathered} \psi(x_1, x_2, x_3) = \left(\frac{2}{l}\right)^{\frac{3}{2}} \prod_{i=1}^{3} sin\left(\frac{\pi x_i m_i}{l}\right), K_l = \{x_1, x_2, x_3 : |x_\alpha| \leq l, \alpha = 1, 2, 3\}, \\ \lambda_s = \frac{\pi^2(m_1^2 + m_2^2 + m_3^2)}{l^2}, s = 1, 2, \ldots \\ \boldsymbol{m} = (m_1, m_2, m_3), m_\alpha > 0, \alpha = 1, 2, 3. \end{gathered} \tag{4}$$

The multiplicity of the eigenvalue λ is equal to the number of points with whole coordinates lying on the sphere of radius $\sqrt{\lambda}\frac{l}{\pi}$.

We should stress here that there exists an infinite number of eigenfunctions.

Let us now consider the subdomains of the cube K_l.

Similarly to Equation (4), let us write the eigenfunctions and eigenvalues for each of the small cubes SK_m. The m-th cube is centred at the point with coordinates $x_{\alpha 0}^{(m)}$, $\alpha = 1, 2, 3$; $m = 1, \ldots, L$.

Then, we have

$$\begin{gathered}\psi_m(x_1,x_2,x_3)=\left(\frac{2}{l_m}\right)^{\frac{3}{2}}\prod_{\alpha=1}^{3}sin\left(\frac{\pi\left(x_\alpha-x_\alpha^{(m)}\right)n_\alpha}{l_m}\right),\\ \lambda_n^{(m)}=\frac{\pi^2\left(n_1^2+n_2^2+n_3^2\right)}{l_m^2},m=1,\ldots,L;\ \boldsymbol{n}=(n_1,n_2,n_3),n_\alpha>0,\ n_\alpha\in\mathbb{Z}\end{gathered}\tag{5}$$

In (5), the vector $\boldsymbol{n}=(n_1,n_2,n_3),n_\alpha>0,\ n_\alpha\in\mathbb{Z}$ can have different coordinates for each small cube.

It can be seen from (6) that the eigenfunction $\psi(x_1,x_2,x_3)$ for a large cube is also the eigenfunction for all subdomains in the case that the following relation holds:

$$\frac{\left(l-x_{\alpha0}^{(m)}\right)n_\alpha}{l_m}=m_\alpha,\alpha=1,2,3;m_\alpha\in\mathbb{N};m=1,\ldots,L.\tag{6}$$

When does (6) hold? Obviously, if the numbers $\frac{\left(l-x_{\alpha0}^{(m)}\right)}{l_m}$ are rational and positive for all m = 1,…,L. Therefore, the smaller the small cube's edge is, the larger the eigenvalue number for the function $\psi(x_1,x_2,x_3)$ should be.

Note 2. If we consider the Neumann boundary condition with respect to some variable (say, variable x_1) in a small cube $SK_i,i=1,\ldots,L$, then Equation (6) takes the following form:

$$\begin{gathered}\psi_i(x_1,x_2,x_3)=\left(\frac{2}{l_i}\right)^{3/2}\prod_{\alpha=2}^{3}sin\left(\frac{\pi\left(x_\alpha-x_\alpha^{(i)}\right)n_\alpha}{l_m}\right)\cdot cos\left(\frac{\pi\left(x_1-x_1^{(i)}\right)n_1}{l_i}\right),\\ \boldsymbol{n}=(n_1,n_2,n_3),n_\alpha>0,\alpha=1,2,3.\end{gathered}\tag{7}$$

The same statement is also true for the large cube, which has an edge of length l.

2.2. Estimates of the Continual Laplace Operator Eigenvalues for the Dirichlet Problem in a Three-Dimensional Domain Represented as a Direct Product of a Rectangle with Several Cutouts and Infinite Line

Let us now consider a three-dimensional domain that can be represented as the product of a square Q with an edge of length l and m cutouts SK_m, located in the OXY plane and the OZ axis. Then, the eigenfunctions of the Laplace operator do not depend on the variable z and have the following form:

$$\begin{gathered}\psi_m(x,y)=\frac{2}{l_m}\sum_{i=1}^{2}sin\left(\frac{\pi\left(x-x_0^{(m)}\right)m_i}{l_m}\right)\cdot sin\left(\frac{\pi\left(y-y_0^{(m)}\right)n_i}{l_m}\right),\\ SK_m=\left\{x,y:\left|x-x_0^{(m)}\right|\le l_m,\ \left|y-y_0^{(m)}\right|\le l_m\right\},\ m=1,\ldots,L.\end{gathered}\tag{8}$$

We can also set an analoguous eigenvalue problem for other pairs of Cartesian variables (i.e., x and z or y and z).

Let us now consider the discrete potential and the eigenvalue problem for the Dirichlet setting in a cube with several cutouts, which can be characterized as a multiconnected domain in three-dimensional grid space.

2.3. Basic Principles of the Discrete Potential Theory

The fundamental principles of the discrete gravitational potential theory were elaborated by V.N. Strakhov. A vector with coordinates in Cartesian space is replaced by its grid analog. We should stress here that these are two different study objects. The coordinates of the grid vector or grid function can take only a discrete set of values. We consider a finite number of grids, namely 2^n. One of them is set to be the basic grid, and the calculated

values of the gravitational potential of the grid are assigned to the nodes of this grid and have the form $V_S(x^{(S)})$, where $x^{(S)}$ is the grid analogue of a continuous vector:

$$x^{(S)} = (x_1^{(S)}, x_2^{(S)}, \ldots, x_n^{(S)})^T,\ x_k^{(S)} = ph_k; p = 0, \pm 1, \pm 2, \ldots; h_k = const,$$
$$h = (h_1, h_2, \ldots, h_n)^T.$$

The values of potential derivatives of a higher order are located at the nodes of auxiliary ("nonbasic") grids. The first derivative can be written as follows:

$$\frac{\partial V_S(x^{(S)})}{\partial x_k^{(S)}} \approx \frac{V_S(x^{(S)} + h_k e_k) - V_S(x^{(S)})}{h_k}$$

The details of the discrete gravitational potential theory can be found in our previous work [12] and in [9].

In a similar way, we can write the following expressions for the second derivatives of the discrete potential:

$$\frac{\partial^2 V_S(x^{(S)})}{\partial x_p^{(S)} \partial x_q^{(S)}} = \frac{1}{h_p}\left[\frac{\partial V_S(x^{(S)} + h_q e_q)}{\partial x_q^{(S)}} - \frac{\partial V_S(x^{(S)})}{\partial x_q^{(S)}}\right],$$

Here, $\frac{\partial^2 V_S(x^{(S)})}{\partial x_p^{(S)} \partial x_q^{(S)}}$ at $p \neq q$ is assigned to nodes of the third auxiliary grid:

$$x^{(S,3)} = x^{(S)} + \frac{h_p e_p}{2} + \frac{h_q e_q}{2};$$

if $p = q$, the finite-differenced analogues of the second derivatives are calculated at the nodes of the basic grids.

The discrete gravity potential $V_S(x^{(S)})$ can be determined as a function that can be represented as follows:

$$V^{(S)}(x^{(s)}) = \sum_{\xi^{(S)} \in J^{(S)}} m_S(\xi^{(S)}) \Omega_n^{(S)}(\xi^{(S)} - x^{(s)}); \tag{9}$$

$$\Delta_x(\Omega_n^{(S)}(x^{(s)})) = -\frac{a^2 C_n}{H^{n-2}} e(x^{(S)}), \tag{10}$$

$$e(x^{(S)}) = \begin{cases} 1, & \left|x^{(S)}\right| = 0 \\ 0, & \left|x^{(S)}\right| \neq 0. \end{cases} \tag{11}$$

Here, $\Omega_n^{(S)}(x^{(s)})$ is called the fundamental solution of the discrete Laplace equation in R^n; Δ_x is the finite-differenced analogue of the Laplace operator in three-dimensional coordinate space; $H = \sqrt[3]{h_1 h_2 h_3}$; and C_n is a constant depending on the dimension of space, for example $C_3 = 4\pi$. The discrete gravity potential (as a function of discrete masses $m_S(\xi^{(S)})$ distributed in the support $J^{(S)}$) should also give the minimum of the following regularization functional $\Psi\left[V_S(x^{(S)})\right]$:

$$\Psi\left[V_S(x^{(S)})\right] \equiv \left|\Lambda_2\left\{V_S(x^{(S)})\right\}\right|^2_{E(D_S)} + \alpha\Phi\left(V_S(x^{(S)}), g_S(x^{(S)})\right) = \min_{V_S(x^{(S)}), x^{(S)} \in IntD_S}. \tag{12}$$

Here, $\alpha \geq 0$ is a regularization parameter [7]); D_S is a given grid domain in n-dimensional grid space; $x^{(S)} \in D_S$; $E(D_S)$ is the Euclidian norm in this grid domain

D_S; and $\Phi(\mathbf{t}, \mathbf{z})$ can be described as a non-negative functional depending on two vectors of the same length: $\boldsymbol{t}$ и $\boldsymbol{z}$. Function $g_S(x^{(S)})$ in (12) is a known signal (gravity potential or higher derivatives of this potential). Λ_2 is a finite-differenced approximation of the differential operator in the Laplace equation defined on another template than the one mentioned in (10). The variational problem in Equations (9)–(12) with respect to unknown masses $m_S(\xi^{(S)})$, $\xi^{(S)} \in J_S$, can be solved using standard methods, described, for example, in [11]. However, we have proved in our previous works [6,7,12] that special methods for finding stable approximate (regularized) solutions of badly conditioned linear algebraic equation systems (LAESs) are yet to be elaborated.

System (9)–(12) is not close in a general sense. Some additional constraints on the solution are to be formulated. The question regarding the uniqueness of the solution to LAESs is crucial for an adequate interpretation of all geophysical data. Without any a priori information about the sources of physical fields, no variational settings can be physically correct. In this case, the source of a physical field will be a "black box" with undefined properties.

2.4. Estimates for Eigenvalues of Finite–Discrete Analogue of the Laplace Operator in a Dirichlet Setting—Multiply Connected Domains

Let us now consider a Dirichlet boundary value problem in a three-dimensional grid domain D_S. Let D_S be a parallelepiped belonging to the grid ω and having the sides d_1, d_2, d_3. A grid function *y(x)* can be determined from the following setting:

$$\begin{array}{l} \Delta y(x) + \lambda_l y(x) = 0, x = (x_1, x_2, x_3) \in D_S, \Lambda = \Lambda_x + \Lambda_y + \Lambda_z, \\ y(x) = 0, x = (x_1, x_2, x_3) \in \partial D_S; \\ y_{ijk} = y(x_{1,i}, x_{2,j}, x_{3,k});\ x_{1,i} = ih_1, x_{2,j} = jh_2, x_{3,k} = kh_3. \end{array} \tag{13}$$

Equation (13) can be "decomposed" into three independent boundary problems as follows:

$$\begin{array}{l} \Lambda_x \mu_{l_1}^{(1)} + \lambda_{l_1}^{(1)} \mu_{l_1}^{(1)} = 0,\ 1 \le i \le N_1 - 1, \\ \Lambda_y \mu_{l_2}^{(2)} + \lambda_{l_2}^{(2)} \mu_{l_2}^{(2)} = 0, 1 \le j \le N_2 - 1, \\ \Lambda_z \mu_{l_3}^{(3)} + \lambda_{l_3}^{(3)} \mu_{l_3}^{(3)} = 0, 1 \le k \le N_3 - 1, \\ \mu_{l_1}^{(1)}(0) = \mu_{l_1}^{(1)}(N_1) = 0 = \mu_{l_2}^{(2)}(0) = \mu_{l_2}^{(2)}(N_2) = \mu_{l_3}^{(3)}(0) = \mu_{l_3}^{(3)}(N_3), \\ \lambda_{l_\alpha}^{(\alpha)} = \frac{4}{h_\alpha^2} sin^2\left(\frac{l_\alpha \pi}{2N_\alpha}\right) = \frac{4}{h_\alpha^2} sin^2\left(\frac{l_\alpha \pi h_\alpha}{2d_\alpha}\right), l_\alpha = 1, 2, \ldots, N_\alpha - 1, \\ \mu_{l_1}^{(1)}(i) = \sqrt{\frac{2}{d_1}} sin\left(\frac{l_1 \pi i}{N_1}\right), l_1 = 1, \ldots, N_1 - 1; \\ \mu_{l_2}^{(2)}(j) = \sqrt{\frac{2}{d_2}} sin\left(\frac{l_2 \pi j}{N_2}\right), l_2 = 1, \ldots, N_2 - 1 \\ \mu_{l_3}^{(3)}(k) = \sqrt{\frac{2}{d_3}} sin\left(\frac{l_3 \pi k}{N_3}\right), l_3 = 1, \ldots, N_3 - 1 \end{array} \tag{14}$$

Then, we can represent *y(x)* as the product:

$$y(x_{1,i}, x_{2,j}, x_{3,k}) = \mu_{l_1}^{(1)}(i) \cdot \mu_{l_2}^{(2)}(j) \cdot \mu_{l_3}^{(3)}(k).$$

The finite-differenced approximation of the Laplace operator is given on the template named "cross":

$$\begin{array}{c} \Lambda_x y_{ijk} = \frac{1}{h_1^2}\left[y_{i+1,j,k} - 2y_{ijk} + y_{i-1,j,k}\right], \Lambda_y y_{ijk} = \frac{1}{h_2^2}\left[y_{i,j+1,k} - 2y_{ijk} + y_{i,j-1,k}\right], \\ \Lambda_z y_{ijk} = \frac{1}{h_3^2}\left[y_{ij,k+1} - 2y_{ijk} + y_{ij,k-1}\right]. \Delta = \Lambda_x + \Delta_y + \Delta_z\ . \end{array} \tag{15}$$

If D_S is a cube, then $d_1 = d_2 = d_3 \equiv d$.

However, the steps $h_k, k = 1, 2, 3$ can be different:

$$h_k = \frac{d}{N_k},\ k = 1, 2, 3.$$

The eigenvalue is defined as follows:

$$\lambda_l = 16\pi^2 \cdot \sum_{\alpha=1}^{3} \lambda_{l_\alpha}^{(\alpha)}. \tag{16}$$

For each small cube from the multiply connected domain $D_{S,L}$, the eigenvalues and eigenfunctions have the following forms:

$$\begin{array}{l} \lambda_{l_{\alpha,s}}^{(\alpha)} = \frac{4}{h_\alpha^2} sin^2\left(\frac{l_{\alpha,s}\pi}{2N_{\alpha,s}}\right) = \frac{4}{h_\alpha^2} sin^2\left(\frac{l_{\alpha,s}\pi h_\alpha}{2d_{\alpha,s}}\right), l_{\alpha,s} = 1, 2, \ldots, N_{\alpha,s} - 1, \\ \mu_{l_{1,s}}^{(1)}(i) = \sqrt{\frac{2}{d_{1,s}}} sin\left(\frac{l_{1,s}\pi i}{N_{1,s}}\right), l_{1,s} = 1, \ldots, N_{1,s} - 1; \\ \mu_{l_{2,s}}^{(2)}(j) = \sqrt{\frac{2}{d_{2,s}}} sin\left(\frac{l_{2,s}\pi j}{N_{2,s}}\right), l_{2,s} = 1, \ldots, N_{2,s} - 1; \\ \mu_{l_{3,s}}^{(3)}(k) = \sqrt{\frac{2}{d_{3,s}}} sin\left(\frac{l_{3,s}\pi j}{N_{3,s}}\right), l_{3,s} = 1, \ldots, N_{3,s} - 1,\ s = 1, \ldots, L. \end{array} \tag{17}$$

We assume that this domain has L cutouts that have the form of a cube. Therefore, we should select only the eigenfunctions in Equation (17) (in other words, only those fractions $\frac{l_\alpha}{N_\alpha}$, $\alpha = 1, 2, 3$) for the whole multiply connected domain for which the following relations hold:

$\frac{l_{\alpha,s}}{N_{\alpha,s}} = const = \frac{l_\alpha}{N_\alpha}$, $\alpha = 1, 2, 3$; $s = 1, \ldots, L$. For example, if the sides of the small and the large cubes relate to each other as $1/3$, then the eigenfunctions are only those for which the following relations hold: $\frac{l_{\alpha,s}}{N_{\alpha,s}} = \frac{1}{3}$, $\alpha = 1, 2, 3$; $s = 1, \ldots, L$. Note that grid steps $h_\alpha, \alpha = 1, 2, 3$ for the multiply connected domain $D_{S,L}$ are the same for each connection component.

Note 3. If we have to solve the Neumann boundary problem with respect to the discrete variable $x_1^{(S)}$ (for example), then Equation (12) transforms into the following expression:

$$\begin{array}{l} \lambda_{l_{\alpha,s}}^{(\alpha)} = \frac{4}{h_\alpha^2} sin^2\left(\frac{l_{\alpha,s}\pi}{2N_{\alpha,s}}\right) = \frac{4}{h_\alpha^2} sin^2\left(\frac{l_{\alpha,s}\pi h_\alpha}{2d_{\alpha,s}}\right), l_{\alpha,s} = 1, 2, \ldots, N_{\alpha,s} - 1, \\ \mu_{l_{1,s}}^{(1)}(i) = \sqrt{\frac{2}{d_{1,s}}} cos\left(\frac{l_{1,s}\pi i}{N_{1,s}}\right), l_{1,s} = 1, \ldots, N_{1,s} - 1; \\ \mu_{l_{2,s}}^{(2)}(j) = \sqrt{\frac{2}{d_{2,s}}} sin\left(\frac{l_{2,s}\pi j}{N_{2,s}}\right), l_{2,s} = 1, \ldots, N_{2,s} - 1; \\ \mu_{l_{3,s}}^{(3)}(k) = \sqrt{\frac{2}{d_{3,s}}} sin\left(\frac{l_{3,s}\pi j}{N_{3,s}}\right), l_{3,s} = 1, \ldots, N_{3,s} - 1,\ s = 1, \ldots, L. \end{array}$$

2.5. Fundamental Principles of the Theory of Discrete Heat Potentials

In both this paper and in [12], we suppose that one of discrete grid coordinates is time. For convenience, let us assume $R^{n+1} = R_x^n \times R_t$. The first item in the direct product corresponds to the spatial coordinates while the second one is time. The form of finite-differenced analogues of the differential operators does not change if such representation is introduced. However, we should take into account that one of the variables is selected.

The differential operator for the grid heat (diffusion) equation can be written in the following form:

$$a^2 \Delta_x \left\{ V_S(x^{(S)}) \right\} = \nabla_t \left\{ V_S(x^{(S)}) \right\}; x^{(S)} \in R^{n+1}. \tag{18}$$

In Equation (18), Δ_x is the given finite-differenced analogue of the Laplace operator over spatial coordinates, and ∇_t is the finite-differenced approximation of the first derivative with respect to time.

Further, let us designate the grid operator of the heat equation as follows:

$$L^{(h)} \left\{ V_S(x^{(S)}) \right\} \equiv a^2 \Delta_x \left\{ V_S(x^{(S)}) \right\} - \nabla_t \left\{ V_S(x^{(S)}) \right\}. \tag{19}$$

The steps over spatial variables are designated as $h_p, p = 1, \ldots, N$, whereas the step over time is τ.

a^2 is a known parameter in the continual heat equation. This parameter represents a physical element, i.e., the heat conductivity or diffusion coefficient.

We assume that the Laplace operator is given on the sample named "cross" (compare with Equation (15)). Then, for the three-dimensional grid, we have

$$\begin{gathered}
a^2\Lambda^+(V^{(S)}(x^{(S),j+1})) - \frac{V^{(S)}(x^{(S),j+1}) - V^{(S)}(x^{(S),j})}{\tau} = f(x^{(S),j});\\
x^{(S),j} = (x_1^{(S)}, x_2^{(S)}, \ldots, x_n^{(S)}; x_{n+1} = j\tau),\\
x^{(S),j+1} = (x_1^{(S)}, x_2^{(S)}, \ldots, x_n^{(S)}; x_{n+1} = (j+1)\tau), j = 1, \ldots, J-1.\\
\Lambda^+(V^{(S)}(x^{(S),j})) = \sum_{p=1}^{3} C_p(\left\{V^{(S)}(x^{(S),j} - h_p e_p) + V^{(S)}(x^{(S),j} + h_p e_p)\right\} - 2V^{(S)}(x^{(S),j})),\\
C_p = \frac{H^2}{h_p^2}, H = \sqrt[3]{h_1 h_2 h_3},\ x^{(S),j} = (x_1^{(s)}, x_2^{(S)}, x_3^{(S)}, t = j\tau), j = 1, \ldots, J-1;\\
x_p^{(S)} = kh_p, -K \le k \le +K; h_p = const, p = 1,2,3; \left|x_p^{(S)}\right| \le d_p = Kh_p, p = 1,2,3.
\end{gathered} \tag{20}$$

Here, $f(x^{(S),j})$ is a grid function depending, in the general case, on the spatial coordinates and the time.

The index j in Equation (20) corresponds to the discrete time. We suppose that the grid solution to the grid heat equation exists on the interval $x_4 \doteq t \in (0, J\tau) = (0, T)$.

In this paper, we will consider the boundary problems on a set of expanding compacts [13] (which are limited point sets in four-dimensional infinite space):

$$\begin{gathered}
x^{(S)} = (x_1^{(S)}, x_2^{(S)}, \ldots, x_n^{(S)}, x_{n+1}^{(S)})^T,\ x_k^{(S)} = ph_k; -\infty < p < +\infty, p = 0, \pm 1, \pm 2, \ldots;\\
h_k = const,\ x_{n+1}^{(S)} \in (0, T), \left|x_p^{(S)}\right| \le d_p = K_n h_p, p = 1,2,3,
\end{gathered}$$

where d_p, K_n, and $h_p, p = 1,2,3$, are some positive constants, and $K_n \to \infty$ at $n \to \infty$. As earlier, the fourth coordinate (time) takes the values from the fixed interval $(0, T)$. Thus, those expanding compacts now play the role of the large cubes mentioned above if $h_k = h, k = 1, \ldots, n$.

Equation (20) can be solved in the direct product of a rectangular parallelipeped and the interval $(0, T)$: $K_{n_q,t} = K_{n_q} \times (0, T)$. In order to correctly set the problem, we should define the boundary and initial values of the heat potential $V^{(S)}(x^{(S)})$ in the mentioned domain. Further, we assume that the values of the heat potential are equal to zero on the sides of the parallelepiped K_{n_q} with number n_q, which belongs to the set of expanding compacts defined before:

$$\begin{gathered}
V^{(S)}(x^{(S)}) = 0,\ x^{(S)} \in \Gamma K_{n_q};\\
V^{(S)}(x^{(S)}) = V_0^{(S)}(\bar{x}^{(S)}),\ x_4^{(S)} = 0,\ \bar{x}^{(S)} = (x_1^{(S)}, x_2^{(S)}, x_3^{(S)}).
\end{gathered} \tag{21}$$

In Equation (16), $V_0^{(S)}(\bar{x}^{(S)})$ denotes in some sense the "initial" distribution of the grid heat potential at $x_4^{(S)} = 0$.

Problems (20) and (21) can be solved by means of the matrix sweep method if there will be an implicit calculation scheme. The matrices of the linear algebraic equation systems (LAESs) will have the three-diagonal or block three-diagonal form in each layer over time (i.e., at each value $x_4^{(S)} = j\tau, j = 1, \ldots, J-1$).

3. Construction of a Unique Solution to the MHD Discrete Equation System

In this section, we outline a constructive approach to solving the MHD equation system (2). First of all, we should stress that it is hydrodynamics where the so-called Beltrami fields play the crucial role. This means that a field is proportional to its rotor. Of course, Beltrami fields are solenoidal.

In the previous part of this paper, we described different types of the eigenvalue settings, in both the continuous and discrete spaces of functions and variables. We have already mentioned that, given an initial distribution of the magnetic field, the Dirichlet boundary problem for the magnetic induction can be solved in a cube with several cutouts in the continuous and discrete spaces. The initial distribution should be equal to zero on the boundary in the case of the Dirichlet setting. If we want to meet the condition of solenoidality, then we can deduce that both the magnetic induction vector and the velocity vector should "touch" the boundary of the domain in which those fields are considered. This means, in the case of the cube with several cutouts, that the Neumann boundary problem with respect to the discrete variable should be set for the corresponding component of the magnetic field. For example, the expansion of the component $B_{x_1^{(S)}}$ into the series (in our case, it is a finite sum of harmonics) of individual harmonics will have the following form:

$$\begin{aligned}
&B_{x_1^{(S)}}(\bar{x}^{(S)}) = \sum_{l_1=1}^{N_1-1}\sum_{l_2=1}^{N_2-1}\sum_{l_3=1}^{N_3-1} B_{l_1 l_2 l_3} B_{l_1}^{(1)}(i)\cdot B_{l_2}^{(2)}(j)\cdot B_{l_3}^{(3)}(k)\\
&B_{l_1}^{(1)}(i) = \sqrt{\frac{2}{d_1}}cos\left(\frac{l_1\pi i}{N_1}\right), l_1 = 1,\ldots,N_1-1;\\
&B_{l_2}^{(2)}(j) = \sqrt{\frac{2}{d_2}}sin\left(\frac{l_2\pi j}{N_2}\right), l_2 = 1,\ldots,N_2-1;\\
&B_{l_3}^{(3)}(k) = \sqrt{\frac{2}{d_3}}sin\left(\frac{l_3\pi k}{N_3}\right), l_3 = 1,\ldots,N_3-1.
\end{aligned} \tag{22}$$

Expansion (22) is similar to the "ordinary" spectral expansion of the magnetic field in three-dimensional space. It should be taken into account, however, that the grid domain $D_{S,L}$ is not fixed. We consider an ensemble of expanding compacts in which the variational settings described in Section 2.4 are solved in order to determine the discrete vector function $B_{x_1^{(S)}}(\bar{x}^{(S)})$.

Let us consider, instead of system (2), a homogeneous equation system with non-zero initial conditions for both the magnetic induction and velocity vectors:

$$\begin{cases}
\frac{\partial \bar{B}_0}{\partial t} = \eta\Delta\bar{B}_0,\\
div\bar{B} = 0,\ div\bar{v} = 0,\\
\frac{\partial \bar{v}}{\partial t} = -(\bar{v},\bar{\nabla})\bar{v} + \bar{v}\times\bar{B},\\
\bar{B}_0\ (x,y,z,0) = \bar{\phi}_0(x,y,z) - \bar{B}_1\ (x,y,z,0);\\
\bar{v}(x,y,z,0) = \bar{\psi}_0(x,y,z);\ B_n|_{\partial D_{S,L}} = 0;\\
v_n|_{\partial D_{S,L}} = 0;\ \bar{B}_\tau|_{\partial D_{S,L}} = \bar{\phi}_1(x,y,z,t);\ \bar{v}_\tau|_{\partial D_{S,L}} = \bar{\psi}_1(x,y,z,t).
\end{cases} \tag{23}$$

We can propose the following algorithm for finding solutions to problem (23).

(1) First of all, let us transfer all equations in (23) into the finite-differenced form. We obtain

$$\begin{array}{l}\eta\Lambda^{+}(\bar{B}_0^{(S)}(x^{(S),j+1})) - \frac{\bar{B}_0^{(S)}(x^{(S),j+1}) - \bar{B}_0^{(S)}(x^{(S),j})}{\tau} = 0;\\ \frac{\bar{v}^{(S)}(x^{(S),j+1}) - \bar{v}^{(S)}(x^{(S),j})}{\tau} = \left[\bar{v}^{(S)}, \bar{B}^{(S)}\right],\\ x^{(S),j} = (x_1^{(S)}, x_2^{(S)}, \ldots, x_n^{(S)}; x_{n+1} = j\tau), n = 3.\\ x^{(S),j+1} = (x_1^{(S)}, x_2^{(S)}, \ldots, x_n^{(S)}; x_{n+1} = (j+1)\tau), j = 1, \ldots, J-1.\end{array} \tag{24}$$

Here, Λ^{+} is defined as in (14). $\bar{B}_0^{(S)}(x^{(S),j})$ and $\bar{v}^{(S)}(x^{(S),j})$ are the grid functions in $D_{S,L} \times (0,T)$.

We also assume that the magnetic viscosity coefficient η is constant in $D_{S,L} \times (0,T)$. We consider the grid space of dimension 3 ($n = 4$).

(2) Let us represent the vector function $\bar{B}_0^{(S)}(x^{(S)})$ in the form of the product

$$\bar{B}_0^{(S)}(x^{(S)}) = \bar{B}_{0X}(\bar{x}^{(S)}) \cdot g(x_{n+1}^{(S)}),\ \bar{x}^{(S)} = (x_1^{(S)}, \ldots, x_n^{(S)})$$

Here, $\bar{B}_{0X}(\bar{x}^{(S)})$ satisfies the equation $\eta\Lambda^{+}\bar{B}_{0X}(\bar{x}^{(S)}) = -\lambda\bar{B}_{0X}(\bar{x}^{(S)})$, and for $g(x_{n+1}^{(S)})$ it holds that $\frac{g(x_{n+1}^{(S),j+1}) - g(x_{n+1}^{(S),j})}{\tau} = -\lambda g(x_{n+1}^{(S),j})$.

The argument $\bar{x}^{(S)}$ belongs to the spatial item of the product $D_{S,L}$, whereas the time variable changes on the interval $(0,T)$.

For $\bar{B}_{0X}(\bar{x}^{(S)})$, we obtain the following Dirichlet boundary setting for the determination of the eigenvalues and eigenfunctions:

$$\begin{array}{l}\Lambda^{+}\bar{B}_{0X}(\bar{x}^{(S)}) = -\lambda\bar{B}_{0X}(\bar{x}^{(S)}),\\ B_{0X,n}(\bar{x}^{(S)})\Big|_{\partial D_{S,L}} = 0.\end{array} \tag{25}$$

Here, $B_{0X,n}(\bar{x}^{(S)})$ denotes the normal component of magnetic induction. The solutions to (25) are similar to the corresponding functions described in (4) and (7). For example, the component $B_{0X,x_1^{(S)}}$ will look like the following:

$$\begin{array}{l}B_{l_1}^{(1)}(i) = \sqrt{\frac{2}{d_1}}cos\left(\frac{l_1\pi i}{N_1}\right), l_1 = 1, \ldots, N_1 - 1;\\ B_{l_2}^{(2)}(j) = \sqrt{\frac{2}{d_2}}sin\left(\frac{l_2\pi j}{N_2}\right), l_2 = 1, \ldots, N_2 - 1;\\ B_{l_3}^{(3)}(k) = \sqrt{\frac{2}{d_3}}sin\left(\frac{l_3\pi k}{N_3}\right), l_3 = 1, \ldots, N_3 - 1\\ B_{0X,x_1^{(S)}}(\bar{x}^{(S)}) = \sum\limits_{l_1=1}^{N_1-1}\sum\limits_{l_2=1}^{N_2-1}\sum\limits_{l_3=1}^{N_3-1} B_{l_1l_2l_3}B_{l_1}^{(1)}(i) \cdot B_{l_2}^{(2)}(j) \cdot B_{l_3}^{(3)}(k).\end{array} \tag{26}$$

In Equation (26), $B_{l_1l_2l_3}$ are coefficients to be found from system (24).

The eigenvalue has the form

$$\begin{array}{c}\lambda_{l_\alpha}^{(\alpha)} = \frac{4\eta}{h_\alpha^2}sin^2\left(\frac{l_\alpha\pi}{2N_\alpha}\right) = \frac{4\eta}{h_\alpha^2}sin^2\left(\frac{l_\alpha\pi h_\alpha}{2d_\alpha}\right),\\ \frac{l_\alpha}{N_\alpha} = k_\alpha \in \Xi: \left\{\frac{l_\alpha}{N_\alpha} : \frac{l_{\alpha,s}}{N_{\alpha,s}} = \frac{l_\alpha}{N_\alpha}, s = 1, \ldots, L, \alpha = 1,2,3\right\};\\ \lambda_l = 16\pi^2 \cdot \sum_{\alpha=1}^{3}\lambda_{l_\alpha}^{(\alpha)}.\end{array} \tag{27}$$

If we consider the system (24) in a cube with L small cubes removed, then it holds that

$$d_\alpha = d,\ d_{s,\alpha} = l,\ \alpha = 1,2,3;\ s = 1, \ldots, L.$$

Theorem 1. *System (2) has a unique solution in the grid domain* $D_{S,L} \times (0,T)$ *provided that the initial and boundary conditions for the magnetic field and the initial values for the velocity field are given. Here,* $D_{S,L}$ *is a cube with the edge of length l with L small cubes removed, each with edges of length* $l_m, m = 1, \ldots, L$:

$$SK_m = \left\{ x_1, x_2, x_3 : \left| x_\alpha - x_{\alpha 0}^{(m)} \right| \le l_m, \alpha = 1,2,3 \right\}, m = 1, \ldots, L.$$

Proof of Theorem 1. Let the representation $\bar{B}_0^{(S)}(x^{(S)}) = \bar{B}_{0X}(\bar{x}^{(S)}) \cdot g(x_{n+1}^{(S)})$, $\bar{x}^{(S)} = (x_1^{(S)}, \ldots, x_n^{(S)})$ take place.

The function $g(x_{n+1}^{(S)})$ satisfies the functional equation of the first order. It can be solved as follows:

$$g(x_{n+1}^{(S),j}) = (1+\tau\lambda)^j g\left(x_{n+1}^{(S),0}\right), j = 1, \ldots, J.$$
$$\bar{B}_0^{(S)}(x^{(S),0}) = g(x_{n+1}^{(S),0}) = \phi_0(x_1^{(S)}, \ldots, x_n^{(S)}) - \bar{B}_1(x_1^{(S)}, \ldots, x_n^{(S)}, 0), n = 1,2,3.$$

To find the solution of setting (24), we should expand the initial vector function $\bar{B}_0^{(S)}(x^{(S),0})$ into the series of eigenfunctions of setting (26):

$$\bar{B}_0^{(S)}(x^{(S),0}) = \sum_l a_l \bar{B}_l, \ \bar{B}_l \equiv \bar{B}_{ijk}, l \in \Xi. \tag{28}$$

Thus, we obtain

$$\begin{aligned} &\bar{B}_0^{(S)}(x^{(S)}) = \sum_{l \in \Xi} a_l \bar{B}_{0X}^{(\lambda_l)}(\bar{x}^{(S)}) \cdot g^{(\lambda_l)}(x_{n+1}^{(S)}), \\ &\bar{x}^{(S)} = \left(x_1^{(S)}, \ldots, x_3^{(S)}\right), x_4^{(S)} = j\tau, j = 0, \ldots, J. \end{aligned} \tag{29}$$

After finding $\bar{B}_{0X}(x^{(S)})$, we can formulate the setting for the determination of the discrete vector field $\bar{v}^{(S)}(x^{(S),j})$.

Let us consider the second equation in (24) to find the discrete vector function $\bar{v}^{(S)}$ provided that the "whole" discrete vector function $\bar{B}^{(S)}$ is given:

$$\frac{\bar{v}^{(S)}(x^{(S),j+1}) - \bar{v}^{(S)}(x^{(S),j})}{\tau} = \left[\bar{v}^{(S)}, \bar{B}^{(S)}\right]. \tag{30}$$

Let us rewrite (30) in the following form (in order to simplify the designations of unknown vector functions in the grid space):

$$\begin{aligned} &v_1(i) - v_1(i-1) = \tau\{v_2(i)B_3(i) - v_3(i)B_2(i)\}, \\ &v_2(i) - v_2(i-1) = \tau\{v_3(i)B_1(i) - v_1(i)B_3(i)\}, \\ &v_3(i) - v_3(i-1) = \tau\{v_1(i)B_2(i) - v_2(i)B_1(i)\}, \\ &\bar{v}^{(S)}(x^{(S),i}) = (v_1(i), v_2(i), v_3(i)), \bar{B}^{(S)}(x^{(S),i}) = (B_1(i), B_2(i), B_3(i)), \\ &i = 0, \ldots, J. \end{aligned} \tag{31}$$

The solution to system (29) (a system of functional equations) can be expressed through the values of the corresponding functions calculated in the previous step:

$$\begin{aligned} v_1(i) &= \tau\left\{ v_1(0) + \sum_{k=1}^{i} B_3(k)v_2(0) + \sum_{k=1}^{i} v_3(k)\left[B_1(k)\sum_{j=k}^{i} B_3(j) - B_2(k)\right] - \sum_{k=1}^{i} v_1(k)B_3(k)\sum_{j=k}^{i} B_3(j)\right\}, \\ v_3(i) &= \tau\left\{ v_3(0) + \sum_{k=1}^{i} B_1(k)v_2(0) + \sum_{k=1}^{i} v_1(k)\left[B_3(k)\sum_{j=k}^{i} B_1(j) + B_2(k)\right] - \sum_{k=1}^{i} v_3(k)B_1(k)\sum_{j=k}^{i} B_1(j)\right\}, \\ &i = 0, \ldots, J. \end{aligned} \tag{32}$$

In turn, system (32) can be transformed as follows:

$$\begin{array}{l} a_{ii}v_1(i) + \sum\limits_{k=1}^{i-1} a_{ik}v_1(k) = f_1(i) + \sum\limits_{k=1}^{i} c_{ki}v_3(k); \\ b_{ii}v_3(i) + \sum_{k=1}^{i-1} b_{ik}v_3(k) = f_2(i) + \sum_{k=1}^{i} d_{ki}v_1 \\ a_{ii} = 1 + \tau B_3^2(i), b_{ii} = 1 + \tau B_1^2(i), \\ c_{ii} = \tau[B_3(i)B_1(i) - B_2(i)]; \\ d_{ii} = \tau[B_3(i)B_1(i) + B_2(i)] \\ a_{ii}v_1(i) - c_{ii}v_3(i) = \widetilde{f}_1(i), \\ b_{ii}v_3(i) - d_{ii}v_1(i) = \widetilde{f}_2(i). \end{array} \tag{33}$$

If the determinant $D = \begin{vmatrix} a_{ii} & -c_{ii} \\ -d_{ii} & b_{ii} \end{vmatrix}$ is not equal to zero, then we can express $v_1(i), v_3(i), i = 1, \ldots, J$ through the previous values of these functions $v_1(k), v_3(k)$, k = $0, \ldots, i-1$. It is easily seen from (33) that this determinant can be equal to zero only if all the components of $\bar{B}^{(S)}(x^{(S),i})$ have such value.

Thus, we obtain a solution of the simplified MHD system in a specific domain $D_{S,L} \times (0,T)$, with the Dirichlet boundary setting being solved in the corresponding spatial domain $D_{S,L}$. We can also consider an ensemble of the expanding compacts $D_{S,L}^{(n)}, n \to \infty$, and construct the product $D_{S,L}^{(n)} \times (0,T), n \to \infty$.

Acting in such a way, one can "comprise" the upper subspace in R^3.

Thus, we have proved the following. □

Note 4. If $\{\bar{v}, \bar{B}\} \neq 0$, then, instead of relation (20), in the case of discrete functions we will have

$$\begin{array}{l} \eta\Lambda^+\bar{B}_{0X}(\bar{x}^{(S)}) + curl^+\left[\bar{v}^{(S)}, \bar{B}_{0X}\right] = -\lambda\bar{B}_{0X}(\bar{x}^{(S)}), \\ B_{0X,n}(\bar{x}^{(S)})\Big|_{\partial D_{S,L}} = 0. \end{array} \tag{34}$$

Here, $curl^+\left[\bar{v}^{(S)}, \bar{B}_{0X}\right]$ is a finite-differenced approximation of the corresponding continuous operator. The Dirichlet boundary setting can now be considered in the domain described in Section 2.4. The magnetic induction should have only one non-zero component, which is parallel to one of the axes. If we have to find $B_{x_3^{(S)},0X}$, then the setting will be studied in the direct product of a rectangle (cube) in the XOY coordinate plane and the OZ axis (see Section 2.4). $B_{x_3^{(S)},0X}$ should depend on $x_2^{(S)}, x_3^{(S)}$.

In this case, we can assume that the vector field $\bar{v}^{(S)}$ is a constant vector field, $\bar{v}^{(S)} = \bar{v}_0^{(S)}$, and is determined in the specified domain $D_{S,L}$. Then, relations (28) will have the following form:

$$\begin{array}{l} \eta\Lambda^+ B_{x_3^{(S)}0X}(x_{1,i}^{(S)}, x_{2,j}^{(S)}) - \frac{v_{x_1^{(S)}0}^{(S)}}{2h_1}(B_{x_3^{(S)}0X}(x_{1,i+1}^{(S)}, x_{2,j}^{(S)}) - B_{x_3^{(S)}0X}(x_{1,i-1}^{(S)}, x_{2,j}^{(S)})) - \\ - \frac{v_{x_2^{(S)}0}^{(S)}}{2h_2}(B_{x_3^{(S)}0X}(x_{1,i}^{(S)}, x_{2,j+1}^{(S)}) - B_{x_3^{(S)}0X}(x_{1,i}^{(S)}, x_{2,j-1}^{(S)})) = -\lambda B_{x_3^{(S)}0X}(x_{1,i}^{(S)}, x_{2,j}^{(S)}),\ i = 1, \ldots, N_1 - 1; \\ j = 1, \ldots, N_2 - 1; \\ B_{0X,n}(\bar{x}^{(S)})\Big|_{\partial D_{S,L}} = 0. \end{array} \tag{35}$$

Here, Λ^+, h_1, and h_2 are the finite-differenced analogues of the Laplace operator and the grid steps along two coordinate axes respectively defined in Section 2.4. We should stress that Λ^+ can contain the finite derivatives with respect to the third coordinate, but $B_{x_3^{(S)}0X}(x_{1,i}^{(S)}, x_{2,j}^{(S)})$ does not depend on it in the case considered here. For simplicity, let

us suppose that Λ^+ can be written as the sum $\Lambda^+ = \Lambda_{x_1^{(S)}} + \Lambda_{x_2^{(S)}} + \Lambda_{x_3^{(S)}}$, similarly to in Equation (15).

The boundary condition $B_{0X,n}(\bar{x}^{(S)})\Big|_{\partial D_{S,L}} = 0$ means that the component $B_{x_3^{(S)}0X}$ of the magnetic induction vector is equal to zero on the infinite edges of $D_{S,L}$, which can be represented as direct products of the sides of the "large cube" and all small cutouts and infinite lines parallel to the OZ axis. The eigenfunction of problem (35) can be expressed as follows:

$$
\begin{aligned}
&B^{(\lambda)}_{x_3^{(S)}0X}(x^{(S)}_{1,i}, x^{(S)}_{2,j}) = w_1^{(\lambda_1)}(x^{(S)}_{1,i}) w_2^{(\lambda_2)}(x^{(S)}_{2,j});\ \lambda = \lambda_1 + \lambda_2;\\
&w_1^{(\lambda_1)}(x^{(S)}_{1,i}) = \sum_{k=0}^{i} C_i^k a_1^k U_{i-k-1}(x^{(S)}_{1,i});\ w_2^{(\lambda_2)}\left(x^{(S)}_{2,j}\right) = \sum_{k=0}^{j} C_j^k a_2^k U_{j-k-1}(x^{(S)}_{2,j});\\
&a_1 = 1 - \frac{\lambda_1 h_1^2}{2\eta} - \sqrt{\left(1 - \frac{\lambda_1 h_1^2}{2\eta}\right)^2 + \frac{h_1^2 v^2_{0x_1^{(S)}}}{4}};\ a_2 = 1 - \frac{\lambda_2 h_2^2}{2\eta} - \sqrt{\left(1 - \frac{\lambda_2 h_2^2}{2\eta}\right)^2 + \frac{h_2^2 v^2_{0x_2^{(S)}}}{4}}\\
&U_N(y) = \frac{sin((N+1)(arccosy))}{sin(arccosy)},\\
&C^k_{N_1} = \frac{N_1!}{k!(N_1-k)!}; C^k_{N_2} = \frac{N_2!}{k!(N_2-k)!}.
\end{aligned}
\tag{36}
$$

Here, $U_N(y)$ is Chebyshev's polynomial of the second type, $C^k_{N_1}$ are binomial coefficients, and λ_1,λ_2 are the roots of the following equation in (35):

$$\sum_{k=0}^{N_1} C^k_{N_1} a_1^k U_{N_1-k-1}(\lambda_1) = 0;\ \sum_{k=0}^{N_2} C^k_{N_2} a_2^k U_{N_2-k-1}(\lambda_2) = 0.$$

Numerical Experiment

We implemeted the algorithm proposed in the previous section to reconstruct the magnetic field of Mercury using the remote sensing data obtained by Messenger [14]. Unfortunately, we do not have any information about the real velocity vector field in Mercury's outerspace. Therefore, we supposed that the zeroth approximation to this vector field is a constant vector field. The value of this constant can be set to around 400 km/s (the mean value of the velocity in solar wind). The first-order approximation of the velocity can be found from the solution of (30)–(34) after determining the magnetic induction. The dimensionless viscosity and magnetic viscosity coefficients were set to be $1.5{\cdot}10^{-1}$ and $3.2{\cdot}10^{-1}$, respectively. It should be stressed here that no sound information on the values of these physical features was included. Our computations should be considered as synthetic samples. In [15], some dimensionless parameters characterizing the viscosity and magnetic viscosity are given. If the Reynolds number is equal to 10^5 and the magnetic field has a value of around 2 nTsl, then we can assume that the parameters mentioned above do not exceed 1.

We worked in a Cartesian patch centred in the planet's centre of mass. The isolyne map and the surface of the values of magnetic induction are shown in Figure 1. The entire polygon contained four zones which were "thrown off" from the consideration. So, three model samples were generated (the four zones are marked with bold black lines in Figure 1). The discrete magnetic potential was computed in the area that can be represented as a cube without one, two, or four zones. The eigenfunctions of the discrete Laplace operator look like (12). We set an initial distribution of the magnetic induction, which was built according to basic principles of the linear integral representation method [7]. Thus, an analytical model of the stationary magnetic field of Mercury was constructed from the Messenger data with a relative accuracy of $6.1{\cdot}10^{-5}$. After building this initial distribution, we solved system (24) for the components of magnetic induction in the grid domain, which was a cube with one, two, or four cutouts. The large cube has an edge length of 20,000 km.

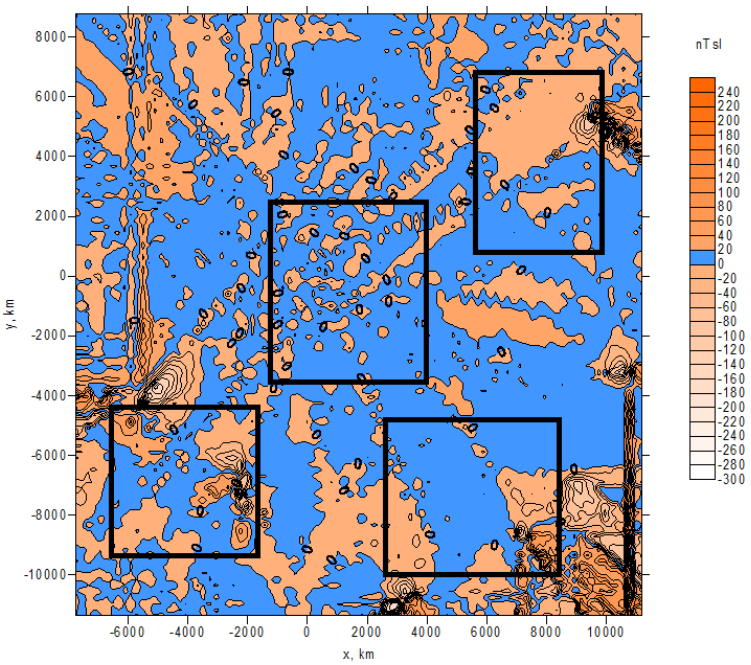


Figure 1. Isolyne map of Mercury's magnetic field.

By means of the S-approximation technique described in [7], a harmonic function which satisfied the boundary conditions was computed. Further, we subtracted this function from the "full" magnetic induction. Thus, we have a boundary setting for the discrete functions such as (24)–(25). It should be emphasized that now the magnetic induction is zero on the boundary.

The relative accuracy of the restored magnetic field was $1.2 \cdot 10^{-4}$, $3.7 \cdot 10^{-4}$, $5.1 \cdot 10^{-3}$ when considering the cube without one, two, or four zones, respectively. The magnetic field obtained using the technique proposed above is shown in Figure 2. The residual map between the approximated magnetic induction and the real data is given in Figure 3.

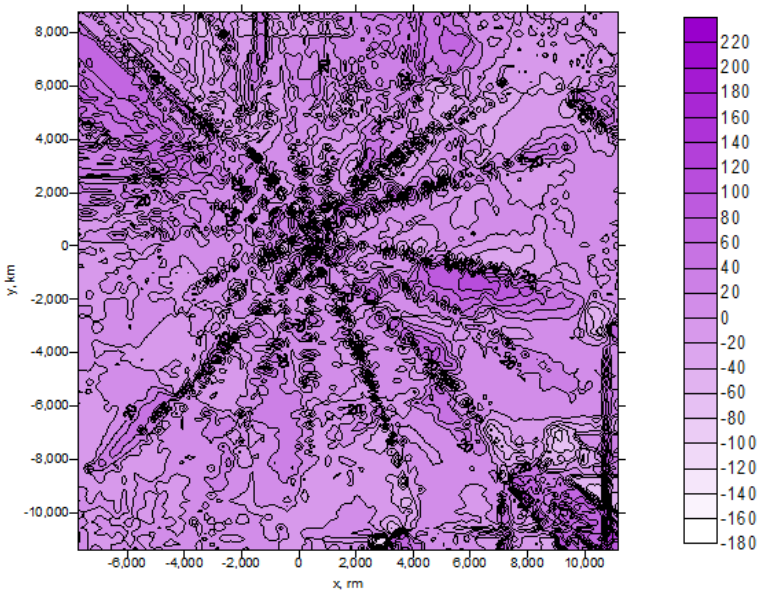


Figure 2. The reconstructed magnetic field of Mercury.

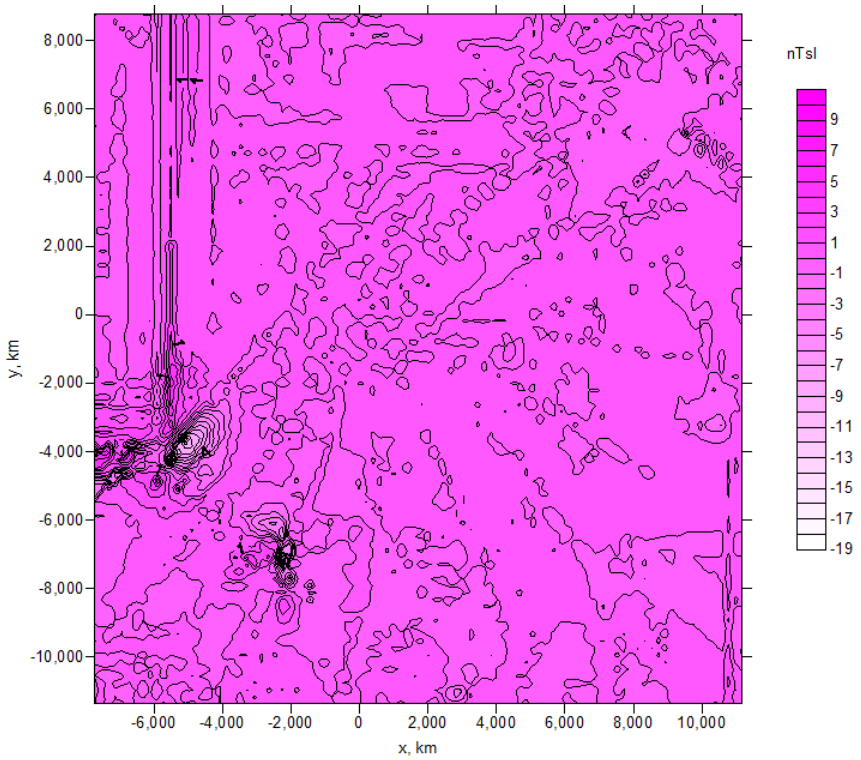


Figure 3. Residual map between the real and reconstructed magnetic field of Mercury.

4. Discussion

1. This paper deals with a simplified version of the MHD equation system in the specific domains of R^{n+1}. The direct setting is solved within the framework of the discrete potential theory in Cartesian coordinates.

2. The algorithm for simplifying (reducing) the whole MHD equation system can be extrapolated into a larger scope: we can consider an ensemble of expanding compacts [13]. We constructed an analytical approximation of both the magnetic field and velocity field of a charged fluid in a specific domain of R^{n+1} within the framework of the theory of discrete potentials. This theory contains some significant differences in comparison with the finite-differenced approximations of the physical fields: all the problems are initially set in infinite grid space. Some additional constraints are needed to reveal the unique solution to those settings.

5. Conclusions

In this paper, a fundamental problem of finding solutions of magnetic hydrodynamic equation systems in specific domains within the framework of discrete potential theory is considered. We analyzed the possibility of the existence of solenoidal vector fields satisfying the equations mentioned above. An algorithm is proposed for determining discrete (grid) functions in some bounded domains in three-dimensional space. Some questions concerning building analytical models of the magnetic field and velocity vector field are also discussed. In the discrete case, there are a finite number of eigenfunctions and eigenvalues of the boundary problems in specific domains. This can mean that some extraordinary features of moving fluid can be revealed on the quantum level.

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Abbreviations

The following abbreviations are used in this manuscript:

LAES	Linear algebraic equation system
MHD	Magnetic hydrodynamics

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