mathematics

MDPI

Article

On the Topological Classification of Four-Dimensional Steady Gradient Ricci Solitons with Nonnegative Sectional Curvature

Yuehan Hao

School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China; 3120221441@bit.edu.cn

Abstract

In this paper, we study the topology of steady gradient Ricci solitons with nonnegative sectional curvature. We apply a characterization theorem for the fundamental group of a positively curved steady gradient Ricci soliton that admits a critical point. As an application of the characterization theorem, we give a classification of the topology of complete four-dimensional steady gradient Ricci solitons with nonnegative sectional curvature.

Keywords: Ricci flow; Ricci soliton; topology

MSC: 53E20; 57R18

1. Introduction

A steady gradient Ricci soliton (M, g, f) consists of a smooth manifold M, a Riemannian metric g, and a potential function f, if these variables satisfy the equation

$$\mathrm{Ric}_M + \nabla^2 f = 0, \tag{1}$$

on M, where Ric_M denotes the Ricci tensor of (M, g), and ∇^2 denotes the Hessian operator. The notation can be simplified by omitting f or g when they can be determined by context. A point $x \in M$ is called a critical point of M if $|\nabla f|(x) = 0$. If M is not Ricci flat, then up to scaling,

$$R + |\nabla f|^2 = 1, \tag{2}$$

where R denotes the scalar curvature.

Steady gradient Ricci solitons are important since they may arise as the Type II singularities of the Ricci flow. In dimension 2, the unique non-flat steady gradient Ricci soliton is the Cigar soliton [1], which is asymptotic to a cylinder and has $O(2)$-symmetry. In dimension $n \geq 3$, Bryant [2] found a family of rotationally symmetric steady gradient Ricci solitons. Examples of 3-dimensional $O(2) \times \mathbb{Z}_2$-symmetric steady gradient Ricci solitons that are all asymptotic to a sector with angle $\nu \in (0, \pi)$ were constructed in [3], which confirmed Hamilton's conjecture on the existence of flying wings. Later, Lai [4] proved that any 3-dimensional steady gradient Ricci soliton with positive curvature is either a flying wing or the Bryant soliton. For more examples of steady gradient Ricci solitons, see [5–12].

How to classify steady gradient Ricci solitons with nonnegative sectional curvature is still an open problem of significant interest. There are many works on classifying positively curved steady gradient Ricci solitons; see [13–18] and references therein. For most of these works, they essentially deal with simply connected steady Ricci solitons. However, the topology of a steady gradient Ricci soliton with nonnegative Ricci curvature is still

check for updates

Academic Editor: Jay M. Jahangiri

Received: 26 September 2025
Revised: 13 November 2025
Accepted: 17 November 2025
Published: 18 November 2025

Citation: Hao, Y. On the Topological Classification of Four-Dimensional Steady Gradient Ricci Solitons with Nonnegative Sectional Curvature. *Mathematics* **2025**, *13*, 3702. https://doi.org/10.3390/math13223702

Copyright: © 2025 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

unknown yet. The work of [19] characterized n-dimensional steady gradient solitons of nonnegative sectional curvature with curvature in L^1. The universal cover of such a soliton can split isometrically as the Cigar soliton and the Euclidean space, and the fundamental group is a Bieberbach group of rank $n-2$. In [20], Deng and the author have studied the fundamental group of a steady gradient Ricci soliton with nonnegative sectional curvature and proved that it is either trivial or infinite. In particular, any noncollapsed complete steady gradient Ricci soliton in dimension n with nonnegative sectional curvature has the trivial fundamental group and is diffeomorphic to $\mathbb{R}^n$. However, very little is known about the case where the soliton may not be simply connected. In this paper, we would like to further explore the topology of a positively curved steady gradient Ricci soliton that has an infinite fundamental group. Note that many important results have been obtained in the classification of non-steady gradient Ricci solitons. Complete expanding gradient solitons with nonnegative Ricci curvature must be diffeomorphic to the Euclidean space; see Lemma 5.5 in [21]. Any complete shrinking gradient Ricci soliton has a finite fundamental group [22].

Our main theorem gives a topological classification of steady gradient Ricci solitons with nonnegative sectional curvature in dimension 4.

Theorem 1. *Suppose that (M, g, f) is a complete 4-dimensional steady gradient Ricci soliton with nonnegative sectional curvature. Then the topological structure can be one of Type* $\mathbf{I}$, $\mathbf{T}$, $\mathbf{S}$, $\mathbf{S}^C_{R(\theta)}$, $\mathbf{S}^I_{R(\theta)}$, $\mathbf{S}^C_S$, $\mathbf{X}_i$, $\mathbf{Y}_j$, *defined in Section 3.2; and* $\mathbf{Z}^C_{R_{(\gamma,p)}}$, $\mathbf{Z}^I_{R_{(\gamma,p)}}$, $\mathbf{Z}^C_{S_{(\cdot,q)}}$, $\mathbf{O}^C_{R_v(\theta)}$, $\mathbf{O}^I_{R_v(\theta)}$, $\mathbf{O}^C_A$ *defined in Theorem 7. Here,* $i = 1,2,3,4,5,6,7$, *and* $j = 1,2,3,4,5,6,7,8,9,10$.

Throughout this paper, we suppose that (M, g, f) is a complete steady gradient Ricci soliton with nonnegative sectional curvature and $\pi : (\tilde{M}, \tilde{g}, \tilde{f}) \to (M, g, f)$ is its universal covering map with $\pi_*(f) = \tilde{f}$, if there is no special emphasis. Note that a simply connected steady gradient Ricci soliton with nonnegative sectional curvature is always diffeomorphic to $\mathbb{R}^n$. It suffices to classify the quotients of a simply connected steady gradient Ricci soliton with nonnegative sectional curvature. So, it is necessary to classify the fundamental groups of a steady gradient Ricci soliton with nonnegative sectional curvature. We have the following characterization of the fundamental group of a steady gradient Ricci soliton that admits a critical point.

Theorem 2. *Suppose* $(\tilde{M}, \tilde{g}, \tilde{f}) = (N, g_N) \times (\mathbb{R}^k, g_{Euclid})$ *is a complete simply connected steady gradient Ricci soliton with nonnegative sectional curvature and* $\mathrm{Ric}_N > 0$. *Assume $\tilde{M}$ has a critical point. Let Γ be a subgroup of* $\mathrm{Iso}(\tilde{M})$. *Then, the following is equivalent:*

(a) Γ *is the fundamental group of a complete steady gradient Ricci soliton, whose universal cover is $\tilde{M}$, up to isometries.*

(b) $\Gamma := \{(\Phi(\sigma), \sigma) : \sigma \in G\}$. *Here,* $\Phi : G \to \mathrm{Iso}(N)$ *is a group homomorphism from G to* $\mathrm{Iso}(N)$, *where G is a discrete Lie group that acts smoothly, freely, properly and isometrically on* $\mathbb{R}^k$.

For positively curved steady gradient Ricci solitons of dimension 4, they are diffeomorphic to $\mathbb{R}^n$ or admit a critical point. Therefore, Theorem 2 will be used to study the topology of steady gradient Ricci solitons of dimension 4. For higher dimensions, it is still open whether the critical point exists on a steady gradient Ricci soliton with nonnegative sectional curvature and positive Ricci curvature.

The paper is organized as follows: Lemma 1 showed $\tilde{M}$ has a splitting property with $\tilde{M} = N \times \mathbb{R}^k$, where $\mathrm{Ric}_N > 0$. Moreover, it induces an alternative characterization of the fundamental group $\pi_1(M)$ (see Theorem 2), which will be introduced in Section 2. Inspired

by results in [4], we study the topology of the cases where N is $O(2)$-symmetric in Section 3. Finally, Theorem 1 will be proved in Section 4.

2. Action Behaviors of Fundamental Groups

In this section, we mainly introduce an alternative characterization of the fundamental group (see Theorem 2) and explain its rationale in detail.

Let $\mathrm{Iso}(M,g)$ be the isometry group of the Riemannian manifold (M,g) and one can omit g if it can be determined by context. Let g_{Euclid} be the standard geometry of Euclidean spaces. Then, we have the following lemma, which shows that every gradient Ricci soliton with nonnegative sectional curvature has an isometrically splitting Riemannian universal cover whose isometry group inherits its splitting property.

Lemma 1. *Suppose that (M,g,f) is a gradient Ricci soliton with nonnegative sectional curvature, and $\pi:(\tilde{M},\tilde{g})\to(M,g)$ is its universal Riemannian covering map. Then, $\tilde{M}=(N,g_N)\times(\mathbb{R}^k,g_{Euclid})$, splits isometrically with $\mathrm{Ric}_N>0$ and $k\in\mathbb{N}$. Moreover, $\mathrm{Iso}(\tilde{M})=\mathrm{Iso}(N)\times\mathrm{Iso}(\mathbb{R}^k)$ and the fundamental group $\pi_1(M)\subseteq\mathrm{Iso}(N)\times\mathrm{Iso}(\mathbb{R}^k)$ can be regarded as a discrete Lie group that acts smoothly, freely, properly and isometrically on $\tilde{M}$.*

Proof. By Theorem 1.1 in [23], the universal cover $\tilde{M}=(N,g_N)\times(\mathbb{R}^k,g_{Euclid})$ splits isometrically. Here, $\mathrm{Ric}_N>0$ and $k\in\mathbb{N}$. Since N has positive Ricci curvature, N contains no line. It follows from Corollary 6.2 in [24] that $\mathrm{Iso}(\tilde{M})=\mathrm{Iso}(N)\times\mathrm{Iso}(\mathbb{R}^k)$. Considering the fact that $\pi_1(M)$ is isomorphic to the deck transformation group of π [25,26], we complete the proof. □

Remark 1. *Note that the proof of the splitting property of isometry groups above is inspired by the argument of Corollary 3.1 in [19], which proved the rigidity of a complete steady breather with nonnegative curvature operator and curvature in L^1, and showed that the universal cover of its underlying manifold can split isometrically as the Cigar soliton and the Euclidean space.*

Moreover, the splitting property of universal covers given by the above lemma is a fundamental property that holds for all steady gradient Ricci solitons with nonnegative sectional curvature. In the remaining sections of this paper, we will discuss the classification of the topological structures according to the value of k.

As the above lemma, every element in $\pi_1(M)$ can be written as (ϕ,σ), where $\phi\in\mathrm{Iso}(N)$, $\sigma\in\mathrm{Iso}(\mathbb{R}^k)$. Let $\tilde{\pi}_N,\tilde{\pi}_{\mathbb{R}^k}$ be the natural projections from $\pi_1(M)$ to $\mathrm{Iso}(N)$ and $\mathrm{Iso}(\mathbb{R}^k)$ respectively; that is, $\tilde{\pi}_N:=\{\phi\in\mathrm{Iso}(N):(\phi,\sigma)\in\pi_1(M)\}$ and $\tilde{\pi}_{\mathbb{R}^k}:=\{\sigma\in\mathrm{Iso}(\mathbb{R}^k):(\phi,\sigma)\in\pi_1(M)\}$. If $\{(\phi_i,\sigma_i)\}_{i\in I}$ is a generator set of $\pi_1(M)$, then $\{\phi_i\}_{i\in I}$ and $\{\sigma_i\}_{i\in I}$ generate $\tilde{\pi}_N$ and $\tilde{\pi}_{\mathbb{R}^k}$ respectively.

We can preliminarily conclude that $\pi_1(M)$ can be determined through a suitable combination of generators of $\tilde{\pi}_N$ and $\tilde{\pi}_{\mathbb{R}^k}$; in particular, we will adopt it to characterize the fundamental group in this paper. In the following, we shall explain this approach by studying the property of $\tilde{\pi}_N$ and $\tilde{\pi}_{\mathbb{R}^k}$.

Lemma 2. *Suppose that (M,g,f) is a complete steady gradient Ricci soliton with nonnegative sectional curvature, and its universal cover $(\tilde{M},\tilde{g})$ is isometric to $(N,g_N)\times(\mathbb{R}^k,g_{Euclid})$, where N has positive Ricci curvature. Assume that the critical point of M exists. Then $\tilde{\pi}_{\mathbb{R}^k}$ is a discrete Lie subgroup of $\mathrm{Iso}(\mathbb{R}^k)$, acting smoothly, freely, properly and isometrically on $\mathbb{R}^k$.*

Proof. By definition, it is easy to verify that $(\tilde{M},\tilde{g},\tilde{f})$ is a steady gradient Ricci soliton, which can split as N and $\mathbb{R}^k$ isometrically. Then by Lemma 2.1 in [27], we can directly obtain that $\tilde{f}$ can split so that N and $\mathbb{R}^k$ are both the steady gradient Ricci soliton, say

(N, g_N, f_N) and $(\mathbb{R}^k, g_{Euclid}, f_k)$ respectively. If the critical point of M exists, then the scalar curvature of M can attain the maximum, and so does N, denoted by $\bar{x} \in N$. It follows from (1) and (2) that $\bar{x}$ is the unique critical point of N. Since every isometry of N keeps the scalar curvature invariant, $\bar{x}$ is also a fixed point of $\mathrm{Iso}(N)$.

By definition, $\tilde{\pi}_{\mathbb{R}^k}$ is both a natural projection of the discrete group $\pi_1(M)$ and a subgroup of the isometry group of $\mathbb{R}^k$. Thus, $\tilde{\pi}_{\mathbb{R}^k}$ is a discrete Lie subgroup of $\mathrm{Iso}(\mathbb{R}^k)$, acting smoothly and isometrically on $\mathbb{R}^k$.

Subsequently, $\tilde{\pi}_{\mathbb{R}^k}$ must act freely on $\mathbb{R}^k$; otherwise, if there is a non-identity element $\sigma \in \tilde{\pi}_{\mathbb{R}^k}$ having a fixed point $\bar{y} \in \mathbb{R}^k$, then each $(\phi, \sigma) \in \pi_1(M)$ fixes the point $(\bar{x}, \bar{y})$, which contradicts Lemma 1.

The group action is proper. To argue by contradiction, we suppose that $\tilde{\pi}_{\mathbb{R}^k}$ is not a proper action. Due to Proposition 21.5 in [25], there exists a compact subset $V \subseteq \mathbb{R}^k$, and the set $\tilde{\pi}^V_{\mathbb{R}^k} = \{\sigma \in \tilde{\pi}_{\mathbb{R}^k} : (\sigma \cdot V) \cap V \neq \emptyset\}$ is noncompact. Choose a compact neighborhood U of $\bar{x}$ in N. Consider $U \times V$. It is a compact subset of $\tilde{M}$. However, $\pi_1^{U\times V} = \{\varphi \in \pi_1(M) : (\varphi \cdot (U \times V)) \cap (U \times V) \neq \emptyset\}$ contains all elements such that $(\phi, \sigma) \in \pi_1(M)$ with $\sigma \in \tilde{\pi}^V_{\mathbb{R}^k}$ and hence is noncompact, contradicting the properness of $\pi_1(M)$. Now, we complete the proof. □

Here, the condition that the critical point exists is necessary because it ensures that the fixed point of $\mathrm{Iso}(N)$ exists and is unique. In particular, it is the essence of the freeness and properness of group action $\tilde{\pi}_{\mathbb{R}^k}$. Now, we turn to group action $\tilde{\pi}_N$ in the following lemma and find that $\tilde{\pi}_N$ is controlled by $\tilde{\pi}_{\mathbb{R}^k}$.

Lemma 3. *Under assumptions in Lemma 2, there exists a group homomorphism $\Phi : \tilde{\pi}_{\mathbb{R}^k} \to \tilde{\pi}_N = \Phi(\tilde{\pi}_{\mathbb{R}^k})$ defined by $(\Phi(\sigma), \sigma) \in \pi_1(M)$ for any $\sigma \in \tilde{\pi}_{\mathbb{R}^k}$, and then $\tilde{\pi}_N$ is isomorphic to a quotient group of $\tilde{\pi}_{\mathbb{R}^k}$.*

Proof. By definition, $\tilde{\pi}_N$ is a group that acts on N. If $(\phi_1, \sigma), (\phi_2, \sigma) \in \pi_1(M)$, then $\phi_1 = \phi_2$; otherwise $(\phi_1^{-1}\phi_2, I) \in \pi_1(M)$ is a nontrivial element with a fixed point, which contradicts the freeness of $\pi_1(M)$ stated in Lemma 1. Hence, we can define a group homomorphism $\Phi : \tilde{\pi}_{\mathbb{R}^k} \to \tilde{\pi}_N$ satisfying $\Phi(\sigma) = \phi_\sigma$ with $(\phi_\sigma, \sigma) \in \pi_1(M)$. Obviously, Φ is an onto mapping; that is, $\mathrm{Im}(\Phi) = \tilde{\pi}_N$, where $\mathrm{Im}(\Phi)$ denotes the image of Φ.

Moreover, the first isomorphism theorem for groups implies that $\tilde{\pi}_N$ is isomorphic to a quotient group of $\tilde{\pi}_{\mathbb{R}^k}$. In particular, $\tilde{\pi}_N \cong \tilde{\pi}_{\mathbb{R}^k}/\mathrm{Ker}(\Phi)$, where $\mathrm{Ker}(\Phi)$ represents the kernel of Φ. Hence, we complete the proof. □

Up to now, we have learned the properties of $\tilde{\pi}_N$ and $\tilde{\pi}_{\mathbb{R}^k}$ and found the quotient group relation between them if the fundamental group is fixed. Naturally, one would like to know whether the converse statement holds or not. The next lemma gives a positive answer to this.

Lemma 4. *Suppose $(\tilde{M}, \tilde{g}) = (N, g_N) \times (\mathbb{R}^k, g_{Euclid})$ is a complete simply connected steady gradient Ricci soliton with nonnegative sectional curvature and $\mathrm{Ric}_N > 0$. Given a discrete Lie subgroup G of $\mathrm{Iso}(\mathbb{R}^k)$, which acts smoothly, freely, properly and isometrically on $\mathbb{R}^k$. Assume that there is a group homomorphism $\Phi : G \to \mathrm{Iso}(N)$ from G to $\mathrm{Iso}(N)$. Then, $F := \{(\Phi(\sigma), \sigma) : \sigma \in G\}$ induced by Φ is a discrete Lie group that acts smoothly, freely, properly and isometrically on $\tilde{M}$.*

Proof. According to our construction, F is a discrete Lie group acting smoothly and isometrically on $\tilde{M}$.

F is a free group action on $\tilde{M}$. Otherwise, suppose there is a non-identity element $(\phi, \sigma) \in F$ having a fixed point $(\bar{x}, \bar{y})$, that is, $(\phi(\bar{x}), \sigma(\bar{y})) = (\bar{x}, \bar{y})$. Since G is a free action

and Φ maps the identity of G to the identity of $\mathrm{Iso}(N)$, σ must be the identity map of $\mathbb{R}^k$ and then ϕ is the identity, contradiction.

F is a proper group action on $\tilde{M}$. Also use characterizations of proper actions in [25]. Let $K \subseteq \tilde{M}$ be any compact subset of $\tilde{M}$ and $K_{\mathbb{R}^k} := \{v \in \mathbb{R}^k : (u, v) \in K\}$. Note that $K_{\mathbb{R}^k}$ is a compact set of $\mathbb{R}^k$. Since G is a proper group action on $\mathbb{R}^k$, $G^{K_{\mathbb{R}^k}} = \{\sigma \in G : (\sigma \cdot K_{\mathbb{R}^k}) \cap K_{\mathbb{R}^k} \neq \emptyset\}$ is compact. It follows from $F^K = \{\varphi \in F : (\varphi \cdot K) \cap K \neq \emptyset\} \subseteq G^{K_{\mathbb{R}^k}}$ that F^K is compact and hence F is proper by Proposition 21.5 in [25].

We complete the proof. □

The above argument provides a proof of the existence of the fundamental group induced by a quotient group of some given $\tilde{\pi}_{\mathbb{R}^k}$. Based on Lemmas 2–4, we now have the sufficient techniques to prove Theorem 2.

Proof of Theorem 2. Assume first Γ is the fundamental group of a complete steady gradient Ricci soliton $M \cong \tilde{M}/\Gamma$. Then, Lemma 2 proves that $\Gamma_{\mathbb{R}^k} := \{\sigma \in \mathrm{Iso}(\mathbb{R}^k) : (\phi, \sigma) \in \Gamma\}$ is a discrete Lie group acting smoothly, freely, properly and isometrically on $\mathbb{R}^k$. The map $\Phi : \Gamma_{\mathbb{R}^k} \to \Gamma_N := \{\phi \in \mathrm{Iso}(N) : (\phi, \sigma) \in \Gamma\}$ induced by Γ and $\Gamma_{\mathbb{R}^k}$ in the same way as in Lemma 3 is a well-defined group homomorphism from $\Gamma_{\mathbb{R}^k}$ to $\mathrm{Iso}(N)$. And Γ can be written as $\{(\Phi(\sigma), \sigma) : \sigma \in \Gamma_{\mathbb{R}^k}\}$.

Conversely, assume that (b) holds. Lemma 4 implies that the group $\Gamma := \{(\Phi(\sigma), \sigma) : \sigma \in G\}$ induced by the group homomorphism Φ is a discrete Lie group that acts smoothly, freely, properly and isometrically on $\tilde{M}$. Due to the quotient manifold theorem for Riemannian manifolds, the orbit space $\tilde{M}/\Gamma$ is a complete Riemannian manifold and its fundamental group is Γ. Lemma 3.1.2 in [28] and the work of [27] imply G and Γ are both the infinite group and any element of Γ acts on the level set of $\tilde{f}$. Since (1) is a local property, $\tilde{M}/\Gamma$ must be a steady gradient Ricci soliton.

Hence, we complete the proof. □

Remark 2. *It is noticeable that without the existence of critical points and knowing the symmetry of N, it will be quite difficult to characterize $\tilde{\pi}_N$ and then $\pi_1(M)$. Because in this case, neither the action $\tilde{\pi}_{\mathbb{R}^k}$ nor $\tilde{\pi}_N$ is necessarily free, and we are hardly able to characterize one by the other.*

However, in higher dimensions, these questions mentioned above are still open.

For a general case, Gromov [29] has shown that the fundamental group of a complete manifold with nonnegative sectional curvature can be generated by finite generators, with an upper bound on the size of the generating set (also see [30] for more details). As a consequence of Lemma 3, we have the following result.

Theorem 3. *Suppose that (M, g) is an n-dimensional steady gradient Ricci soliton with nonnegative sectional curvature and at least one critical point. Suppose that its Riemannian cover $(\tilde{M}, \tilde{g}) = (N, g_N) \times (\mathbb{R}^k, g_{Euclid})$, where (N, g_N) has positive Ricci curvature. Then $\pi_1(M)$ is finitely generated and the minimal number of its generators is no more than $C(k)$.*

Here, $C(k) = \frac{v(k-1,1,\pi)}{v(k-1,1,\frac{\pi}{6})}$ and $v(n, m, r)$ denotes the volume of a ball of radius r in the constant-curvature space form S^n_m.

Proof. Lemma 2 shows that $\tilde{\pi}_{\mathbb{R}^k}$ is a discrete Lie group acting freely, properly, smoothly and isometrically on $\mathbb{R}^k$. Then $\mathbb{R}^k/\tilde{\pi}_{\mathbb{R}^k}$ is a complete flat manifold. The work in [29,30] implies that $\tilde{\pi}_{\mathbb{R}^k}$ is finitely generated and the minimum number of its generators is no more than $C(k) = \frac{v(k-1,1,\pi)}{v(k-1,1,\frac{\pi}{6})}$. Then, the result follows from Lemma 3. □

Remark 3. *Although the existence of critical points allows us to reduce the upper bound from the original $C(n)$ to $C(k)$, $k \leq n$, the estimate in Theorem 3 is still not sharp. In particular, $C(2) = 6$ which is larger than the bounds we obtain below.*

3. Classification with O(2)-Symmetric Factor Manifolds

Based on the definition of the fundamental group and Theorem 2, we can conclude that, the symmetry of N plays a crucial role in studying the topological classification, and understanding $\mathrm{Iso}(N)$ is necessary if we want to determine the specific topology. By considering that all 2, 3-dimensional steady gradient Ricci solitons have $O(2)$-symmetry, one can realize that $O(2)$ is a common and important symmetry for steady gradient Ricci solitons. Therefore, it is reasonable to talk about the situation where N is $O(2)$-invariant.

Recall that $O(2) = SO(2) \rtimes \mathbb{Z}_2$ consists of rotations and reflections. In dimension $n \geq 3$, we say a complete Riemannian manifold (M^n, g) is $O(2)$-symmetric if it admits an isometric $O(2)$-action and these actions act trivially on the orthogonal complement. For convenience, we denote the rotation through the origin o by angle θ and the reflection across the line through o at angle α, by $R(\theta) \in SO(2)$ and $S(\alpha)$ respectively, where $\theta \in (0, 2\pi]$ and $\alpha \in (0, \pi]$.

Throughout the remainder of this paper, we denote the topological type of M (or precisely the type of group action $\pi_1(M)$) by $\mathcal{T}$, which will be classified by generators of fundamental groups inspired by Theorem 2.

3.1. When $\tilde{M} = N \times \mathbb{R}$

In this subsection, we mainly use the characterization of the fundamental group to study the topology of M when the Euclidean factor manifold of $\tilde{M}$ is 1-dimensional, especially when the isometry group of N is $O(2)$. First, we give a simple classification of $\pi_1(M)$ in this case.

Lemma 5. *Suppose (M, g, f) is a complete steady gradient Ricci soliton with nonnegative sectional curvature and at least one critical point. Suppose that its universal cover $(\tilde{M}, \tilde{g}) = (N, g_N) \times (\mathbb{R}, g_{Euclid})$, where $\mathrm{Ric}_N > 0$.*

Then, $\pi_1(M)$ is either the trivial group $\{I\}$, denoted by Type **I***; or $\langle(\phi, T)\rangle$ for some $\phi \in \mathrm{Iso}(N)$ and nontrivial translation T of $\mathbb{R}$, denoted by Type* **J***.*

Proof. It suffices to determine a group homomorphism induced by $\pi_1(M)$ by Theorem 2.

Since $\tilde{\pi}_{\mathbb{R}}$ is a free group action on $\mathbb{R}$ (see Lemma 2) and any reflection of $\mathbb{R}$ has a fixed point, each element of $\tilde{\pi}_{\mathbb{R}}$ must be a translation of $\mathbb{R}$. Using Lemma 2.5.4 in [28], $\tilde{\pi}_{\mathbb{R}}$ has at most one non-identity generator, and then is either $\{I\}$ or $\langle T\rangle$ for some fixed nontrivial translation T. If $\tilde{\pi}_{\mathbb{R}} = \{I\}$, then $\tilde{\pi}_N = \{I\}$ and $\pi_1(M)$ must be $\{I\}$.

Now, let $\tilde{\pi}_{\mathbb{R}} = \langle T\rangle$. It is easy to see that for any $\phi \in \mathrm{Iso}(N)$, there is a uniquely well-defined group homomorphism $\Phi : \langle T\rangle \to \mathrm{Iso}(N)$ induced by $\Phi(T) = \phi$. It follows that $\tilde{\pi}_N = \langle\phi\rangle$ and $\pi_1(M) = \{(\Phi(T^n), T^n) : n \in \mathbb{Z}\} = \langle(\phi, T)\rangle$.

Now, we complete the proof. □

As an example, Bryant $\times\, \mathbb{S}^1$ is of Type **J**, whose universal cover is Bryant $\times\, \mathbb{R}$ and the fundamental group is $\langle(I, T)\rangle$ for some nontrivial translation T. Let $D_n := \langle r, s : r^n = s^2 = I,\ srs^{-1} = r^{-1}\rangle$ be the dihedral group of order $2n$ and $D_\infty := \langle r, s : s^2 = I,\ srs^{-1} = r^{-1}\rangle$ be the infinite dihedral group. And we will have the following classification.

Theorem 4. *Suppose (M, g, f) is a complete steady gradient Ricci soliton with nonnegative sectional curvature and at least one critical point. Suppose that its universal cover $(\tilde{M}, \tilde{g}) = (N, g_N) \times (\mathbb{R}, g_{Euclid})$, where N has positive Ricci curvature and its isometry group $\mathrm{Iso}(N) =$*

$O(2)$. *Let T denote some nontrivial translation of $\mathbb{R}$. Then $\pi_1(M)$ must be one of the types in Table 1. Here $\theta \in (0, 2\pi)$, $\alpha \in (0, \pi]$.*

Table 1. Topological classification when $\tilde{M}$ splits as $O(2)$-symmetric N and $\mathbb{R}$.

$\mathcal{T}$	$\pi_1(M)$	$\tilde{\pi}_N$	$\tilde{\pi}_{\mathbb{R}}$
$\mathbf{I}$	$\{I\}$	$\{I\}$	$\{I\}$
$\mathbf{T}$	$\langle (I, T) \rangle$	$\{I\}$	$\mathbb{Z}$
$\mathbf{T}^C_{R(\theta)}$	$\langle (R(\theta), T) \rangle, \theta/2\pi \in \mathbb{Q}$	$\mathbb{Z}_m$	$\mathbb{Z}$
$\mathbf{T}^I_{R(\theta)}$	$\langle (R(\theta), T) \rangle, \theta/2\pi \notin \mathbb{Q}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbf{T}^C_S$	$\langle (S(\alpha), T) \rangle$	$D_1 \cong \mathbb{Z}_2$	$\mathbb{Z}$

Proof. Based on the proof of Lemma 5, we only need to extend its argument and further refine Type **J**, knowing $\mathrm{Iso}(N) = O(2)$. Here, we use the notation in Lemma 5 still. It remains to determine the explicit form of group homomorphism Φ satisfying $\mathrm{Im}(\Phi) \subseteq O(2)$, as well as the corresponding fundamental group.

Let $\Phi(T)$ be a rotation $R(\theta)$, then $\pi_1 = \langle (R(\theta), T) \rangle$. In particular, $\tilde{\pi}_N$ is finite cyclic if $\theta/2\pi \in \mathbb{Q}$; or infinite cyclic if $\theta/2\pi \notin \mathbb{Q}$.

If $\Phi(T)$ is a reflection $S(\alpha)$, then $\pi_1 = \langle (S(\alpha), T) \rangle$ and $\tilde{\pi}_N \cong D_1$ holds. For different angles α_1, α_2, $\langle (S(\alpha_1), T) \rangle$ and $\langle (S(\alpha_2), T) \rangle$ is equivalent, up to coordinate change. Therefore, we denote this type by $\mathbf{T}^C_S$. In the remainder of this paper, we will not further explain the equivalence in similar cases below.

The existence of these types has been stated in Lemma 5. Then, we have completed all the classification. □

Remark 4. *Note that the group $\tilde{\pi}_N$ of Type $\mathbf{T}^C_{R(\pi)}$ is isomorphic to that of Type $\mathbf{T}^C_S$; that is, $\langle (R(\pi)) \rangle \cong \mathbb{Z}_2 \cong D_1 \cong \langle (S(\alpha)) \rangle$. However, the diffeomorphism of $\tilde{M}$, compatible with the fundamental groups of these types, does not necessarily exist. In fact, although these groups may be isomorphic as abstract groups, their induced group actions on $\tilde{M}$ often differ in invariants, such as orientation properties, preventing the existence of a diffeomorphism conjugating one group action to the other.*

3.2. When $\tilde{M} = N \times \mathbb{R}^2$

In this section, we shall continue to calculate the topological classification when $\tilde{M} = N \times \mathbb{R}^2$. Before that, recall the well-known classification of discrete, free, proper and isometric group action on $\mathbb{R}^2$ (see [28] for more details).

Lemma 6 ([28])**.** *Let Γ be a discrete subgroup of $\mathrm{Iso}(\mathbb{R}^2)$, acting freely, smoothly and properly on $\mathbb{R}^2$. Then, Γ can be one of the following classes:*

(1) *Γ is the trivial group $\{I\}$.*
(2) *$\Gamma \cong \mathbb{Z}$ and is generated by a nontrivial translation $T(x) = x + t$ for some fixed nonzero t in $\mathbb{R}^2$.*
(3) *$\Gamma \cong \mathbb{Z}$ and is generated by $Q(x) = Bx + b$, $x \in \mathbb{R}^2$, for some fixed $b \in \mathbb{R}^2$, $B \in O(2)$ (the same below), satisfying $B^2 = I \neq B$ and $B(b) = b \neq 0$.*
(4) *$\Gamma \cong \mathbb{Z} \times \mathbb{Z}$ and is generated by 2 linearly independent translations $T_1(x) = x + t_1$, $T_2(x) = x + t_2$, for some fixed $t_1, t_2 \in \mathbb{R}^2$.*
(5) *$\Gamma \cong \mathbb{Z} \rtimes \mathbb{Z}$ is generated by $W(x) = Bx + t_1$ and $T_2(x) = x + t_2$, satisfying $B^2 = I \neq B$, $B(t_1) = t_1 \neq 0$, $B(t_2) = -t_2$ and linearly independent $t_1, t_2 \in \mathbb{R}^2$.*

The projection $\tilde{\pi}_{\mathbb{R}^2}$ can be any group listed in Lemma 6. For Type (1) or Type (2), we have the following result in the same way as Theorem 4.

Lemma 7. *Suppose that (M, g, f) is a complete steady gradient Ricci soliton with nonnegative sectional curvature and at least one critical point. Suppose that its universal cover $(\tilde{M}, \tilde{g})$ can split as $(N, g_N) \times (\mathbb{R}^2, g_{Euclid})$, where N has positive Ricci curvature and $\mathrm{Iso}(N) = O(2)$.*

If $\tilde{\pi}_{\mathbb{R}^2}$ is of Type (1) or Type (2) in Lemma 6, then $\pi_1(M)$ can be classified in Table 2. Here $\theta \in (0, 2\pi)$ and $\alpha \in (0, \pi]$.

Table 2. Topological classification with respect to Type (1) and (2) in Lemma 6.

$\mathcal{T}$	$\pi_1(M)$	Condition of $\pi_1(M)$	$\tilde{\pi}_N$
$\mathbf{I}$	$\{I\}$		$\{I\}$
$\mathbf{T}$	$\langle (I, T) \rangle$		$\{I\}$
$\mathbf{T}^C_{R(\theta)}$	$\langle (R(\theta), T) \rangle$	$\theta/2\pi \in \mathbb{Q}$	$\mathbb{Z}_m$
$\mathbf{T}^I_{R(\theta)}$	$\langle (R(\theta), T) \rangle$	$\theta/2\pi \notin \mathbb{Q}$	$\mathbb{Z}$
$\mathbf{T}^C_S$	$\langle (S(\alpha), T) \rangle$		$D_1 \cong \mathbb{Z}_2$

Proof. Under Theorem 2, it suffices to find all possible group homomorphisms from $\tilde{\pi}_{\mathbb{R}^2}$ of Type (1) or (2) in Lemma 6 to some subgroup of $O(2)$, denoted by $\Phi : \tilde{\pi}_{\mathbb{R}^2} \to \mathrm{Iso}(N) = O(2)$. We will use the same approach as in Theorem 4 to determine Φ.

If $\tilde{\pi}_{\mathbb{R}^2}$ is the trivial group $\{I\}$, then $\pi_1(M)$ can only be $\{I\}$. Consider $\tilde{\pi}_{\mathbb{R}^2} = \langle T \rangle \cong \mathbb{Z}$, where T is a nontrivial translation of $\mathbb{R}$. If $\Phi(T)$ is a rotation $R(\theta)$, then we obtain Type $\mathbf{T}^C_{R(\theta)}$ and Type $\mathbf{T}^I_{R(\theta)}$. Type $\mathbf{T}^C_S$ can be obtained by mapping $\Phi(T)$ to a reflection $S(\alpha)$.

Hence, we complete the proof. □

Remark 5. *Notice that there are some types denoted by the same notation in Theorem 4 and Lemma 7, but their universal covers are different up to isometries; that is, Type $\mathbf{I}$, $\mathbf{T}^C_{R(\theta)}$, $\mathbf{T}^I_{R(\theta)}$ and $\mathbf{T}^C_S$. This is due to the fact that these types are the same in essence.*

Suppose $\tilde{M}_1 = N_1 \times \mathbb{R}$ and $\tilde{M}_2 = N_2 \times \mathbb{R}^2$ are both n-dimensional simply connected steady gradient Ricci solitons with nonnegative sectional curvature and critical points, satisfying $\mathrm{Ric}(N_1) > 0$ and $\mathrm{Ric}(N_2) > 0$, respectively. Let x_1 and x_2 be the critical points of N_1 and N_2, respectively. One can define a diffeomorphism $\varphi : \tilde{M}_2 \to \tilde{M}_1$ with $\varphi((x_2, 0, 0)) = (x_1, 0)$, which maps the translational component in $\mathbb{R}^2$ of $\tilde{M}_2$ into the $\mathbb{R}$-factor of $\tilde{M}_1$, and arranges the correspondence so that it is equivariant with respect to the $O(2)$-action. Then, it is easy to see that φ is compatible with the group actions denoted by the same notation in Theorem 4 and Lemma 7.

For Type (3), exactly as the discussion in Theorem 4, we obtain the following lemma:

Lemma 8. *Assume the same hypotheses as in Lemma 7. If $\tilde{\pi}_{\mathbb{R}^2}$ is of Type (3) in Lemma 6, then the topology of M can be classified in Table 3. Here $\theta \in (0, 2\pi)$ and $\alpha \in (0, \pi]$.*

Table 3. Topological classification with respect to Type (3) in Lemma 6.

$\mathcal{T}$	$\pi_1(M)$	Condition of $\pi_1(M)$	$\tilde{\pi}_N$
$\mathbf{S}$	$\langle (I, Q) \rangle$		$\{I\}$
$\mathbf{S}^C_{R(\theta)}$	$\langle (R(\theta), Q) \rangle$	$\theta/2\pi \in \mathbb{Q}$	$\mathbb{Z}_m$
$\mathbf{S}^I_{R(\theta)}$	$\langle (R(\theta), Q) \rangle$	$\theta/2\pi \notin \mathbb{Q}$	$\mathbb{Z}$
$\mathbf{S}^C_S$	$\langle (S(\alpha), Q) \rangle$		$D_1 \cong \mathbb{Z}_2$

The cases with respect to Type (4) and Type (5) will be more involved, since $\tilde{\pi}_{\mathbb{R}^2}$ is no longer generated by only one generator. Also, there is some additional relation between generators so that some restrictions arise.

Lemma 9. *Assume the same hypotheses as in Lemma 7. If $\tilde{\pi}_{\mathbb{R}^2}$ is of Type (4) in Lemma 6, then the topology of M can be classified in Table 4. Here $\theta_1, \theta_2 \in (0, 2\pi]$, $\alpha \in (0, \pi]$ and $(\theta_1, \theta_2) \neq (2\pi, 2\pi)$.*

Table 4. Topological classification with respect to Type (4) in Lemma 6.

$\mathcal{T}$	$\pi_1(M)$	Condition of $\pi_1(M)$	$\tilde{\pi}_N$
$\mathbf{X}_1$	$\langle (I, T_1), (I, T_2) \rangle$		$\{I\}$
$\mathbf{X}_2$	$\langle (R(\theta_1), T_1), (R(\theta_2), T_2) \rangle$	$\theta_1/2\pi, \theta_2/2\pi \in \mathbb{Q}$	$\mathbb{Z}_m$
$\mathbf{X}_3$		$\theta_1/2\pi \notin \mathbb{Q}, \theta_2/2\pi \in \mathbb{Q}$	$\mathbb{Z} \times \mathbb{Z}_m$
$\mathbf{X}_4$		$\theta_1/2\pi, \theta_2/2\pi \notin \mathbb{Q}, \theta_1/\theta_2 \in \mathbb{Q}$	$\mathbb{Z}$
$\mathbf{X}_5$		$\theta_1/2\pi, \theta_2/2\pi, \theta_1/\theta_2 \notin \mathbb{Q}$	$\mathbb{Z} \times \mathbb{Z}$
$\mathbf{X}_6$	$\langle (S(\alpha), T_1), (R(\theta), T_2) \rangle$	$\theta = 2\pi$	D_1
$\mathbf{X}_7$		$\theta = \pi$	D_2

Proof. Basically, $\tilde{\pi}_{\mathbb{R}^2} = \langle T_1, T_2 \rangle \cong \mathbb{Z} \times \mathbb{Z}$ for two linearly independent translations T_1, T_2 and $\tilde{\pi}_{\mathbb{R}^2}$ is abelian. Similarly to the argument in Lemma 7, we should identify the group homomorphism $\Phi : \langle T_1, T_2 \rangle \to O(2)$. It is natural to let T_1 and T_2 randomly map to a rotation or a reflection to receive all possibilities. We can easily verify that group homomorphism Φ is well-defined if and only if $\Phi(T_1)$ and $\Phi(T_2)$ commute. Therefore, it suffices to determine commutative $\Phi(T_1)$ and $\Phi(T_2)$ in $O(2)$.

Since $SO(2)$ is abelian, $\Phi(T_1)$ and $\Phi(T_2)$ can be $R(\theta_1)$ and $R(\theta_2)$ for any θ_1, θ_2.

If $\Phi(T_1) = S(\alpha)$ and $\Phi(T_2) = R(\theta)$, then $S(\alpha)R(\theta) = R(\theta)S(\alpha)$ must hold, which implies that $S(\alpha - \theta/2) = S(\alpha + \theta/2)$. It yields that $\theta = 2\pi, \pi$, which are of Type $\mathbf{X}_6$ and $\mathbf{X}_7$ respectively.

If $\Phi(T_1) = S(\alpha_1)$ and $\Phi(T_2) = S(\alpha_2)$, then $\pi_1(M) = \langle (S(\alpha_1), T_1), (S(\alpha_2), T_2) \rangle = \langle (S(\alpha_1), T_1), (R(2\alpha_1 - 2\alpha_2), T_1T_2) \rangle$, which is equivalent to the above situation. Therefore, $\pi_1(M)$ must be Type $\mathbf{X}_6$ or Type $\mathbf{X}_7$. Now, we complete the proof. □

When considering Type (5), the topology will be more diverse since the group generated by a reflection and a translation has a more delicate structure.

Lemma 10. *Assume the same hypotheses as in Lemma 7. If $\tilde{\pi}_{\mathbb{R}^2}$ is of Type (5) in Lemma 6, then the topology of M can be classified in Table 5. Here $\theta_1, \theta_2 \in (0, 2\pi]$, $\alpha, \alpha_1, \alpha_2 \in (0, \pi]$, $\beta = |\alpha_1 - \alpha_2|$ and $(\theta_1, \theta_2) \neq (2\pi, 2\pi)$.*

Table 5. Topological classification with respect to Type (5) in Lemma 6.

$\mathcal{T}$	$\pi_1(M)$	Condition of $\pi_1(M)$	$\tilde{\pi}_N$
$\mathbf{Y}_1$	$\langle (I, W), (I, T_2) \rangle$		$\{I\}$
$\mathbf{Y}_2$	$\langle (R(\theta_1), W), (R(\theta_2), T_2) \rangle$	$\theta_1/2\pi \in \mathbb{Q}, \theta_2 = \pi, 2\pi$	$\mathbb{Z}_m$
$\mathbf{Y}_3$		$\theta_1/2\pi \notin \mathbb{Q}, \theta_2 = 2\pi$	$\mathbb{Z}$
$\mathbf{Y}_4$		$\theta_1/2\pi \notin \mathbb{Q}, \theta_2 = \pi$	$\mathbb{Z} \times \mathbb{Z}_2$
$\mathbf{Y}_5$	$\langle (S(\alpha), W), (R(\theta), T_2) \rangle$	$\theta/2\pi \in \mathbb{Q}$	D_m
$\mathbf{Y}_6$		$\theta/2\pi \notin \mathbb{Q}$	D_∞
$\mathbf{Y}_7$	$\langle (R(\theta), W), (S(\alpha), T_2) \rangle$	$\theta = 2\pi$	D_1
$\mathbf{Y}_8$		$\theta = \pi$	D_2
$\mathbf{Y}_9$	$\langle (S(\alpha_1), W), (S(\alpha_2), T_2) \rangle$	$\beta = 0$	D_1
$\mathbf{Y}_{10}$		$\beta = \pi/2$	D_2

Proof. $\tilde{\pi}_{\mathbb{R}^2} \cong \mathbb{Z} \rtimes \mathbb{Z}$ is generated by $W(x) = Bx + t_1$ and $T_2(x) = x + t_2$, satisfying $B^2 = I \neq B$, $B(t_1) = t_1 \neq 0$, $B(t_2) = -t_2$ and linearly independent $t_1, t_2 \in \mathbb{R}^2$. Note

that W and T_2 satisfy $WT_2W^{-1}T_2 = I$. The group homomorphism $\Phi : \langle W, T_2\rangle \to O(2)$ is well-defined if and only if $\Phi(W)\Phi(T_2)(\Phi(W))^{-1}\Phi(T_2) = I$.

If $\Phi(W) = R(\theta_1)$ and $\Phi(T_2) = R(\theta_2)$, then $R(2\theta_2) = I$ must hold, which implies that θ_2 must be π or 2π. If $\theta_1/2\pi \in \mathbb{Q}$, then $\tilde{\pi}_N$ is finite cyclic; if $\theta_1/2\pi \notin \mathbb{Q}$ and $\theta_2 = 2\pi$, then $\tilde{\pi}_N \cong \mathbb{Z}$ is infinite cyclic; and if $\theta_1/2\pi \notin \mathbb{Q}$ and $\theta_2 = \pi$, then $\tilde{\pi}_N \cong \mathbb{Z} \times \mathbb{Z}_2$. They correspond to Type $\mathbf{Y}_2$, $\mathbf{Y}_3$ and $\mathbf{Y}_4$ respectively.

Recall that $S(\alpha)R(\theta)S(\alpha)R(\theta) = I$ for any θ, α. Therefore, we can let $\Phi(W) = S(\alpha)$ and $\Phi(T_2) = R(\theta)$ for any θ, α. It follows that $\langle S(\alpha), R(\theta)\rangle$ is the dihedral group.

Suppose that $\Phi(W) = R(\theta)$ and $\Phi(T_2) = S(\alpha)$. $\Phi(W)\Phi(T_2)(\Phi(W))^{-1}\Phi(T_2) = I$ implies that $S(\alpha)R(\theta) = R(\theta)S(\alpha)$ must hold, that is, $S(\alpha - \theta/2) = S(\alpha + \theta/2)$. It yields that $\theta = 2\pi, \pi$, which represent Type $\mathbf{Y}_7$ and $\mathbf{Y}_8$ respectively.

If $\Phi(W) = S(\alpha_1)$ and $\Phi(T_2) = S(\alpha_2)$, then we have $R(4\alpha_1 - 4\alpha_2) = I$, which implies that β must be $l\pi/2$, l integer. Hence, β can only be 0 or $\pi/2$. If $\beta = 0$, then $\langle S(\alpha_1), S(\alpha_2)\rangle \cong D_1$; otherwise $\langle S(\alpha_1), S(\alpha_2)\rangle \cong D_2$.

Now we complete the proof. □

In conclusion, we obtain the following theorem on the topological classification.

Theorem 5. *Suppose that (M, g, f) is a complete steady gradient Ricci soliton with nonnegative sectional curvature and at least one critical point. Suppose that its universal cover $(\tilde{M}, \tilde{g})$ can split as $(N, g_N) \times (\mathbb{R}^2, g_{Euclid})$, where N has positive Ricci curvature and* $\mathrm{Iso}(N) = O(2)$. *Then, $\pi_1(M)$ should be one of Type* $\mathbf{I}$, $\mathbf{T}$, $\mathbf{T}^C_{R(\theta)}$, $\mathbf{T}^I_{R(\theta)}$, $\mathbf{T}^C_S$, $\mathbf{S}$, $\mathbf{S}^C_{R(\theta)}$, $\mathbf{S}^I_{R(\theta)}$, $\mathbf{S}^C_S$, $\mathbf{X}_i$, $\mathbf{Y}_j$, *defined in this subsection above, where $i = 1, 2, 3, 4, 5, 6, 7$, and $j = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.*

3.3. Higher Dimensions

With current studies, the topological classification of the case where $\tilde{M} = N \times \mathbb{R}^k$ and $k \geq 3$ may not be determined or requires a large amount of computation. For completeness, we briefly remark on this point, although these classifications are not in the scope of this paper.

In this paper, the key to calculating the fundamental group of a positively curved steady gradient Ricci soliton with a critical point is understanding the discrete free and proper group action of $\mathrm{Iso}(\mathbb{R}^k)$ and ensuring the isometry group of N.

The former is equivalent to studying the topology of flat Riemannian manifolds. Chapter 3 in [28] gave a complete affine diffeomorphism classification of 3-dimensional Euclidean space forms, yielding a total of 18 types. In dimension 4, Levine [31] has used the computer to give 75 homomorphism equivalences of closed flat manifolds and there exists a duplication in the non-oriented equivalences, which were found in [32].

As for the latter, the geometry of steady gradient Ricci solitons in higher dimension has attracted attention and study in recent years; see [33–36] and references therein.

4. Four Dimensional Topological Classification

In this section, we will prove the main theorem. First, we give a complete topological classification of 3-dimensional steady gradient Ricci solitons.

Theorem 6. *Suppose that (M, g, f) is a complete 3-dimensional steady gradient Ricci soliton. Then the topological structure is one of the following:* $\mathbf{I}$, $\mathbf{T}$, $\mathbf{T}^C_{R(\theta)}$, $\mathbf{T}^C_S$ *and* $\mathbf{T}^I_{R(\theta)}$, *defined in Theorem 4. If the fundamental group is of Type* $\mathbf{I}$, *then M is flat,* Cigar $\times \mathbb{R}$ *or has positive sectional curvature.*

Proof. Theorem 5.6 in [37] states that any 3-dimensional complete steady gradient Ricci soliton (M, g, f) has one of the following properties: g is flat, g has positive sectional curvature, or its universal cover $\tilde{M} =$ Cigar $\times \mathbb{R}$. If g has positive sectional curvature, then

M is diffeomorphic to $\mathbb{R}^3$ by the work of Schoen and Yau in [38]. Consider $\tilde{M} = \text{Cigar} \times \mathbb{R}$, and the classification follows from Theorem 4.

Thus, we complete the proof. □

As the geometric classification in dimensions less than four and the existence of critical points have been essentially proved, we are in a position to study the topological classification in dimension 4, by considering Theorem 2.

Recall from [4] that any 3-dimensional positively curved steady gradient Ricci soliton must be $O(2)$-invariant and admit a critical point, whenever it is either the Bryant soliton or a flying wing.

However, these solitons may have symmetry stronger than $O(2)$-symmetry. In fact, the Bryant soliton is rotationally symmetric. Although flying wings are not $O(3)$-invariant, Lai [3] showed the existence of the reflection on the complement of $O(2)$-orbits ($O(2) \times \mathbb{Z}_2$-symmetry) and constructed a family of such solitons.

For clarity, we denote $O(2) \times \mathbb{Z}_2$-action by $\{R_{(\gamma,p)},\ S_{(\psi,q)} : \psi \in (0,\pi],\ \gamma \in (0,2\pi],\ p,q \in \{I,-I\}\}$, where

$$R_{(\gamma,p)} := \begin{pmatrix} R(\gamma) & 0 \\ 0 & p \end{pmatrix}, \text{ and } S_{(\psi,q)} := \begin{pmatrix} S(\alpha) & 0 \\ 0 & q \end{pmatrix}. \tag{3}$$

Now, we prove the main theorem.

Theorem 7. *Suppose that (M,g,f) is a complete 4-dimensional steady gradient Ricci soliton with nonnegative sectional curvature. Then the topological structure must be one of Type* **I**, **T**, **S**, $\mathbf{S}^C_{R(\theta)}$, $\mathbf{S}^I_{R(\theta)}$, $\mathbf{S}^C_S$, $\mathbf{X}_i$ *($i = 1,2,3,4,5,6,7$),* $\mathbf{Y}_j$ *($j = 1,2,3,4,5,6,7,8,9,10$), defined in Section 3.2 and the following types. Here, $R_v(\theta) \in SO(3)$ denotes the rotation of the rotation axis $v \in \mathbb{R}^3$ and angle $\theta \in (0,2\pi)$; $A \in O(3)$, $\det(A) = -1$; $(\gamma,p) \neq (2\pi,I)$; T denotes some nontrivial translation on the Euclidean factor manifold.*

Type	$\pi_1(M)$	$\tilde{\pi}_N$	$\tilde{\pi}_{\mathbb{R}}$
$\mathbf{Z}^C_{R_{(\gamma,p)}}$	$\langle (R_{(\gamma,p)},T)\rangle$, $\gamma/2\pi \in \mathbb{Q}$	$\mathbb{Z}_m$	$\mathbb{Z}$
$\mathbf{Z}^I_{R_{(\gamma,p)}}$	$\langle (R_{(\gamma,p)},T)\rangle$, $\gamma/2\pi \notin \mathbb{Q}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbf{Z}^C_{S_{(\cdot,q)}}$	$\langle (S_{(\psi,q)},T)\rangle$	D_1	$\mathbb{Z}$
$\mathbf{O}^C_{R_v(\theta)}$	$\langle (R_v(\theta),T)\rangle$, $\theta/2\pi \in \mathbb{Q}$	$\mathbb{Z}_m$	$\mathbb{Z}$
$\mathbf{O}^I_{R_v(\theta)}$	$\langle (R_v(\theta),T)\rangle$, $\theta/2\pi \notin \mathbb{Q}$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbf{O}^C_A$	$\langle (A,T)\rangle$	D_1	$\mathbb{Z}$

Proof. From Lemma 1, the universal cover $\tilde{M}$ of M can split as N and $\mathbb{R}^k$. Here, $\text{Ric}_N > 0$ and $k = 0,1,2,4$. If $k = 4$, then M is flat.

Consider the case where $k = 0$. Then M is a complete 4-dimensional steady gradient Ricci soliton with nonnegative sectional curvature and positive Ricci curvature. If the critical point exists, Proposition 2.3 in [15] implies that M is diffeomorphic to $\mathbb{R}^4$. We now focus on the case where M admits no critical point and then the potential function f is an open map. Directly, we obtain that $f(M)$ is a connected open subset of $\mathbb{R}$, say $f(M) = (t_1,t_2)$ with $-\infty \leq t_1 < t_2 \leq +\infty$. In this case, the level set of f must be noncompact. Otherwise, M is diffeomorphic to $(t_1,t_2) \times M'$ and has two ends, which contradicts Theorem 4.2 in [39]. Here, M' is any level set of the potential function f and is proved to be a 3-dimensional manifold with positive Ricci curvature in the argument in Lemma 2.1 of [35]. Hence, the

level set is diffeomorphic to $\mathbb{R}^3$ and then M is diffeomorphic to $\mathbb{R}^4$. Thus, we can conclude that M is diffeomorphic to $\mathbb{R}^4$ when $k = 0$.

For $k = 2$ (i.e. $\tilde{M} = \text{Cigar} \times \mathbb{R}^2$), Theorem 5 gives a complete classification of all possible $\pi_1(M)$.

If $k = 1$, then $\tilde{M} = N \times \mathbb{R}$, and N is either a flying wing or a Bryant soliton. Lemma 5 shows that $\tilde{\pi}_{\mathbb{R}}$ can only be $\{I\}$ or $\langle T \rangle$, where T is a nontrivial translation in $\mathbb{R}$.

Let (N, g_N, f_N) be a flying wing. Then by definition, it is a complete steady gradient Ricci soliton asymptotic, in the pointed Gromov-Hausdorff sense to the sector with angle $\nu \in (0, \pi)$. Lemma 2.1 in [40] showed that the level set of f_N is diffeomorphic to $\mathbb{S}^2$ and every isometry of N acts on the level set. Then, the isometry group of flying wings can only be $O(2)$ or $O(2) \times \mathbb{Z}_2$. If $\text{Iso}(N) = O(2)$, then all the types of quotients can be determined by Lemma 4: Type **I**, **T**, $\mathbf{T}^C_{R(\theta)}$, $\mathbf{T}^I_{R(\theta)}$ and $\mathbf{T}^C_S$. Otherwise, if its isometry group is $O(2) \times \mathbb{Z}_2$, then more types of $\pi_1(M)$ occur. Except for Type **I** and **T**, we have Type $\mathbf{Z}^C_{R_{(\gamma,p)}}$, $\mathbf{Z}^I_{R_{(\gamma,p)}}$ and $\mathbf{Z}^C_{S_{(\cdot,q)}}$, using the same method as that in Theorem 4. Moreover, It is easy to verify that Type $\mathbf{Z}^C_{R_{(\gamma,p)}}$, $\mathbf{Z}^I_{R_{(\gamma,p)}}$ and $\mathbf{Z}^C_{S_{(\cdot,q)}}$ contain Type $\mathbf{T}^C_{R(\theta)}$, $\mathbf{T}^I_{R(\theta)}$ and $\mathbf{T}^C_S$, respectively.

Consider $\tilde{M} = \text{Bryant} \times \mathbb{R}$. If $\tilde{\pi}_{\mathbb{R}} = \{I\}$, then $\pi_1(M)$ must be Type **I**. Otherwise, $\tilde{\pi}_{\mathbb{R}} = \langle T \rangle$. Let $\Phi : \langle T \rangle \to O(3)$ is a group homomorphism with respect to some fundamental group. There is no restriction on $\Phi(T)$. Assume $\Phi(T) = R_v(\theta)$, and then $\pi_1(M) = \langle (R_v(\theta), T) \rangle$. If $\theta/2\pi \in \mathbb{Q}$, $\tilde{\pi}_N \cong \mathbb{Z}_m$; otherwise, $\tilde{\pi}_N \cong \mathbb{Z}$, denoted by Type $\mathbf{O}^C_{R_v(\theta)}$ and $\mathbf{O}^I_{R_v(\theta)}$, respectively. If $\Phi(T) = A \in O(3)$ with $\det(A) = -1$, then $\pi_1(M) = \langle (A, T) \rangle$ exits and $\tilde{\pi}_N \cong D_1$. Hence, we complete the proof. □

Remark 6. *It is still open that whether all nonrotationally symmetric 3-dimensional steady gradient Ricci soliton with positive sectional curvature should be $O(2) \times \mathbb{Z}_2$-symmetric.*

5. Conclusions

As shown in [20], the fundamental group of a steady gradient Ricci soliton with nonnegative sectional curvature has been proved to be either trivial or infinite. If we additionally assume that it is noncollapsed, the fundamental group must be trivial. However, the topology of steady gradient Ricci solitons in the infinite case is still not well understood.

In this paper, we mainly focus on the infinite case and give a classification of the topology of complete 4-dimensional steady gradient Ricci solitons with nonnegative sectional curvature.

The whole argument in the paper starts with Lemma 1, which shows that for gradient Ricci solitons, the nonnegative sectional curvature condition ensures the splitting property of universal covers. If the curvature condition is weakened, the splitting structure is unclear.

The key result in this paper is the characterization theorem for the fundamental group of a positively curved steady gradient Ricci soliton that admits a critical point; see Theorem 2. This is achieved by a core idea, which is to characterize the fundamental group by using the existence of the critical point and analyzing the isometry subgroup (more precisely, the deck transformation group) of the universal cover.

Note that the existence of critical points and the symmetry of universal covers are essential in the proofs; see Remark 2. For any steady gradient Ricci soliton with positive Ricci curvature, the existence of the critical point alone implies that the isometry group of its universal cover has a unique fixed point. This is the only extra assumption for the potential function required in our proof and is a crucial technique in this paper. However, it remains unknown whether any n-dimensional steady gradient Ricci soliton with nonnegative sectional curvature, positive Ricci curvature and $n \geq 4$ admits a critical point. This is a

widely open problem and future study may focus on the topological classification without assuming this condition.

As for the symmetry of universal covers, we essentially require a complete classification and isometry groups of complete simply connected manifolds satisfying certain conditions. In particular, this paper needs the complete classification of complete steady gradient Ricci solitons with nonnegative sectional curvature and positive Ricci curvature in dimension $n \leq 3$. Now, the known results only show that complete 4-dimensional steady gradient Ricci solitons with positive Ricci curvature are diffeomorphic to $\mathbb{R}^4$, see [20]. Replacing positive Ricci curvature by nonnegative isotropic curvature, Theorem 1.2 in [18] gives a classification in dimension 4. In case (i), the solitons have only 2-positive Ricci curvature, which is weaker than positive Ricci curvature, and we are currently unable to conclude that they are diffeomorphic to the Euclidean space. However, when the soliton is Kähler, 2-positive Ricci curvature is equivalent to positive Ricci curvature, and hence the soliton is diffeomorphic to the Euclidean space.

Moreover, the core idea mentioned above is not restricted to steady solitons and can also be applied to shrinking gradient Ricci solitons with nonnegative curvature operators. This extension is mainly supported by the work of [41], which has given a complete classification of complete simply connected shrinking gradient Ricci solitons with nonnegative curvature operator. However, for the case of nonnegative sectional curvature, the classification remains unknown.

Funding: This research was supported by NSFC 12571059.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Hamilton, R.S. The Ricci Flow on Surfaces. In *Mathematics and General Relativity (Santa Cruz, CA, 1986)*; American Mathematical Society: Providence, RI, USA, 1988; Volume 71, pp. 237–262. [CrossRef]
2. Bryant, R.L. Ricci Flow Solitons in Dimension Three with SO(3)-Symmetries. Available online: https://services.math.duke.edu/~bryant/3DRotSymRicciSolitons.pdf (accessed on 3 December 2006).
3. Lai, Y. A family of 3D steady gradient solitons that are flying wings. *J. Differ. Geom.* **2024**, *126*, 297–328. [CrossRef]
4. Lai, Y. $O(2)$-symmetry of 3D steady gradient Ricci solitons. *Geom. Topol.* **2025**, *29*, 687–789. [CrossRef]
5. Buttsworth, T. $SU(2)$-invariant steady gradient Ricci solitons on four-manifolds. *Comm. Anal. Geom.* **2025**, *33*, 365–401. [CrossRef]
6. Buzano, M.; Dancer, A.S.; Gallaugher, M.P.B.; Wang, M.Y. A family of steady Ricci solitons and Ricci flat metrics. *Comm. Anal. Geom.* **2015**, *23*, 611–638. [CrossRef]
7. Conlon, R.J.; Deruelle, A. Steady gradient Kähler-Ricci solitons on crepant resolutions of Calabi-Yau cones. *Mem. Am. Math. Soc.* **2025**, *307*, v+119. [CrossRef]
8. Wink, M. Cohomogeneity one Ricci solitons from Hopf fibrations. *Comm. Anal. Geom.* **2023**, *31*, 625–676. [CrossRef]
9. Appleton, A. A family of non-collapsed steady Ricci solitons in even dimensions greater or equal to four. *arXiv* **2022**, arXiv:1708.00161. [CrossRef]
10. Dancer, A.S.; Wang, M.Y. Some new examples of non-Kähler Ricci solitons. *Math. Res. Lett.* **2009**, *16*, 349–363. [CrossRef]
11. Dancer, A.S.; Wang, M.Y. On Ricci solitons of cohomogeneity one. *Ann. Glob. Anal. Geom.* **2008**, *39*, 259–292. [CrossRef]
12. Chan, P.Y.; Conlon, R.J.; Lai, Y. A family of Kähler flying wing steady Ricci solitons. *arXiv* **2024**, arXiv:2403.04089. [CrossRef]
13. Brendle, S. Rotational symmetry of self-similar solutions to the Ricci flow. *Invent. Math.* **2013**, *194*, 731–764. [CrossRef]
14. Brendle, S. Rotational symmetry of Ricci solitons in higher dimensions. *J. Differ. Geom.* **2014**, *97*, 191–214. [CrossRef]
15. Cao, H.D.; Chen, Q. On locally conformally flat gradient steady Ricci solitons. *Trans. Am. Math. Soc.* **2012**, *364*, 2377–2391. [CrossRef]
16. Deng, Y.; Zhu, X. Higher dimensional steady Ricci solitons with linear curvature decay. *J. Eur. Math. Soc.* **2020**, *22*, 4097–4120. [CrossRef]
17. Deng, Y.; Zhu, X. Classification of gradient steady Ricci solitons with linear curvature decay. *Sci. China Math.* **2020**, *63*, 135–154. [CrossRef]

18. Cao, H.D.; Xie, J. Four-dimensional gradient Ricci solitons with (half) nonnegative isotropic curvature. *J. Math. Pures Appl.* **2025**, *197*, 103686. [CrossRef]
19. Deruelle, A. Steady gradient Ricci soliton with curvature in L^1. *Comm. Anal. Geom.* **2012**, *20*, 31–53. [CrossRef]
20. Deng, Y.; Hao, Y. On the fundamental group of steady gradient Ricci solitons with nonnegative sectional curvature. *arXiv* **2025**, arXiv:2412.07452.
21. Cao, H.D.; Catino, G.; Chen, Q.; Mantegazza, C.; Mazzieri, L. Bach-flat gradient steady Ricci solitons. *Calc. Var. Partial Differ. Equations* **2014**, *49*, 125–138. [CrossRef]
22. Wylie, W. Complete shrinking Ricci solitons have finite fundamental group. *Proc. Am. Math. Soc.* **2008**, *136*, 1803–1806. [CrossRef]
23. Guan, P.; Lu, P.; Xu, Y. A rigidity theorem for codimension one shrinking gradient Ricci solitons in $\mathbb{R}^{n+1}$. *Calc. Var. Partial. Differ. Equ.* **2015**, *54*, 4019–4036. [CrossRef]
24. Cheeger, J.; Gromoll, D. On the Structure of Complete Manifolds of Nonnegative Curvature. *Ann. Math.* **1972**, *96*, 413–443. [CrossRef]
25. Lee, J.M. *Introduction to Smooth Manifolds*, 2nd ed.; Graduate Texts in Mathematics; Springer: New York, NY, USA, 2013; Volume 218, p. xvi+708.
26. Lee, J.M. *Introduction to Riemannian Manifolds*, 2nd ed.; Graduate Texts in Mathematics; Springer: Cham, Switzerland, 2018; Volume 176, p. xiii+437.
27. Petersen, P.; Wylie, W. On gradient Ricci solitons with symmetry. *Proc. Am. Math. Soc.* **2009**, *137*, 2085–2092. [CrossRef]
28. Wolf, J.A. *Spaces of Constant Curvature*, 6th ed.; AMS Chelsea Publishing: Providence, RI, USA, 2011; p. xviii+424. [CrossRef]
29. Gromov, M. Almost flat manifolds. *J. Differ. Geom.* **1978**, *13*, 231–241. [CrossRef]
30. Petersen, P. *Riemannian Geometry*, 2nd ed.; Graduate Texts in Mathematics; Springer: New York, NY, USA, 2006; Volume 171, p. xvi+401.
31. Levine, R.D. *The Compact Euclidean Space Forms of Dimension Four*; ProQuest LLC: Ann Arbor, MI, USA, 1970; p. 121.
32. Lambert, T.P. *On the Classification of Closed Flat Four-Manifolds*; ProQuest LLC: Ann Arbor, MI, USA, 2007; p. 76.
33. Bamler, R.H.; Chow, B.; Deng, Y.; Ma, Z.; Zhang, Y. Four-dimensional steady gradient Ricci solitons with 3-cylindrical tangent flows at infinity. *Adv. Math.* **2022**, *401*, 108285. [CrossRef]
34. Chow, B.; Deng, Y.; Ma, Z. On four-dimensional steady gradient Ricci solitons that dimension reduce. *Adv. Math.* **2022**, *403*, 108367. [CrossRef]
35. Chan, P.Y.; Ma, Z.; Zhang, Y. Dimension reduction for positively curved steady solitons. *Ann. Global Anal. Geom.* **2025**, *67*, 20. [CrossRef]
36. Ma, Z.; Mahmoudian, H.; Sesum, N. Unique Asymptotics of Steady Ricci Solitons with Symmetry. *arXiv* **2023**, arXiv:2311.09405. [CrossRef]
37. Chow, B. *Ricci Solitons in Low Dimensions*; Graduate Studies in Mathematics; American Mathematical Society: Providence, RI, USA, 2023; Volume 235, p. xvi+339. [CrossRef]
38. Schoen, R.; Yau, S.T. Complete Three-Dimensional Manifolds with Positive Ricci Curvature and Scalar Curvature. In *Seminar on Differential Geometry*; Princeton Univ. Press: Princeton, NJ, USA, 1982; Volume 102, pp. 209–228.
39. Munteanu, O.; Wang, J. Smooth metric measure spaces with non-negative curvature. *Comm. Anal. Geom.* **2011**, *19*, 451–486. [CrossRef]
40. Deng, Y.; Zhu, X. Steady Ricci solitons with horizontally ϵ-pinched Ricci curvature. *Sci. China Math.* **2021**, *64*, 1411–1428. [CrossRef]
41. Munteanu, O.; Wang, J. Positively curved shrinking Ricci solitons are compact. *J. Differ. Geom.* **2017**, *106*, 499–505. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.