

## Article

# On Approximate Solutions for Nonsmooth Interval-Valued Multiobjective Optimization Problems with Vanishing Constraints

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## Abstract

The purpose of this research is to develop approximate weak and strong stationary conditions for interval-valued multiobjective optimization problems with vanishing constraints (IVMOPVC) involving nonsmooth functions. In many real-world situations, the exact values of objectives are uncertain or imprecise; hence, interval-valued formulations are used to model such uncertainty more effectively. The proposed approximate weak and strong stationarity conditions provide a robust framework for deriving meaningful optimality results even when the usual constraint and data qualifications fail. We first introduce approximate variants of these qualifications and establish their relationships. Secondly, we establish some approximate KKT type necessary optimality conditions in terms of approximate weak strongly stationary points and approximate strong strongly stationary points to identify *type-2*  $\mathcal{E}$ -quasi weakly Pareto and *type-1*  $\mathcal{E}$ -quasi Pareto solutions of the IVMOPVC. Lastly, we show that the approximate weak and strong strongly stationary conditions are sufficient for optimality under some approximate convexity assumptions. All the outcomes are well illustrated by examples.

**Keywords:** interval-valued multiobjective optimization; vanishing constraints; approximate constraint qualifications; approximate stationary conditions; approximate efficient solutions

**MSC:** 90C29; 90C46; 90C70; 58E17; 49J52



Academic Editors: Cristina Flaut, Cristina Sburlan and Dragos Sburlan

Received: 8 October 2025

Revised: 12 November 2025

Accepted: 14 November 2025

Published: 18 November 2025

**Citation:** Dwivedi, A.; Laha, V.; Beldiman, M.-M.; Halanay, A.-D. On Approximate Solutions for Nonsmooth Interval-Valued Multiobjective Optimization Problems with Vanishing Constraints. *Mathematics* **2025**, *13*, 3699. <https://doi.org/10.3390/math13223699>

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## 1. Introduction

The idea of mathematical programs with vanishing constraints (MPVC), was first introduced by Achtziger and Kanzow [1], which is an extension of mathematical programming with equilibrium constraints (MPEC); see, e.g., [2–5]. However, under standard constraint qualifications, these models fall short of meeting optimality standards. Consequently, Achtziger and Kanzow [1] proposed a modified CQ and established related optimality results. Moreover, Hoheisel and Kanzow [6] investigated different CQs for a subset of MPVC. These types of problems constitute a novel class of optimization problems that have significant applications in the field of topology design for mechanical structures. Izmailov and Pogosyan [7] established standard optimality conditions for an optimization problem involving constraints that become negligible or vanish under certain conditions.

Additionally, they proposed a solution method based on Newton's approach for solving such optimization problems with vanishing constraints. Inspired by the work of Izmailov and Pogosyan [7], Dorsch et al. [8] formulated a critical point theory for mathematical models subject to vanishing constraints. Achtziger et al. [9] suggested a regularization method to solve smooth optimization problems involving vanishing constraints. Many researchers have subsequently investigated the dual models corresponding to MPVCs; for more detailed information on these works, see [10–12]. The mathematical concept of MPVC has become quite popular and attracted significant interest recently. Numerous authors have studied and worked on MPVC models. Initially, the functions involved in MPVC are considered to be continuously differentiable, see e.g., [1,6,12–14]. Recently, there is research focused on nonsmooth or non-differentiable cases of MPVC, which is covered in [15–17].

Interval-valued programming is a type of optimization problem where some or all of the parameters are intervals or ranges rather than precise values. In these problems, the goal is to find the optimal solution that satisfies the constraints and optimizes the objective function over the given intervals. Solving interval-valued programming problems typically involves techniques from interval analysis, which deals with the arithmetic operations and computations involving intervals. Different approaches have been developed, including: interval-valued linear programming, interval-valued non-linear programming. The main challenge in interval-valued programming lies in the presence of interval parameters, which can lead to an infinite number of possible scenarios within the feasible region. As a result, the solution methods aim to find solutions that are optimal or near optimal for all possible realizations of the intervals. Wu [18] developed two solution concepts by considering two partial orderings on the set of all closed intervals. The development of these solution concepts resulted in the establishment of the Karush-Kuhn-Tucker (KKT) conditions for a differentiable interval-valued scalar optimization problem. In order to establish duality results, Wu [19] formulated four different types of interval-valued optimization problems and derived the corresponding KKT optimality conditions for each type. Various researchers [20–27] contributed to developing formulations, deriving optimality conditions, and exploring interval-valued optimization problems, particularly related to Lagrangian duality, KKT conditions, and multiobjective optimization.

Rani et al. [28] derived optimality conditions and duality results for multiobjective interval-valued programming problems involving vanishing constraints. Yadav and Gupta [29] established the necessary and sufficient conditions, under certain assumptions of constraint qualifications and convexity, for solving a multiobjective interval-valued semi-infinite optimization problem with vanishing constraints. They also formulated Wolfe's and Mond-Weir type dual models for such problems. Van Su and Hang [30] studied nonsmooth semi-infinite interval-valued multiobjective programming problems with vanishing constraints and derived Karush-Kuhn-Tucker (KKT) type necessary and sufficient optimality conditions and duality results based on Hadamard derivatives. Subsequently, Upadhyay et al. [31] developed a Newton-type algorithm for interval-valued multiobjective programming problems. Recent research efforts in the area of multiobjective mathematical programming problems involving interval-valued functions and vanishing constraint Ahmad et al. [32], Joshi et al. [33,34] have significantly broadened and advanced this field of study.

The natural intersection of these two areas interval-valued optimization and vanishing constraints has recently attracted attention. Despite this progress, one critical gap remains unaddressed: the role of approximation in deriving strongly stationary conditions for IVMOPVC. Existing studies largely focus on exact solutions, which, while mathematically rigorous, often fail to capture the approximate nature of real-world decision-making, where data uncertainty and computational limitations make exact solutions impractical. This

paper aims to bridge that gap. Motivated by the aforementioned limitations, we study approximate strong stationarity conditions for (IVMOPVC) within the framework of the Clarke subdifferential. The key contributions of our work are as follows:

- **Approximate Strong Stationarity:** We formulate necessary conditions that remain valid when the solution obtained is approximate rather than exact, thereby broadening the applicability of MPVC theory to more realistic settings.
- **New Constraint and Data Qualifications:** We introduce several approximate constraint and data qualifications specifically designed for IVMOPVC and examine their interconnections.
- **Theoretical Advancement with Approximation:** By embedding approximation into the stationarity framework, we ensure that the results are both mathematically rigorous and adaptable to uncertain optimization contexts.
- **Sufficient Conditions:** In addition to necessity, we establish conditions under which approximate strong stationarity leads to approximate optimality, thereby strengthening the practical relevance of the framework.

In essence, our work takes IVMOPVC beyond exact mathematical theory and develops approximation-based results that are both rigorous and practical. This innovation opens new avenues for solving complex interval-valued multiobjective optimization problems with vanishing constraints, especially in applications where exact solutions are unattainable but high-quality approximations are essential.

This paper is organized as follows: in Section 2, the essential definitions and preliminary results are provided that will serve as the foundation for the studies and findings given in this paper. Section 3 presents the main results of this paper. It introduces several approximate constraint qualifications and data qualifications for the interval-valued multiobjective optimization problem with vanishing constraints (IVMOPVC) and explores the relationships among them. Utilizing these constraint qualifications, the section establishes approximate necessary strongly stationary conditions for the IVMOPVC, with the Clarke subdifferential serving as the primary tool. Section 4 explores the conditions under which the necessary strongly stationary conditions derived in the previous section become sufficient for obtaining an approximate solution to the IVMOPVC. We conclude our paper in Section 5. The abbreviations used in this paper are listed in the Abbreviations section at the end of the paper.

## 2. Preliminaries

In this section, we recall several definitions and preliminary results that will be used throughout the paper.

Let  $\mathbb{R}^n$  be the  $n$ -dimensional Euclidean space. For any two vectors  $\check{x}, z \in \mathbb{R}^n$ , we write  $\check{x} < z$  (or  $\check{x} \leq z$ ) when  $\check{x}_i < z_i$  (or  $\check{x}_i \leq z_i$  and  $\check{x} \neq z$ ) for all  $i = 1, \dots, n$ . The zero vector in  $\mathbb{R}^n$  is represented by  $0_{\mathbb{R}^n}$ .

Let  $\Omega$  be a nonempty subset of  $\mathbb{R}^n$ , the *polar cone* of  $\Omega$ , the *strict polar cone* of  $\Omega$ , and the *orthogonal set* to  $\Omega$  are denoted by  $\Omega^-$ ,  $\Omega^s$  and  $\Omega^\perp$ , respectively, and are defined as:

$$\Omega^- := \{\check{x} \in \mathbb{R}^n \mid \langle \check{x}, \omega \rangle \leq 0, \forall \omega \in \Omega\},$$

$$\Omega^s := \{\check{x} \in \mathbb{R}^n \mid \langle \check{x}, \omega \rangle < 0, \forall \omega \in \Omega\},$$

$$\Omega^\perp := \{\check{x} \in \mathbb{R}^n \mid \langle \check{x}, \omega \rangle = 0, \forall \omega \in \Omega\},$$

where  $\langle \cdot, \cdot \rangle$  indicates the standard inner product in  $\mathbb{R}^n$ .

We recall that in  $\mathbb{R}^n$ , a polyhedral convex set is defined as the intersection of a finite collection of closed half-spaces. Additionally, a bounded convex set that is generated by a finite number of elements is referred to as a polytope.

The closure of set  $\Omega$ , its convex cone (containing the origin), the convex hull, and the closed convex cone generated by  $\Omega$  are denoted by  $\bar{\Omega}$ ,  $\text{cone}(\Omega)$ ,  $\text{conv}(\Omega)$ , and  $\overline{\text{cone}}(\Omega)$ , respectively.

Also, the *contigent cone* to  $\Omega$  at  $\zeta \in \bar{\Omega}$  is denoted by

$$\Gamma(\Omega, \zeta) := \{d \in \mathbb{R}^n \mid \exists \{(t_k, d_k)\} \rightarrow (0^+, d) : \zeta + t_k d_k \in \Omega, \forall k \in \mathbb{N}\}.$$

Notice that  $\Gamma(\Omega, \zeta)$  is a closed cone (generally nonconvex) in  $\mathbb{R}^n$ .

**Definition 1** ([35]). Let  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  be a locally Lipschitz function at  $\zeta \in \mathbb{R}^n$ . The Clarke directional derivative of  $h$  at  $\zeta$  in a direction  $v \in \mathbb{R}^n$  is defined by

$$h^\circ(\zeta; v) := \limsup_{z \rightarrow \zeta, t \downarrow 0} \frac{h(z + tv) - h(z)}{t}$$

and the Clarke subdifferential of  $h$  at  $\zeta$  is defined by

$$\partial^\circ h(\zeta) := \{\zeta^* \in \mathbb{R}^n \mid \langle \zeta^*, v \rangle \leq h^\circ(\zeta; v), \forall v \in \mathbb{R}^n\}.$$

**Remark 1.** If the function  $h$  is continuously differentiable at  $\zeta$ , then  $\partial^\circ h(\zeta) = \{\nabla h(\zeta)\}$ . Moreover, if the function  $h$  is convex, then the Clarke subdifferential  $\partial^\circ h(\zeta)$  coincides with the subdifferential  $\partial h(\zeta)$  in the sense of convex analysis given by

$$\partial h(\zeta) := \{\zeta^* \in \mathbb{R}^n \mid h(\check{x}) \geq h(\zeta) + \langle \zeta^*, \check{x} - \zeta \rangle, \forall \check{x} \in \mathbb{R}^n\}.$$

The proposition below highlights important aspects of the Clarke directional derivative and subdifferential from [35], which we will often use ahead.

**Proposition 1** ([35]). Let  $h_1, h_2 : \mathbb{R}^n \rightarrow \mathbb{R}$  be functions that are Lipschitz near  $\zeta \in \mathbb{R}^n$ . Then, the following assertions hold:

- (a)  $h^\circ(\zeta; v) = \max\{\langle \zeta^*, v \rangle \mid \zeta^* \in \partial^\circ h(\zeta)\}$  for any  $v \in \mathbb{R}^n$ ;
- (b)  $\partial^\circ(\alpha_1 h_1 + \alpha_2 h_2)(\zeta) \subseteq \alpha_1 \partial^\circ h_1(\zeta) + \alpha_2 \partial^\circ h_2(\zeta), \forall \alpha_1, \alpha_2 \in \mathbb{R}$ ;
- (c)  $\partial^\circ(h_1 h_2)(\zeta) \subseteq h_1(\zeta) \partial^\circ h_2(\zeta) + h_2(\zeta) \partial^\circ h_1(\zeta)$ .

**Definition 2** (Definition 2.6, [36]). Let  $K \subseteq \mathbb{R}^n$  be an open convex set. A locally Lipschitz function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be pseudoconvex at  $\zeta$  over  $K \subseteq \mathbb{R}^n$  iff for each  $\check{x} \in K$  one has

$$h(\check{x}) < h(\zeta) \implies \langle \zeta^*, \check{x} - \zeta \rangle < 0, \forall \zeta^* \in \partial^\circ h(\zeta)$$

or equivalently

$$\exists \zeta^* \in \partial^\circ h(\zeta) : \langle \zeta^*, \check{x} - \zeta \rangle \geq 0 \implies h(\check{x}) \geq h(\zeta)$$

The function  $h$  is said to be pseudoconcave at  $\zeta$  over  $K$ , iff  $-h$  is pseudoconvex at  $\zeta$  over  $K$ . The function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be pseudolinear at  $\zeta$  over  $K \subseteq \mathbb{R}^n$ , iff  $h$  is both pseudoconvex and pseudoconcave at  $\zeta$  over  $K$ .

Now, consider a multiobjective optimization problem with vanishing constraints as follows:

$$\begin{aligned} \min f(\tilde{x}) &:= (f_1(\tilde{x}), \dots, f_p(\tilde{x})) \\ \text{s.t. } \mathcal{H}_i(\tilde{x}) &\geq 0, \quad \forall i \in \mathcal{M}, \\ \mathcal{H}_i(\tilde{x})\mathcal{G}_i(\tilde{x}) &\leq 0, \quad \forall i \in \mathcal{M}, \end{aligned} \quad (\text{MOPVC})$$

where the functions  $f_j$ ,  $\mathcal{H}_i$  and  $\mathcal{G}_i$  are locally Lipschitz from  $\mathbb{R}^n$  to  $\mathbb{R}$  for all  $i \in \mathcal{M} := \{1, \dots, m\}$  and  $j \in \mathcal{P} := \{1, \dots, p\}$ . The feasible set  $\mathcal{S}$  of the problem MOPVC is defined as

$$\mathcal{S} := \{\tilde{x} \in \mathbb{R}^n \mid \mathcal{H}_i(\tilde{x}) \geq 0, \mathcal{H}_i(\tilde{x})\mathcal{G}_i(\tilde{x}) \leq 0, i \in \mathcal{M}\} \neq \emptyset.$$

A point  $\zeta \in \mathcal{S}$  is said to be an efficient solution (or weakly efficient solution) for MOPVC iff there exists no  $\tilde{x} \in \mathcal{S}$  satisfying

$$f(\tilde{x}) \leq f(\zeta) \text{ (or } f(\tilde{x}) < f(\zeta)).$$

We borrow the following symbols from [15] to represent the whole of this article.

The index set  $\mathcal{M}$  can be divided into  $\mathcal{M} = \mathcal{M}_+ \cup \mathcal{M}_0$  at any  $\zeta \in \mathcal{S}$ , where

$$\begin{aligned} \mathcal{M}_+ &:= \{i \in \mathcal{M} \mid \mathcal{H}_i(\zeta) > 0\}, \\ \mathcal{M}_0 &:= \{i \in \mathcal{M} \mid \mathcal{H}_i(\zeta) = 0\}. \end{aligned}$$

Also, the index sets  $\mathcal{M}_+$  and  $\mathcal{M}_0$  can be divided as

$$\begin{aligned} \mathcal{M}_{+0} &:= \{i \in \mathcal{M}_+ \mid \mathcal{G}_i(\zeta) = 0\}, \\ \mathcal{M}_{+-} &:= \{i \in \mathcal{M}_+ \mid \mathcal{G}_i(\zeta) < 0\}, \\ \mathcal{M}_{0+} &:= \{i \in \mathcal{M}_0 \mid \mathcal{G}_i(\zeta) > 0\}, \\ \mathcal{M}_{00} &:= \{i \in \mathcal{M}_0 \mid \mathcal{G}_i(\zeta) = 0\}, \\ \mathcal{M}_{0-} &:= \{i \in \mathcal{M}_0 \mid \mathcal{G}_i(\zeta) < 0\}. \end{aligned}$$

For each  $i \in \mathcal{M}$ , set  $\theta_i := \mathcal{H}_i\mathcal{G}_i$  and for each  $K \subseteq \mathcal{M} \cup \mathcal{P}$  and  $\psi \in \{\mathcal{G}, \mathcal{H}, f, \theta\}$ , we have

$$\sigma_K^\psi := \bigcup_{k \in K} \partial^\circ \psi_k(\zeta).$$

Since  $\mathcal{H}_i(\zeta) = 0$  when  $i \in \mathcal{M}_0$ , and  $\theta_i(\zeta) = 0$  when  $i \in \mathcal{M}_0 \cup \mathcal{M}_{+0}$ , the classic linearized cone of problem (MOPVC) at  $\zeta$  is

$$(\sigma_{\mathcal{P}}^f)^- \cap (-\sigma_{\mathcal{M}_0}^{\mathcal{H}})^- \cap (\sigma_{\mathcal{M}_0 \cup \mathcal{M}_{+0}}^\theta)^-.$$

Following [15], consider the linearized cone  $\mathcal{L}^-$ , where

$$\mathcal{L} := \sigma_{\mathcal{M}_{0+}}^{\mathcal{H}} \cup (-\sigma_{\mathcal{M}_0}^{\mathcal{H}}) \cup \sigma_{\mathcal{M}_{+0}}^{\mathcal{G}}.$$

Additionally, for each  $k \in \mathcal{P}$ , let  $\mathcal{P}_k := \mathcal{P} \setminus \{k\}$  and

$$\mathcal{Q}_k := \begin{cases} \mathcal{S} \cap \{\tilde{x} \in \mathbb{R}^n \mid f_j(\tilde{x}) \leq f_j(\zeta), j \in \mathcal{P}_k\}, & \text{if } p > 1, \\ \mathcal{S}, & \text{if } p = 1. \end{cases}$$

Sadeghieh et al. [15] define the following Abadie type data qualification for (MOPVC).

**Definition 3** ([15]). *The problem (MOPVC) satisfies*

- the Abadie data qualification, denoted by ADQ, at  $\zeta$  iff

$$(\sigma_{\mathcal{P}}^f)^- \cap (-\sigma_{\mathcal{M}_0}^{\mathcal{H}})^- \cap (\sigma_{\mathcal{M}_0 \cup \mathcal{M}_{+0}}^{\mathcal{G}})^- \subseteq \Gamma(\mathcal{S}, \zeta);$$

- the weak ADQ, denoted by WADQ, at  $\zeta$  iff

$$(\sigma_{\mathcal{P}}^f)^- \cap \mathcal{L}^- \subseteq \Gamma(\mathcal{S}, \zeta);$$

- the refined WADQ, denoted by RWADQ, at  $\zeta$  iff

$$(\sigma_{\mathcal{P}}^f)^s \cap \mathcal{L}^- \subseteq \Gamma(\mathcal{S}, \zeta);$$

- the extended ADQ, denoted by EADQ, at  $\zeta$  iff

$$(\sigma_{\mathcal{P}}^f)^- \cap \mathcal{L}^- \subseteq \cap_{k=1}^p \Gamma(\mathcal{Q}_k, \zeta).$$

**Lemma 1** (Lemma 1, [15]). The diagram below illustrates the relationships between the data qualifications listed above:

$$\begin{array}{ccccc} & & \text{ADQ} & & \\ & & \Downarrow & & \\ \text{EADQ} & \implies & \text{WADQ} & \implies & \text{RWADQ} \end{array}$$

Moreover, when  $p = 1$   $\text{ADQ} \implies \text{EADQ}$ .

**Definition 4** (Definition 3, [15]). The problem (MOPVC) satisfies Mangasarian-Fromovitz data qualification, denoted by MFDQ, at  $\zeta$  iff

- $0_{\mathbb{R}^n} \in \sum_{i \in \mathcal{M}_{0+}} \alpha_i \partial^{\circ} \mathcal{H}_i(\zeta) \implies \alpha_i = 0, i \in \mathcal{M}_{0+},$
- $(\sigma_{\mathcal{P}_k}^f)^s \cap (\sigma_{\mathcal{M}_{0+}}^{\mathcal{H}})^{\perp} \cap (-\sigma_{\mathcal{M}_{00} \cup \mathcal{M}_{0-}}^{\mathcal{H}})^s \cap (\sigma_{\mathcal{M}_{+0}}^{\mathcal{G}})^s \neq \emptyset, \forall k \in \mathcal{P}.$

**Definition 5** (Definition 4, [15]). The problem (MOPVC) satisfies Mangasarian-Fromovitz constraint qualification, denoted MFCQ, at  $\zeta$  iff

- $0_{\mathbb{R}^n} \in \sum_{i \in \mathcal{M}_{0+}} \alpha_i \partial^{\circ} \mathcal{H}_i(\zeta) \implies \alpha_i = 0, i \in \mathcal{M}_{0+},$
- $(\sigma_{\mathcal{M}_{0+}}^{\mathcal{H}})^{\perp} \cap (-\sigma_{\mathcal{M}_{00} \cup \mathcal{M}_{0-}}^{\mathcal{H}})^s \cap (\sigma_{\mathcal{M}_{+0}}^{\mathcal{G}})^s \neq \emptyset.$

Sadeghieh et al. [15] derived the following necessary weak strongly stationary conditions and strong strongly stationary conditions for (MOPVC), respectively.

**Theorem 1** (Theorem 2, [15]). Suppose that  $\zeta$  is a weakly efficient solution for (MOPVC) such that RWADQ holds at  $\zeta$ . If the cone ( $\mathcal{L}$ ) is closed, then there exist scalars  $\lambda_j^f, \lambda_i^{\mathcal{H}},$  and  $\lambda_i^{\mathcal{G}}$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$ , satisfying

$$0_{\mathbb{R}^n} \in \sum_{j \in \mathcal{P}} \lambda_j^f \partial^{\circ} f_j(\zeta) + \sum_{i \in \mathcal{M}} \left( \lambda_i^{\mathcal{G}} \partial^{\circ} \mathcal{G}_i(\zeta) - \lambda_i^{\mathcal{H}} \partial^{\circ} \mathcal{H}_i(\zeta) \right), \quad (1)$$

$$\lambda_i^{\mathcal{H}} = 0 \ (i \in \mathcal{M}_{+0} \cup \mathcal{M}_{+-}), \ \lambda_i^{\mathcal{H}} \geq 0 \ (i \in \mathcal{M}_{0-} \cup \mathcal{M}_{00}), \ \lambda_i^{\mathcal{H}} \text{ free } (i \in \mathcal{M}_{0+}), \quad (2)$$

$$\lambda_i^{\mathcal{G}} = 0 \ (i \in \mathcal{M}_0 \cup \mathcal{M}_{+-}), \ \lambda_i^{\mathcal{G}} \geq 0 \ (i \in \mathcal{M}_{+0}), \quad (3)$$

$$(\lambda_1^f, \dots, \lambda_p^f) \geq 0_{\mathbb{R}^p}, \ \sum_{j=1}^p \lambda_j^f = 1. \quad (4)$$

**Theorem 2** (Theorem 4, [15]). Suppose that  $\zeta$  is an efficient solution for (MOPVC) and EADQ is satisfied at  $\zeta$ . If  $\partial^\circ f_j(\zeta)$ ,  $\partial^\circ \mathcal{H}_i(\zeta)$ , and  $\partial^\circ \mathcal{G}_i(\zeta)$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$  are polytopes, then we can find some scalars  $\lambda_j^f$ ,  $\lambda_i^{\mathcal{H}}$ , and  $\lambda_i^{\mathcal{G}}$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$ , such that (1)–(3) hold and

$$(\lambda_1^f, \dots, \lambda_p^f) > 0_{\mathbb{R}^p}, \quad \sum_{i=1}^p \lambda_j^f = 1. \quad (5)$$

**Remark 2.** A point  $\zeta \in \mathcal{S}$  satisfies

- a weak strongly stationary condition, denoted by W-S-SC, iff there exist scalars satisfying (1)–(4).
- a strong strongly stationary condition, denoted by S-S-SC, iff there exist scalars satisfying (1)–(3) and (5).

For  $p = 1$ , both the conditions will coincide.

Theorems 3 below shows that MFDQ is sufficient to satisfy Abadie type DQs without assuming  $\mathcal{M}_{00} = \emptyset$ .

**Theorem 3** (Theorem 5, [15]). Suppose that the MFDQ holds at  $\zeta$  for (MOPVC) and  $\mathcal{H}_i$  is a pseudolinear function for each  $i \in \mathcal{M}_{0+}$ , then, EADQ also holds at  $\zeta$ .

Theorem 4 ensures that the MFCQ (resp. MFDQ) is indeed a valid qualification condition, and it leads to the W-S-SC (resp. S-S-SC) at weakly efficient solution of the (MOPVC) when the set  $\mathcal{M}_{00} = \emptyset$ .

**Theorem 4.** Suppose that  $\zeta$  is a weakly efficient solution for (MOPVC) such that  $\mathcal{M}_{00} = \emptyset$ . Then,

- (Theorem 7, [15]) if MFCQ is satisfied at  $\zeta$ , then W-S-SC also holds at  $\zeta$ .
- (Theorem 8, [15]) if MFDQ is satisfied at  $\zeta$ , then S-S-SC also holds at  $\zeta$ .

Now, we briefly revisit the essential notations from interval-valued analysis, see e.g., [37–39].

Let  $\mathcal{K}_c := \{[s^L, s^U] : s^L, s^U \in \mathbb{R}, s^L \leq s^U\}$  be the class of all closed and bounded intervals in  $\mathbb{R}$ . Let  $S := [s^L, s^U]$  and  $T := [t^L, t^U]$  be in  $\mathcal{K}_c$ . Then,

- $S + T := \{s + t : s \in S, t \in T\} = [s^L + t^L, s^U + t^U]$ ;
- $S - T := \{s - t : s \in S, t \in T\} = [s^L - t^U, s^U - t^L]$ ;
- for each  $\alpha \in \mathbb{R}$ , one has

$$\alpha S := \{\alpha s : s \in S\} = \begin{cases} [\alpha s^L, \alpha s^U], & \text{if } \alpha \geq 0, \\ [\alpha s^U, \alpha s^L], & \text{if } \alpha < 0. \end{cases}$$

If  $s^L = s^U = s$ , then  $S = [s, s] = \{s\}$ .

The different LU-orderings between two intervals are defined as follows:

**Definition 6** (Definition 3, [40]). Let  $S = [s^L, s^U], T = [t^L, t^U] \in \mathcal{K}_c$ . We say that:

- $S \leq_{LU} T$  iff  $s^L \leq t^L$  and  $s^U \leq t^U$ ,
- $S <_{LU} T$  iff  $S \leq_{LU} T$  and  $S \neq T$ ,

or, equivalently,

$S <_{LU} T$  iff

$$\begin{cases} s^L < t^L \\ s^U \leq t^U \end{cases} \text{ or } \begin{cases} s^L \leq t^L \\ s^U < t^U \end{cases} \text{ or } \begin{cases} s^L < t^L \\ s^U < t^U \end{cases}$$

(iii)  $S <_{LU}^s T$  iff  $s^L < t^L$  and  $s^U < t^U$ .

Now, we consider the following interval-valued multiobjective optimization problem with vanishing constraints:

$$\begin{aligned} \min F(\tilde{x}) &:= (F_1(\tilde{x}), \dots, F_p(\tilde{x})) \\ \text{s.t. } \mathcal{H}_i(\tilde{x}) &\geq 0, \quad i \in \mathcal{M} \\ \mathcal{H}_i(\tilde{x})\mathcal{G}_i(\tilde{x}) &\leq 0, \quad i \in \mathcal{M}, \end{aligned} \quad (\text{IVMOPVC})$$

where  $F_j : \mathbb{R}^n \rightarrow \mathcal{K}_c$ ,  $j \in \mathcal{P}$  are interval-valued functions defined by  $F_j(\tilde{x}) := [f_j^L(\tilde{x}), f_j^U(\tilde{x})]$ , where the functions  $f_j^L, f_j^U, \mathcal{H}_i$  and  $\mathcal{G}_i$ ,  $i \in \mathcal{M}$  are locally Lipschitz function from  $\mathbb{R}^n$  to  $\mathbb{R}$  such that  $f_j^L(\tilde{x}) \leq f_j^U(\tilde{x})$ ,  $j \in \mathcal{P}$  for every  $\tilde{x} \in \mathcal{S}$ . The feasible region of the (IVMOPVC) is denoted by  $\mathcal{S}$ .

Following Definition 3.1 of [41], we write the concepts of approximate Pareto efficient solutions for the (IVMOPVC) as follows.

**Definition 7.** Let  $\mathcal{E}_j^L, \mathcal{E}_j^U, j \in \mathcal{P}$  be real-numbers satisfying  $0 \leq \mathcal{E}_j^L \leq \mathcal{E}_j^U$  with  $\mathcal{E}_j := [\mathcal{E}_j^L, \mathcal{E}_j^U]$  for all  $j \in \mathcal{P}$  and let  $\mathcal{E} := (\mathcal{E}_1, \dots, \mathcal{E}_p)$ . Then,  $\zeta \in \mathcal{S}$  is a

(i) type-1  $\mathcal{E}$ -quasi Pareto solution of the IVMOPVC, denoted by  $\zeta \in \mathcal{E} - \mathcal{S}_1^q$ , iff there is no  $\tilde{x} \in \mathcal{S}$  such that

$$F_j(\tilde{x}) + \mathcal{E}_j \|\tilde{x} - \zeta\| \leq_{LU} F_j(\zeta), \quad \forall j \in \mathcal{P},$$

and

$$F_r(\tilde{x}) + \mathcal{E}_r \|\tilde{x} - \zeta\| <_{LU} F_r(\zeta), \quad \text{for at least one } r \in \mathcal{P};$$

(ii) type-2  $\mathcal{E}$ -quasi Pareto solution of the IVMOPVC, denoted by  $\zeta \in \mathcal{E} - \mathcal{S}_2^q$ , iff there is no  $\tilde{x} \in \mathcal{S}$  such that

$$F_j(\tilde{x}) + \mathcal{E}_j \|\tilde{x} - \zeta\| \leq_{LU} F_j(\zeta), \quad \forall j \in \mathcal{P},$$

and

$$F_r(\tilde{x}) + \mathcal{E}_r \|\tilde{x} - \zeta\| <_{LU}^s F_r(\zeta), \quad \text{for at least one } r \in \mathcal{P};$$

(iii) type-1  $\mathcal{E}$ -quasi weakly Pareto solution of the IVMOPVC, denoted by  $\zeta \in \mathcal{E} - \mathcal{S}_1^{qw}$  iff there is no  $\tilde{x} \in \mathcal{S}$  such that

$$F_j(\tilde{x}) + \mathcal{E}_j \|\tilde{x} - \zeta\| <_{LU} F_j(\zeta), \quad \forall j \in \mathcal{P};$$

(iv) type-2  $\mathcal{E}$ -quasi weakly Pareto solution of the IVMOPVC, denoted by  $\zeta \in \mathcal{E} - \mathcal{S}_2^{qw}$  iff there is no  $\tilde{x} \in \mathcal{S}$  such that

$$F_j(\tilde{x}) + \mathcal{E}_j \|\tilde{x} - \zeta\| <_{LU}^s F_j(\zeta), \quad \forall j \in \mathcal{P}.$$

**Remark 3.** If  $\mathcal{E}_j = 0$ , i.e., for any  $j \in \mathcal{P}$ ,  $\mathcal{E}_j^L = \mathcal{E}_j^U = 0$ , then the concepts of a type-1  $\mathcal{E}$ -quasi Pareto solution, a type-2  $\mathcal{E}$ -quasi Pareto solution coincides with a type-1 Pareto solution, a type-2 Pareto solution, respectively, and a type-1  $\mathcal{E}$ -quasi-weakly Pareto solution and a type-2  $\mathcal{E}$ -quasi-weakly Pareto solution coincides with a type-1 weakly Pareto solution and a type-2 weakly Pareto solution, respectively, which were given by Tung [40].

The following inclusion relations hold:

- $\mathcal{E} - \mathcal{S}_1^q \subset \mathcal{E} - \mathcal{S}_2^q \subset \mathcal{E} - \mathcal{S}_2^{qw}$ .
- $\mathcal{E} - \mathcal{S}_1^q \subset \mathcal{E} - \mathcal{S}_1^{qw} \subset \mathcal{E} - \mathcal{S}_2^{qw}$ .

The following result interrelate a type-2 weakly Pareto solution of the (IVMOPVC) with a weakly efficient solution of a multiobjective optimization problem.

**Theorem 5** (Lemma 4, [40]). A feasible point  $\zeta \in \mathcal{S}$  is a type-2 weakly Pareto solution of the (IVMOPVC) iff  $\zeta$  is a weakly efficient solution of the (MOPVC1) which is given as

$$\min f(\tilde{x}) := (f_1^L(\tilde{x}), \dots, f_p^L(\tilde{x}), f_1^U(\tilde{x}), \dots, f_p^U(\tilde{x})) \text{ subject to } \tilde{x} \in \mathcal{S}. \quad (\text{MOPVC1})$$

### 3. Approximate KKT Conditions for IVMOPVC

This section presents the approximate necessary strongly stationary conditions that a feasible point must satisfy to be considered as an approximate solution of problem (IVMOPVC). Here, we extend the results of [15] to the setting of interval-valued objective functions and focus on approximate solutions instead of exact ones. To achieve this, first we introduce several approximate constraint and data qualifications for the problem (IVMOPVC) and examine the relationships among them. Using these qualifications, we establish the approximate necessary strongly stationary conditions for the IVMOPVC, employing the Clarke subdifferential as the main analytical tool.

The notations listed below will be used in the subsequent discussion.

Let  $\mathcal{E}_j^L, \mathcal{E}_j^U, j \in \mathcal{P}$  be real-numbers satisfying  $0 \leq \mathcal{E}_j^L \leq \mathcal{E}_j^U$  with  $\mathcal{E}_j := [\mathcal{E}_j^L, \mathcal{E}_j^U]$  for all  $j \in \mathcal{P}$  and let  $\mathcal{E} := (\mathcal{E}_1, \dots, \mathcal{E}_p) \in \mathcal{K}_c^p$ . For any  $\zeta \in \mathcal{S}$  and  $\mathcal{E} \in \mathcal{K}_c^p$ , define

$$\begin{aligned} \sigma_{\mathcal{P}, \mathcal{E}}^{F^L}(\zeta) &:= \bigcup_{j \in \mathcal{P}} \partial^\circ (f_j^L + \mathcal{E}_j^L \|\cdot - \zeta\|)(\zeta); \\ \sigma_{\mathcal{P}, \mathcal{E}}^{F^U}(\zeta) &:= \bigcup_{j \in \mathcal{P}} \partial^\circ (f_j^U + \mathcal{E}_j^U \|\cdot - \zeta\|)(\zeta); \\ \sigma_{\mathcal{P}_k, \mathcal{E}}^{F^L}(\zeta) &:= \bigcup_{j \in \mathcal{P}, j \neq k} \partial^\circ (f_j^L + \mathcal{E}_j^L \|\cdot - \zeta\|)(\zeta); \\ \sigma_{\mathcal{P}_k, \mathcal{E}}^{F^U}(\zeta) &:= \bigcup_{j \in \mathcal{P}, j \neq k} \partial^\circ (f_j^U + \mathcal{E}_j^U \|\cdot - \zeta\|)(\zeta). \end{aligned}$$

We introduce approximate version of various data qualifications which will be used to derive approximate KKT conditions for the (IVMOPVC).

**Definition 8.** Let  $\zeta \in \mathcal{S}$  and let  $\mathcal{E} \in \mathcal{K}_c^p$ . The (IVMOPVC) satisfies

- the  $\mathcal{E}$ -Abadie data qualification, denoted by  $\mathcal{E}$ -IV-ADQ, at  $\zeta$  iff

$$(\sigma_{\mathcal{P}, \mathcal{E}}^{F^L} \cup \sigma_{\mathcal{P}, \mathcal{E}}^{F^U})^- \cap (-\sigma_{\mathcal{M}_0}^{\mathcal{H}})^- \cap (\sigma_{\mathcal{M}_0 \cup \mathcal{M}_{+0}}^{\theta})^- \subseteq \Gamma(\mathcal{S}, \zeta);$$

- the weak  $\mathcal{E}$ -ADQ, denoted by  $\mathcal{E}$ -IV-WADQ, at  $\zeta$  iff

$$(\sigma_{\mathcal{P}, \mathcal{E}}^{F^L} \cup \sigma_{\mathcal{P}, \mathcal{E}}^{F^U})^- \cap \mathcal{L}^- \subseteq \Gamma(\mathcal{S}, \zeta);$$

- the refined  $\mathcal{E}$ -WADQ, denoted by  $\mathcal{E}$ -IV-RWADQ, at  $\zeta$  iff

$$(\sigma_{\mathcal{P}, \mathcal{E}}^{F^L} \cup \sigma_{\mathcal{P}, \mathcal{E}}^{F^U})^s \cap \mathcal{L}^- \subseteq \Gamma(\mathcal{S}, \zeta);$$

- the extended  $\mathcal{E}$ -ADQ, denoted by  $\mathcal{E}$ -IV-EADQ, at  $\zeta$  iff

$$(\sigma_{\mathcal{P}, \mathcal{E}}^{F^L} \cup \sigma_{\mathcal{P}, \mathcal{E}}^{F^U})^- \cap \mathcal{L}^- \subseteq \bigcap_{k=1}^p \Gamma(\mathcal{Q}_k, \zeta).$$

**Remark 4.** Some special cases are given as follows:

- If  $\mathcal{E}_j^L = \mathcal{E}_j^U = 0$  for all  $j \in \mathcal{P}$ , then (IVMOPVC) satisfies IV-ADQ, IV-WADQ, IV-RWADQ, IV-EADQ, respectively, at  $\zeta$ .

- If  $f_j^L = f_j^U = f_j$  for all  $j \in \mathcal{P}$ , then (MOPVC) satisfies  $\mathcal{E}$ -ADQ,  $\mathcal{E}$ -WADQ,  $\mathcal{E}$ -RWADQ and  $\mathcal{E}$ -EADQ, respectively, at  $\zeta$ .
- If  $f_j^L = f_j^U = f_j$  and  $\mathcal{E}_j^L = \mathcal{E}_j^U = 0$  for all  $j \in \mathcal{P}$ , then (MOPVC) satisfies ADQ, WADQ, RWADQ, EADQ, respectively, at  $\zeta$  as given by Sadeghieh et al. (Definition 2, [15]).

**Remark 5.** It is easy to observe that (IVMOPVC) satisfies  $\mathcal{E}$ -IV-ADQ,  $\mathcal{E}$ -IV-WADQ,  $\mathcal{E}$ -IV-RWADQ, and  $\mathcal{E}$ -IV-EADQ at  $\zeta \in \mathcal{S}$  iff ADQ, WADQ, RWADQ, and EADQ, respectively, is satisfied at  $\zeta$  for the problem  $\mathcal{E}$ -MOPVC1, which is given by

$$\begin{aligned} \min \quad & (f_1^L(\check{x}) + \mathcal{E}_1^L \|\check{x} - \zeta\|, \dots, f_p^L(\check{x}) + \mathcal{E}_p^L \|\check{x} - \zeta\|, \\ & f_1^U(\check{x}) + \mathcal{E}_1^U \|\check{x} - \zeta\|, \dots, f_p^U(\check{x}) + \mathcal{E}_p^U \|\check{x} - \zeta\|) \\ \text{s.t.} \quad & \mathcal{H}_i(\check{x}) \geq 0, i \in \mathcal{M} \\ & \mathcal{H}_i(\check{x})\mathcal{G}_i(\check{x}) \leq 0, i \in \mathcal{M}. \end{aligned} \quad (\mathcal{E}\text{-MOPVC1})$$

**Remark 6.** Based on Lemma 1, the relation among various approximate data qualifications is given as follows.

$$\begin{array}{ccccc} & & \mathcal{E} - IV - ADQ & & \\ & & \Downarrow & & \\ \mathcal{E} - IV - EADQ & \implies & \mathcal{E} - IV - WADQ & \implies & \mathcal{E} - IV - RWADQ \end{array}$$

Moreover,  $\mathcal{E} - IV - ADQ \implies \mathcal{E} - IV - EADQ$  when  $p = 1$ .

Now we give an example where (MOPVC) does not satisfy RWADQ but satisfies  $\mathcal{E}$ -RWADQ at  $\zeta \in \mathcal{S}$  for a suitable choice of  $\mathcal{E}$ .

**Example 1.** Consider the following MOPVC:

$$\begin{aligned} \min \quad & f(x_1, x_2) := (f_1(x_1, x_2), f_2(x_1, x_2)) \\ \text{s.t.} \quad & \mathcal{H}_1(x_1, x_2) \geq 0, \\ & \mathcal{H}_1(x_1, x_2)\mathcal{G}_1(x_1, x_2) \leq 0, \end{aligned}$$

where

$$\begin{aligned} f_1(x_1, x_2) &:= |x_1| - x_2, \quad f_2(x_1, x_2) := x_1^2 - x_2, \\ \mathcal{H}_1(x_1, x_2) &:= x_2, \quad \mathcal{G}_1(x_1, x_2) := x_2 - |x_1|. \end{aligned}$$

At a point  $\zeta = 0_{\mathbb{R}^2}$ , one has

$$\begin{aligned} \partial^\circ f_1(\zeta) &= [-1, 1] \times \{-1\}, \quad \partial^\circ f_2(\zeta) = \{(0, -1)\}, \\ \partial^\circ \mathcal{H}_1(\zeta) &= \{(0, 1)\}, \quad \partial^\circ \mathcal{G}_1(\zeta) = [-1, 1] \times \{1\}, \\ \mathcal{M} = \mathcal{M}_{00} &= \{1\}, \quad \mathcal{P} = \{1, 2\}. \end{aligned}$$

Now, observe that

$$\begin{aligned} \sigma_{\mathcal{P}}^f &= \bigcup_{j=1,2} \partial^\circ f_j(\zeta) = [-1, 1] \times \{-1\}, \\ (\sigma_{\mathcal{P}}^f)^s &= \{(d_1, d_2) \in \mathbb{R}^2 : d_1 + d_2 > 0, d_1 - d_2 < 0, d_2 > 0\}, \\ \mathcal{L} &= \sigma_{\mathcal{M}_{0+}}^{\mathcal{H}} \cup (-\sigma_{\mathcal{M}_0}^{\mathcal{H}}) \cup (\sigma_{\mathcal{M}_{+0}}^{\mathcal{G}}) = -\{(0, 1)\} = \{(0, -1)\}, \\ \mathcal{L}^- &= \{(d_1, d_2) \in \mathbb{R}^2 : d_2 \geq 0\}, \\ \Gamma(\mathcal{S}, \zeta) &= \{(d_1, d_2) \in \mathbb{R}^2 : d_2(d_2 - |d_1|) \leq 0, d_2 \geq 0\}, \end{aligned}$$

and

$$(\sigma_p^f)^s \cap \mathcal{L}^- \not\subseteq \Gamma(\mathcal{S}, \zeta).$$

Hence, (MOPVC) does not satisfy RWADQ at  $\zeta$  as given in (Example 1, [15]).

If we choose  $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2) = (0, 2)$ , then

$$\partial^\circ(f_1 + \mathcal{E}_1 \|\cdot - \zeta\|)(\zeta) = [-1, 1] \times \{-1\},$$

$$\partial^\circ(f_2 + \mathcal{E}_2 \|\cdot - \zeta\|)(\zeta) = \{(0, -1)\} + 2\mathbb{B}_{\mathbb{R}^2}$$

and

$$(\sigma_{\mathcal{P}, \mathcal{E}}^f)^s = \emptyset,$$

which implies that

$$(\sigma_{\mathcal{P}, \mathcal{E}}^f)^s \cap \mathcal{L}^- \subseteq \Gamma(\mathcal{S}, \zeta).$$

Hence, (MOPVC) satisfies  $\mathcal{E}$ -RWADQ at  $\zeta$ .

### 3.1. Type-2 $\mathcal{E}$ -Quasi Weakly Pareto Solutions

In this subsection, we derive an approximate KKT conditions for the (IVMOPVC) by utilizing the approximate data qualifications given in Definition 8.

**Theorem 6.** Suppose that  $\zeta \in \mathcal{E} - S_2^{qw}$  such that  $\mathcal{E}$ -IV-RWADQ holds at  $\zeta$ . If the cone( $\mathcal{L}$ ) is closed, then there exist scalars  $\lambda_j^{f^L}$ ,  $\lambda_j^{f^U}$ ,  $\lambda_i^{\mathcal{H}}$  and  $\lambda_i^{\mathcal{G}}$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$  satisfying

$$\begin{aligned} 0_{\mathbb{R}^n} \in & \sum_{j \in \mathcal{P}} \lambda_j^{f^L} \partial^\circ f_j^L(\zeta) + \sum_{j \in \mathcal{P}} \lambda_j^{f^U} \partial^\circ f_j^U(\zeta) + \sum_{i \in \mathcal{M}} (\lambda_i^{\mathcal{G}} \partial^\circ \mathcal{G}_i(\zeta) - \lambda_i^{\mathcal{H}} \partial^\circ \mathcal{H}_i(\zeta)) \\ & + \sum_{j \in \mathcal{P}} (\lambda_j^{f^L} \mathcal{E}_j^L + \lambda_j^{f^U} \mathcal{E}_j^U) \mathbb{B}_{\mathbb{R}^n}, \end{aligned} \quad (6)$$

$$\lambda_i^{\mathcal{H}} = 0 \ (i \in \mathcal{M}_{+0} \cup \mathcal{M}_{+-}), \ \lambda_i^{\mathcal{H}} \geq 0 \ (i \in \mathcal{M}_{0-} \cup \mathcal{M}_{00}), \ \lambda_i^{\mathcal{H}} \text{ free } (i \in \mathcal{M}_{0+}), \quad (7)$$

$$\lambda_i^{\mathcal{G}} = 0 \ (i \in \mathcal{M}_0 \cup \mathcal{M}_{+-}), \ \lambda_i^{\mathcal{G}} \geq 0 \ (i \in \mathcal{M}_{+0}), \quad (8)$$

$$(\lambda_1^{f^L} \dots \lambda_p^{f^L}, \lambda_1^{f^U} \dots \lambda_p^{f^U}) \geq 0_{\mathbb{R}^{2p}}, \ \sum_{j=1}^p (\lambda_j^{f^L} + \lambda_j^{f^U}) = 1. \quad (9)$$

**Proof.** Since  $\zeta \in \mathcal{E} - S_2^{qw}$ , therefore  $\zeta$  is a type-2 quasi weakly Pareto solution of the problem  $\mathcal{E}$ -IVMOPVC, where  $\mathcal{E}$ -IVMOPVC is given by

$$\begin{aligned} \min & F(\check{x}) + \mathcal{E} \|\check{x} - \zeta\| := (F_1(\check{x}) + \mathcal{E}_1 \|\check{x} - \zeta\|, \dots, F_p(\check{x}) + \mathcal{E}_p \|\check{x} - \zeta\|) \\ \text{s.t. } & \mathcal{H}_i(\check{x}) \geq 0, i \in \mathcal{M} \\ & \mathcal{H}_i(\check{x}) \mathcal{G}_i(\check{x}) \leq 0, i \in \mathcal{M}. \end{aligned} \quad (\mathcal{E}\text{-IVMOPVC})$$

By Theorem 5, it follows that  $\zeta \in S$  is a weakly efficient solution of the ( $\mathcal{E}$ -MOPVC1).

Since IVMOPVC satisfies  $\mathcal{E}$ -IV-RWADQ at  $\zeta$ , therefore  $\mathcal{E}$ -IVMOPVC satisfies IV-RWADQ at  $\zeta$  and hence ( $\mathcal{E}$ -MOPVC1) satisfies RWADQ at  $\zeta$ . By Theorem 1, there exist  $\lambda_j^{f^L}$ ,  $\lambda_j^{f^U}$ ,  $j \in \mathcal{P}$ ,  $\lambda_i^{\mathcal{H}}$ , and  $\lambda_i^{\mathcal{G}}$ ,  $i \in \mathcal{M}$  such that

$$0_{\mathbb{R}^n} \in \sum_{j \in \mathcal{P}} \lambda_j^L \partial^\circ(f_j^L + \mathcal{E}_j^L \|\cdot - \zeta\|)(\zeta) + \sum_{j \in \mathcal{P}} \lambda_j^U \partial^\circ(f_j^U + \mathcal{E}_j^U \|\cdot - \zeta\|)(\zeta) + \sum_{i \in \mathcal{M}} (\lambda_i^{\mathcal{G}} \partial^\circ \mathcal{G}_i(\zeta) - \lambda_i^{\mathcal{H}} \partial^\circ \mathcal{H}_i(\zeta)), \quad (10)$$

$$\lambda_i^{\mathcal{H}} = 0 \ (i \in \mathcal{M}_{+0} \cup \mathcal{M}_{+-}), \ \lambda_i^{\mathcal{H}} \geq 0 \ (i \in \mathcal{M}_{0-} \cup \mathcal{M}_{00}), \ \lambda_i^{\mathcal{H}} \text{ free } (i \in \mathcal{M}_{0+}), \quad (11)$$

$$\lambda_i^{\mathcal{G}} = 0 \ (i \in \mathcal{M}_0 \cup \mathcal{M}_{+-}), \ \lambda_i^{\mathcal{G}} \geq 0 \ (i \in \mathcal{M}_{+0}), \quad (12)$$

$$(\lambda_1^{f^L} \dots \lambda_p^{f^L}, \lambda_1^{f^U} \dots \lambda_p^{f^U}) \geq 0_{\mathbb{R}^{2p}}, \ \sum_{j=1}^p (\lambda_j^{f^L} + \lambda_j^{f^U}) = 1. \quad (13)$$

By the property of the Clarke subdifferentials in Proposition 1, one has

$$\partial^\circ(f_j^L + \mathcal{E}_j^L \|\cdot - \zeta\|)(\zeta) \subseteq \partial^\circ f_j^L(\zeta) + \mathcal{E}_j^L \partial^\circ \|\cdot - \zeta\|(\zeta)$$

and

$$\partial^\circ(f_j^U + \mathcal{E}_j^U \|\cdot - \zeta\|)(\zeta) \subseteq \partial^\circ f_j^U(\zeta) + \mathcal{E}_j^U \partial^\circ \|\cdot - \zeta\|(\zeta).$$

Since the Clarke subdifferential of the norm function  $\partial^\circ \|\cdot - \zeta\|(\zeta) = \mathbb{B}_{\mathbb{R}^n}$  (see (Example 4, p. 198, [42])), we have the required result.  $\square$

**Note 1.** In many real-life multiobjective problems with vanishing constraints such as supply chain management with uncertain suppliers [43] or investment planning under changing policies [44], uncertainty and variability are unavoidable. Classical exact optimality conditions ( $\mathcal{E} = 0$ ) require perfect precision, which is unrealistic when data is noisy and constraints shift over time. In such cases, insisting on exact solutions may lead to no feasible results. Instead, approximate solutions provide a more practical approach, offering near-optimal decisions under uncertainty.

We now present an example that satisfies the above theorem and demonstrates that, although no point exactly solves the problem, an approximate  $\mathcal{E}$ -solution does exist and meets the optimality conditions.

**Example 2.** Consider an IV-MOPVC in  $\mathbb{R}^2$  as follows:

$$\begin{aligned} \min F(\check{x}) &:= (F_1(\check{x}) := [f_1^L(\check{x}), f_1^U(\check{x})], F_2(\check{x}) := [f_2^L(\check{x}), f_2^U(\check{x})]) \\ \text{s. t. } \mathcal{H}_1(\check{x}) &\geq 0, \\ \mathcal{H}_1(\check{x}) \mathcal{G}_1(\check{x}) &\leq 0, \end{aligned}$$

where  $\check{x} := (x_1, x_2) \in \mathbb{R}^2$  and

$$\begin{cases} f_1^L(\check{x}) := |x_1| + x_2 - 2\sqrt{x_1^2 + x_2^2}, & f_1^U(\check{x}) := 8|x_1| - x_2 - 3\sqrt{x_1^2 + x_2^2}, \\ f_2^L(\check{x}) := -|x_1| - 2x_2 - \sqrt{x_1^2 + x_2^2}, & f_2^U(\check{x}) := 3|x_1| - x_2 - 2\sqrt{x_1^2 + x_2^2}. \\ \mathcal{H}_1(\check{x}) := x_2, & \mathcal{G}_1(\check{x}) := x_2 - 2|x_1|. \end{cases}$$

For  $\zeta = 0_{\mathbb{R}^2}$ , the system

$$\begin{cases} f_1^L(\tilde{x}) - f_1^L(\zeta) = |x_1| + x_2 - 2\sqrt{x_1^2 + x_2^2} < 0, \\ f_1^U(\tilde{x}) - f_1^U(\zeta) = 8|x_1| - x_2 - 3\sqrt{x_1^2 + x_2^2} < 0, \\ f_2^L(\tilde{x}) - f_2^L(\zeta) = -|x_1| - 2x_2 - \sqrt{x_1^2 + x_2^2} < 0, \\ f_2^U(\tilde{x}) - f_2^U(\zeta) = 3|x_1| - x_2 - 2\sqrt{x_1^2 + x_2^2} < 0, \\ \mathcal{H}_1(\tilde{x}) = x_2 \geq 0, \\ \mathcal{H}_1(\tilde{x})\mathcal{G}_1(\tilde{x}) = x_2(x_2 - 2|x_1|) \leq 0, \end{cases} \quad (14)$$

is solvable for some  $\tilde{x} \in \mathbb{R}^2$  shown by the shaded region in the Figure 1. Since such points exist, the point  $\zeta$  does not satisfy the required condition of type-2 weakly Pareto solution. Hence,  $\zeta \notin S_2^{qw}$ .

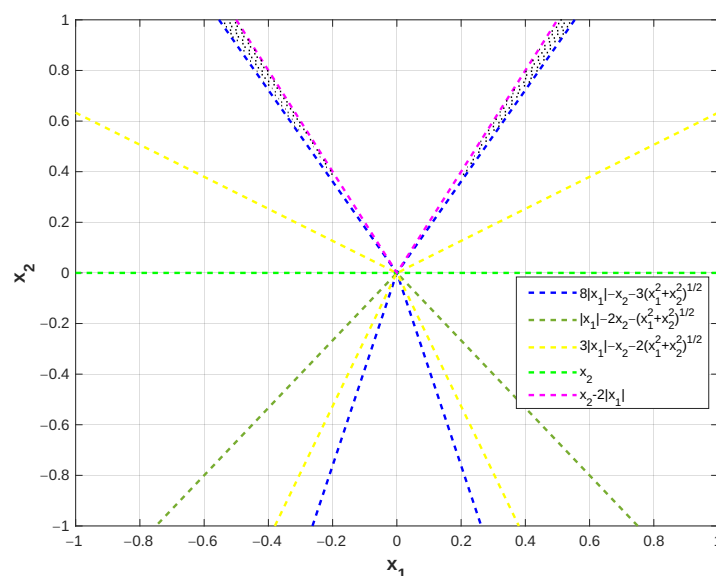


**Figure 1.** The shaded region represents the points where the system (14) of Example 2 is solvable.

Now, if we choose  $\mathcal{E}_1^L = 2$ ,  $\mathcal{E}_1^U = 3$ ,  $\mathcal{E}_2^L = 1$ , and  $\mathcal{E}_2^U = 2$ , then the system

$$\begin{cases} f_1^L(\tilde{x}) - f_1^L(\zeta) + \mathcal{E}_1^L \|\tilde{x} - \zeta\| = |x_1| + x_2 < 0, \\ f_1^U(\tilde{x}) - f_1^U(\zeta) + \mathcal{E}_1^U \|\tilde{x} - \zeta\| = 8|x_1| - x_2 < 0, \\ f_2^L(\tilde{x}) - f_2^L(\zeta) + \mathcal{E}_2^L \|\tilde{x} - \zeta\| = -|x_1| - 2x_2 < 0, \\ f_2^U(\tilde{x}) - f_2^U(\zeta) + \mathcal{E}_2^U \|\tilde{x} - \zeta\| = 3|x_1| - x_2 < 0, \\ \mathcal{H}_1(\tilde{x}) = x_2 \geq 0, \\ \mathcal{H}_1(\tilde{x})\mathcal{G}_1(\tilde{x}) = x_2(x_2 - 2|x_1|) \leq 0. \end{cases} \quad (15)$$

is not solvable for any  $\tilde{x} \in \mathbb{R}^2$ . Hence,  $\zeta \in \mathcal{E} - S_2^{qw}$ . Moreover, it is easy to see that

$$\begin{aligned} \partial^\circ(f_1^L + \mathcal{E}_1^L \|\cdot - \zeta\|)(\zeta) &= [-1, 1] \times \{1\}, \\ \partial^\circ(f_1^U + \mathcal{E}_1^U \|\cdot - \zeta\|)(\zeta) &= [-8, 8] \times \{-1\}, \\ \partial^\circ(f_2^L + \mathcal{E}_2^L \|\cdot - \zeta\|)(\zeta) &= [-1, 1] \times \{-2\}, \\ \partial^\circ(f_2^U + \mathcal{E}_2^U \|\cdot - \zeta\|)(\zeta) &= [-3, 3] \times \{-1\}, \\ \partial^\circ \mathcal{H}_1(\zeta) &= \{(0, 1)\}, \\ \partial^\circ \mathcal{G}_1(\zeta) &= [-2, 2] \times \{1\}, \\ \mathcal{M} = \mathcal{M}_{00} &= \{1\}, \mathcal{P} = \{1, 2\}, \end{aligned}$$

and

$$\sigma_{\mathcal{P},\mathcal{E}}^{FL}(\zeta) = \bigcup_{j \in \mathcal{P}} \partial^\circ(f_j^L + \mathcal{E}_j^L \|\cdot - \zeta\|)(\zeta) = [-1, 1] \times \{1\} \cup [-1, 1] \times \{-2\},$$

$$\sigma_{\mathcal{P},\mathcal{E}}^{FU}(\zeta) = \bigcup_{j \in \mathcal{P}} \partial^\circ(f_j^U + \mathcal{E}_j^U \|\cdot - \zeta\|)(\zeta) = [-8, 8] \times \{-1\},$$

$$\mathcal{L} = \sigma_{\mathcal{M}_{0+}}^{\mathcal{H}} \cup (-\sigma_{\mathcal{M}_0}^{\mathcal{H}}) \cup (\sigma_{\mathcal{M}_{+0}}^{\mathcal{G}}) = \{(0, -1)\},$$

$$\Gamma(\mathcal{S}, \zeta) = \{(d_1, d_2) \in \mathbb{R}^2 : d_2(d_2 - 2|d_1|) \leq 0, d_2 \geq 0\},$$

which implies that

$$\emptyset = (\sigma_{\mathcal{P},\mathcal{E}}^{FL}(\zeta) \cup \sigma_{\mathcal{P},\mathcal{E}}^{FU}(\zeta))^s \cap \mathcal{L}^- \subseteq \Gamma(\mathcal{S}, \zeta).$$

Therefore,  $\mathcal{E}$ -IV-RWADQ is satisfied at  $\zeta = 0_{\mathbb{R}^2}$  for IVMOPVC.

By taking  $\lambda_1^L = 1, \lambda_1^U = 0, \lambda_2^L = 0, \lambda_2^U = 0, \lambda_1^{\mathcal{G}} = 0, \lambda_1^{\mathcal{H}} = 1$ , we get

$$\begin{aligned} 0_{\mathbb{R}^2} &\in \sum_{j=1}^2 \lambda_j^{fL} \partial^\circ(f_j^L + \mathcal{E}_j^L \|\cdot - \zeta\|)(\zeta) + \sum_{j=1}^2 \lambda_j^{fU} \partial^\circ(f_j^U + \mathcal{E}_j^U \|\cdot - \zeta\|)(\zeta) \\ &\quad + (\lambda_1^{\mathcal{G}} \partial^\circ \mathcal{G}_1(\zeta) - \lambda_1^{\mathcal{H}} \partial^\circ \mathcal{H}_1(\zeta)), \\ &= 1([-1, 1] \times \{1\}) - 1\{(0, 1)\}, \end{aligned}$$

which verifies Theorem 6.

A corollary of Theorem 6 with  $\mathcal{E}_j^L = \mathcal{E}_j^U = 0$  for every  $j \in \mathcal{P}$  is given as follows:

**Corollary 1.** Suppose that  $\zeta \in S_2^{qw}$  such that IV-RWADQ holds at  $\zeta$ . If the cone( $\mathcal{L}$ ) is closed, then there exist scalars  $\lambda_j^{fL}, \lambda_j^{fU}, \lambda_i^{\mathcal{H}}$  and  $\lambda_i^{\mathcal{G}}$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$  satisfying

$$0_{\mathbb{R}^n} \in \sum_{j \in \mathcal{P}} \lambda_j^{fL} \partial^\circ f_j^L(\zeta) + \sum_{j \in \mathcal{P}} \lambda_j^{fU} \partial^\circ f_j^U(\zeta) + \sum_{i \in \mathcal{M}} (\lambda_i^{\mathcal{G}} \partial^\circ \mathcal{G}_i(\zeta) - \lambda_i^{\mathcal{H}} \partial^\circ \mathcal{H}_i(\zeta)) \quad (16)$$

with (7)–(9).

Here,  $\mathcal{E}_j^L = \mathcal{E}_j^U = 0$  for every  $j \in \mathcal{P}$ , that is,  $\sum_{j \in \mathcal{P}} (\lambda_j^{fL} \mathcal{E}_j^L + \lambda_j^{fU} \mathcal{E}_j^U) \mathbb{B}_{\mathbb{R}^n} = 0$ . Therefore, the conditions represent exact weak strongly stationarity for (IVMOPVC) rather than approximate ones.

For  $f_j^L = f_j^U = f_j$  for every  $j \in \mathcal{P}$ , a corollary of Theorem 6 is given as follows:

**Corollary 2.** Suppose that  $\zeta$  is a  $\mathcal{E}$ -weakly efficient solution for (MOPVC) such that  $\mathcal{E}$ -RWADQ holds at  $\zeta$ . If the cone( $\mathcal{L}$ ) is closed, then there exist scalars  $\lambda_j^f, \lambda_i^{\mathcal{H}}$ , and  $\lambda_i^{\mathcal{G}}$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$ , satisfying

$$0_{\mathbb{R}^n} \in \sum_{j \in \mathcal{P}} \lambda_j^f \partial^\circ f_j(\zeta) + \sum_{i \in \mathcal{M}} (\lambda_i^{\mathcal{G}} \partial^\circ \mathcal{G}_i(\zeta) - \lambda_i^{\mathcal{H}} \partial^\circ \mathcal{H}_i(\zeta)) + \sum_{j \in \mathcal{P}} \lambda_j^f \mathcal{E}_j \mathbb{B}_{\mathbb{R}^n} \quad (17)$$

with (7) and (8) and

$$(\lambda_1^f \dots \lambda_p^f) \geq 0_{\mathbb{R}^p}, \sum_{j=1}^p \lambda_j^f = 1. \quad (18)$$

Here,  $f_j^L = f_j^U = f_j$  for every  $j \in \mathcal{P}$ , that is,  $\sum_{j \in \mathcal{P}} \lambda_j^{fL} \partial^\circ f_j^L(\zeta) = \sum_{j \in \mathcal{P}} \lambda_j^{fU} \partial^\circ f_j^U(\zeta) = \sum_{j \in \mathcal{P}} \lambda_j^f \partial^\circ f_j(\zeta)$  and  $(\lambda_1^{fL} \dots \lambda_p^{fL}, \lambda_1^{fU} \dots \lambda_p^{fU}) = (\lambda_1^f \dots \lambda_p^f)$ . Therefore, the conditions represent the approximate weak strongly stationarity conditions for (MOPVC).

The following example illustrates the Corollary 2.

**Example 3.** Consider an MOPVC in  $\mathbb{R}^2$  as follows:

$$\begin{aligned} \min f(\tilde{x}) &:= (f_1(\tilde{x}), f_2(\tilde{x})) \\ \text{s. t. } \mathcal{H}_1(\tilde{x}) &\geq 0, \\ \mathcal{H}_1(\tilde{x})\mathcal{G}_1(\tilde{x}) &\leq 0 \end{aligned}$$

where

$$\begin{aligned} f_1(\tilde{x}) &:= 2x_1^2 - x_2 - \sqrt{x_1^2 + x_2^2}, & f_2(\tilde{x}) &:= x_1^2 + 2x_2 - 2\sqrt{x_1^2 + x_2^2}. \\ \mathcal{H}_1(\tilde{x}) &:= x_2, & \mathcal{G}_1(\tilde{x}) &:= x_2 - 2|x_1|. \end{aligned} \quad (19)$$

For  $\zeta = 0_{\mathbb{R}^2}$ , the system

$$\begin{cases} f_1(\tilde{x}) - f_1(\zeta) = 2x_1^2 - x_2 - \sqrt{x_1^2 + x_2^2} < 0, \\ f_2(\tilde{x}) - f_2(\zeta) = x_1^2 + 2x_2 - 2\sqrt{x_1^2 + x_2^2} < 0, \\ \mathcal{H}_1(\tilde{x}) = x_2 \geq 0, \\ \mathcal{H}_1(\tilde{x})\mathcal{G}_1(\tilde{x}) = x_2(x_2 - 2|x_1|) \leq 0, \end{cases} \quad (20)$$

is solvable for some  $\tilde{x} \in \mathbb{R}^2$  shown by the shaded region in the Figure 2. Since such points exist, the point  $\zeta$  does not satisfy the required condition of weakly efficient solution. Hence,  $\zeta$  is not a weakly efficient solution of the given problem MOPVC.

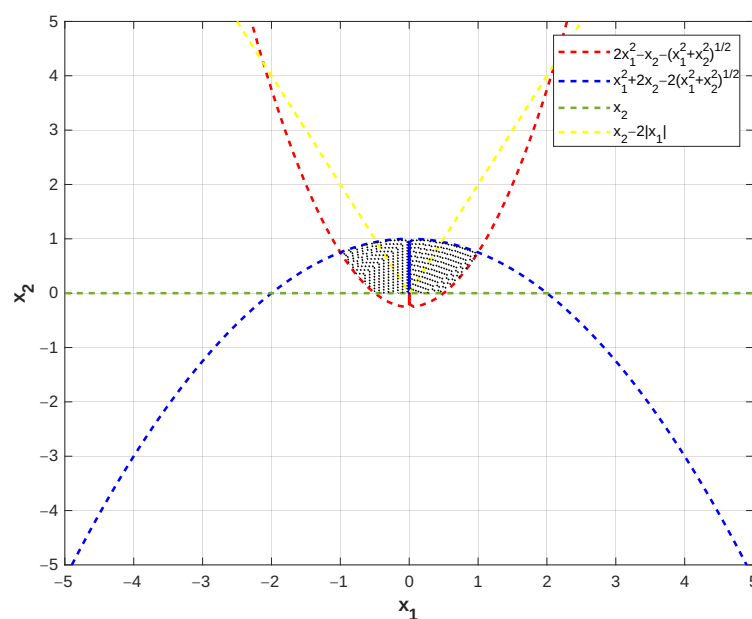


**Figure 2.** The shaded region represents the points where the system (20) of Example 3 is solvable.

Now, if we choose  $\mathcal{E}_1 = 1$ , and  $\mathcal{E}_2 = 2$ , then  $\zeta$  is  $\mathcal{E}$ -weakly efficient solution of the problem MOPVC as the system

$$\begin{cases} f_1(\tilde{x}) - f_1(\zeta) + \mathcal{E}_1 \|\tilde{x} - \zeta\| = 2x_1^2 - x_2 < 0, \\ f_2(\tilde{x}) - f_2(\zeta) + \mathcal{E}_2 \|\tilde{x} - \zeta\| = x_1^2 + 2x_2 < 0, \\ \mathcal{H}_1(\tilde{x}) = x_2 \geq 0, \\ \mathcal{H}_1(\tilde{x})\mathcal{G}_1(\tilde{x}) = x_2(x_2 - 2|x_1|) \leq 0, \end{cases} \quad (21)$$

is not solvable for any  $\check{x} \in \mathbb{R}^2$ . Moreover, it is easy to see that

$$\begin{aligned}\partial^\circ(f_1 + \mathcal{E}_1\|\cdot - \zeta\|)(\zeta) &= \{(0, -1)\}, \\ \partial^\circ(f_2 + \mathcal{E}_2\|\cdot - \zeta\|)(\zeta) &= \{(0, 2)\}, \\ \partial^\circ \mathcal{H}_1(\zeta) &= \{(0, 1)\}, \\ \partial^\circ \mathcal{G}_1(\zeta) &= [-2, 2] \times \{1\}, \\ \mathcal{M} = \mathcal{M}_{00} &= \{1\}, \mathcal{P} = \{1, 2\},\end{aligned}$$

and

$$\begin{aligned}\sigma_{\mathcal{P}, \mathcal{E}}^f(\zeta) &= \bigcup_{j \in \mathcal{P}} \partial^\circ(f_j + \mathcal{E}_j\|\cdot - \zeta\|)(\zeta) = \{(0, -1)\} \cup \{(0, 2)\}, \\ \mathcal{L} &= \sigma_{\mathcal{M}_{0+}}^{\mathcal{H}} \cup (-\sigma_{\mathcal{M}_0}^{\mathcal{H}}) \cup (\sigma_{\mathcal{M}_{+0}}^{\mathcal{G}}) = \{-(0, 1)\}, \\ \Gamma(\mathcal{S}, \zeta) &= \{(d_1, d_2) \in \mathbb{R}^2 : d_2(d_2 - 2|d_1|) \leq 0, d_2 \geq 0\},\end{aligned}$$

which implies that

$$\emptyset = (\sigma_{\mathcal{P}, \mathcal{E}}^f(\zeta))^s \cap \mathcal{L}^- \subseteq \Gamma(\mathcal{S}, \zeta).$$

Therefore,  $\mathcal{E}$ -RWADQ is satisfied at  $\zeta = 0_{\mathbb{R}^2}$  for MOPVC.

By taking  $\lambda_1 = 0, \lambda_2 = 1, \lambda_1^{\mathcal{G}} = 0, \lambda_1^{\mathcal{H}} = 2$ , we get

$$\begin{aligned}0_{\mathbb{R}^2} &\in \sum_{j=1}^2 \lambda_j^f \partial^\circ(f_j + \mathcal{E}_j\|\cdot - \zeta\|)(\zeta) + (\lambda_1^{\mathcal{G}} \partial^\circ \mathcal{G}_1(\zeta) - \lambda_1^{\mathcal{H}} \partial^\circ \mathcal{H}_1(\zeta)). \\ &= 1\{(0, 2)\} - 2\{(0, 1)\},\end{aligned}$$

which verifies Corollary 2.

### 3.2. Type-1 $\mathcal{E}$ -Quasi Pareto Solutions

In this subsection, first we give a relation to interrelate an efficient solution of (MOPVC1) and a type-1 Pareto solution of (IVMOPVC).

**Theorem 7.** A point  $\zeta \in \mathcal{S}$  is a type-1 Pareto solution of the (IVMOPVC) iff  $\zeta$  is an efficient solution of the (MOPVC1).

**Proof.** Let  $\zeta$  be a type-1 Pareto solution of the (IVMOPVC). Then, there is no  $\check{x} \in \mathcal{S}$  satisfying

$$F_j(\check{x}) \leq_{LU} F_j(\zeta), \forall j \in \mathcal{P},$$

and

$$F_r(\check{x}) <_{LU} F_r(\zeta), \text{ for at least one } r \in \mathcal{P};$$

or equivalently

$$f_j^L(\check{x}) \leq f_j^L(\zeta), \quad f_j^U(\check{x}) \leq f_j^U(\zeta), \quad \forall j \in \mathcal{P}$$

and for at least one  $r \in \mathcal{P}$ , one has

$$\begin{cases} f_r^L(\check{x}) \leq f_r^L(\zeta) \\ f_r^U(\check{x}) < f_r^U(\zeta) \end{cases} \quad \text{or} \quad \begin{cases} f_r^L(\check{x}) < f_r^L(\zeta) \\ f_r^U(\check{x}) \leq f_r^U(\zeta) \end{cases} \quad \text{or} \quad \begin{cases} f_r^L(\check{x}) < f_r^L(\zeta) \\ f_r^U(\check{x}) < f_r^U(\zeta) \end{cases}. \quad (22)$$

Contrarily suppose that  $\zeta$  is not an efficient solution of the (MOPVC1). Then, there is  $\hat{x} \in \mathcal{S}$  such that

$$f(\hat{x}) \leq f(\zeta),$$

or equivalently,

$$f_j^L(\hat{x}) \leq f_j^L(\zeta), \quad f_j^U(\hat{x}) \leq f_j^U(\zeta), \quad \forall j \in \mathcal{P}$$

and for at least one  $r \in \mathcal{P}$

$$\text{either } f_r^L(\hat{x}) < f_r^L(\zeta) \quad \text{or} \quad f_r^U(\hat{x}) < f_r^U(\zeta),$$

which is a contradiction with (22).

Conversely, let  $\zeta$  be an efficient solution of the (MOPVC1). Then, there is no  $\check{x} \in \mathcal{S}$  such that

$$f_j(\check{x}) \leq f_j(\zeta), \quad \forall j \in \mathcal{P}, \quad (23)$$

with strict inequality for at least one  $j \in \mathcal{P}$ . Suppose to the contrary that  $\zeta$  is not a type-1 Pareto solution of the (IVMOPVC). Then, there exists  $\hat{x} \in \mathcal{S}$  satisfying

$$F_j(\hat{x}) \leq_{LU} F_j(\zeta), \quad \forall j \in \mathcal{P},$$

and

$$F_r(\hat{x}) <_{LU} F_r(\zeta), \quad \text{for at least one } r \in \mathcal{P};$$

or equivalently,

$$f_j^L(\hat{x}) \leq f_j^L(\zeta), \quad f_j^U(\hat{x}) \leq f_j^U(\zeta), \quad \forall j \in \mathcal{P}$$

and for at least one  $r \in \mathcal{P}$ , one has

$$\begin{cases} f_r^L(\hat{x}) \leq f_r^L(\zeta) \\ f_r^U(\hat{x}) < f_r^U(\zeta) \end{cases} \quad \text{or} \quad \begin{cases} f_r^L(\hat{x}) < f_r^L(\zeta) \\ f_r^U(\hat{x}) \leq f_r^U(\zeta) \end{cases} \quad \text{or} \quad \begin{cases} f_r^L(\hat{x}) < f_r^L(\zeta) \\ f_r^U(\hat{x}) < f_r^U(\zeta) \end{cases}$$

which contradicts (23).  $\square$

Now, we state an approximate KKT condition to identify type-1  $\mathcal{E}$ -quasi Pareto solution of the (IVMOPVC) under  $\mathcal{E}$ -IV-EADQ.

**Theorem 8.** Suppose that  $\zeta \in \mathcal{E} - S_1^q$  such that  $\mathcal{E}$ -IV-EADQ holds at  $\zeta$ . If  $\partial^\circ(f_j^L + \mathcal{E}_j^L \|\cdot - \zeta\|)(\zeta)$ ,  $\partial^\circ(f_j^U + \mathcal{E}_j^U \|\cdot - \zeta\|)(\zeta)$ ,  $\partial^\circ \mathcal{H}_i(\zeta)$ , and  $\partial^\circ \mathcal{G}_i(\zeta)$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$  are polytopes, then there exist scalars  $\lambda_j^{f^L}$ ,  $\lambda_j^{f^U}$ ,  $\lambda_i^{\mathcal{H}}$ , and  $\lambda_i^{\mathcal{G}}$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$ , such that (6)–(8) hold and

$$(\lambda_1^{f^L}, \dots, \lambda_p^{f^L}, \lambda_1^{f^U}, \dots, \lambda_p^{f^U}) > 0_{\mathbb{R}^{2p}}, \quad \sum_{j=1}^p (\lambda_j^{f^L} + \lambda_j^{f^U}) = 1. \quad (24)$$

**Proof.** Since  $\zeta \in \mathcal{E} - S_1^q$ , therefore  $\zeta \in S_1^q$  for the problem ( $\mathcal{E}$ -IVMOPVC). Hence by Theorem 7  $\zeta$  is an efficient solution of the ( $\mathcal{E}$ -MOPVC1). Moreover, since  $\mathcal{E}$ -IV-EADQ is satisfied at  $\zeta$  for (IVMOPVC), therefore IV-EADQ is satisfied at  $\zeta$  for ( $\mathcal{E}$ -IVMOPVC) and hence EADQ is satisfied at  $\zeta$  for ( $\mathcal{E}$ -MOPVC1). Further, since  $\partial^\circ(f_j^L + \mathcal{E}_j^L \|\cdot - \zeta\|)(\zeta)$ ,  $\partial^\circ(f_j^U + \mathcal{E}_j^U \|\cdot - \zeta\|)(\zeta)$ ,  $\partial^\circ \mathcal{H}_i(\zeta)$  and  $\partial^\circ \mathcal{G}_i(\zeta)$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$  are polytopes, therefore by Theorem 2, there exist scalars  $\lambda_j^{f^L}$ ,  $\lambda_j^{f^U}$ ,  $\lambda_i^{\mathcal{H}}$ , and  $\lambda_i^{\mathcal{G}}$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$  such that (10)–(12) hold along with (24). The remaining part of the proof is similar to the proof of Theorem 6 and hence the result.  $\square$

**Remark 7.** A point  $\zeta \in \mathcal{S}$  satisfies

- an approximate weak strongly stationary condition for the (IVMOPVC), denoted by  $\mathcal{E}$ -IV-W-S-SC, iff there exist scalars satisfying (6)–(9).

- A point  $\zeta \in \mathcal{S}$  satisfies an approximate strong strongly stationary condition, denoted by  $\mathcal{E}$ -IV-S-S-SC, iff there exist scalars satisfying (6)–(8) and (24).

For  $p = 1$ , both the conditions will coincide.

A corollary of the Theorem 8 with  $\mathcal{E}_j^L = \mathcal{E}_j^U = 0$  for every  $j \in \mathcal{P}$  is given as follows:

**Corollary 3.** Suppose that  $\zeta \in \mathcal{S}_1^q$  such that IV-EADQ holds at  $\zeta$ . If  $\partial^\circ f_j^L(\zeta), \partial^\circ f_j^U(\zeta), \partial^\circ \mathcal{H}_i(\zeta)$ , and  $\partial^\circ \mathcal{G}_i(\zeta)$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$  are polytopes, then there exist scalars  $\lambda_j^{f^L}, \lambda_j^{f^U}, \lambda_i^{\mathcal{H}}$ , and  $\lambda_i^{\mathcal{G}}$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$ , such that (7) and (8) along with (16) and (24) hold.

**Remark 8.** If  $\mathcal{E}_j^L = \mathcal{E}_j^U = 0$  for every  $j \in \mathcal{P}$ , then a point  $\zeta \in \mathcal{S}$  satisfies

- a weak strongly stationary condition for the (IVMOPVC), denoted by IV-W-S-SC, iff there exist scalars satisfying (7)–(9) along with (16).
- a strong strongly stationary condition for (IVMOPVC), denoted by IV-S-S-SC, iff there exist scalars satisfying (7) and (8) along with (16) and (24).

For  $p = 1$ , both the conditions will coincide.

For  $f_j^L = f_j^U = f_j$  for every  $j \in \mathcal{P}$ , a corollary of Theorem 8 is given as follows:

**Corollary 4.** Suppose that  $\zeta$  is a  $\mathcal{E}$ -efficient solution for (MOPVC) such that  $\mathcal{E}$ -EADQ holds at  $\zeta$ . If  $\partial^\circ(f_j + \mathcal{E}_j \|\cdot - \zeta\|)(\zeta), \partial^\circ \mathcal{H}_i(\zeta)$ , and  $\partial^\circ \mathcal{G}_i(\zeta)$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$  are polytopes, then there exist scalars  $\lambda_j^f, \lambda_i^{\mathcal{H}}$ , and  $\lambda_i^{\mathcal{G}}$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$ , such that (7) and (8) along with (17) hold and

$$(\lambda_1^f, \dots, \lambda_p^f) > 0_{\mathbb{R}^p}, \quad \sum_{j=1}^p \lambda_j^f = 1. \quad (25)$$

**Remark 9.** If  $f_j^L = f_j^U = f_j$  for every  $j \in \mathcal{P}$ , then a point  $\zeta \in \mathcal{S}$  satisfies

- an approximate weak strongly stationary condition for the (MOPVC), denoted by  $\mathcal{E}$ -W-S-SC, iff there exist scalars satisfying (7) and (8) along with (17) and (18).
- an approximate strong strongly stationary condition for (MOPVC), denoted by  $\mathcal{E}$ -S-S-SC, iff there exist scalars satisfying (7) and (8) along with (17) and (25).

For  $p = 1$ , both the conditions will coincide.

We are now going to introduce an approximate version of the Mangasarian-Fromovitz data qualification for the (IVMOPVC), denoted by  $\mathcal{E}$ -IV-MFDQ as follows:

**Definition 9.** We say that the  $\mathcal{E}$ -IV-MFDQ is satisfied at  $\zeta$  iff

- $0_{\mathbb{R}^n} \in \sum_{i \in \mathcal{M}_{0+}} \alpha_i \partial^\circ \mathcal{H}_i(\zeta) \implies \alpha_i = 0, \quad i \in \mathcal{M}_{0+},$
- $(\sigma_{\mathcal{P}_k, \mathcal{E}}^{f^L} \cup \sigma_{\mathcal{P}_k, \mathcal{E}}^{f^U})^s \cap (\sigma_{\mathcal{M}_{0+}}^{\mathcal{H}})^\perp \cap (-\sigma_{\mathcal{M}_{00} \cup \mathcal{M}_{0-}}^{\mathcal{H}})^s \cap (\sigma_{\mathcal{M}_{+0}}^{\mathcal{G}})^s \neq \emptyset, \quad \forall k \in \mathcal{P},$   
 $(\sigma_{\mathcal{P}, \mathcal{E}}^{f^L} \cup \sigma_{\mathcal{P}_k, \mathcal{E}}^{f^U})^s \cap (\sigma_{\mathcal{M}_{0+}}^{\mathcal{H}})^\perp \cap (-\sigma_{\mathcal{M}_{00} \cup \mathcal{M}_{0-}}^{\mathcal{H}})^s \cap (\sigma_{\mathcal{M}_{+0}}^{\mathcal{G}})^s \neq \emptyset, \quad \forall k \in \mathcal{P}.$

**Remark 10.** The  $\mathcal{E}$ -IV-MFDQ is satisfied at  $\zeta$  for (IVMOPVC) iff MFDQ is satisfied at  $\zeta$  for ( $\mathcal{E}$ -MOPVC1).

**Remark 11.** Some special cases are given as follows:

- If  $\mathcal{E}_j^L = \mathcal{E}_j^U = 0$  for all  $j \in \mathcal{P}$ , then (IVMOPVC) satisfies IV-MFDQ at  $\zeta$ .
- If  $f_j^L = f_j^U = f_j$  for all  $j \in \mathcal{P}$ , then (MOPVC) satisfies  $\mathcal{E}$ -MFDQ at  $\zeta$ .

- If  $f_j^L = f_j^U = f_j$  and  $\mathcal{E}_j^L = \mathcal{E}_j^U = 0$  for all  $j \in \mathcal{P}$ , then (MOPVC) satisfies MFDQ at  $\zeta$  as given by Sadeghieh et al. (Definition 2, [15]).

Now we give an example where MFDQ is not satisfied for the (MOPVC) at  $\zeta$ , but  $\mathcal{E}$ -MFDQ is satisfied at  $\zeta$ .

**Example 4.** Consider the following MOPVC:

$$\begin{aligned} \min f(x_1, x_2) &= (f_1(x_1, x_2), f_2(x_1, x_2)) \\ \text{s.t. } \mathcal{H}_1(x_1, x_2) &\geq 0, \\ \mathcal{H}_1(x_1, x_2) \mathcal{G}_1(x_1, x_2) &\leq 0, \end{aligned}$$

where

$$\begin{aligned} f_1(x_1, x_2) &= |x_1| - x_2, \quad f_2(x_1, x_2) = x_1^2 - x_2 + 2\sqrt{x_1^2 + x_2^2}, \\ \mathcal{H}_1(x_1, x_2) &= x_2, \quad \mathcal{G}_1(x_1, x_2) = x_2 - |x_1|. \end{aligned}$$

At a point  $\zeta = 0_{\mathbb{R}^2}$ , one has

$$\begin{aligned} \partial^\circ f_1(\zeta) &= [-1, 1] \times \{-1\}, \quad \partial^\circ f_2(\zeta) = \{(0, -1)\} + 2\mathbb{B}_{\mathbb{R}^2}, \\ \partial^\circ \mathcal{H}_1(\zeta) &= \{(0, 1)\}, \quad \partial^\circ \mathcal{G}_1(\zeta) = [-1, 1] \times \{1\}, \\ \mathcal{M} &= \mathcal{M}_{00} = \{1\}, \quad \mathcal{P} = \{1, 2\}. \end{aligned}$$

Now, observe that

$$\begin{aligned} \sigma_{\mathcal{P}_1}^f &= \partial^\circ f_2(\zeta) = \{(0, -1)\} + 2\mathbb{B}_{\mathbb{R}^2}, \\ \sigma_{\mathcal{P}_2}^f &= \partial^\circ f_1(\zeta) = [-1, 1] \times \{-1\}, \\ (\sigma_{\mathcal{P}_1}^f)^s &= \emptyset, \\ (\sigma_{\mathcal{P}_2}^f)^s &= \{(d_1, d_2) \in \mathbb{R}^2 : d_1 + d_2 > 0, d_1 - d_2 < 0, d_2 > 0\}, \\ (-\sigma_{\mathcal{M}_{00} \cup \mathcal{M}_{0-}}^{\mathcal{H}}) &= \{(0, -1)\}, \\ (-\sigma_{\mathcal{M}_{00} \cup \mathcal{M}_{0-}}^{\mathcal{H}})^s &= \{(d_1, d_2) \in \mathbb{R}^2 : d_2 > 0\}, \end{aligned}$$

Observe that

$$(\sigma_{\mathcal{P}_1}^f)^s \cap (-\sigma_{\mathcal{M}_{00} \cup \mathcal{M}_{0-}}^{\mathcal{H}})^s = \emptyset.$$

Hence, MFDQ doesn't hold at  $\zeta$ .

If we take  $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2) = (0, 2)$ , then

$$\begin{aligned} \partial^\circ (f_1 + \mathcal{E}_1 \|\cdot - \zeta\|)(\zeta) &= [-1, 1] \times \{-1\}, \\ \partial^\circ (f_2 + \mathcal{E}_2 \|\cdot - \zeta\|)(\zeta) &= \{(0, -1)\}, \\ (\sigma_{\mathcal{P}_1, \mathcal{E}}^f)^s &= \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}, \\ (\sigma_{\mathcal{P}_2, \mathcal{E}}^f)^s &= (\sigma_{\mathcal{P}_2}^f)^s. \end{aligned}$$

Moreover,

$$\begin{aligned} (\sigma_{\mathcal{P}_1, \mathcal{E}}^f)^s \cap (-\sigma_{\mathcal{M}_{00} \cup \mathcal{M}_{0-}}^{\mathcal{H}})^s &\neq \emptyset, \\ (\sigma_{\mathcal{P}_2, \mathcal{E}}^f)^s \cap (-\sigma_{\mathcal{M}_{00} \cup \mathcal{M}_{0-}}^{\mathcal{H}})^s &\neq \emptyset. \end{aligned}$$

Hence  $\mathcal{E}$ -MFDQ satisfied at  $\zeta$ .

Theorem 9 shows that  $\mathcal{E}$ -IV-MFDQ is sufficient to satisfy  $\mathcal{E}$ -Abadie type DQs without assuming  $\mathcal{M}_{00} = \emptyset$ .

**Theorem 9.** Suppose that the  $\mathcal{E}$ -IV-MFDQ holds at  $\zeta$  for (IVMOPVC) and  $\mathcal{H}_i$  is a pseudolinear function for each  $i \in \mathcal{M}_{0+}$ , then,  $\mathcal{E}$ -IV-EADQ also holds at  $\zeta$ .

**Proof.** Since the  $\mathcal{E}$ -IV-MFDQ holds at  $\zeta$  for problem (IVMOPVC), it follows from Remark 10 that MFDQ also holds at  $\zeta$  for problem ( $\mathcal{E}$ -MOPVC1). Further, as each  $\mathcal{H}_i$  is a pseudolinear function for all  $i \in \mathcal{M}_{0+}$ , Theorem 3 implies that EADQ holds at  $\zeta$  for ( $\mathcal{E}$ -MOPVC1). Hence, by Remark 5, the  $\mathcal{E}$ -IV-EADQ also holds at  $\zeta$  for (IVMOPVC).  $\square$

The following connections exist among the discussed qualification conditions:

$$\begin{array}{ccccc} \mathcal{E} - \text{IV} - \text{MFDQ} & & & & \\ \text{(Theorem 9)} \Downarrow & & & & \\ \mathcal{E} - \text{IV} - \text{EADQ} & \xrightarrow{\text{(Remark 6)}} & \mathcal{E} - \text{IV} - \text{WADQ} & \xrightarrow{\text{(Remark 6)}} & \mathcal{E} - \text{IV} - \text{RWADQ} \\ & & \text{(Remark 6)} \Uparrow & & \\ & & \mathcal{E} - \text{IV} - \text{ADQ} & & \end{array}$$

Theorem 10 (resp. Theorem 11) ensures that the MFCQ (resp.  $\mathcal{E}$ -IV-MFDQ) is indeed a valid qualification condition, and it leads to the  $\mathcal{E}$ -W-S-SC (resp.  $\mathcal{E}$ -S-S-SC) at  $\mathcal{E}$ -quasi Pareto solution of the (IVMOPVC) when the set  $\mathcal{M}_{00} = \emptyset$ .

**Theorem 10.** Suppose that  $\zeta \in \mathcal{E} - S_2^{qw}$  such that MFCQ holds at  $\zeta$ . If  $\mathcal{M}_{00} = \emptyset$ , there exist scalars  $\lambda_j^{f^L}, \lambda_j^{f^U}, \lambda_i^{\mathcal{H}}$  and  $\lambda_i^{\mathcal{G}}$ , for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$ , such that  $\mathcal{E}$ -IV-W-S-SC holds.

**Proof.** Since  $\zeta \in \mathcal{E} - S_2^{qw}$ , it follows that  $\zeta \in S_2^{qw}$  for the problem ( $\mathcal{E}$ -IVMOPVC). Hence, by Theorem 5,  $\zeta$  is an efficient solution of ( $\mathcal{E}$ -MOPVC1). Moreover, since the MFCQ is satisfied at  $\zeta$  for (IVMOPVC), it is also satisfied at  $\zeta$  for ( $\mathcal{E}$ -MOPVC1) by the definition of MFCQ. Further, as  $\mathcal{M}_{00} = \emptyset$ , Theorem 4(a) ensures the existence of scalars  $\lambda_j^{f^L}, \lambda_j^{f^U}, \lambda_i^{\mathcal{H}}$ , and  $\lambda_i^{\mathcal{G}}$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$  such that the W-S-SC hold for ( $\mathcal{E}$ -MOPVC1), that is, (10)–(13) are satisfied. The remaining part of the proof follows in the same manner as the proof of Theorem 6, and hence the result is established.  $\square$

**Theorem 11.** Suppose that  $\zeta \in \mathcal{E} - S_2^{qw}$  such that  $\mathcal{E}$ -IV-MFDQ holds at  $\zeta$ . If  $\mathcal{M}_{00} = \emptyset$ , there exist scalars  $\lambda_j^{f^L}, \lambda_j^{f^U}, \lambda_i^{\mathcal{H}}$  and  $\lambda_i^{\mathcal{G}}$ , for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$ , such that  $\mathcal{E}$ -IV-S-S-SC holds.

**Proof.** Since  $\zeta \in \mathcal{E} - S_2^{qw}$ , it follows that  $\zeta \in S_2^{qw}$  for the problem ( $\mathcal{E}$ -IVMOPVC). Hence, by Theorem 5,  $\zeta$  is an efficient solution of ( $\mathcal{E}$ -MOPVC1). Moreover, since  $\mathcal{E}$ -IV-MFDQ is satisfied at  $\zeta$  for (IVMOPVC), therefore by Remark 10 IV-MFDQ is satisfied at  $\zeta$  for ( $\mathcal{E}$ -IVMOPVC) and hence MFDQ is satisfied at  $\zeta$  for ( $\mathcal{E}$ -MOPVC1). Further, as  $\mathcal{M}_{00} = \emptyset$ , Theorem 4(b) ensures the existence of scalars  $\lambda_j^{f^L}, \lambda_j^{f^U}, \lambda_i^{\mathcal{H}}$ , and  $\lambda_i^{\mathcal{G}}$  for  $j \in \mathcal{P}$  and  $i \in \mathcal{M}$  such that the S-S-SC hold for ( $\mathcal{E}$ -MOPVC1), that is, (10)–(12) hold along with (24). The remaining part of the proof follows in the same manner as the proof of Theorem 6, and hence the result is established.  $\square$

#### 4. Sufficient Optimality Conditions for Approximate Stationary Points

In this section, we derive sufficient conditions for an approximate stationary point to be an approximate Pareto efficient solution for (IVMOPVC). To support our results,

we present an example and state several corollaries obtained by considering different conditions in the sufficient conditions.

The following approximate generalized convexity assumptions from [45,46] will be helpful to identify an approximate Pareto efficient solution for (IVMOPVC).

**Definition 10.** Let  $\epsilon \geq 0$ . A locally Lipschitz function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be

- (a)  $\epsilon - \partial^\circ$ -pseudoconvex at  $\zeta \in \mathcal{S} \subset \mathbb{R}^n$  iff the function  $\phi := h + \epsilon \|\cdot - \zeta\|$  is  $\partial^\circ$ -pseudoconvex at  $\zeta$ , i.e., for every  $\check{x} \in \mathbb{R}^n$  and  $\check{x} \neq \zeta$ , one has

$$\phi(\check{x}) < \phi(\zeta) \implies \langle \zeta_\phi^*, \check{x} - \zeta \rangle < 0, \quad \forall \zeta_\phi^* \in \partial^\circ \phi(\zeta);$$

- (b)  $\epsilon - \partial^\circ$ -quasiconvex at  $\zeta \in \mathcal{S} \subset \mathbb{R}^n$  iff the function  $\phi := h + \epsilon \|\cdot - \zeta\|$  is  $\partial^\circ$ -quasiconvex at  $\zeta$  i.e., if for every  $\check{x} \in \mathbb{R}^n$ , one has

$$\phi(\check{x}) \leq \phi(\zeta) \implies \langle \zeta_\phi^*, \check{x} - \zeta \rangle \leq 0, \quad \forall \zeta_\phi^* \in \partial^\circ \phi(\zeta),$$

where  $\partial^\circ \phi(\zeta)$  is the Clarke subdifferential of  $\phi$  at  $\zeta$ ,

Now, let  $\zeta \in \mathcal{S}$  is satisfied in  $\mathcal{E}$ -S-S-SC or  $\mathcal{E}$ -W-S-SC with corresponding multipliers  $\lambda_j^{f^L}, \lambda_j^{f^U}, \lambda_i^{\mathcal{H}}$ , and  $\lambda_i^{\mathcal{G}}$ . The following index sets will be utilize in this section:

$$\begin{aligned} \mathcal{M}_{00}^+ &:= \{i \in \mathcal{M}_{00} \mid \lambda_i^{\mathcal{H}} > 0\}. \\ \mathcal{M}_{00}^0 &:= \{i \in \mathcal{M}_{00} \mid \lambda_i^{\mathcal{H}} = 0\}. \\ \mathcal{M}_{0-}^+ &:= \{i \in \mathcal{M}_{0-} \mid \lambda_i^{\mathcal{H}} > 0\}. \\ \mathcal{M}_{0-}^0 &:= \{i \in \mathcal{M}_{0-} \mid \lambda_i^{\mathcal{H}} = 0\}. \\ \mathcal{M}_{0+}^+ &:= \{i \in \mathcal{M}_{0+} \mid \lambda_i^{\mathcal{H}} > 0\}. \\ \mathcal{M}_{0+}^- &:= \{i \in \mathcal{M}_{0+} \mid \lambda_i^{\mathcal{H}} < 0\}. \\ \mathcal{M}_{0+}^0 &:= \{i \in \mathcal{M}_{0+} \mid \lambda_i^{\mathcal{H}} = 0\}. \\ \mathcal{M}_{+0}^{0+} &:= \{i \in \mathcal{M}_{+0} \mid \lambda_i^{\mathcal{H}} = 0, \lambda_i^{\mathcal{G}} > 0\}. \\ \mathcal{M}_{+0}^{00} &:= \{i \in \mathcal{M}_{+0} \mid \lambda_i^{\mathcal{H}} = 0, \lambda_i^{\mathcal{G}} = 0\}. \end{aligned}$$

In fact, one can write

$$\begin{aligned} \mathcal{M}_{00} &= \mathcal{M}_{00}^+ \cup \mathcal{M}_{00}^0, \quad \mathcal{M}_{0-} = \mathcal{M}_{0-}^+ \cup \mathcal{M}_{0-}^0. \\ \mathcal{M}_{+0} &= \mathcal{M}_{+0}^{0+} \cup \mathcal{M}_{+0}^{00}, \quad \mathcal{M}_{0+} = \mathcal{M}_{0+}^+ \cup \mathcal{M}_{0+}^- \cup \mathcal{M}_{0+}^0. \end{aligned} \quad (26)$$

The following theorems demonstrate that  $\mathcal{E}$ -IV-W-S-SC and  $\mathcal{E}$ -IV-S-S-SC serve as sufficient optimality conditions for identifying an approximate Pareto efficient solution under the assumptions of approximate pseudoconvexity and quasiconvexity.

**Theorem 12.** Let  $0 \leq \mathcal{E}_j^L \leq \mathcal{E}_j^U$ . Suppose that there exist multipliers  $\lambda_j^{f^L}, \lambda_j^{f^U}, \lambda_i^{\mathcal{H}}$  and  $\lambda_i^{\mathcal{G}}$  such that  $\mathcal{E}$ -IV-W-S-SC is satisfied at  $\zeta \in \mathcal{S}$ . Furthermore, assume that  $\mathcal{G}_i(i \in \mathcal{M}_{+0}^{0+}), \mathcal{H}_i(i \in \mathcal{M}_{0+}^-), -\mathcal{H}_i(i \in \mathcal{M}_{0+}^+ \cup \mathcal{M}_{00}^+ \cup \mathcal{M}_{0-}^+)$  are  $\partial^\circ$ -quasiconvex at  $\zeta$  and  $f_j^L, f_j^U$  are  $\mathcal{E}_j^L - \partial^\circ$ -pseudoconvex,  $\mathcal{E}_j^U - \partial^\circ$ -pseudoconvex at  $\zeta$  for  $j \in \mathcal{P}$ , respectively.

- (i) Then  $\zeta$  is a locally type-2  $\mathcal{E}$ -quasi weakly Pareto solution for (IVMOPVC).  
(ii) If  $\mathcal{M}_{0+}^- \cup \mathcal{M}_{+0}^{0+} = \emptyset$ , then  $\zeta$  is a type-2  $\mathcal{E}$ -quasi weakly Pareto solution for (IVMOPVC).

**Proof.**

- (i) Firstly, we mention that the continuity of functions  $\mathcal{G}_i (i \in \mathcal{M}_{0+})$  and  $\mathcal{H}_i (i \in \mathcal{M}_{+0})$  implies the existence of two neighbourhoods  $\mathcal{C}$  and  $\mathcal{D}$  for  $\zeta$  such that

$$\begin{aligned} \mathcal{G}_i(\tilde{x}) &> 0, \mathcal{H}_i(\tilde{x}) = 0, \quad \forall \tilde{x} \in \mathcal{S} \cap \mathcal{C}, \quad \forall i \in \mathcal{M}_{0+}, \\ \mathcal{H}_i(\tilde{x}) &> 0, \mathcal{G}_i(\tilde{x}) \leq 0, \quad \forall \tilde{x} \in \mathcal{S} \cap \mathcal{D}, \quad \forall i \in \mathcal{M}_{+0}. \end{aligned} \quad (27)$$

Secondly, since  $\zeta \in \mathcal{S}$  is satisfied  $\mathcal{E}$ -IV-W-S-SC therefore, there exist some  $\chi_j^{f^L} \in \partial^\circ(f_j^L + \mathcal{E}_j^L \|\cdot - \zeta\|)(\zeta)$ ,  $\chi_j^{f^U} \in \partial^\circ(f_j^U + \mathcal{E}_j^U \|\cdot - \zeta\|)(\zeta)$  for  $j \in \mathcal{P}$ ,  $\chi_i^{\mathcal{H}} \in \partial^\circ \mathcal{H}_i(\zeta)$  for  $i \in \mathcal{M}_0$  and  $\chi_i^{\mathcal{G}} \in \partial^\circ \mathcal{G}_i(\zeta)$  for  $i \in \mathcal{M}_{+0}$ , such that

$$\sum_{j \in \mathcal{P}} \lambda_j^{f^L} \chi_j^{f^L} + \sum_{j \in \mathcal{P}} \lambda_j^{f^U} \chi_j^{f^U} + \sum_{i \in \mathcal{M}_0} -\lambda_i^{\mathcal{H}} \chi_i^{\mathcal{H}} + \sum_{i \in \mathcal{M}_{+0}} \lambda_i^{\mathcal{G}} \chi_i^{\mathcal{G}} = 0_{\mathbb{R}^n}. \quad (28)$$

Now, assume on the contrary that  $\zeta$  is not a local type-2  $\mathcal{E}$ -quasi weakly Pareto solution for (IVMOPVC). Thus, there exists  $\hat{x} \in \mathcal{S} \cap \mathcal{C} \cap \mathcal{D}$  such that

$$F(\hat{x}) + \mathcal{E} \|\hat{x} - \zeta\| <_{LU}^s F(\zeta).$$

Given that  $f_j^L, j \in \mathcal{P}$  and  $f_j^U, j \in \mathcal{P}$  are  $\mathcal{E}_j^L - \partial^\circ$ -pseudoconvex and  $\mathcal{E}_j^U - \partial^\circ$ -pseudoconvex at  $\zeta$ , respectively, then

$$\langle \chi_j^{f^L}, \hat{x} - \zeta \rangle < 0, \quad \forall \chi_j^{f^L} \in \partial^\circ(f_j^L + \mathcal{E}_j^L \|\cdot - \zeta\|)(\zeta), \quad \forall j \in \mathcal{P},$$

$$\langle \chi_j^{f^U}, \hat{x} - \zeta \rangle < 0, \quad \forall \chi_j^{f^U} \in \partial^\circ(f_j^U + \mathcal{E}_j^U \|\cdot - \zeta\|)(\zeta), \quad \forall j \in \mathcal{P}.$$

From this and (9) we conclude that

$$\sum_{j \in \mathcal{P}} \lambda_j^{f^L} \langle \chi_j^{f^L}, \hat{x} - \zeta \rangle < 0, \quad \forall \chi_j^{f^L} \in \partial^\circ(f_j^L + \mathcal{E}_j^L \|\cdot - \zeta\|)(\zeta), \quad \forall j \in \mathcal{P},$$

$$\sum_{j \in \mathcal{P}} \lambda_j^{f^U} \langle \chi_j^{f^U}, \hat{x} - \zeta \rangle < 0, \quad \forall \chi_j^{f^U} \in \partial^\circ(f_j^U + \mathcal{E}_j^U \|\cdot - \zeta\|)(\zeta), \quad \forall j \in \mathcal{P}.$$

and hence

$$\sum_{i \in \mathcal{M}_0} -\lambda_i^{\mathcal{H}} \langle \chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle + \sum_{i \in \mathcal{M}_{+0}} \lambda_i^{\mathcal{G}} \langle \chi_i^{\mathcal{G}}, \hat{x} - \zeta \rangle > 0, \quad (29)$$

by (28). Also, the  $\partial^\circ$ -quasiconvexity of  $\mathcal{G}_i (i \in \mathcal{M}_{+0}^+)$  and  $\mathcal{H}_i (i \in \mathcal{M}_{0+}^-)$  functions and (27) deduce that:

$$\mathcal{G}_i(\hat{x}) \leq 0 = \mathcal{G}_i(\zeta) \implies \langle \chi_i^{\mathcal{G}}, \hat{x} - \zeta \rangle \leq 0, \quad \forall i \in \mathcal{M}_{+0}^+, \quad (30)$$

$$\mathcal{H}_i(\hat{x}) = 0 = \mathcal{H}_i(\zeta) \implies \langle \chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle \leq 0, \quad \forall i \in \mathcal{M}_{0+}^-. \quad (31)$$

On the other hand, since  $-\mathcal{H}_i(\hat{x}) \leq 0 = -\mathcal{H}_i(\zeta)$  and  $-\chi_i^{\mathcal{H}} \in \partial^\circ(-\mathcal{H}_i)(\zeta)$  for  $i \in \mathcal{M}_{0+}^+ \cup \mathcal{M}_{00}^+ \cup \mathcal{M}_{0-}^+$ , the  $\partial^\circ$ -quasiconvexity of  $-\mathcal{H}_i$  implies that

$$\langle -\chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle \leq 0, \quad i \in \mathcal{M}_{0+}^+ \cup \mathcal{M}_{00}^+ \cup \mathcal{M}_{0-}^+. \quad (32)$$

Involving (26), we get the following four inequalities:

$$\begin{aligned} \sum_{i \in \mathcal{M}_{0+}} -\lambda_i^{\mathcal{H}} \langle \chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle &= \sum_{i \in \mathcal{M}_{0+}^+} \overbrace{\lambda_i^{\mathcal{H}} \langle -\chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle}^{>0} + \sum_{i \in \mathcal{M}_{0+}^-} \overbrace{-\lambda_i^{\mathcal{H}} \langle \chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle}^{<0} \\ &\quad + \sum_{i \in \mathcal{M}_{0+}^0} \overbrace{-\lambda_i^{\mathcal{H}} \langle \chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle}^{=0} \leq 0, \\ \sum_{i \in \mathcal{M}_{00}} -\lambda_i^{\mathcal{H}} \langle \chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle &= \sum_{i \in \mathcal{M}_{00}^+} \overbrace{\lambda_i^{\mathcal{H}} \langle -\chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle}^{>0} + \sum_{i \in \mathcal{M}_{00}^0} \overbrace{-\lambda_i^{\mathcal{H}} \langle \chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle}^{=0} \leq 0, \\ \sum_{i \in \mathcal{M}_{0-}} -\lambda_i^{\mathcal{H}} \langle \chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle &= \sum_{i \in \mathcal{M}_{0-}^+} \overbrace{\lambda_i^{\mathcal{H}} \langle -\chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle}^{>0} + \sum_{i \in \mathcal{M}_{0-}^0} \overbrace{-\lambda_i^{\mathcal{H}} \langle \chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle}^{=0} \leq 0, \end{aligned}$$

and

$$\sum_{i \in \mathcal{M}_{+0}} \lambda_i^{\mathcal{G}} \langle \chi_i^{\mathcal{G}}, \hat{x} - \zeta \rangle = \sum_{i \in \mathcal{M}_{+0}^+} \overbrace{\lambda_i^{\mathcal{G}} \langle \chi_i^{\mathcal{G}}, \hat{x} - \zeta \rangle}^{>0} + \sum_{i \in \mathcal{M}_{+0}^0} \overbrace{\lambda_i^{\mathcal{G}} \langle \chi_i^{\mathcal{G}}, \hat{x} - \zeta \rangle}^{=0} \leq 0.$$

Since  $\mathcal{M}_0 = \mathcal{M}_{0+} \cup \mathcal{M}_{00} \cup \mathcal{M}_{0-}$ , adding these four inequalities, we deduce that

$$\sum_{i \in \mathcal{M}_0} -\lambda_i^{\mathcal{H}} \langle \chi_i^{\mathcal{H}}, \hat{x} - \zeta \rangle + \sum_{i \in \mathcal{M}_{+0}} \lambda_i^{\mathcal{G}} \langle \chi_i^{\mathcal{G}}, \hat{x} - \zeta \rangle \leq 0,$$

which contradicts (29). This contradiction proves (i).

- (ii) By assumption of  $\mathcal{M}_{0+} \cup \mathcal{M}_{+0}^0 = \emptyset$ , we can remove the neighbourhoods  $\mathcal{C}$  and  $\mathcal{D}$  from the proof of (i), which will allow us to obtain the desired result.

□

**Example 5.** In Example 2, Observe that

- $\zeta = (0, 0)$  is a  $\mathcal{E}$ -IV-weak strongly stationary point with  $\mathcal{M} = \mathcal{M}_{00} = \{1\}$ , and other indexing are empty.
- $-\mathcal{H}_1(1 \in \mathcal{M}_{00}^+)$  is  $\partial^\circ$ -quasiconvex at  $\zeta$ .
- for  $\mathcal{E}_1^L = 2, \mathcal{E}_1^U = 3, \mathcal{E}_2^L = 1$ , and  $\mathcal{E}_2^U = 2$  the functions  $f_1^L$  is  $\mathcal{E}_1^L - \partial^\circ$ -pseudoconvex,  $f_1^U$  is  $\mathcal{E}_1^U - \partial^\circ$ -pseudoconvex,  $f_2^L$  is  $\mathcal{E}_2^L - \partial^\circ$ -pseudoconvex,  $f_2^U$  is  $\mathcal{E}_2^U - \partial^\circ$ -pseudoconvex.

Hence by Theorem 12,  $\zeta$  is a locally type-2  $\mathcal{E}$ -quasi weakly Pareto solution for the given problem. Also, since  $\mathcal{M}_{0+}^- \cup \mathcal{M}_{+0}^0 = \emptyset$  therefore  $\zeta$  is a type-2  $\mathcal{E}$ -quasi weakly Pareto solution for the given problem.

A corollary of the Theorem 12 for  $\mathcal{E}_j^L = \mathcal{E}_j^U = 0$  for every  $j \in \mathcal{P}$  is given as follows:

**Corollary 5.** Suppose that there exist multipliers  $\lambda_j^{f^L}, \lambda_j^{f^U}, \lambda_i^{\mathcal{H}}$  and  $\lambda_i^{\mathcal{G}}$  such that IV-W-S-SC is satisfied at  $\zeta \in \mathcal{S}$ . Furthermore, assume that  $\mathcal{G}_i(i \in \mathcal{M}_{+0}^0), \mathcal{H}_i(i \in \mathcal{M}_{0+}^-), -\mathcal{H}_i(i \in \mathcal{M}_{0+}^+ \cup \mathcal{M}_{00}^+ \cup \mathcal{M}_{0-}^+)$  are  $\partial^\circ$ -quasiconvex at  $\zeta$  and  $f_j^L, f_j^U$  are  $\partial^\circ$ -pseudoconvex at  $\zeta$  for  $j \in \mathcal{P}$ .

- (i) Then  $\zeta$  is a locally type-2-quasi weakly Pareto solution for (IVMOPVC).  
(ii) If  $\mathcal{M}_{0+}^- \cup \mathcal{M}_{+0}^0 = \emptyset$ , then  $\zeta$  is a type-2-quasi weakly Pareto solution for (IVMOPVC).

For  $f_j^L = f_j^U = f_j$  for every  $j \in \mathcal{P}$ , a corollary of Theorem 12 is given as follows:

**Corollary 6.** Let  $0 \leq \mathcal{E}_j^L \leq \mathcal{E}_j^U$ . Suppose that there exist multipliers  $\lambda_j^f, \lambda_i^{\mathcal{H}}$  and  $\lambda_i^{\mathcal{G}}$  such that  $\mathcal{E}$ -W-S-SC is satisfied at  $\zeta \in \mathcal{S}$ . Furthermore, assume that  $\mathcal{G}_i(i \in \mathcal{M}_{+0}^{0+}), \mathcal{H}_i(i \in \mathcal{M}_{0+}^-), -\mathcal{H}_i(i \in \mathcal{M}_{0+}^+ \cup \mathcal{M}_{00}^+ \cup \mathcal{M}_{0-}^+)$  are  $\partial^\circ$ -quasiconvex at  $\zeta$  and  $f_j$  are  $\mathcal{E}_j - \partial^\circ$ -pseudoconvex at  $\zeta$  for  $j \in \mathcal{P}$ , respectively.

- (i) Then  $\zeta$  is a locally  $\mathcal{E}$ -weakly efficient solution for (MOPVC).
- (ii) If  $\mathcal{M}_{0+}^- \cup \mathcal{M}_{+0}^{0+} = \emptyset$ , then  $\zeta$  is a  $\mathcal{E}$ -weakly efficient solution for (MOPVC).

The proof of the following sufficient condition is similar to that of Theorem 12, therefore, it is omitted here.

**Theorem 13.** Let  $0 \leq \mathcal{E}_j^L \leq \mathcal{E}_j^U$ . Suppose that there exist multipliers  $\lambda_j^{f^L}, \lambda_j^{f^U}, \lambda_i^{\mathcal{H}}$  and  $\lambda_i^{\mathcal{G}}$  such that  $\mathcal{E}$ -IV-S-S-SC is satisfied at  $\zeta \in \mathcal{S}$ . Furthermore, assume that  $\mathcal{G}_i(i \in \mathcal{M}_{+0}^{0+}), \mathcal{H}_i(i \in \mathcal{M}_{0+}^-), -\mathcal{H}_i(i \in \mathcal{M}_{0+}^+ \cup \mathcal{M}_{00}^+ \cup \mathcal{M}_{0-}^+)$  are  $\partial^\circ$ -quasiconvex at  $\zeta$  and  $f_j^L, f_j^U$  are  $\mathcal{E}_j^L - \partial^\circ$ -pseudoconvex,  $\mathcal{E}_j^U - \partial^\circ$ -pseudoconvex at  $\zeta$  for  $j \in \mathcal{P}$ , respectively.

- (i) Then  $\zeta$  is a locally type-1  $\mathcal{E}$ -quasi Pareto solution for (IVMOPVC).
- (ii) If  $\mathcal{M}_{0+}^- \cup \mathcal{M}_{+0}^{0+} = \emptyset$ , then  $\zeta$  is a type-1  $\mathcal{E}$ -quasi Pareto solution for (IVMOPVC).

## 5. Conclusions

In this paper, we consider a nonsmooth interval-valued multiobjective optimization problem with vanishing constraints (IVMOPVC). We derive approximate versions of some CQs given in [15] for (IVMOPVC), and use them to establish approximate necessary strongly stationary conditions. Further, by employing approximate generalized convexity assumptions, we obtain approximate strongly sufficient conditions. The results presented here extend those in [15], and several corollaries are provided to highlight the relationships between our findings and the existing results. The theoretical developments are well supported by examples. The results of this paper can be applied to other types of optimization problems, such as: semidefinite programming [34] with interval-valued objective functions, robust (MOPVC) under data uncertainty [47] with interval-valued objective functions, robust multiobjective fractional semi-infinite programming [48] with interval-valued objective functions, interval-valued multiobjective problems based on convex cones [49] and other related problems. Moreover, it would be an interesting research problem to investigate constraint qualifications and optimality conditions for (IVMOPVC) in the framework of Hadamard manifold [50].

**Author Contributions:** Conceptualization, A.D., V.L., M.-M.B. and A.-D.H.; Methodology, A.D., V.L., M.-M.B. and A.-D.H.; Formal analysis, A.D.; Investigation, A.D., V.L., M.-M.B. and A.-D.H.; Writing—original draft, A.D., V.L., M.-M.B. and A.-D.H. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research of the first author is supported by the Joint CSIR-UGC Junior Research Fellowship funded by University Grants Commission, India via NTA reference No. 201610153253.

**Data Availability Statement:** The authors affirm that data sharing does not apply to this article since no datasets were generated or analyzed during the current study.

**Conflicts of Interest:** The authors declare that there are no actual or potential conflicts of interest in relation to this article.

## Abbreviations

The following abbreviations are used in this manuscript:

|                          |                                                                                |
|--------------------------|--------------------------------------------------------------------------------|
| CQ                       | Constraint Qualification                                                       |
| DQ                       | Data Qualification                                                             |
| MOPVC                    | Multiobjective optimization problem with vanishing constraints                 |
| IVMOPVC                  | Interval-valued multiobjective optimization problem with vanishing constraints |
| ADQ                      | Abadie data qualification                                                      |
| WADQ                     | Weak Abadie data qualification                                                 |
| RWADQ                    | Refined weak Abadie data qualification                                         |
| EADQ                     | Extended Abadie data qualification                                             |
| MFDQ                     | Mangasarian-Fromovitz data qualification                                       |
| MFCQ                     | Mangasarian-Fromovitz constraint qualification                                 |
| W-S-SC                   | Weak strongly stationary condition for (MOPVC)                                 |
| S-S-SC                   | Strong strongly stationary condition for (MOPVC)                               |
| $\mathcal{E}$ -IV-ADQ    | $\mathcal{E}$ -Abadie data qualification for (IVMOPVC)                         |
| $\mathcal{E}$ -IV-WADQ   | Weak $\mathcal{E}$ -Abadie data qualification for (IVMOPVC)                    |
| $\mathcal{E}$ -IV-RWADQ  | Refined $\mathcal{E}$ -weak Abadie data qualification for (IVMOPVC)            |
| $\mathcal{E}$ -IV-EADQ   | Extended $\mathcal{E}$ -Abadie data qualification for (IVMOPVC)                |
| $\mathcal{E}$ -IV-MFDQ   | $\mathcal{E}$ -Mangasarian-Fromovitz data qualification for (IVMOPVC)          |
| $\mathcal{E}$ -IV-W-S-SC | Approximate weak strongly stationary condition for (IVMOPVC)                   |
| $\mathcal{E}$ -IV-S-S-SC | Approximate strong strongly stationary condition for (IVMOPVC)                 |

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