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Chaos and Bifurcations in the Dynamics of the Variable-Order Fractional Rössler System

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Abstract

This article investigates the chaotic features of a novel variable-order fractional Rössler system built with Liouville–Caputo derivatives of variable order. Variable-order fractional (VOF) operators incorporated in the system render its dynamics more flexible and richer with memory and hereditary effects. We run numerical simulations to see how different fractional-order functions alter the qualitative behavior of the system. We demonstrate this via phase portraits and time-series responses. The research analyzes bifurcation development, chaotic oscillations, and stability transition and demonstrates dynamic patterns impossible to describe with integer-order models. Lyapunov exponent analysis also demonstrates system sensitivity to initial conditions and small disturbances. The outcomes confirm that the variable-order procedure provides a faithful representation of nonlinear and intricate processes of engineering and physical sciences, pointing out the dominant role of memory effects on the transitions among periodic, quasi-periodic, and chaotic regimes.

Keywords: fractional derivatives; chaos; bifurcation; simulation; dynamic system

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1. Introduction

Fractional calculus has, in recent years, attracted considerable attention for the purpose of extending the routine calculus operations of integration and differentiation from integer orders to fractional orders. Fractional calculus is a mathematical theory that yields more realistic descriptions of extremely nonlinear dynamic behavior than classical mathematical models [1,2].

Fractional-order systems allow for enhanced capability and closer realism for the modeling of biological [3], engineering [4], and physical processes where memory effects or long-range dependence [5,6] are relevant. Unlike their integer-order counterparts, fractional formulations naturally include inherent memory and long-range dependence features of a process and, thus, are able to capture the real dynamic behavior more accurately. Fractional models have been found to be very effective in describing complex dynamics, including chaotic dynamics, with better fidelity than their classical counterparts [7,8].

Fractional-order systems exhibit a notable ability to capture memory impacts and high-range dynamics. The use of a fixed-order derivative may be inappropriate for systems exhibiting time-varying memory characteristics [9]. To avoid this drawback, the concept of variable-order fractional derivatives developed, which enables the order of differentiation to vary with time, space, or other governing variables [10,11]. This advancement allows for the better depiction of systems with non-uniform memory characteristics [12,13], thereby broadening the utilization of fractional calculus in various fields [14,15].

To examine the stability, bifurcations, and chaotic dynamics for variable-order fractional systems [16], numerical solutions must be achieved through various methods [17,18]. Fractional chaotic systems are being utilized more and more in practical applications, which reflects how useful they are becoming [19,20].

The flowchart in Figure 1 is a simple way to contemplate the fractional Rössler system. The procedure is separated into different phases of analysis that make it easier to analyze dynamical systems thoroughly. The diagram's left part shows how to methodically explore the Variable Fractional Dynamical System (VFDS), i.e., stability checking and formulation of research hypotheses.

An overall overview of chaotic behavior, the computation of Lyapunov foundations, bifurcation studies, phase trajectories, and time-series analysis are presented on the right side of the chart. This side of the chart is typically used to present the information. The combination of theoretical modeling, computational modeling, and empirical verification through expert analytical approaches is what makes this methodical approach so effective at enhancing multimodal comprehension.

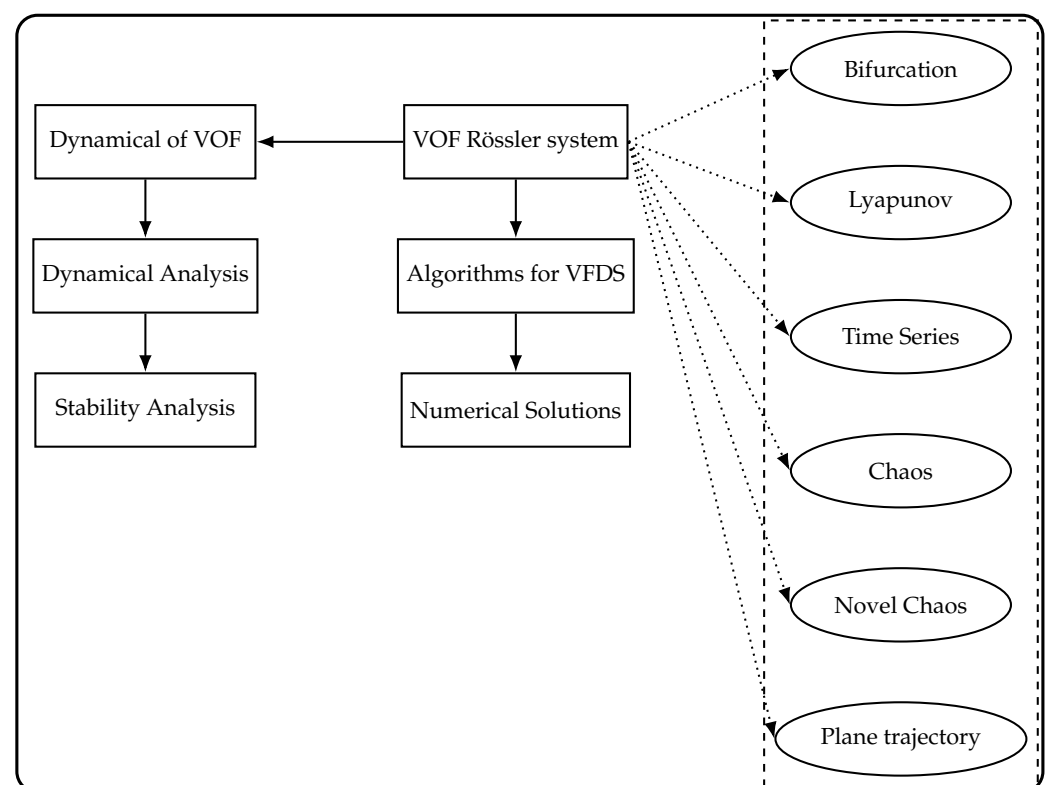


Figure 1. The procedure for analyzing the dynamic properties of the Rössler system.

Table 1 and Figure 1 show the stunning variety of various research methods employed to explore the four-dimensional hyperchaotic fractional Rössler system.

Many tools are available to examine dynamic behavior [21,22], such as system definitions [23], bifurcation (BIF) phenomena [24], broad characterizations of chaotic phenomena (Chaos) [25], and novel chaotic phenomena (NChaos) [26]. Subsequent investigations, es-

pecially [27], have incorporated more sophisticated analyses, e.g., symmetry analysis (SA) and time-series (TSER) analyses. This is the only research that encompasses a broad array of dynamical properties, including symmetry analysis, time series, Lyapunov exponents (LY), bifurcation (BIF), numerical simulations (NSIM), stability analysis (SAN), and a phase portrait (PHPR), making it broad in its methodological scope. It differs from the available references in the depth of its comprehensiveness and variety of dynamic analyses.

Table 1. Overview of research on fractional-order Rössler systems and chaotic dynamics.

References	FDS	BIF	VFDS	LY	Chaos	TSER	NChaos	SA	NSIM	SAN	PHPR
[21]	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗
[22]	✓	✗	✗	✓	✗	✗	✗	✗	✗	✗	✗
[23]	✓	✗	✗	✗	✗	✗	✗	✗	✓	✗	✗
[24]	✓	✓	✗	✓	✗	✗	✗	✗	✗	✗	✗
[25]	✓	✓	✗	✗	✓	✗	✗	✗	✗	✗	✗
[26]	✓	✓	✗	✗	✓	✗	✗	✗	✗	✗	✗
[27]	✓	✗	✗	✗	✓	✓	✓	✓	✗	✗	✗
This study	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

This paper introduces a variable-order fractional Rössler system with Liouville–Caputo derivatives, in which the fractional order (α) varies with time to show memory effects. The main innovation is the time-dependent order, which enables the model to capture nonstationary dynamics and hereditary behavior that constant-order systems cannot. This allows for analysis of nonlinear systems with time-dependent memory intensity in a more realistic and flexible way.

The paper is structured as follows: Section 1 talks about the subject. Sections 2–5 talk about the variable-order fractional Rössler system, how to analyze it, and how to use numbers to do so. Sections 6–9 talk about bifurcation, chaos, Lyapunov exponents, and results from time series. Section 10 presents a discussion, and last section reports conclusions.

2. Variable-Order Fractional Rössler System

This part presents formulations of the constant and variable fractional derivatives in the Liouville–Caputo fractional sense, which constitute the foundation of the suggested model.

Definition 1 ([28,29]). *The constant-order Liouville–Caputo (LC) fractional derivative of a function ($\sigma(t)$) is defined as*

$${}^{\text{LC}}D_t^\alpha \sigma(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\sigma'(\xi)}{(t-\xi)^\alpha} d\xi, \quad 0 < \alpha < 1, \quad (1)$$

where $\sigma'(t) = d\sigma/dt$, α is the constant fractional order, $\Gamma(\cdot)$ is the Gamma function, and ξ is the dummy variable of integration.

Definition 2 ([30]). *The variable-order Liouville–Caputo (LCV) fractional derivative of a function ($\sigma(t)$) with time-dependent order ($\alpha(t)$) is defined as*

$${}^{\text{LCV}}D_0^{\alpha(t)} \sigma(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t \frac{\sigma'(\xi)}{(t-\xi)^{\alpha(t)}} d\xi, \quad 0 < \alpha(t) < 1, \quad (2)$$

where $\sigma'(t) = d\sigma/dt$, $\alpha(t)$ is the variable fractional order, $\Gamma(\cdot)$ is the Gamma function, and ξ is the dummy variable of integration.

Here, we consider Equation (3), the variable-order fractional Rössler system [31] in which the fractional derivative order ($\alpha(t)$) varies with time to simulate time-varying memory and heredity of the system. In the usual Rössler model, whose dynamics are governed by constant-order derivatives, rendering the fractional order time-dependent enables the real-time modulation of the strength of its memory. This ability makes it possible for the model to account for transitions between different dynamical regimes, i.e., periodic, quasi-periodic, and chaotic oscillations, based on the time evolution of $\alpha(t)$. The time-dependent order actually modulates the system's dissipation and resonates or annihilates chaotic oscillations, providing an improved methodology for handling nonstationary and memory-dependent processes in complex dynamical systems.

$$\begin{aligned}\mathcal{D}^{\alpha(t)}u_1(t) &= -u_2(t) - u_3(t), \\ \mathcal{D}^{\alpha(t)}u_2(t) &= u_1(t) + a_1 u_2(t), \\ \mathcal{D}^{\alpha(t)}u_3(t) &= a_2 + (u_1(t) - a_3)u_3(t).\end{aligned}\tag{3}$$

We have state variables ($u_1(t), u_2(t), u_3(t)$) and three control parameters ($a_1 = 0.45$, $a_2 = 2$, and $a_3 = 4$) in [32].

3. Dynamical Analysis

Dynamical systems of the fractional order have garnered increasing interest in recent years due to the fact that they are capable of simulating complicated real life more effectively than conventional integer-order systems. They are especially suitable for describing hereditary characteristics and memory effects inherent in biological and physical phenomena. This section explains a dynamical system controlled by fractional Caputo derivatives represented by Equation (3). For a dynamical system E(uation (3)) with strictly positive ICs ($u_1(0) > 0$, $u_2(0) > 0$, and $u_3(0) > 0$), there exists a unique solution on the entire domain ($t \in [0, \infty)$). We define the state vector as $\mathbf{U} = (u_1, u_2, u_3)$, with the vector field expressed as

$$\mathbf{Q}(\mathbf{U}) = (Q_1(\mathbf{U}), Q_2(\mathbf{U}), Q_3(\mathbf{U})),$$

given by

$$\begin{aligned}Q_1(\mathbf{U}) &= -u_2(\tau) - u_3(\tau), \\ Q_2(\mathbf{U}) &= u_1(\tau) + a_1 u_2(\tau), \\ Q_3(\mathbf{U}) &= a_2 + u_3(\tau)(u_1(\tau) - a_3).\end{aligned}\tag{4}$$

This system can be expressed in autonomous form as $\dot{\mathbf{U}} = \mathbf{Q}(\mathbf{U})$, where $\mathbf{Q} : \mathbb{R}^+ \rightarrow \mathbb{R}^3$ maps the positive real numbers into three-dimensional space and each component (Q_i) is infinitely differentiable (i.e., $Q_i \in \mathcal{C}^\infty(\mathbb{R})$). The $\mathbf{Q}$ function is continuous and locally Lipschitz in the positive orthant.

$$W = \{(u_1, u_2, u_3) \mid u_i > 0, i = 1, 2, 3\}.$$

Consequently, according to the theorem of Picard–Lindelöf, the system containing preliminary data (W) possesses a unique global solution.

4. Analysis of a Fractional-Order System

This section examines the qualitative dynamics of the system in Equation (3) under diverse initial conditions, parameter values, and fractional orders ($\alpha(t)$), focusing on identifying chaotic attractors that reveal the complex oscillatory behavior inherent to the system.

The Jacobian matrix of the system in a generic state (u_1, u_2, u_3) is

$$J(u_1, u_2, u_3) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a_1 & 0 \\ u_3 & 0 & u_1 - a_3 \end{bmatrix},$$

with parameters of $a_1 = 0.45$, $a_2 = 2$, and $a_3 = 4$.

4.1. Equilibria and Stability

The equilibrium points can first be expressed parametrically in terms of the system parameters (a_1, a_3) :

$$E_0 = (u_1^0, u_2^0, u_3^0), \quad E_1 = (u_1^1, u_2^1, u_3^1), \quad (5)$$

where u_i^j represents functions of the system parameters. With the chosen parameters, the equilibria become

$$E_0 = (0.007, -0.035, 0.035), \quad E_1 = \left(\frac{283}{50}, -\frac{283}{10}, \frac{283}{10}\right). \quad (6)$$

The corresponding Jacobians are

$$J(E_0) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0.035 & 0 & -5.693 \end{bmatrix}, \quad J(E_1) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 28.3 & 0 & -0.04 \end{bmatrix}.$$

The characteristic polynomials are

$$\lambda^3 + 5.493\lambda^2 - 0.1036\lambda + 5.686 = 0 \quad \text{at } E_0,$$

$$-\lambda^3 + \frac{4}{25}\lambda^2 - \frac{7323}{250}\lambda + \frac{281}{50} = 0 \quad \text{at } E_1.$$

The approximate eigenvalues are

$$\lambda_1 \approx -5.7, \quad \lambda_{2,3} \approx 0.052 \pm 0.998i \quad \text{at } E_0,$$

$$\lambda_1 \approx -0.016 + 5.413i, \quad \lambda_2 \approx -0.016 - 5.413i, \quad \lambda_3 \approx 0.192 \quad \text{at } E_1,$$

indicating that both equilibria are unstable, with E_1 exhibiting saddle-focus behavior.

For the fractional-order system, stability can be analyzed using

$$\Delta(\lambda) = \text{diag}(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \lambda^{M\alpha_3}) - J,$$

where α_i denotes the fractional orders, J is the Jacobian at the equilibrium, and M is the least common multiple of their denominators. The system is stable if

$$|\arg(\lambda_i)| > \frac{\pi}{2M} \quad \forall i.$$

If this condition is violated, the system exhibits chaotic dynamics:

$$\frac{\pi}{2M} - \min_i |\arg(\lambda_i)| \geq 0 \Rightarrow \text{chaos}.$$

Results in Table 2 were obtained by first calculating equilibria of the fractional-order system parametrically, then substituting the chosen control parameters, i.e., $a_1 = 0.45$, $a_2 = 2$, and $a_3 = 4$. Then, the Jacobian matrix was evaluated at each of those equilibria, and its eigenvalues were computed to assess the local stability. For $\alpha < 0.96$, all eigenvalues

of E_0 have negative real parts, yielding a stable spiral sink. For $\alpha = 0.91$, E_1 possesses one eigenvalue with a positive real part and complex conjugates, resulting in bounded chaotic motion. When $\alpha = 1.0$, the system exhibits two-scroll chaos because the presence of unstable eigenvalues combines with the full integer-order dynamics.

Table 2. Dynamics of a fractional system for different fractional orders (α).

α	Equilibrium	Eigenvalues	Stability	Observed Dynamics
<0.96	E_0	$-5.7, 0.052 \pm 0.998i$	Stable	Spiral sink
1.0	E_1	$-0.016 \pm 5.413i, 0.192$	Unstable	Two-scroll chaos
0.91	E_1	$-0.016 \pm 5.413i, 0.192$	Chaotic	Bounded chaos

4.2. Analysis of Variable-Order Fractional System

This section investigates the stability analysis of the following three-dimensional variable-order fractional system in Equation (3) with parameters of $a_1 = 0.45$, $a_2 = 2$, and $a_3 = 4$ and subject to the ICs of $u_1(0) = 0.1$, $u_2(0) = 0.1$, and $u_3(0) = 0.1$. System (3) can be expressed in compact matrix form as

$${}^{\text{LC}}D_t^{\alpha(t)} \mathbf{u}(t) = A\mathbf{u}(t) + F(\mathbf{u}(t)), \quad (7)$$

where $\mathbf{u}(t) = (u_1, u_2, u_3)^T$,

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a_1 & 0 \\ 0 & 0 & -a_3 \end{bmatrix}, \quad F(\mathbf{u}(t)) = \begin{bmatrix} 0 \\ 0 \\ a_2 + u_1 u_3 \end{bmatrix}.$$

Here, the linear dynamics are governed by the matrix (A), whereas $F(\mathbf{u}(t))$ represents the nonlinear terms. By analogy with the general stability framework for variable-order fractional systems, the stability of system (7) can be analyzed through the spectral properties of A . In particular, the equilibrium point is asymptotically stable if all eigenvalues (λ_j) of A satisfy

$$|\arg(\lambda_j)| > \frac{\pi}{2} \sup_t \alpha(t), \quad (8)$$

for each $\lambda_j \in \text{spec}(A)$.

From this condition, it follows that the variable fractional order ($\alpha(t)$) will not destabilize the linearized dynamics, and the memory effects will remain bounded. The behavior of the system will then be dominated by the linear part, local perturbations will decay in time, and asymptotic stability will be realized under the above spectral constraint.

Theoretical and applied studies confirm that variable-order operators significantly enhance the modeling capability of fractional systems, particularly for nonlinear and chaotic dynamics. Comprehensive analyses on the existence, uniqueness, numerical schemes, and stability of variable-order systems can be found in [33–36].

5. Numerical Algorithms for Fractional Systems of Variable Order

In this section, we use numerical methods from [37] to approximate the solutions of Liouville–Caputo models. The system is defined as

$${}^*D_{0,t}^{\alpha(t)} q(t) = \mathcal{Q}(q(t)), \quad t \in [0, a], \quad q(0) = q_0, \quad (9)$$

where $*$ represents a variable-order fractional derivative that has been selected, q_0 is the starting value, and q_r approximates $q(t)$ at $t = t_r$ with a uniform step size of $\Delta t = 0.05$ over $[0, a]$.

According to the Algorithms in [37], Equation (9) can be written in the following form:

$$q(t) - q(0) = \frac{1}{\Gamma(q(t))} \int_0^t \phi(\xi, q(\xi))(t - \xi)^{q(t)-1} d\xi. \quad (10)$$

Evaluating at $t = t_{n+1}$, this becomes

$$q(t_{n+1}) - q(0) = \frac{1}{\Gamma(q(t_{n+1}))} \sum_{r=0}^n \int_{t_r}^{t_{r+1}} \phi(\xi, q(\xi))(t_{n+1} - \xi)^{q(t)-1} d\xi. \quad (11)$$

The $\phi(\xi, q(\xi))$ on the interval $[t_{\xi}, t_{\xi+1}]$ can be approximated using second-order Lagrange interpolation:

$$\mathcal{L}_{\xi}(\xi) \approx \frac{\phi(t_{\xi}, q_{\xi})}{h} (\xi - t_{\xi-1}) - \frac{\phi(t_{\xi-1}, q_{\xi-1})}{h} (\xi - t_{\xi}), \quad (12)$$

where $h = t_{\xi+1} - t_{\xi}$.

Substituting $\mathcal{L}_{\xi}(\xi)$ into (11), we obtain the following:

$$q_{n+1}(t) = q_0 + \frac{1}{\Gamma(q(t_{n+1}))} \sum_{\xi=0}^n \left[\frac{\phi(t_{\xi}, q_{\xi})}{h} \int_{t_{\xi}}^{t_{\xi+1}} (\xi - t_{\xi-1})(t_{n+1} - \xi)^{q(t)-1} d\xi - \frac{\phi(t_{\xi-1}, q_{\xi-1})}{h} \int_{t_{\xi}}^{t_{\xi+1}} (\xi - t_{\xi})(t_{n+1} - \xi)^{q(t)-1} d\xi \right]. \quad (13)$$

For computational convenience, we define

$$\mathcal{A}_{q(t), \xi, 1} = \frac{h^{q(t)+1}}{q(t) \Gamma(q(t) + 1)} \left[(n + 1 - \xi)^{q(t)} (n - \xi + 2 + q(t)) - (n - \xi)^{q(t)} (n - \xi + 2 + 2q(t)) \right]. \quad (14)$$

$$\mathcal{A}_{q(t), \xi, 2} = \frac{h^{q(t)+1}}{q(t) \Gamma(q(t) + 1)} \left[(n + 1 - \xi)^{q(t)+1} - (n - \xi)^{q(t)} (n - \xi + 1 + q(t)) \right]. \quad (15)$$

Using these, the solution turns becomes

$$q_{n+1}(t) = q(0) + \frac{1}{\Gamma(q(t_{n+1}))} \sum_{\xi=0}^n \left[\frac{h^{q(t)} \phi(t_{\xi}, q_{\xi})}{q(t) \Gamma(q(t) + 1)} \left((n + 1 - \xi)^{q(t)} (n - \xi + 2 + q(t)) - (n - \xi)^{q(t)} (n - \xi + 2 + 2q(t)) \right) - \frac{h^{q(t)} \phi(t_{\xi-1}, q_{\xi-1})}{q(t) \Gamma(q(t) + 1)} \left((n + 1 - \xi)^{q(t)+1} - (n - \xi)^{q(t)} (n - \xi + 1 + q(t)) \right) \right]. \quad (16)$$

Consequently, the numerical approximation for the variable-order LC fractional system components is formulated as follows:

$${}^{\text{LCV}}D_0^{\alpha(t)} u_{1,n+1}(t) = u_1(0) + \frac{1}{\Gamma(q(t_{n+1}))} \sum_{\xi=0}^n \frac{h^{q(t)} \vartheta_1(t_{\xi}, u_{1,\xi}, u_{2,\xi}, u_{3,\xi})}{q(t) \Gamma(q(t) + 1)} \times \left((n + 1 - \xi)^{q(t)} - (n - \xi)^{q(t)} (n - \xi + 2 + 2q(t)) \right). \quad (17)$$

where $\vartheta_1(t, u_1) = -u_2 - u_3$

$${}_0^{\text{LCV}}D_0^{\alpha(t)}u_{2,n+1}(t) = u_2(0) + \frac{1}{\Gamma(\varrho(t_{n+1}))} \sum_{\xi=0}^n \frac{h^{\varrho(t)}\vartheta_1(t_\xi, u_{2,\xi}, u_{2,\xi}, u_{3,\xi})}{\varrho(t)\Gamma(\varrho(t) + 1)} \\ \times \left((n+1-\xi)^{\varrho(t)} - (n-\xi)^{\varrho(t)}(n-\xi+2+2\varrho(t)) \right). \quad (18)$$

where $\vartheta_2(t, u_2) = u_1 + a_1u_2$

$${}_0^{\text{LCV}}D_0^{\alpha(t)}z_{1,n+1}(t) = z_1(0) + \frac{1}{\Gamma(\varrho(t_{n+1}))} \sum_{\xi=0}^n \frac{h^{\varrho(t)}\vartheta_2(t_\xi, u_{1,\xi}, u_{2,\xi}, u_{3,\xi})}{\varrho(t)\Gamma(\varrho(t) + 1)} \\ \times \left((n+1-\xi)^{\varrho(t)} - (n-\xi)^{\varrho(t)}(n-\xi+2+2\varrho(t)) \right), \quad (19)$$

where $\vartheta_3(t, u_3) = a_2 + (u_1(t) - a_3)u_3(t)$.

Tables 3–6 summarize the effects of three variable-order functions— $\alpha(t) = \tanh(t^2 + 3)$, $\alpha(t) = \frac{1}{1+e^{-t}}$, and $\alpha(t) = 1 - \frac{\cos^2(t)}{8}$ —on the system dynamics. All cases start off similarly yet start to diverge slightly after $t = 0.1$. The tanh function produces smooth, stable oscillations; the logistic introduces mild damping and slight phase distortion; and the cosine-modulated order amplifies periodic oscillations, with u_2 reaching its minimum around $t \approx 1$. In general, the tanh($t^2 + 3$) function yields stable transients; the logistic form introduces light damping, while the cosine-modulated order enhances oscillatory behavior. The NVOF scheme is numerically stable for $\Delta t = 0.1$, but smaller time steps may accurately capture the fast variations of $\alpha(t)$. In the future, we intend solve some new fractional models and make comparisons with other numerical methods.

Table 3. System outcomes with $\alpha(t) = \tanh(t^2 + 3)$ using NVOF.

Time	$u_1(t)$	$u_2(t)$	$u_3(t)$
0.0000	0.100000000000	0.100000000000	0.100000000000
0.1000	−0.385601382560	−0.050023318099	0.452750309550
0.2000	−0.455966624652	−0.685523939972	0.442990350469
0.3000	0.139417092907	−1.314411017562	0.493585054011
0.4000	0.950668499460	−1.308640989281	0.609395497123
0.5000	1.197508901679	−0.559135285493	0.700844541816
0.6000	0.557575660518	0.319454592508	0.621380167451
0.7000	−0.479923221414	0.512054044574	0.473159118294
0.8000	−1.025776010469	−0.289376972052	0.403316568224
0.9000	−0.444538705871	−1.515120250771	0.432209537989
1.0000	0.950304114197	−2.039011363424	0.582691277756

Table 4. System outcomes with $\alpha(t) = \frac{1}{1+e^{-t}}$ using NVOF.

Time	$u_1(t)$	$u_2(t)$	$u_3(t)$
0.0000	0.100000000000	0.100000000000	0.100000000000
0.1000	−0.387823171894	−0.028115998831	0.456326706417
0.2000	−0.500402639849	−0.656590130078	0.440825492461
0.3000	0.067674652726	−1.303138677254	0.486481093580
0.4000	0.900139606471	−1.322085506098	0.600357300558
0.5000	1.197573424131	−0.572465489938	0.699492191626
0.6000	0.577753108724	0.333635076314	0.626776556223
0.7000	−0.492232371075	0.546704105768	0.473167032461
0.8000	−1.093145177911	−0.270427096891	0.398318681448
0.9000	−0.532821514864	−1.541938037549	0.424022513290
1.0000	0.912927131093	−2.094929460522	0.574327765445

Table 5. System outcomes with $\alpha(t) = 1 - \frac{\cos^2(t)}{8}$ using NVOF.

Time	$u_1(t)$	$u_2(t)$	$u_3(t)$
0.0000	0.100000000000	0.100000000000	0.100000000000
0.1000	−0.389108233825	−0.005849282625	0.460168239492
0.2000	−0.544144741650	−0.628168348171	0.438709955974
0.3000	−0.004280194336	−1.293861329663	0.479475095809
0.4000	0.849509484527	−1.338741533972	0.591416973232
0.5000	1.200116045538	−0.588230793588	0.698400389244
0.6000	0.602376846005	0.349084195672	0.633156112430
0.7000	−0.502727069117	0.586169054615	0.473504294107
0.8000	−1.164537667343	−0.247609818848	0.393128226215
0.9000	−0.628883631971	−1.571167172706	0.415426369551
1.0000	0.872126405314	−2.160088755527	0.565369568968

Table 6. Discussion of system outcomes computed with NVOF for three cases of $\alpha(t)$.

Aspect	$\alpha(t) = \tanh(t^2 + 3)$	$\alpha(t) = \frac{1}{1 + e^{-t}}$	$\alpha(t) = 1 - \frac{\cos^2(t)}{8}$
Response to the initial condition (at $t = 0$)	The initial sensitivity is consistent across all three $\alpha(t)$ selections, as the short-term departure from the starting state is of equal size.	Same thing; small differences show up at $t = 0.1$, which is when the local fractional-order values are different.	Mid-time behavior ($0.3 < t \leq 0.7$)
Late-time behavior (near $t = 1$)	At $t = 1.0$, u_1 returns to a positive value ≈ 0.95 , u_2 is strongly negative (≈ -2.04), and u_3 is moderately positive (≈ 0.58). This suggests an attracting regime for u_1, u_3 while u_2 drifts negative.	At $t = 1.0$, $u_1 \approx 0.91$, $u_2 \approx -2.09$, and $u_3 \approx 0.57$. Very close to the tanh case but with a slightly more negative u_2 corresponds to increased damping or drift introduced by this sigmoidal ($\alpha(t)$).	At $t = 1.0$, $u_1 \approx 0.87$, $u_2 \approx -2.16$, and $u_3 \approx 0.56$. This case produces the most negative u_2 at $t = 1$, suggesting the cosine-modulated $\alpha(t)$ amplifies the negative drift in u_2 .
The qualitative effect of $\alpha(t)$	$\tanh(t^2 + 3)$ is smooth and relatively large (α closer to 1), producing stable-looking oscillatory transients with strong negative drift in u_2 .	Sigmoidal $\alpha(t)$ increases with t from 0.5 toward 1; this gradual change slightly accentuates the negative u_2 drift by $t = 1$ and shifts the timing of u_1 oscillations.	Cosine-modulated $\alpha(t)$ injects a mild periodic modulation in the effective order; this correlates with the largest negative excursions in u_2 and slightly stronger u_1 negativity.
Numerical remarks (NVOF)	NVOF appears stable for the given step ($\Delta t = 0.1$). Small differences among cases are numerically consistent.	Because $\alpha(t)$ changes more rapidly near $t \sim 0-1$ for the logistic map, the time step is refined to confirm the observed extra negativity in u_2 is not a numerical artifact.	Cosine modulation may require a smaller step or higher-order interpolation of $\alpha(t)$ inside each time step to accurately capture its effect on memory kernels.

6. Bifurcation

Figure 2 shows how the dynamics of a variable-order fractional Rössler system change as the fractional order (α) gets closer to one. It clearly shows how oscillations that happen at regular or almost regular intervals can cause chaos. The central bifurcation diagram shows how the highest and lowest values of the system variables change as α changes. This shows period-doubling bifurcations and the beginning of chaotic behavior as α rises. The phase portraits around it, which show different values of α , also show this change: for lower α values, the trajectories are regular and closed, but for higher α values, they get more complicated and eventually form a chaotic attractor. This study shows how the variable-order fractional Rössler system reacts to changes in the fractional order. It

also shows how important fractional modeling is for systems with memory and strange dynamics and how variable order can be used to cause complex dynamic behaviors.

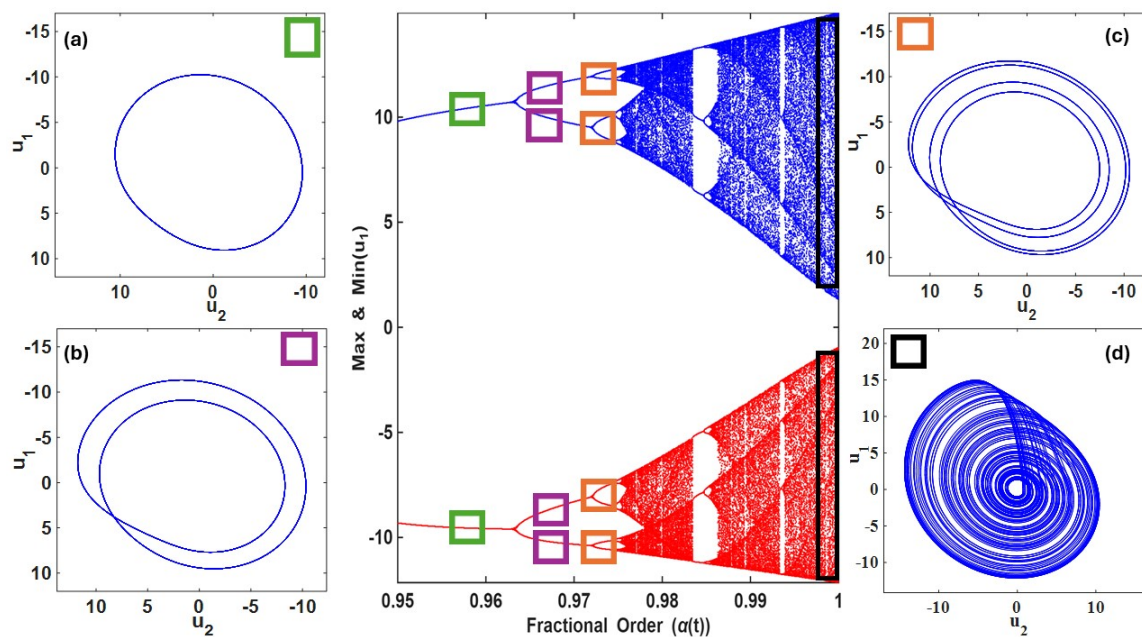


Figure 2. Bifurcation diagram and phase portraits of u_1 versus fractional-order $\alpha(t)$. Subplots (a–d), indicated by the green, purple, orange, and black boxes, respectively, illustrate the corresponding chaotic regions in the main bifurcation plot using the same color mapping.

7. Chaos

Figures 3–5 show the phase portraits of the proposed variable-order fractional system when the $\alpha(t)$ is defined in different ways. In Figure 3, for the case of ${}_1\alpha(t) = \tanh(t^2 + 3)$, the trajectories create smooth spiral attractors, which means that the hyperbolic tangent variation controls a stable but oscillatory motion. Figure 4 shows that in the case of ${}_2\alpha(t) = \frac{1}{1+e^{-t}}$, the system switches trajectories and makes sudden changes, which suggests that it is chaotic and very sensitive to the logistic-type change in $\alpha(t)$. On the other hand, Figure 5, which corresponds to the case of ${}_3\alpha(t) = 1 - \frac{\cos^2(t)}{8}$, shows regular closed loops that show a quasi-periodic or weakly stable response, where oscillations stay within certain limits. Overall, the results show that the fractional order changes over time and has a big effect on the system's stability and the switch between periodic and chaotic regimes.

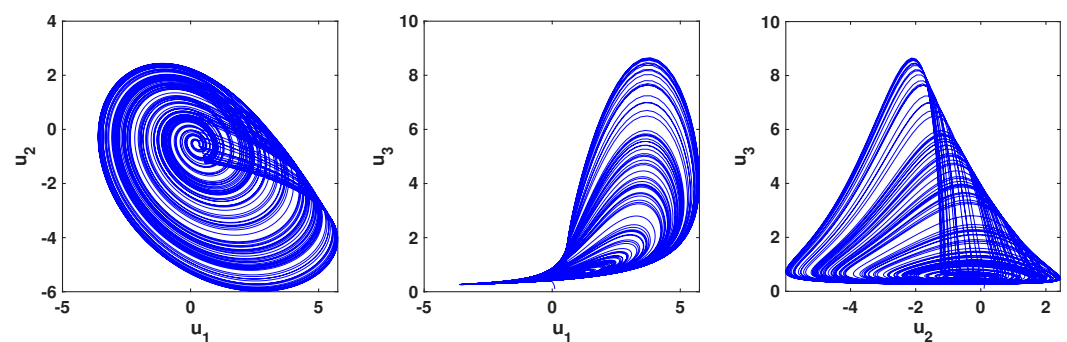


Figure 3. Chaos plots for different system state variables for $\alpha(t) = \tanh(t^2 + 3)$.

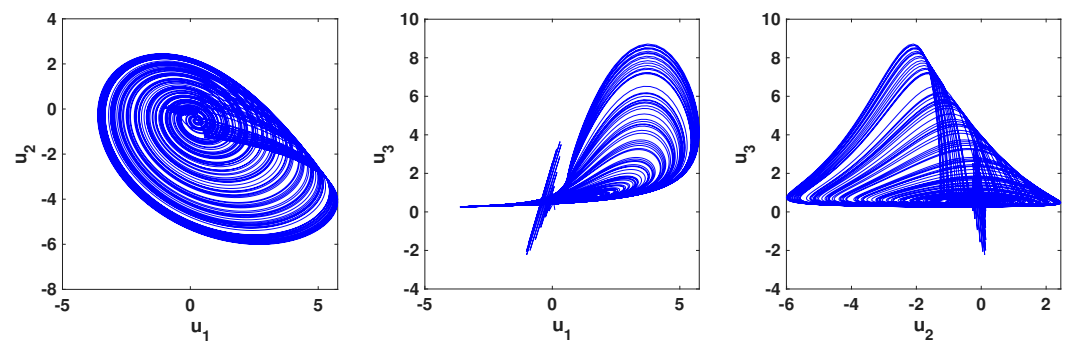


Figure 4. Chaos plots for different system state variables for $\alpha(t) = \frac{1}{1+e^{-t}}$.

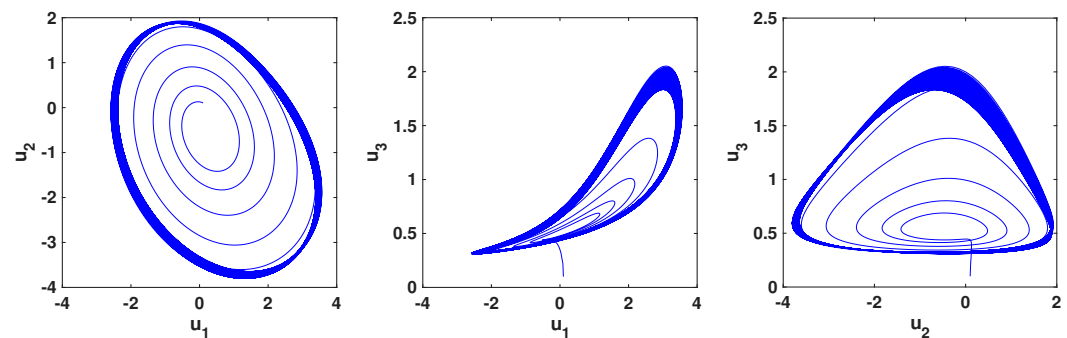


Figure 5. Chaos plots for different system state variables for $\alpha(t) = 1 - \frac{\cos^2(t)}{8}$.

8. Lyapunov Exponents

In this section, a close-up of the three plots in Figure 6, from right to left, is a clear and consistent demonstration of the system dynamics. The plot on the right is the time evolution of the three Lyapunov exponents, which tend towards steady values after some transient behavior. The largest exponent, LE_1 (blue line), tends towards a value close to zero, and the second exponent, LE_2 (orange line), tends towards a slightly negative value. The third exponent, LE_3 (yellow line), converges to a strongly negative value of about -3.8 . The Lyapunov spectrum so obtained $(0, -, -)$ is characteristic for a system converging towards a stable limit cycle, and the negative sum of the exponents ($\sum LE_i < 0$) shows that the system is dissipative. The middle plot is virtually the same, with exponents LE_1 , LE_2 , and LE_3 all converging to the same characteristic values ($LE_1 \approx 0$, $LE_2 < 0$, and $LE_3 \approx -3.6$), again providing strong evidence for a periodic attractor. The leftmost plot verifies this, with the exponents converging as $LE_1 \rightarrow 0$, LE_2 to a small negative number, and LE_3 to around -3.3 .

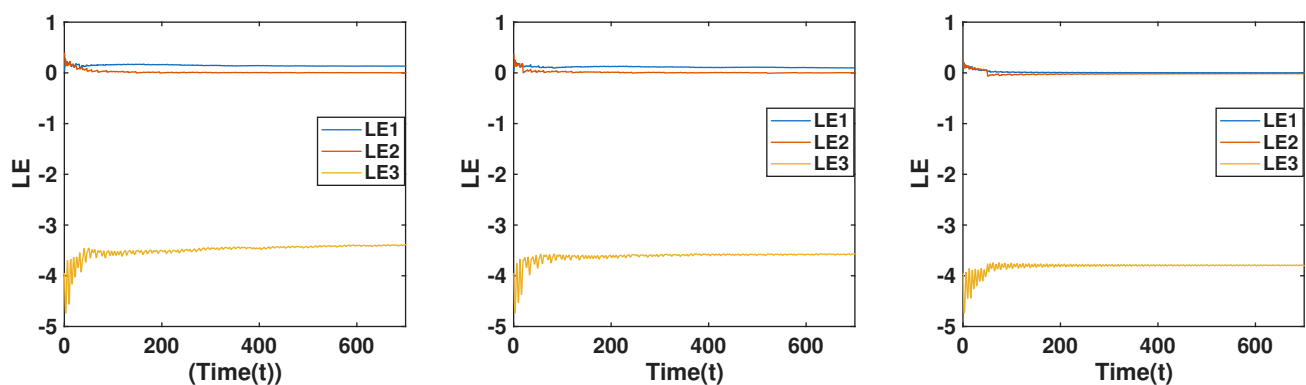


Figure 6. Lyapunov exponent plots for different system state variables for three cases.

9. Time Series

Here, Figure 7 shows how state variables u_1 , u_2 , and u_3 change in time for case 1 of the variable-order fractional system. The time series shows strong, nonperiodic oscillations with sudden changes in amplitude and nonrepeating patterns, all typical of chaotic motion. The u_1 variable oscillates in a complex pattern close to the equilibrium, with u_2 undergoing smaller amplitude variations interrupted by short bursts. Alternatively, u_3 shows clear spikes and sudden changes, which indicate that it is very sensitive to the initial conditions and the connectivity between the system states. These results verify the existence of a chaotic attractor and showcase the variable fractional order's role in creating complex and stochastic temporal behavior.

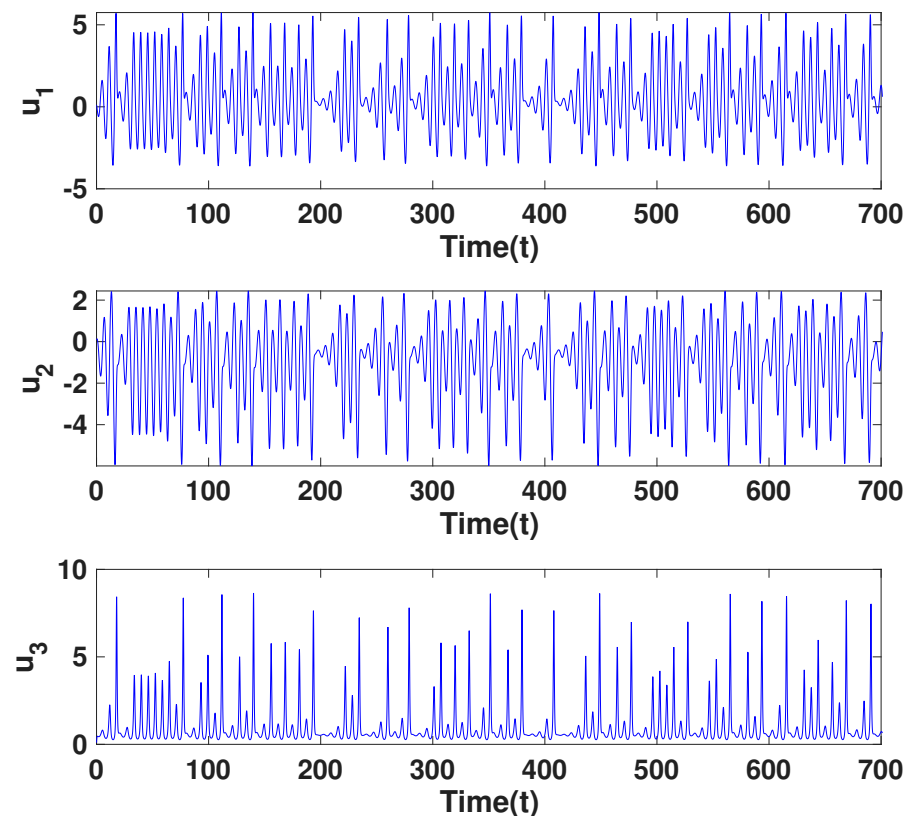


Figure 7. Time-series plots for different system state variables for three cases. case₁

Figure 8 shows how the system variables (u_1 , u_2 , and u_3) change over time for case₂. It shows that all three variables exhibit persistent oscillatory and complex dynamic behavior over the interval of $0 \leq t \leq 700$. u_1 has the largest amplitude, followed by the narrower oscillations of u_2 and the burst-like, intermittent spikes of u_3 . This pattern suggests that there are strong nonlinear interactions and possibly chaotic dynamics in the system. This is because there is no damping and the fluctuations are sustained and aperiodic. This suggests that the parameter settings favor persistent energy exchange and highlights the possibility of complex multi-scale phenomena in these kinds of models.

In Figure 9, the time responses of u_1 , u_2 , and u_3 contain clear, periodic oscillations that continue over the interval of $0 \leq t \leq 700$. This is in contrast to the more intricate dynamic behaviors demonstrated with previous examples. All three variables quickly come to oscillatory, periodic motion of small amplitudes and fixed waveforms once the transients have decayed after the fast initial response. This means that, here, parameters in the system exert strong regularity and predictability upon the dynamics. This is characteristic of greater stability and less sensitivity to initial conditions, and nonlinearities are minimized

or precisely counterbalanced, generating stable periodic behavior that suggests the presence of limit-cycle phenomena in the modeled system.

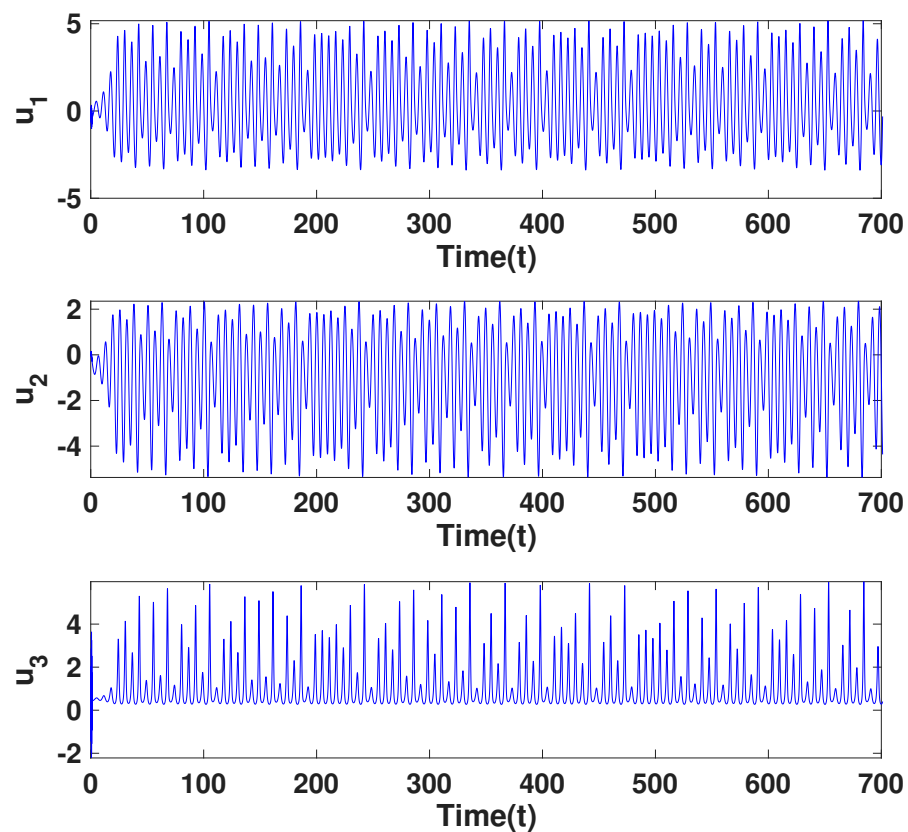


Figure 8. Time-series plots for different system state variables for three cases. case₂

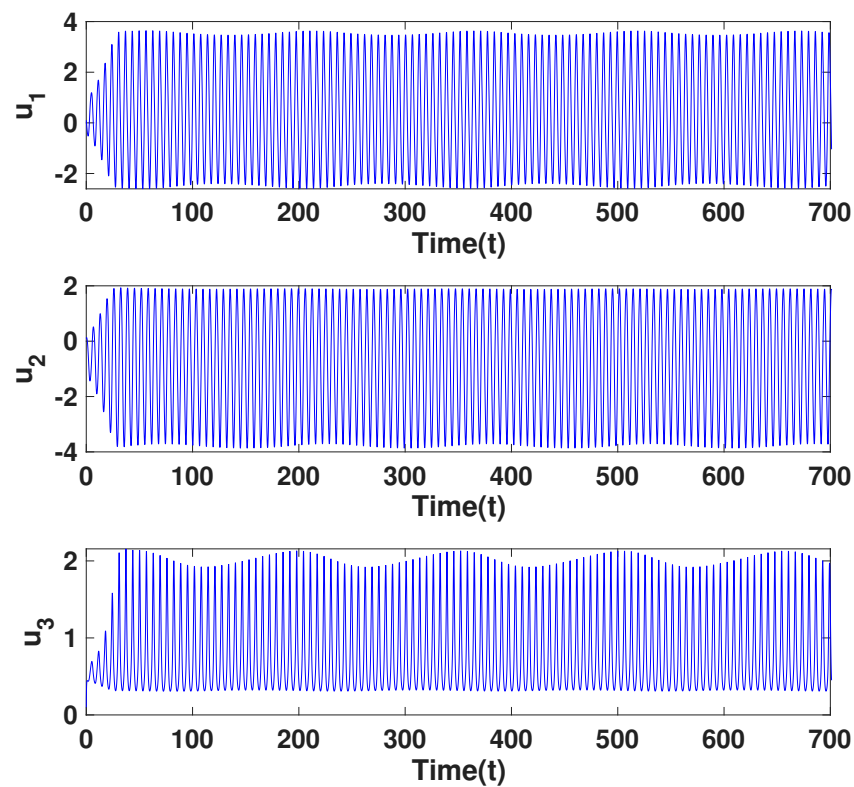


Figure 9. Time-series plots for different system state variables for three cases. case₃

10. Discussion

This paper presents a new analysis of the variable-order fractional Rössler system, expressed in terms of Liouville–Caputo variable-order derivatives. The innovation is the addition of time-varying fractional orders to permit memory and hereditary effects that change with time. This improves the modelability beyond conventional constant-order representations. By varying the fractional-order function ($\alpha(t)$), the system displays an extremely wide range of dynamic responses that are analogous to real-world processes and engineering with long-range dependence and changing dynamics. This method not only generalizes classical fractional dynamics but also provides a computational framework that can study systems with changing memory intensity. The study proves the usefulness of variable-order derivatives in investigating the transitions among stable, quasi-periodic, and chaotic behaviors, thereby enhancing understanding of the structure and sensitivity of nonlinear systems.

The extended investigation uncovered that tiny perturbations in $\alpha(t)$ can potentially induce monumental changes in the qualitative dynamics of the system, such as bifurcations, chaotic oscillations, and stable quasi-periodic motions. Phase portraits and time-series responses show that different fractional-order functions result in different attractor geometries, from bounded quasi-periodic loops to dense chaotic spirals. Case 2, for example, shows persistent irregular oscillations and infrequent bursts, which are characteristic of multiscale coupling. Case 3, in contrast, goes into quasi-periodic motion, showing how some forms of $\alpha(t)$ can eliminate chaos. The numerical computations, backed by Lyapunov exponent computation, validate that the system exhibits high sensitivity to initial conditions and variations in memory. This paper validates that variable-order fractional modeling offers a strong and versatile paradigm for simulation of nonlinear physical phenomena in the real world with dynamic memory characteristics, underscoring its applicability in analysis and control of dynamical complex systems in physics and engineering.

11. Conclusions

The present research demonstrates the way inclusion of variable-order fractional derivatives substantially improves the dynamic behavior of the fractional Rössler system through memory and hereditary effects that can be adjusted. Numerical simulations demonstrate that adjustment of the fractional-order function ($\alpha(t)$) has a principal influence on the qualitative dynamics of the system, resulting in bifurcations, chaotic oscillations, and changes in stability that can be impossible to describe using conventional models of integer order. Lyapunov exponent analysis indicates the impact of initial conditions on the system, and it indicates that it is of a complex, chaotic nature. These results indicate the success of the variable-order fractional strategy in properly simulating nonlinear and complex physical and engineering phenomena. In general, this research illustrates how important memory effects are in controlling transitions between periodic, quasi-periodic, and chaotic regimes. It also shows the use of variable-order fractional calculus in simulating complex dynamical systems. In future work, we plan to solve new fractional models [38–40] and compare them with other numerical methods [41,42].

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