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Subsymmetric Polynomials on Banach Spaces and Their Applications

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Abstract

We investigate algebraic and topological properties of subsymmetric polynomials on finite- and infinite-dimensional spaces. In particular, we focus on the problem of the existence of an algebraic basis in the algebra of subsymmetric polynomials, as well as possible extensions of subsymmetric polynomials and analytic functions to larger spaces. We consider algebras of subsymmetric analytic functions of bounded type and their spectra, and study linear subspaces in the zero-sets of subsymmetric polynomials, as well as subspaces where a subsymmetric polynomial is symmetric. In addition, we propose some possible applications of subsymmetric polynomials in cryptography and in operator theory.

Keywords: symmetric polynomials; subsymmetric polynomials on Banach spaces; algebraic basis of polynomials; spectrum of algebra; extension of polynomials; subsymmetric analytic functions; polynomial hash function

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1. Introduction

The concept of symmetric polynomials on a finite-dimensional linear space was very successful in mathematics and important for the development of algebraic geometry, invariant theory and combinatorics (see e.g., [1]). This concept admits a natural generalization for the infinite-dimensional space of eventually finite sequences $c_{00} = \bigcup_{n=1}^{\infty} \mathbb{C}^n$. Symmetric polynomials on ℓ_p for $1 \leq p < \infty$ were considered first in [2,3]. In particular, it was observed that polynomials

$$P_k(x) = \sum_{i=1}^{\infty} x_i^k, \quad k \geq \lceil p \rceil$$

form an algebraic basis in the algebra of symmetric polynomials on ℓ_p . Here $\lceil p \rceil$ is the ceiling of p , and the Banach space ℓ_p consists of the vectors $x = (x_1, \dots, x_i, \dots)$ such that

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty.$$

In other words, for every symmetric polynomial P of degree n on ℓ_p there is a unique polynomial $q(t_1, \dots, t_{n-\lceil p \rceil+1})$ of $n - \lceil p \rceil + 1$ variables such that

$$P(x) = q(P_{\lceil p \rceil}(x), \dots, P_n(x)), \quad x \in \ell_p.$$

Algebras of symmetric analytic functions on ℓ_p were considered in many works (see, e.g., [4–8] and references therein). Some other generalizations of symmetric polynomials and analytic functions for the various cases of Banach spaces and (semi)groups of symmetry can be found in [9–15]. Algebras of subsymmetric polynomials and their generators were investigated in [16–19].

Let X be a complex sequence space c_{00} or ℓ_p for $1 \leq p < \infty$. A function F on X is *subsymmetric* if it is invariant with respect to the following operators $\mathcal{C}_j$ on X :

$$\mathcal{C}_j: (x_1, \dots, x_n, \dots) \mapsto (x_1, \dots, x_{j-1}, 0, x_j, x_{j+1}, \dots), \quad j \in \mathbb{N}. \quad (1)$$

That is, $F(\mathcal{C}_j(x)) = F(x)$ for every $j \in \mathbb{N}$. Clearly, the operators $\mathcal{C}_j$ are continuous on ℓ_p . It is known [19] that the following (so-called *standard*) subsymmetric polynomials

$$P_{\alpha_1, \dots, \alpha_n}(x) = \sum_{i_1 < \dots < i_n} x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n}, \quad \alpha_j \geq [p]$$

form a linear basis in the linear space of subsymmetric polynomials on ℓ_p . In particular, any m -homogeneous subsymmetric polynomial can be uniquely represented as a finite linear combination of the standard subsymmetric polynomials $P_{\alpha_1, \dots, \alpha_n}$ such that $\alpha_1 + \dots + \alpha_n = m$. We denote by $\mathfrak{S}$ the minimal semigroup generated by operators $\mathcal{C}_j$ and by $\mathcal{P}_{\mathfrak{S}}(X)$ the algebra of all subsymmetric polynomials on X .

In this paper, we consider properties of subsymmetric polynomials on finite- and infinite-dimensional spaces, the problem of the existence of an algebraic basis in the algebra of subsymmetric polynomials, extensions of subsymmetric polynomials and analytic functions to larger spaces, algebras of subsymmetric analytic functions of bounded type and their spectra, linear subspaces in zero-sets of subsymmetric polynomials, and subspaces where a subsymmetric polynomial is symmetric. In addition, we consider some possible applications of subsymmetric polynomials in cryptography and in operator theory. Algebras of all analytic functions of bounded type on Banach spaces were investigated in [20–24]. Linear subspaces in zero sets of polynomials on complex Banach spaces and related questions were studied in [25–31].

It is well-known [32] that there is no Hahn–Banach theorem for polynomials in the general case. In [9], the problem of the existence of a symmetry-preserving extension of symmetric polynomials on a Banach space was considered. In this paper, we investigate extensions of subsymmetric polynomials and find some general conditions when the operator of extension is a homomorphism of corresponding algebras. Using this approach, we obtain some advances for the description of the spectrum of algebras of subsymmetric functions on ℓ_p . These results generalise results in [6,7,20] for the case of subsymmetric functions.

It is known [19] that for a large class of spaces X , in particular for ℓ_p , $1 \leq p < \infty$, for every polynomial Q on X there is $\varepsilon > 0$ and an infinite-dimensional subspace $X_0 \subset X$ and a subsymmetric polynomial P on X_0 such that $|Q(x) - P(x)| < \varepsilon$ for every $x \in X_0$. This fact gives some kind of approximation of polynomials by subsymmetric polynomials. In the paper, we considered linear subspaces of subsymmetric polynomials and some subspaces V such that the restriction of a subsymmetric polynomial to V is symmetric with respect to a linear basis.

Let $\mathcal{P}$ be an algebra of polynomials. Let us recall that a finite or infinite sequence $(Q_1, Q_2, \dots)$ of nonzero polynomials is *algebraically dependent* if there is a number m and a nontrivial polynomial of m variables $q(t_1, \dots, t_m)$ such that $q(Q_1, \dots, Q_m) \equiv 0$. If the sequence (Q_i) is not algebraically dependent, then it is *algebraically independent*. An algebraically independent sequence (Q_i) is an *algebraic basis* in $\mathcal{P}$ if every polynomial in $\mathcal{P}$ can

be represented as an algebraic combination of polynomials Q_i . Note that an algebraic basis does not always exist even if $\mathcal{P}$ is a subalgebra of polynomials on a finite-dimensional space. In this case, for any sequence of generators of $\mathcal{P}$, there are algebraic dependencies. The problem of the existence of an algebraic basis in the algebra of subsymmetric polynomials was investigated in [16–18,20]. Unfortunately, this problem is still open, and we propose solutions for some partial cases.

In Section 2, we investigate subsymmetric polynomials on finite-dimensional complex spaces $\mathbb{C}^M$ and show that the algebra of all subsymmetric polynomials on $\mathbb{C}^M$ has no algebraic basis. Also, we propose a general method of calculating algebraic dependencies between symmetric polynomials and a nonsymmetric polynomial. In Section 3, we consider algebraic properties of subsymmetric polynomials on the linear space c_{00} of eventually zero sequences. We described some algebraically independent sequences of subsymmetric polynomials which, however, do not form algebraic bases. Also, we proved that two points, x and y in c_{00} , are equivalent up to actions of operators $\mathcal{C}_j$ if and only if $P(x) = P(y)$ for every subsymmetric P . Moreover, it is enough to check this equality for polynomials P of some special form. Section 4 is devoted to the extension of the obtained results to continuous subsymmetric polynomials on some Banach spaces containing c_{00} . In particular in Section 4.1 we consider subsymmetric polynomials on ℓ_p , $1 \leq p < \infty$. In Section 4.2, we observed that there exists a natural extending topological homomorphism from the algebra of subsymmetric polynomials on ℓ_p to the algebra of subsymmetric polynomials on $\ell_p(\mathfrak{A})$ for any well-ordered set $\mathfrak{A}$. This approach is useful for the investigation of algebras of subsymmetric analytic functions of bounded type and their spectra, which we considered in Section 5. In Section 6, we constructed some subspaces of ℓ_p where subsymmetric polynomials are symmetric with respect to some bases. Also, we considered linear subspaces in zero sets of some subsymmetric polynomials. Section 7 is devoted to some possible applications of the obtained results. In Section 7.1, we propose subsymmetric polynomials for the construction of hash functions that may be of interest in cryptography. In Section 7.2, using subsymmetric polynomials on eigenvalues of a p -trace operator A , we show that A is completely defined by traces of A^n and $(A \wedge A^2)^n$, $n \geq \lceil p \rceil$.

We refer the reader for the general theory of polynomial and analytic mappings on Banach spaces to books [33,34].

2. Subsymmetric Polynomials on $\mathbb{C}^M$

In this section, we prove that there is no algebraic basis in the algebra of subsymmetric polynomials on a finite-dimensional space.

Subsymmetric polynomials were introduced using operators $\mathcal{C}_j$ that are well-defined on an infinite-dimensional vector space. Let $P_{\alpha_1, \dots, \alpha_n}^{(M)}$ be the restriction of $P_{\alpha_1, \dots, \alpha_n}$ to the M -dimensional subspace $\mathbb{C}^M$ of X spanned to $e_1, \dots, e_M$. We say that a polynomial P on $\mathbb{C}^M$ is subsymmetric if it is a linear combination of polynomials $P_{\alpha_1, \dots, \alpha_n}^{(M)}$.

It is well-known in algebraic geometry that every family of nonconstant polynomials $Q_1, Q_2, \dots, Q_{M+1}$ is algebraically dependent on $\mathbb{C}^M$. From here it follows, in particular, that if some algebra of polynomials on $\mathbb{C}^M$ is generated by all symmetric polynomials and a single nonsymmetric polynomial, then it has no algebraic basis because such an algebra must have at least $M + 1$ generators that cannot be algebraically independent. The following example delivers an algebraic dependence between a basis of symmetric polynomials and a subsymmetric polynomial on $\mathbb{C}^3$.

Example 1. Let

$$\xi_1(x) = P_1^{(3)}(x) = x_1 + x_2 + x_3,$$

$$\zeta_2(x) = P_{1,1}^{(3)}(x) = x_1x_2 + x_1x_3 + x_2x_3,$$

$$\zeta_3(x) = P_{1,1,1}^{(3)}(x) = x_1x_2x_3,$$

$$\zeta_4(x) = P_{1,3}^{(3)}(x) = x_1x_2^2 + x_1x_3^2 + x_2x_3^2.$$

Then, by direct calculations we can check that

$$\begin{aligned} & \zeta_4^6 + (9\zeta_3 - 3\zeta_1\zeta_2)\zeta_4^5 + (\zeta_1^3\zeta_3 + 36\zeta_3^2 + 3\zeta_1^2\zeta_2^2 + 3\zeta_2^3 - 28\zeta_1\zeta_2\zeta_3)\zeta_4^4 \\ & + 6\zeta_1^3\zeta_3^2 + 81\zeta_3^3 - \zeta_1^3\zeta_2^3 - 6\zeta_1\zeta_2^4 - 2\zeta_1^4\zeta_2\zeta_3 + 29\zeta_1^2\zeta_2^2\zeta_3 - 105\zeta_1\zeta_2\zeta_3^2 + 18\zeta_2^3\zeta_3\zeta_4^3 \\ & + (108\zeta_3^4 + 13\zeta_1^3\zeta_3^3 + 3\zeta_2^6 + 3\zeta_1^2\zeta_2^5 - 38\zeta_1\zeta_2^4\zeta_3 + 49\zeta_2^3\zeta_3^2 - 8\zeta_1^3\zeta_2^3\zeta_3 + 96\zeta_1^2\zeta_2^2\zeta_3^2 + \zeta_1^5\zeta_2^2\zeta_3 \\ & - 198\zeta_1\zeta_2\zeta_3^3 - 10\zeta_1^4\zeta_2\zeta_3^2)\zeta_4^3 + (81\zeta_3^5 + 12\zeta_1^3\zeta_3^4 - 3\zeta_1\zeta_2^7 + 9\zeta_2^6\zeta_3 + 20\zeta_1^2\zeta_2^5\zeta_3 - 82\zeta_1\zeta_2^4\zeta_3^2 \\ & - 2\zeta_1^4\zeta_2^4\zeta_3 + 66\zeta_2^3\zeta_3^3 - 21\zeta_1^3\zeta_2^3\zeta_3^2 + 135\zeta_1^2\zeta_2^2\zeta_3^3 + 4\zeta_1^5\zeta_2^2\zeta_3^2 - 189\zeta_1\zeta_2\zeta_3^4 - 16\zeta_1^4\zeta_2\zeta_3^3)\zeta_4^2 \\ & + (27\zeta_3^6 + 4\zeta_1^3\zeta_3^5 + \zeta_2^9 - 10\zeta_1\zeta_2^7\zeta_3 + 13\zeta_2^6\zeta_3^2 + \zeta_1^3\zeta_2^6\zeta_3 + 27\zeta_1^2\zeta_2^5\zeta_3^2 - 62\zeta_1\zeta_2^4\zeta_3^3 - 4\zeta_1^4\zeta_2^4\zeta_3^2 \\ & + 31\zeta_2^3\zeta_3^4 - 18\zeta_1^3\zeta_2^3\zeta_3^3 + 71\zeta_1^2\zeta_2^2\zeta_3^4 + 4\zeta_1^5\zeta_2^2\zeta_3^3 - 72\zeta_1\zeta_2\zeta_3^5 - 8\zeta_1^4\zeta_2\zeta_3^4) \equiv 0. \end{aligned} \quad (2)$$

Note that in [6] it is proved that polynomials P_n , $n \in \mathbb{N}$ and $P_{1,2}$ are algebraically independent on ℓ_1 . However, the question about the existence of an algebraic basis in the algebra of all subsymmetric polynomials on ℓ_p or c_{00} is still open.

Now we consider how to find explicitly an algebraic dependence of a non-symmetric polynomial on symmetric polynomials defined on $\mathbb{C}^n$. Let $P(z) = P(z_1, \dots, z_n) \in \mathcal{P}(\mathbb{C}^n)$. We denote by $\text{Orb}(P)$ the orbit of P under permutations of the basis vectors:

$$\text{Orb}(P) = \{\sigma(P)(z) := P(z_{\sigma(1)}, \dots, z_{\sigma(n)}): \sigma \in S_n\}$$

and by $|\text{Orb}(P)|$ the cardinality of $\text{Orb}(P)$. Here S_n is the group of permutations of $\{1, \dots, n\}$. Clearly, $1 \leq |\text{Orb}(P)| \leq n!$ and if $|\text{Orb}(P)| = 1$, then P is symmetric.

Theorem 1. For a given polynomial P on $\mathbb{C}^n$ there is a polynomial of a complex variable t

$$A(t) = t^m - a_1(\cdot)t^{m-1} + a_2(\cdot)t^{m-2} - \dots + (-1)^m a_m(\cdot)$$

of degree $m = |\text{Orb}(P)|$, where coefficients $a_i(\cdot)$ are symmetric polynomials on $\mathbb{C}^n$, such that $A(Q) \equiv 0$ for every $Q \in \text{Orb}(P)$. In particular, $A(P) \equiv 0$.

Proof. Let $\text{Orb}(P) = \{Q_1, \dots, Q_m\}$ and $Q_1 = P$. We denote by

$$a_k(z) = \sum_{i_1 < \dots < i_k} Q_{i_1}(z) \cdots Q_{i_k}(z), \quad 1 \leq k \leq m, \quad z \in \mathbb{C}^n,$$

the elementary symmetric functions of $\{Q_1(z), \dots, Q_m(z)\}$. Since for every $\sigma \in S_n$ the mapping

$$\{Q_1(z), \dots, Q_m(z)\} \mapsto \{Q_1(\sigma(z)), \dots, Q_m(\sigma(z))\}$$

is a bijection, $\sigma(a_k)(z) = a_k(z)$ and so $a_k(z)$ is symmetric for every $1 \leq k \leq m$. By the Vieta Theorem (see e.g., [35], pp. 29–30), for every $z \in \mathbb{C}^n$ the numbers $\{Q_1(z), \dots, Q_m(z)\}$ are zeros of $A(t)$, in particular, $A(P(z)) = 0$, $z \in \mathbb{C}^n$. $\square$

From the proof of Theorem 1 it follows that if P is a nonsymmetric polynomial on $\mathbb{C}^n$, then in order to construct the polynomial $A(t)$, we have to find the orbit $\text{Orb}(P) = \{Q_1, \dots, Q_m\}$ and define $a_k(z)$ as elementary symmetric polynomials of $Q_1, \dots, Q_m$. Then $A(P) = 0$ is an algebraic dependence between symmetric polynomials and P . In particular, identity (2) can be obtained by this way.

Each polynomial on the 1-dimensional space $\mathbb{C}$ is trivially symmetric (and so subsymmetric). If $M \geq 2$, then $P_1^{(M)}$, $(P_1^{(M)})^2$, and $P_2^{(M)}$ form a linear basis in the linear space of subsymmetric polynomials of degree no greater than 2. Thus, each 1-degree and 2-degree subsymmetric polynomial is symmetric. But we have some nonsymmetric 3-degree polynomials. Even if $M = 2$, $P_{1,2}^{(2)}(x) = x_1 x_2^2$ is a nontrivial subsymmetric polynomial that cannot be generated by symmetric polynomials. Thus, any system of generators of the algebra of subsymmetric polynomials on $\mathbb{C}^2$ must have at least 3 elements, and they are always algebraically dependent. In the general case, it is easy to check that the symmetric polynomial $P_k^{(M)}$, $k \leq M$ cannot be represented by subsymmetric polynomials of degree less than k (see [18]) and so any system of generators of the algebra of subsymmetric polynomials on $\mathbb{C}^M$ must have at least one generator of each degree $1 \leq k \leq M$. But as we observed, for 3-degree polynomials we need to have one generator more. Thus, the total number of generators is not less than $M + 1$. As we mentioned above, $M + 1$ polynomials on $\mathbb{C}^M$ are algebraically dependent. Hence, we have the following proposition.

Proposition 1. *For every $M > 1$, the algebra of subsymmetric polynomials on $\mathbb{C}^M$ has no algebraic basis.*

3. Subsymmetric Polynomials on c_{00}

In this section we consider basic algebraic properties of subsymmetric polynomials on the linear space of finite sequences $c_{00} = \cup_{n=1}^{\infty} \mathbb{C}^n$.

Note that Theorem 1 cannot be extended to the infinite-dimensional case because the algebra of symmetric polynomials of infinite many variables is factorial (see e.g., [36]), that is, if a symmetric polynomial P can be represented as a product of two polynomials, $P = P_1 P_2$, then both P_1 and P_2 must be symmetric. We denote by $\mathcal{P}_s(c_{00})$ the algebra of symmetric polynomials on c_{00} and by $\mathcal{P}_{\mathfrak{S}}(c_{00})$ the algebra of subsymmetric polynomials on c_{00} .

Proposition 2. *Let*

$$A(t) = t^m - a_1 t^{m-1} + a_2 t^{m-2} - \dots + (-1)^m a_m$$

be a polynomial with coefficients $a_j \in \mathcal{P}_s(c_{00})$. If P is a polynomial on c_{00} such that $A(P(x)) = 0$ for every $x \in c_{00}$, then $P \in \mathcal{P}_s(c_{00})$.

Proof. The identity $A(P) \equiv 0$ implies that $A(P(T_{\sigma}(x))) = 0$ for every $x \in c_{00}$ and permutation $\sigma \in S_{\mathbb{N}}$, where $T_{\sigma}(x) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots)$. Let $Q_1, \dots, Q_m$ be m different elements in $\text{Orb}(P)$. Then for every $x \in c_{00}$, numbers $\{Q_1(x), \dots, Q_m(x)\}$ are roots of $A(t)(x)$. By the Vieta theorem,

$$a_k(x) = \sum_{i_1 < \dots < i_k} Q_{i_1}(x) \cdots Q_{i_k}(x), \quad 1 \leq k \leq m.$$

Since each a_k is symmetric and $a_m(x) = Q_1(x) \cdots Q_m(x)$ for every $x \in c_{00}$, the polynomial $Q_1 \cdots Q_m$ is symmetric while $Q_1, \dots, Q_m$ are not. This is a contradiction. Thus, the number of different elements in $\text{Orb}(P)$ is less than m . Let $\text{Orb}(P)$ has r different elements. If $r > 1$, then $Q_1 \cdots Q_r$ must be symmetric while the multipliers $Q_1, \dots, Q_r$ are not, and again we have a contradiction. Thus, $r = 1$ and so $P \in \mathcal{P}_s(c_{00})$. $\square$

From the straightforward computations we can see that

$$P_n(x)P_{\alpha_1, \dots, \alpha_m}(x) = P_{n, \alpha_1, \dots, \alpha_m}(x) + P_{\alpha_1 + n, \dots, \alpha_m}(x) + \dots + P_{\alpha_1, \dots, n, \alpha_m}(x) + P_{\alpha_1, \dots, \alpha_m + n}(x). \quad (3)$$

Proposition 3. For every standard subsymmetric polynomial $P_{\alpha_1, \dots, \alpha_m}$ on c_{00} , the polynomial

$$P_{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)}}^s = \sum_{\sigma \in S_m} P_{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)}}$$

is symmetric. Here S_m is the group of permutations of the set $\{1, \dots, m\}$.

Proof. It is enough to check that $P_{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)}}^s$ is invariant with respect to any operator of transposition

$$\sigma_{ij}: (x_1, \dots, x_i, \dots, x_j, \dots) \mapsto (x_1, \dots, x_j, \dots, x_i, \dots).$$

Let us consider the set of terms of $P_{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)}}^s$ containing coordinates x_i and x_j ,

$$\{x_{i_1}^{\alpha_1} \dots x_i^{\alpha_s} \dots x_j^{\alpha_r} \dots x_{i_m}^{\alpha_m}\},$$

where i_k are mutually different. From the definition of $P_{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)}}^s$ it follows that coordinates x_i and x_j may take any places s and r so that $s \neq r$. Clearly, this set is invariant with respect to the interchanging of x_i and x_j and the sum of its elements is invariant as well. Thus, $P_{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)}}^s$ is invariant with respect to every σ_{ij} and so is symmetric. $\square$

Example 2. From (3) it follows that $P_k P_m = P_{k,m} + P_{k+m} + P_{m,k}$ and so

$$P_{k,m}^s = P_{k,m} + P_{m,k} = P_k P_m - P_{k+m} \quad (4)$$

is the representation of $P_{k,m}^s$ by the power symmetric polynomials.

From Proposition 3 it follows that if a polynomial $P \in \mathcal{P}_{\mathfrak{S}}(c_{00})$ is symmetric with respect to the permutation of the indexes $\alpha_1, \dots, \alpha_m$ in its representation as a linear combination of the standard polynomials $P_{\alpha_1, \dots, \alpha_m}$, then P is symmetric. However, for an algebraic combination it is not so. For example, polynomial $P = P_{km} P_{mk}$, $1 \leq k < m$ is invariant with respect to the permutation $k \leftrightarrow m$ but not symmetric because a symmetric polynomial cannot be a product of two nonsymmetric polynomials. By the same reasoning, $P_{km}^2 + P_{mk}^2 = (P_{km} + iP_{mk})(P_{km} - iP_{mk})$ is not symmetric.

For every $x = (x_1, \dots, x_m, 0, 0, \dots) \in c_{00}$ we denote by $\check{x} = (x_m, \dots, x_1, 0, 0, \dots)$, where x_m is the last nonzero coordinate of x . Also, if $y = (y_1, \dots, y_n, 0, 0, \dots) \in c_{00}$, then we denote

$$x \triangleleft y = (x_1, \dots, x_m, y_1, \dots, y_n, 0, 0, \dots),$$

and

$$x \bullet y = (x_1, y_1, x_2, y_2, \dots).$$

Formally, the definitions of $\check{x}$ and $x \triangleleft y$ are dependent on the m that bounds the number of nonzero coordinates of x . But it is easy to check that for every $P \in \mathcal{P}_{\mathfrak{S}}(c_{00})$,

$$P(\check{x}) = \lim_{k \rightarrow \infty} P(x_k, x_{k-1}, \dots, x_1, 0, 0, \dots), \quad (5)$$

and

$$P(x \triangleleft y) = \lim_{k \rightarrow \infty} P(x_1, x_2, \dots, x_k, y_1, y_2, \dots). \quad (6)$$

The right-hand-side parts of (5) and (6) do not depend on m and can be used to extend these definitions for more general spaces (see Section 4).

Let us consider some useful identities. From direct computations (see e.g., [18]) it is possible to check that

$$P_{\alpha_1, \dots, \alpha_m}(x \triangleleft y) = \sum_{j=0}^{m+1} P_{\alpha_1, \dots, \alpha_j}(x) P_{\alpha_{j+1}, \dots, \alpha_m}(y), \quad x, y \in c_{00}. \quad (7)$$

Also, obviously

$$P_{\alpha_1, \dots, \alpha_m}(\check{x}) = P_{\alpha_m, \dots, \alpha_1}(x), \quad x \in c_{00}. \quad (8)$$

For a given polynomial $P \in \mathcal{P}_{\mathfrak{S}}(c_{00})$ we denote by $\check{P} = (P)^\vee$ as the polynomial in $\mathcal{P}_{\mathfrak{S}}(c_{00})$ such that $\check{P}(x) = P(\check{x})$ for every $x \in c_{00}$. Clearly, the map $P \mapsto \check{P}$ is linear and multiplicative. For example, the multiplicativity follows from the computations

$$(PQ)^\vee(x) = (PQ)(\check{x}) = P(\check{x})Q(\check{x}) = \check{P}(x)\check{Q}(x), \quad x \in c_{00}.$$

From (8) we have that $\check{P}_{\alpha_1, \dots, \alpha_m} = P_{\alpha_m, \dots, \alpha_1}$. Thus the map $P \mapsto \check{P}$ is a homomorphism from $\mathcal{P}_{\mathfrak{S}}(c_{00})$ to itself. Moreover, this map is an involution because $(\check{P})^\vee = P$ for every $P \in \mathcal{P}_{\mathfrak{S}}(c_{00})$.

Proposition 4. For every pair $k, m \in \mathbb{N}$ the polynomial

$$Q(x) = P_{k,m}(\check{x} \triangleleft x), \quad x \in c_{00}$$

is symmetric and

$$Q(x) = 2P_k(x)P_m(x) - P_{k+m}(x).$$

Proof. From (7) we have

$$P_{k,m}(\check{x} \triangleleft x) = P_{k,m}(\check{x}) + P_k(x)P_m(x) + P_{k,m}(x) = P_{k,m}(x) + P_{m,k}(x) + P_k(x)P_m(x).$$

From Proposition 3 it follows that $Q(x)$ is symmetric and from (4) we have $Q(x) = 2P_k(x)P_m(x) - P_{k+m}(x)$. $\square$

Note that for every $x \in c_{00}$ we have

$$P_{k,m}(x \bullet x) = 4P_{k,m}(x) + P_{k+m}(x).$$

Let us introduce two types of equivalences on c_{00} . We say that x and y in c_{00} are equivalent with respect to the actions of $\mathfrak{S}$, notation $x \simeq y$, if there are τ_1 and τ_2 in $\mathfrak{S}$ such that $\tau_1(x) = \tau_2(y)$. Similarly, we say that x and y are equivalent with respect to the semigroup S of all permutations (that is, injective mappings) of $\mathbb{N}$ acting on c_{00} as $\sigma(x) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots)$, notation $x \sim y$, if there are σ_1 and σ_2 in S such that $\sigma_1(x) = \sigma_2(y)$. We denote by $\mathcal{M}_0^{\mathfrak{S}}$ the quotient set $c_{00}/\simeq$, and by $\mathcal{M}_0^S$ the quotient set $c_{00}/\sim$. It is known that $\mathcal{M}_0^S$ can be identified with the set of all finite multisets (that is, unordered tuples with possible repetitions) of nonzero complex numbers. Moreover, $x \sim y$ if and only if $P(x) = P(y)$ for every symmetric polynomial P . Since P_n form an algebraic basis in $\mathcal{P}_s(c_{00})$, it is enough to check the equality $P_n(x) = P_n(y)$ for every n . Finally, if we know that both x and y have only m nonzero coordinates each, then it is enough to check $P_n(x) = P_n(y)$ for $1 \leq n \leq m$. We denote by $[x] \in \mathcal{M}_0^S$ the equivalence class containing x and by $[[x]]$ the equivalence class in $\mathcal{M}_0^{\mathfrak{S}}$ containing x . Clearly $x \simeq y$ if and only if after removing all zero coordinates in x and y we have that $(x_{i_1}, \dots, x_{i_m})$ and $(y_{j_1}, \dots, y_{j_m})$ are equal as vectors.

Note that the operation “ $\triangleleft$ ” can be lifted to $\mathcal{M}_0^{\mathfrak{S}}$ by

$$[[x]] \triangleleft [[y]] = [[x \triangleleft y]],$$

while “ $\bullet$ ” cannot. Indeed, let $x' = (0, x_1, x_2, \dots)$. Then $x \simeq x'$, but

$$x' \bullet y = (0, y_1, x_1, y_2, \dots) \not\simeq (x_1, y_1, x_2, y_2, \dots) = x \bullet y.$$

However, the unary operation $x \mapsto x \bullet x$ can be lifted to the equivalence classes by $[[x]] \bullet [[x]] = [[x \bullet x]]$. The restriction of “ $\triangleleft$ ” to the subalgebra of symmetric polynomials coincides with “ $\bullet$ ”, and “ $\bullet$ ” can be lifted to $\mathcal{M}_0^{\mathfrak{S}}$.

Theorem 2. *The equivalence $x \simeq y$ holds for some vectors $x, y \in c_{00}$ if and only if $P(x) = P(y)$ for every subsymmetric polynomial P . Moreover, it is enough to check the equality of subsymmetric polynomials of the form P_k and $P_{k,2k}$, $k \in \mathbb{N}$. More explicitly, if x and y have no more than m nonzero coordinates, then we can check the equalities $P_k(x) = P_k(y)$ for all $k \leq m$ and $P_{k,2k}(x) = P_{k,2k}(y)$ for $k \leq \frac{m(m-1)}{2}$.*

Proof. Clearly, if $x \simeq y$, then $P(x) = P(y)$ for every subsymmetric polynomial P , by the definition of subsymmetric polynomials. Suppose that $P(x) = P(y)$ for every subsymmetric polynomial P . In particular, $P(x) = P(y)$ for every symmetric polynomial P . It is well-known that in this case the multiset $\{x_1, \dots, x_m\}$ of non-zero coordinates of x coincides with the multiset of non-zero coordinates of y , $\{y_1, \dots, y_m\}$. To make sure that $x \simeq y$ we need check that after removing zero coordinates, $(x_1, \dots, x_m) = (y_1, \dots, y_m)$ as vectors. Consider the following multisets

$$M_{1,2}(x) = \{x_i x_j^2 : i < j\} \quad \text{and} \quad M_{1,2}(y) = \{y_i y_j^2 : i < j\}.$$

We claim that if $(x_1, \dots, x_m) \neq (y_1, \dots, y_m)$, then $M_{1,2}(x) \neq M_{1,2}(y)$. Indeed, if $(x_1, \dots, x_m) \neq (y_1, \dots, y_m)$, then there are $x_r \neq x_s$ such that $r < s$ and a bijection $\sigma: \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ such that $(y_1, \dots, y_m) = (x_{\sigma(1)}, \dots, x_{\sigma(m)})$ and $\sigma(r) > \sigma(s)$. Then the symmetric difference $M_{1,2}(x) \triangle M_{1,2}(y)$ is nonempty, because it contains, at least, elements $x_r x_s^2$ and $x_s x_r^2$. Here $M_{1,2}(x) \triangle M_{1,2}(y) = (M_{1,2}(x) \setminus M_{1,2}(y)) \cup (M_{1,2}(y) \setminus M_{1,2}(x))$, and the set-theoretical subtraction “ $\setminus$ ” takes into account the multiplicity of each element. Hence, we have two different multisets $M_{1,2}(x)$ and $M_{1,2}(y)$ of length $N = \frac{m(m-1)}{2}$. As we mentioned above, there exists a symmetric polynomial P of N variables such that

$$P(x_1 x_2^2, x_1 x_3^2, \dots, x_2 x_3^2, \dots, x_{N-1} x_N^2) \neq P(y_1 y_2^2, y_1 y_3^2, \dots, y_2 y_3^2, \dots, y_{N-1} x_N^2).$$

Equivalently, there is $1 \leq k \leq N$ such that

$$\sum_{i < j} (x_i x_j^2)^k \neq \sum_{i < j} (y_i y_j^2)^k,$$

that is, $P_{k,2k}(x) \neq P_{k,2k}(y)$. This is a contradiction. Thus $(x_1, \dots, x_m) = (y_1, \dots, y_m)$ and so $x \simeq y$. $\square$

Let us denote by $\mathcal{P}_{\mathfrak{S}}^{(n)}(c_{00})$ the subalgebra of $\mathcal{P}_{\mathfrak{S}}(c_{00})$, generated by polynomials $P_{\alpha_1, \dots, \alpha_k}$ for $0 \leq k \leq n$. Clearly, $\mathcal{P}_{\mathfrak{S}}^{(1)}(c_{00}) = \mathcal{P}_s(c_{00})$ is the algebra of symmetric polynomials and it is not equal to $\mathcal{P}_{\mathfrak{S}}^{(2)}(c_{00})$. Let $(\Theta(P))(x) = P(\check{x} \triangleleft x)$. From Proposition 4 it follows that the restriction of Θ to $\mathcal{P}_{\mathfrak{S}}^{(2)}(c_{00})$ is a homomorphism from $\mathcal{P}_{\mathfrak{S}}^{(2)}(c_{00})$ to $\mathcal{P}_s(c_{00})$.

The following example shows that $\mathcal{P}_{\mathfrak{S}}^{(2)}(c_{00})$ is a proper subalgebra in $\mathcal{P}_{\mathfrak{S}}^{(3)}(c_{00})$.

Example 3. Let $k \neq n$ be positive integers. We claim that $P_{k,n,k} \notin \mathcal{P}_{\mathfrak{S}}^{(2)}(c_{00})$. Indeed,

$$\begin{aligned}\Theta(P_{k,n,k})(x) &= P_{k,n,k}(\check{x} \triangleleft x) = P_{k,n,k}(\check{x}) + P_k(\check{x})P_{n,k}(x) + P_{k,n}(\check{x})P_k(x) + P_{k,n,k}(x) \\ &= 2P_{k,n,k}(x) + 2P_k(x)P_{n,k}(x).\end{aligned}$$

Polynomial $2P_{k,n,k}(x) + 2P_k(x)P_{n,k}(x)$ is not symmetric because, for example,

$$2P_{k,n,k}(1, 2, 0, \dots) + 2P_k(1, 2, 0, \dots)P_{n,k}(1, 2, 0, \dots) = 10 \cdot 2^k$$

and

$$2P_{k,n,k}(2, 1, 0, \dots) + 2P_k(2, 1, 0, \dots)P_{n,k}(2, 1, 0, \dots) = 10 \cdot 2^n.$$

But, if $P_{k,n,k} \in \mathcal{P}_{\mathfrak{S}}^{(2)}(c_{00})$, then by Proposition 4, $\Theta(P_{k,n,k})$ must be symmetric.

The following theorem delivers some nontrivial sequences of algebraically independent subsymmetric polynomials.

Theorem 3. Let (R_n) , $n \in \mathbb{N}$ be a sequence of subsymmetric polynomials. If there exists an algebra homomorphism $\Phi: \mathcal{P}_{\mathfrak{S}}(c_{00}) \rightarrow \mathcal{P}_{\mathfrak{S}}(c_{00})$ such that the sequence $(\Phi(R_n))$, $n \in \mathbb{N}$ is algebraically independent, then polynomials $R_1, \dots, R_n, \dots$ are algebraically independent as well.

Proof. If (R_n) , $n \in \mathbb{N}$ are not algebraically independent, then for some $r \in \mathbb{N}$ there is a nontrivial polynomial of several variables $q(t_1, \dots, t_r)$ such that

$$q(R_1, \dots, R_r) \equiv 0.$$

Since Φ is a homomorphism,

$$\Phi(q(R_1, \dots, R_r)) = q(\Phi(R_1), \dots, \Phi(R_r)) = \Phi(0) = 0.$$

But it contradicts the algebraic independence of polynomials $\Phi(R_n)$, $n \in \mathbb{N}$. $\square$

Let $Z_1 = P_1$, $Z_2 = P_2$ and for $n > 2$,

$$Z_n(x) = \sum_{\substack{k+m=n, \\ 0 < k < m}} P_{k,m}(x), \quad x \in c_{00}.$$

Corollary 1.

- (i) Polynomials Z_n , $n \in \mathbb{N}$ are algebraically independent in $\mathcal{P}_{\mathfrak{S}}^{(2)}(c_{00})$.
- (ii) Polynomials $P_{1,m}$, $m \in \mathbb{Z}_+$ are algebraically independent in $\mathcal{P}_{\mathfrak{S}}^{(2)}(c_{00})$. Here $P_{1,0} = P_1$.

Proof. (i). Let $\Phi = \Theta$. According to Proposition 4,

$$\begin{aligned}\Theta(Z_1) &= 2P_1, \\ \Theta(Z_2) &= 2P_2, \\ \Theta(Z_3) &= 2P_1P_2 - P_3, \\ &\dots \\ \Theta(Z_n) &= 2P_1P_{n-1} + \dots + 2P_NP_{n-N} - NP_n, \\ &\dots,\end{aligned}\tag{9}$$

where N is the number of pairs (k, m) such that $0 < k < m$ and $k + m = n$. It is easy to check that $N = \lfloor \frac{n-1}{2} \rfloor$. For every fixed n , relations (9) give a polynomial automorphism

$$(P_1, P_2, \dots, P_n) \mapsto (\Theta(Z_1), \Theta(Z_2), \dots, \Theta(Z_n)).$$

Since $P_1, P_2, \dots, P_n$ are algebraically independent for every n , the same is true for $\Theta(Z_1), \Theta(Z_2), \dots, \Theta(Z_n)$. By Theorem 3, polynomials $Z_n, n \in \mathbb{N}$ are algebraically independent.

The item (ii) can be proved by the same way taking into account that

$$\begin{aligned}\Theta(P_1) &= 2P_1, \\ \Theta(P_{1,1}) &= 2(P_1)^2 - P_2, \\ \Theta(P_{1,2}) &= 2P_1P_2 - P_3, \\ &\dots \\ \Theta(P_{1,n}) &= 2P_1P_n - P_{n+1}, \\ &\dots.\end{aligned}$$

□

Note that both sequences (Z_n) and $(P_{1,m})$ of algebraically independent subsymmetric polynomials in Corollary 1 do not contain symmetric polynomials, excepting P_1 and P_2 . If we set $R_n = P_n$ if n is odd and $R_n = Z_n$ if n is even, we can show using Theorem 3 for $\Phi = \Theta$ that polynomials R_n are algebraically independent. But we do not know the following: *Is there an algebraically independent sequence of subsymmetric polynomials containing polynomials P_n for all $n \in \mathbb{N}$ and an infinite subsequence of nonsymmetric polynomials?*

4. Extension of Subsymmetric Polynomials

4.1. Extension to Banach Spaces ℓ_p

Throughout the rest of the paper we consider spaces ℓ_p only for $1 \leq p < \infty$. The linear space c_{00} is dense in the Banach space ℓ_p for every $1 \leq p < \infty$. Thus, if a subsymmetric polynomial P on c_{00} is continuous in the topology of ℓ_p , we can extend it to a continuous subsymmetric polynomial on ℓ_p . As we observed the standard subsymmetric polynomials $P_{\alpha_1, \dots, \alpha_n}$ are well-defined and continuous on ℓ_p if and only if each $\alpha_j \geq \lceil p \rceil$ [19]. In particular, each subsymmetric polynomial on c_{00} can be extended to a subsymmetric polynomial on ℓ_1 so that for every $x \in \ell_1$,

$$P_{\alpha_1, \dots, \alpha_n}(x) = \sum_{i_1 < \dots < i_n} x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n}.$$

We denote by $\mathcal{P}_{\mathfrak{S}}(\ell_p)$ the algebra of all continuous subsymmetric polynomials on ℓ_p .

For every $x \in \ell_p$ the mapping

$$\delta_x: P \mapsto P(x)$$

is called a *point evaluation complex homomorphism* of $\mathcal{P}_{\mathfrak{S}}(\ell_p)$. As in the case of c_{00} , we define the equivalence on ℓ_p so that $x \simeq y$ if there are operators τ_1 and τ_2 in $\mathfrak{S}$ such that $\tau_1(x) = \tau_2(y)$. Similarly, $x \sim y$ if there are operators σ_1 and σ_2 in S such that $\sigma_1(x) = \sigma_2(y)$. The quotient set $\mathcal{M}_p^S = \ell_p / \sim$ can be associated with the set of infinite multisets $[x]$ consisting of nonzero complex numbers x_i such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$. The following proposition was proved in [4].

Proposition 5. *Let $x, y \in \ell_p$. Then $x \sim y$ in ℓ_p if and only if there exists a natural number $m \geq \lceil p \rceil$ such that $P_n(x) = P_n(y)$ for every $n \geq m$. In this case, of course, $P(x) = P(y)$ for every symmetric polynomial P .*

The following result extends Theorem 2 to ℓ_p .

Theorem 4. Let $x, y \in \ell_p$. Then $x \simeq y$ if there exists a natural number $m \geq \lceil p \rceil$ such that $P_n(x) = P_n(y)$ and $P_{n,2n}(x) = P_{n,2n}(y)$ for every $n \geq m$. In this case, $P(x) = P(y)$ for every subsymmetric polynomial P .

Proof. If $x \simeq y$, then it is trivially true that $P(x) = P(y)$ for all $P \in \mathcal{P}_{\mathfrak{S}}(\ell_p)$. Conversely, from Proposition 5 it follows that $x \sim y$, and removing zero coordinates, we may assume x_i are elements of the multiset $[x]$ and y_j are elements of the multiset $[y]$, $i, j \in \mathbb{N}$. As in Theorem 2, we consider the following multisets

$$M_{1,2}(x) = \{x_i x_j^2 : i < j\} \quad \text{and} \quad M_{1,2}(y) = \{y_i y_j^2 : i < j\}.$$

We claim that if $(x_1, x_2, \dots) \neq (y_1, y_2, \dots)$ (as vectors) then $M_{1,2}(x) \neq M_{1,2}(y)$ (as multisets). Indeed, suppose that there are $x_r \neq x_s$ such that $r < s$ and a bijection $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ such that $(y_1, y_2, \dots) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots)$ and $\sigma(r) > \sigma(s)$. Then, as in the proof of Theorem 2, the symmetric difference $M_{1,2}(x) \triangle M_{1,2}(y)$ is nonempty, because it contains elements $x_r x_s^2$ and $x_s x_r^2$. Applying Proposition 5 to multisets $u = M_{1,2}(x)$ and $v = M_{1,2}(y)$ we have that for every $m \geq \lceil p \rceil$ there is $n \geq m$ such that $P_n(u) \neq P_n(v)$. But

$$P_n(u) = \sum_{i < j} (x_i x_j^2)^n = \sum_{i < j} x_i^n x_j^{2n} = P_{n,2n}(x),$$

and so $P_{n,2n}(x) \neq P_{n,2n}(y)$. This is a contradiction. Thus, $M_{1,2}(x) = M_{1,2}(y)$, that is, $x \simeq y$. In particular, $P(x) = P(y)$ for every subsymmetric polynomial P . $\square$

From Theorem 4 it follows that $x \simeq y$ if and only if $\delta_x = \delta_y$ on $\mathcal{P}_{\mathfrak{S}}(\ell_p)$.

It is easy to see that if $x \in \ell_p$ has infinitely many nonzero coordinates, then the element $\check{x}$ does not exist in ℓ_p . But we can define a complex homomorphism $\delta_{\check{x}}$ by

$$\delta_{\check{x}}(P) = \lim_{k \rightarrow \infty} P(x_k, x_{k-1}, \dots, x_1, 0, 0, \dots), \quad P \in \mathcal{P}_{\mathfrak{S}}(\ell_p).$$

Similarly, if x and y are in ℓ_p , the $x \triangleleft y$ may be not defined as a vector in ℓ_p , but we can define

$$\delta_{x \triangleleft y}(P) = \lim_{k \rightarrow \infty} P(x_1, x_2, \dots, x_k, y_1, y_2, \dots), \quad P \in \mathcal{P}_{\mathfrak{S}}(\ell_p).$$

Moreover, we can define

$$\delta_{\check{x} \triangleleft y}(P) = \lim_{k \rightarrow \infty} P(x_k, x_{k-1}, \dots, x_1, y_1, y_2, \dots), \quad P \in \mathcal{P}_{\mathfrak{S}}(\ell_p).$$

Proposition 6. Complex homomorphisms $\delta_{\check{x}}$, $\delta_{x \triangleleft y}$, and $\delta_{\check{x} \triangleleft y}$ are well-defined on $\mathcal{P}_{\mathfrak{S}}(\ell_p)$ for every x and y in ℓ_p . Moreover,

$$\delta_{\check{x}}(P_{\alpha_1, \dots, \alpha_n}) = P_{\alpha_n, \dots, \alpha_1}(x),$$

and

$$\delta_{x \triangleleft y} = \sum_{j=0}^{n+1} P_{\alpha_1, \dots, \alpha_j}(x) P_{\alpha_{j+1}, \dots, \alpha_n}(y)$$

for every standard polynomial $P_{\alpha_1, \dots, \alpha_n} \in \mathcal{P}_{\mathfrak{S}}(\ell_p)$.

Proof. Let $x^{(k)} = (x_1, x_2, \dots, x_k, 0, 0, \dots)$. For every fixed k and a standard polynomial $P_{\alpha_1, \dots, \alpha_n}$,

$$P_{\alpha_1, \dots, \alpha_n}((x^{(k)})^\vee) = P_{\alpha_n, \dots, \alpha_1}(x^{(k)}).$$

Since $x^{(k)} \rightarrow x$ as $k \rightarrow \infty$, from the continuity of $P_{\alpha_n, \dots, \alpha_1}$ it follows

$$\delta_{\check{x}}(P_{\alpha_1, \dots, \alpha_n}) = \lim_{k \rightarrow \infty} P_{\alpha_n, \dots, \alpha_1}(x^{(k)}) = P_{\alpha_n, \dots, \alpha_1}(x).$$

Since every polynomial in $\mathcal{P}_{\mathfrak{S}}(\ell_p)$ is a finite linear combination of standard polynomials, $\delta_{\check{x}}$ is well-defined on $\mathcal{P}_{\mathfrak{S}}(\ell_p)$.

Similarly, using equality (7) and the density of c_{00} in ℓ_p we can prove the rest of the proposition concerning $\delta_{x \triangleleft y}$, and $\delta_{\check{x} \triangleleft y}$. $\square$

For every $P \in \mathcal{P}_{\mathfrak{S}}(\ell_p)$ we define $\check{P}(x) = \delta_{\check{x}}(P)$. Clearly, if $x \in c_{00}$, then $\delta_{\check{x}}(P) = P(\check{x})$.

Corollary 2. *The mapping $P \mapsto \check{P}$ is an involution of the algebra $\mathcal{P}_{\mathfrak{S}}(\ell_p)$ and $\|P\| = \|\check{P}\|$.*

Proof. We already have

$$\check{P}_{\alpha_1, \dots, \alpha_n}(x) = \delta_{\check{x}}(P_{\alpha_1, \dots, \alpha_n}) = P_{\alpha_n, \dots, \alpha_1}(x).$$

Thus, the homomorphism $P \mapsto \check{P}$ is well-defined for every $P \in \mathcal{P}_{\mathfrak{S}}(\ell_p)$. Also, $(\check{P})^\vee = P$, that is, P is an involution. Since c_{00} is dense in ℓ_p and $\|x\| = \|\check{x}\|$ in ℓ_p for every $x \in c_{00} \subset \ell_p$,

$$\|\check{P}\| = \{\sup |\check{P}(x)| : x \in c_{00} \subset \ell_p, \|x\| \leq 1\} = \{\sup |\check{P}(\check{x})| : x \in c_{00} \subset \ell_p, \|x\| \leq 1\} = \|P\|.$$

$\square$

4.2. Extension to $\ell_p(\mathfrak{A})$ for a Well-Ordered Set $\mathfrak{A}$

Let us recall that a set $\mathfrak{A}$ is well-ordered if it is linearly ordered and every nonempty subset of $\mathfrak{A}$ has a minimal element. Throughout this subsection we assume that $\mathfrak{A}$ is an infinite well-ordered set with respect to an order $\preceq$. The Banach space $\ell_p(\mathfrak{A})$ is the space of functions

$$x: \mathfrak{A} \rightarrow \mathbb{C}, \quad x = (x_\gamma)_{\gamma \in \mathfrak{A}}$$

such that

$$\|x\| := \left(\sum_{\gamma \in \mathfrak{A}} |x_\gamma|^p \right)^{1/p} < \infty.$$

Clearly, for any $x \in \ell_p(\mathfrak{A})$ only a countable number of coordinates x_γ is not equal to zero.

Let

$$\iota: \mathbb{N} \rightarrow \mathfrak{A}$$

be an embedding of the set of natural numbers with the natural order $\leq$ to $\mathfrak{A}$ such that if $n_1 \leq n_2$ then $\iota(n_1) \preceq \iota(n_2)$. Thus, ℓ_p can be considered as a subspace of $\ell_p(\mathfrak{A})$ consisting of vectors

$$(x_{\iota(1)}, x_{\iota(2)}, \dots, x_{\iota(n)}, \dots).$$

Let $P_{\alpha_1, \dots, \alpha_n}$ be a standard polynomial. We can write

$$P_{\alpha_1, \dots, \alpha_n}(x_{\iota(1)}, x_{\iota(2)}, \dots) = \sum_{i_1 < \dots < i_n} x_{\iota(i_1)}^{\alpha_1} \cdots x_{\iota(i_n)}^{\alpha_n}.$$

Let us denote by $P_{\alpha_1, \dots, \alpha_n}^{\mathfrak{A}}$ the following polynomial on $\ell_p(\mathfrak{A})$,

$$P_{\alpha_1, \dots, \alpha_n}^{\mathfrak{A}}(x) = \sum_{\gamma_1 < \dots < \gamma_n} x_{\gamma_1}^{\alpha_1} \cdots x_{\gamma_n}^{\alpha_n}. \quad (10)$$

Clearly, $P_{\alpha_1, \dots, \alpha_n}^{\mathfrak{A}}$ is an extension of $P_{\alpha_1, \dots, \alpha_n}$ and this extension does not depend on the embedding ι . Since every subsymmetric polynomial is a linear combination of standard polynomials, we have the following proposition.

Proposition 7. Every subsymmetric polynomial P on ℓ_p can be extended to a polynomial $P^{\mathfrak{A}}$ on $\ell_p(\mathfrak{A})$. The operator of extension $\mathcal{J}: P \mapsto P^{\mathfrak{A}}$ is linear and $P_{\alpha_1, \dots, \alpha_n}^{\mathfrak{A}}$ is defined by (10).

Let us denote by $\mathcal{P}_{\mathfrak{S}}(\ell_p(\mathfrak{A}))$ the range of $\mathcal{P}_{\mathfrak{S}}(\ell_p)$ under the mapping $\mathcal{J}$. We can see that $\mathcal{P}_{\mathfrak{S}}(\ell_p(\mathfrak{A}))$ is an algebra and $\mathcal{J}$ is an algebra isomorphism. In particular, for every point $u \in \ell_p(\mathfrak{A})$, the functional $P \mapsto P^{\mathfrak{A}}(u)$ is a complex homomorphism.

Example 4. As we mentioned above, there is no point z in ℓ_p such that $P(z) = \delta_{x \prec y}(P)$ if $y \neq 0$ and x has infinitely many nonzero coordinates. However, such a point exists in $\ell_p(\mathfrak{A})$ for some set of indexes $\mathfrak{A}$. Indeed, let $\mathfrak{A} = \mathbb{N}_1 \cup \mathbb{N}_2$, where $\mathbb{N}_1$ and $\mathbb{N}_2$ are copies of natural numbers with the usual order, and if $i \in \mathbb{N}_1$ and $j \in \mathbb{N}_2$, then $i \prec j$. For given x and y in ℓ_p we set

$$u = (x_1, \dots, x_i, \dots) + (y_1, \dots, y_j, \dots) \in \ell_p(\mathbb{N}_1 \cup \mathbb{N}_2), \quad i \in \mathbb{N}_1, j \in \mathbb{N}_2.$$

Then $\delta_{x \prec y}(P) = P(u)$. Note that $\|u\|^p = \|x\|^p + \|y\|^p$.

Proposition 8. For every $P \in \mathcal{P}_{\mathfrak{S}}(\ell_p)$, $\|P\| = \|P^{\mathfrak{A}}\|$.

Proof. Since ℓ_p is isometrically embedded into $\ell_p(\mathfrak{A})$, $\|P\| \leq \|P^{\mathfrak{A}}\|$. For every $\varepsilon > 0$ there is a vector $u = (u_{\gamma_1}, \dots, u_{\gamma_n}, \dots)$ in $\ell_p(\mathfrak{A})$, $\|u\| = 1$ such that $|P^{\mathfrak{A}}(u)| \geq \|P^{\mathfrak{A}}\| - \varepsilon$. Let $x \in \ell_p$ be such that $x_n = u_{\gamma_n}$, $n \in \mathbb{N}$. Then

$$\|x\|^p = \sum_{n=1}^{\infty} |u_{\gamma_n}|^p = \|u\|^p = 1.$$

Also, by the definition of the extension operator $\mathcal{J}$, $P(x) = P^{\mathfrak{A}}(u)$ and so $|P(x)| \geq \|P^{\mathfrak{A}}\| - \varepsilon$. Since such a vector exists for every $\varepsilon > 0$, $\|P\| \geq \|P^{\mathfrak{A}}\|$. Thus, $\|P\| = \|P^{\mathfrak{A}}\|$. $\square$

5. Algebras of Subsymmetric Analytic Functions

5.1. General Properties of the Algebra of Subsymmetric Analytic Functions of Bounded Type

Let B_p^r be the ball in ℓ_p of radius $r > 0$ and centered at the origin. Denote by $H_{u\mathfrak{S}}(B_p^r)$ the completion of $\mathcal{P}_{\mathfrak{S}}(\ell_p)$ with respect to the norm

$$\|P\|_r = \sup_{\|x\|_{\ell_p} \leq r} |P(x)|.$$

$H_{u\mathfrak{S}}(B_p^r)$ is a Banach algebra of uniformly continuous subsymmetric analytic functions on B_p^r . It is a closed subalgebra of the algebra of all uniformly continuous analytic functions on B_p^r , $H_u(B_p^r)$. Also, we denote by $H_{b\mathfrak{S}}(\ell_p)$ the projective limit of algebras $H_{u\mathfrak{S}}(B_p^r)$ as $r \rightarrow \infty$. In other words, $H_{b\mathfrak{S}}(\ell_p)$ is a locally convex Fréchet algebra consisting of subsymmetric entire functions of bounded type (i.e., bounded on each bounded subset), and

$$H_{b\mathfrak{S}}(\ell_p) = \bigcap_{r>0} H_{u\mathfrak{S}}(B_p^r).$$

Any function $f \in H_{b\mathfrak{S}}(\ell_p)$ can be expressed using the Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n(x), \quad x \in \ell_p,$$

where f_n are n -homogeneous subsymmetric polynomials. The Taylor series converges to f uniformly on every bounded subset. Moreover, a formal series $\sum_{n=0}^{\infty} f_n(x)$ of n -homogeneous polynomials converges to an entire function of bounded type if and only if the radius of uniform converges at zero,

$$\varrho_0(f) = \left(\limsup_{n \rightarrow \infty} \|f_n\|^{1/n} \right)^{-1} = \infty.$$

Note that $H_{b\mathfrak{S}}(\ell_p)$ is a closed subalgebra of the algebra $H_b(\ell_p)$ of all entire functions of bounded type and $H_{b\mathfrak{S}}(\ell_p)$ contains the closed subalgebra $H_{bs}(\ell_p)$ of all symmetric entire functions of bounded type.

Let $\mathfrak{A}$ be an infinite well-ordered set. For every function $f \in H_{b\mathfrak{S}}(\ell_p)$ we assign

$$f^{\mathfrak{A}} = \sum_{n=0}^{\infty} f_n^{\mathfrak{A}},$$

and denote by $H_{b\mathfrak{S}}(\ell_p(\mathfrak{A}))$ the set $\{f^{\mathfrak{A}} : f \in H_{b\mathfrak{S}}(\ell_p)\}$.

Proposition 9. For every $f \in H_{b\mathfrak{S}}(\ell_p)$, $f^{\mathfrak{A}}$ is an entire function of bounded type on $\ell_p(\mathfrak{A})$ and the mapping $f \mapsto f^{\mathfrak{A}}$ is a topological isomorphism of algebras such that

$$\|f\|_r = \|f^{\mathfrak{A}}\|_r, \quad r > 0.$$

Proof. From Proposition 8 it follows that $\|f_n^{\mathfrak{A}}\|_r = \|f_n\|_r$ for every $r > 0$. Thus $\varrho_0(f^{\mathfrak{A}}) = \varrho_0(f) = \infty$ and so $f^{\mathfrak{A}}$ is an entire function of bounded type. Clearly, $(fg)^{\mathfrak{A}} = f^{\mathfrak{A}}g^{\mathfrak{A}}$ and $(f+g)^{\mathfrak{A}} = f^{\mathfrak{A}} + g^{\mathfrak{A}}$. Also, for every $r > 0$,

$$\|f^{\mathfrak{A}}\|_r = \lim_{m \rightarrow \infty} \left\| \sum_{n=0}^m f_n^{\mathfrak{A}} \right\|_r = \lim_{m \rightarrow \infty} \left\| \sum_{n=0}^m f_n \right\|_r = \|f\|_r.$$

□

5.2. Spectrum of $H_{b\mathfrak{S}}(\ell_p)$

Let us denote by $M_{b\mathfrak{S}}(\ell_p)$ the spectrum (i.e., the set of continuous complex homomorphisms (characters)) of $H_{b\mathfrak{S}}(\ell_p)$. The set $M_{b\mathfrak{S}}(\ell_p)$ is the inductive limit of the spectra of $H_{u\mathfrak{S}}(B_p^r)$, $M_{u\mathfrak{S}}(B_p^r)$ as $r \rightarrow \infty$, in particular,

$$M_{b\mathfrak{S}}(\ell_p) = \bigcup_{r>0} M_{u\mathfrak{S}}(B_p^r)$$

and each $M_{u\mathfrak{S}}(B_p^r)$ is a compact subset of $M_{b\mathfrak{S}}(\ell_p)$ (cf. [20]). Thus, every character $\varphi \in M_{b\mathfrak{S}}(\ell_p)$ belongs to $M_{u\mathfrak{S}}(B_p^r)$ for some $r > 0$. The infimum of $r > 0$ such that $\varphi \in M_{u\mathfrak{S}}(B_p^r)$ is called the *radius function* of φ and denoted by $R(\varphi)$. Thus, for every character φ , the radius function is well-defined and finite. Let φ_n be the restriction of φ to the normed space $\mathcal{P}_{\mathfrak{S}}(n\ell_p)$ of subsymmetric n -homogeneous polynomials and

$$\|\varphi_n\| = \{\sup |\varphi_n(P)| : \|P\|_1 \leq 1\}.$$

It is known that

$$R(\varphi) = \limsup_{n \rightarrow \infty} \|\varphi_n\|^{1/n}. \quad (11)$$

Formula (11) was proved in [20] for characters on the algebra $H_b(X)$ for any Banach space X and in [10] for characters on any subalgebra of $H_b(X)$. From (11) it follows that if φ is a linear multiplicative functional on $\mathcal{P}_{\mathfrak{S}}(\ell_p)$ such that $\limsup_{n \rightarrow \infty} \|\varphi_n\|^{1/n} = r_0 < \infty$, then

it is bounded in $H_{u\mathfrak{S}}(B_p^r)$ for every $0 < r < r_0$. Thus, φ can be extended to a continuous complex homomorphism of $H_{u\mathfrak{S}}(B_p^r)$. In particular, the extension of φ is in $M_{b\mathfrak{S}}(\ell_p)$. This can be formulated as following.

Proposition 10. *A linear multiplicative functional φ on $\mathcal{P}_{\mathfrak{S}}(\ell_p)$ can be extended to a character in $M_{b\mathfrak{S}}(\ell_p)$ if and only if*

$$R(\varphi) = \limsup_{n \rightarrow \infty} \|\varphi_n\|^{1/n} < \infty.$$

Theorem 5. *The following linear multiplicative functionals are continuous on $H_{b\mathfrak{S}}(\ell_p)$.*

- (i) δ_x for every $x \in \ell_p$. Moreover, $R(\delta_x) = \|x\|$.
- (ii) δ_u for every $u \in \ell_p(\mathfrak{A})$ for every well-ordered set $\mathfrak{A}$. Moreover, $R(\delta_u) = \|u\|$.
- (iii) $\delta_{x \triangleleft y}$ for all $x, y \in \ell_p$. Moreover, $R(\delta_{x \triangleleft y}) = \sqrt[p]{\|x\|^p + \|y\|^p}$.
- (iv) $\delta_{\check{x}}$ for every $x \in \ell_p$. Moreover, $R(\delta_{\check{x}}) = \|x\|$.

Proof. According to Proposition 10, it is enough to compute radius functions for the complex homomorphisms.

(i) From the definition of radius function we have $R(\delta_x) \leq \|x\|$. On the other hand, in [7] it is proved that in the algebra of entire symmetric function of bounded type $R(\delta_x) = \|x\|$. Since $H_{bs}(\ell_p) \subset H_{b\mathfrak{S}}(\ell_p)$ we have $R(\delta_x) = \|x\|$ in $H_{b\mathfrak{S}}(\ell_p)$.

(ii) From Propositions 8 and 9 we have that

$$\delta_u: f \mapsto f^{\mathfrak{A}}(u), \quad f \in H_{b\mathfrak{S}}(\ell_p)$$

is a complex homomorphism and $R(\delta_u) = \|u\|$.

(iii) As we observed in Example 4, there is a well-ordered set $\mathfrak{A}$ and $u \in \mathfrak{A}$ such that $\delta_{x \triangleleft y} = \delta_u$, and $\|u\|^p = \|x\|^p + \|y\|^p$. Thus,

$$R(\delta_{x \triangleleft y}) = R(\delta_u) = \|u\| = \sqrt[p]{\|x\|^p + \|y\|^p}.$$

(iv) Let $(\delta_{\check{x}})_n$ be the restriction of $\delta_{\check{x}}$ to the subspace $\mathcal{P}_{\mathfrak{S}}(^n\ell_p)$ of n -homogeneous subsymmetric polynomials. Then

$$(\delta_{\check{x}})_n(P) = (\delta_x)_n(\check{P})$$

and so $\|(\delta_{\check{x}})_n\| = \|(\delta_x)_n\|$. Taking into account (11), $R(\delta_{\check{x}}) = R(\delta_x) = \|x\|$. $\square$

5.3. A Convolution on the Spectrum

For a Banach space X and a vector $x \in X$, the translation operator T_x on $H_b(X)$ is defined by

$$(T_x f)(y) = f(x + y), \quad f \in H_b(X).$$

Using the translation operator in [20] (see also [22]) a convolution operation $\varphi * \theta$ was constructed for linear functionals φ and θ in $H_b^*(X)$ by

$$(\varphi * \theta)(f) = \varphi(\theta(T_x f)) = \varphi(g), \quad \text{where } g(x) = \theta(T_x(f))$$

and proved that $\varphi * \theta \in H_b^*(X)$. Here $H_b^*(X)$ is the space of all continuous linear functionals on $H_b(X)$. Moreover, if φ and θ are characters of $H_b(X)$, then $\varphi * \theta$ is so (Theorem 6.9. in [20]), and by Lemma 6.2. in [20], $R(\varphi * \theta) \leq R(\varphi) + R(\theta)$. Note that the subalgebra $H_{b\mathfrak{S}}(\ell_p) \subset H_b(\ell_p)$ is not invariant with respect to T_x . Thus, T_x cannot to be used for a construction of a convolution on $H_{b\mathfrak{S}}(\ell_p)$. In [6], they proposed a symmetric translation operator $(\mathcal{T}_x f)(y) = f(x \bullet y)$ on the algebra of symmetric entire functions of bounded type.

In this section we consider a convolution on $H_{b\mathfrak{S}}(\ell_p)$ based on the following “subsymmetric translation”, $(\mathfrak{T}_x f) = f(x \triangleleft y)$.

Theorem 6. For every $\theta \in H_{b\mathfrak{S}}^*(\ell_p)$ and $f \in H_{b\mathfrak{S}}(\ell_p)$ the function $g(x) = \theta(\mathfrak{T}_x f)$ is in $H_{b\mathfrak{S}}(\ell_p)$. If $\theta \in M_{b\mathfrak{S}}(\ell_p)$, then the map $f \mapsto g$ is a continuous homomorphism of $H_{b\mathfrak{S}}(\ell_p)$ and for every $\varphi \in M_{b\mathfrak{S}}(\ell_p)$,

$$(\varphi \triangleleft \theta)(f) := \varphi(\theta(\mathfrak{T}_x f)) = \varphi(g) \quad (12)$$

belongs to $M_{b\mathfrak{S}}(\ell_p)$.

Proof. Let $\mathfrak{A} = \mathbb{N}_1 \cup \mathbb{N}_2$ be the well-ordered set as in Example 4. The space $\ell_p(\mathfrak{A})$ has a Schauder basis $\{e_i^1\} \cup \{e_j^2\}$, $i \in \mathbb{N}_1$, $j \in \mathbb{N}_2$,

$$e_i^1 = (\underbrace{0, \dots, 0}_{i \in \mathbb{N}_1}, 1, 0, \dots), \quad e_j^2 = (\underbrace{0, \dots, 0}_{j \in \mathbb{N}_2}, 1, 0, \dots).$$

We denote by $\theta^{\mathfrak{A}}$ the extension of the linear functional θ to a linear functional on $H_{b\mathfrak{S}}(\ell_p(\mathfrak{A}))$ defined as

$$\theta^{\mathfrak{A}}(P_{\alpha_1, \dots, \alpha_m}^{\mathfrak{A}}) = \theta(P_{\alpha_1, \dots, \alpha_m}).$$

Since every homogeneous subsymmetric polynomial is a finite linear combination of standard polynomials,

$$\theta^{\mathfrak{A}}(f^{\mathfrak{A}}) = \lim_{m \rightarrow \infty} \left(\sum_{n=1}^m \theta^{\mathfrak{A}}(f_n^{\mathfrak{A}}) \right) = \lim_{m \rightarrow \infty} \left(\sum_{n=1}^m \theta(f_n) \right) = \theta(f).$$

On the other hand, as we observed in Example 4,

$$(\mathfrak{T}_x f)(y) = \delta_{x \triangleleft y}(f) = f^{\mathfrak{A}}(\tilde{x} + \tilde{y}) = (T_{\tilde{x}} f)(\tilde{y}),$$

where

$$\tilde{x} = \sum_{i \in \mathbb{N}_1} x_i e_i^1, \quad \text{and} \quad \tilde{y} = \sum_{j \in \mathbb{N}_2} y_j e_j^2.$$

Note that Theorem 6.1. in [20] asserts that for fixed $\varphi \in H_b^*(X)$ and $f \in H_b(X)$ the function $x \mapsto \varphi(T_x(f))$ belongs to $H_b(X)$. Thus, by Theorem 6.1. in [20], the function $\tilde{x} \mapsto \theta^{\mathfrak{A}}(T_{\tilde{x}} f^{\mathfrak{A}})$ is an entire function of bounded type on $\ell_p(\mathfrak{A})$. Since $\|\tilde{x}\| = \|x\|$ and $f \mapsto f^{\mathfrak{A}}$ is a topological isomorphism of algebras, the function $g(x) = \theta^{\mathfrak{A}}(T_{\tilde{x}} f^{\mathfrak{A}})$ is an entire function of bounded type on ℓ_p . Clearly, for every operator $\mathcal{C}_j$ as in (1), $g(\mathcal{C}_j(x)) = g(x)$. Hence, $g \in H_{b\mathfrak{S}}(\ell_p)$. So, for every linear continuous functional φ , the functional

$$(\varphi \triangleleft \theta)(f) = \varphi(g) = \varphi(\theta(\mathfrak{T}_x f))$$

is well-defined. On the other hand,

$$(\varphi \triangleleft \theta)(f) = (\varphi^{\mathfrak{A}} * \theta^{\mathfrak{A}})(f^{\mathfrak{A}}).$$

By 6.2. Lemma in [20],

$$R(\varphi^{\mathfrak{A}} * \theta^{\mathfrak{A}}) \leq R(\varphi^{\mathfrak{A}}) + R(\theta^{\mathfrak{A}})$$

and so $\varphi^{\mathfrak{A}} * \theta^{\mathfrak{A}}$ is continuous. Thus, $\varphi \triangleleft \theta$ is continuous as well. Moreover, if both φ and θ are characters, then $\varphi^{\mathfrak{A}}$ and $\theta^{\mathfrak{A}}$ are characters too and, by Theorem 6.9. in [20], $\varphi^{\mathfrak{A}} * \theta^{\mathfrak{A}}$ is a character. Thus, $\varphi \triangleleft \theta$ is a character. Finally, since for every $\varphi \in M_{b\mathfrak{S}}(\ell_p)$ the function $f \mapsto \varphi(g) = \varphi(\theta(\mathfrak{T}_x f))$ is a complex homomorphism of $H_{b\mathfrak{S}}(\ell_p)$ and $H_{b\mathfrak{S}}(\ell_p)$

is a semisimple commutative algebra, the map $f \mapsto g$ is a continuous homomorphism of $H_{b\mathfrak{S}}(\ell_p)$ to itself. $\square$

From the definition of $\varphi \triangleleft \theta$ (Equation (12)) it follows that $\delta_{x \triangleleft y} = \delta_x \triangleleft \delta_y$.

Corollary 3. *The linear multiplicative functional $\delta_{\tilde{x} \triangleleft x} = \delta_{\tilde{x}} \triangleleft \delta_x$ is continuous for every $x \in \ell_p$.*

Proposition 11. *Let $\varphi, \theta \in H_{b\mathfrak{S}}^*(\ell_p)$. Then*

$$(\varphi \triangleleft \theta)(P_{\alpha_1, \dots, \alpha_m}) = \sum_{j=0}^{m+1} \varphi(P_{\alpha_1, \dots, \alpha_j}) \theta(P_{\alpha_{j+1}, \dots, \alpha_m}).$$

Proof. From the density of c_{00} in ℓ_p it follows that formula (7) is true for polynomials in $\mathcal{P}_{\mathfrak{S}}(\ell_p)$. Thus, it is enough to apply $\varphi \triangleleft \theta$ to $P_{\alpha_1, \dots, \alpha_m}$ by the definition and taking into account that $(\mathfrak{T}_x P_{\alpha_1, \dots, \alpha_m})(y) = P_{\alpha_1, \dots, \alpha_m}(x \triangleleft y)$ can be represented by (7). $\square$

5.4. “Exceptional” Characters of $H_{b\mathfrak{S}}(\ell_p)$

The complex homomorphism $\delta_{x \triangleleft y}$ is not a point evaluation functional on $H_{b\mathfrak{S}}(\ell_p)$ but it is the evaluation at a point in $\ell_p(\mathfrak{A})$, and the character $\delta_{\tilde{x}}$ cannot be represented as an evaluation at any point of $\ell_p(\mathfrak{A})$. However, the restriction of $\delta_{\tilde{x}}$ to the subalgebra $H_{bs}(\ell_p)$ coincides with δ_x . It is known (see e.g., [6]) that for a natural number $p > 1$, there exists a family of characters of $H_{bs}(\ell_p)$ (so called exceptional characters) that are not point evaluation functionals. Let us show that the construction of such functionals works for subsymmetric analytic functions as well.

Let us recall the definition of an ultrafilter. A *filter* on a nonempty subset D is a family $\mathfrak{F}$ of subsets of D with the following properties:

- (1) $D \in \mathfrak{F}$ and $\emptyset \notin \mathfrak{F}$;
- (2) If $A, B \in \mathfrak{F}$, then $A \cap B \in \mathfrak{F}$;
- (3) If $A \in \mathfrak{F}$ and $A \subseteq B \subseteq D$, then $B \in \mathfrak{F}$.

An *ultrafilter* on D is a filter on D which is not properly contained in any other filter on D . An ultrafilter is *free* if there is no point $x \in D$ such that $x \in A$ for every $A \in \mathfrak{F}$. It is well-known that for every free ultrafilter $\mathcal{U}$ on the set of natural numbers and for every bounded sequence (x_n) , there exists a unique limit

$$w = \lim_{\mathcal{U}} x_n,$$

which we understand so that for every $\varepsilon > 0$ there is $A \subset \mathfrak{F}$ such that $|w - x_n| < \varepsilon$ if $n \in A$. For details on ultrafilters we refer the reader to [37].

For a given complex number λ and $p \in \mathbb{N}$, we consider a sequence

$$v_n(\lambda) = \left(\underbrace{\frac{\lambda}{\sqrt[p]{n}}, \dots, \frac{\lambda}{\sqrt[p]{n}}}_n, 0, \dots \right).$$

Since $\|v_n(\lambda)\| = |\lambda|$, $R(\delta_{v_n(\lambda)}) = |\lambda|$. The sequence $\delta_{v_n(\lambda)}$ has a cluster point (not unique) in the spectrum $M_b(\ell_p)$ of the algebra of all entire functions of bounded type, $H_b(\ell_p)$. More precisely, for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ there exists a character $\psi_{\lambda}^{\mathcal{U}} \in M_b(\ell_p)$ defined by

$$\psi_{\lambda}^{\mathcal{U}}(f) = \lim_{\mathcal{U}} \delta_{v_n(\lambda)}(f) = \lim_{\mathcal{U}} f(v_n(\lambda)).$$

However, the restriction of any such cluster point to the subalgebra $H_{bs}(\ell_p)$ defines a unique character ψ_{λ} (does not depend on $\mathcal{U}$) such that $\psi_{\lambda}(P_p) = \lambda^p$ and $\psi_{\lambda}(P_k) = 0$ for

$k > p$. It is known [4,6] that ψ_λ is not a point evaluation functional on $H_{bs}(\ell_p)$. Let us consider the action of this character on $H_{b\mathfrak{S}}(\ell_p)$. From simple combinatorial arguments we observe that for $n \geq m$,

$$P_{\alpha_1, \dots, \alpha_m}(\underbrace{1, \dots, 1}_n, 0, \dots) = \binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

Thus, in ℓ_p , $p \in \mathbb{N}$ for $n \geq m$ we have

$$\begin{aligned} P_{\alpha_1, \dots, \alpha_m}(v_n(\lambda)) &= P_{\alpha_1, \dots, \alpha_m}\left(\underbrace{\frac{\lambda}{\sqrt[p]{n}}, \dots, \frac{\lambda}{\sqrt[p]{n}}}_n, 0, \dots\right) \\ &= \lambda^{\alpha_1 + \dots + \alpha_m} \frac{n(n-1) \cdots (n-m+1)}{m! n^{\frac{\alpha_1 + \dots + \alpha_m}{p}}}. \end{aligned} \quad (13)$$

The numerator of the right-hand part of (13) is $O(n^m)$. So, to get a nonzero limit as $n \rightarrow \infty$, we need to have $\alpha_1 + \dots + \alpha_m \leq mp$. Since each $\alpha_j \geq p$, this is possible only if $\alpha_1 = \dots = \alpha_m = p$. In this case, $\alpha_1 + \dots + \alpha_m = mp$ and so

$$P_{\underbrace{p, \dots, p}_m}(v_n(\lambda)) = \lambda^{mp} \frac{n(n-1) \cdots (n-m+1)}{m! n^m} \rightarrow \frac{\lambda^{mp}}{m!} \quad \text{as } n \rightarrow \infty,$$

and the limit does not depend of any ultrafilter. Thus, we have the following theorem.

Theorem 7. *There is a unique restriction ψ_λ of $\psi_\lambda^{\mathcal{U}}$ to the subalgebra $H_{b\mathfrak{S}}(\ell_p) \subset H_{bs}(\ell_p)$, and ψ_λ is defined on the standard polynomials by*

$$\psi_\lambda(P_{\alpha_1, \dots, \alpha_m}) = \begin{cases} \frac{\lambda^{mp}}{m!} & \text{if } \alpha_1 = \dots = \alpha_m = p, \\ 0 & \text{otherwise.} \end{cases}$$

Example 5. *Let us compute $(\psi_\lambda \triangleleft \delta_x)(P_{p,k})$ for some $k > p$, $P \in \mathcal{P}_{\mathfrak{S}}(\ell_p)$, and $p \in \mathbb{N}$. Using Proposition 11 and Theorem 7 we have*

$$(\psi_\lambda \triangleleft \delta_x)(P_{p,k}) = \delta_x(P_{p,k}) + \psi_\lambda(P_p)\delta_x(P_k) + \psi_\lambda(P_{p,k}) = P_{p,k}(x) + \lambda^p P_k(x).$$

Note that $(\psi_\lambda \triangleleft \delta_x)(P_{k,p}) = P_{k,p}(x)$ because, for $k > p$, $\psi_\lambda(P_k) = 0$. Thus, if $A_{p,k} = P_{p,k} - P_{k,p}$, $k > p$, then

$$(\psi_\lambda \triangleleft \delta_x)(A_{p,k}) = A_{p,k}(x) + \lambda^p P_k(x).$$

Proposition 12. *Let $p \in \mathbb{N}$. Polynomials $A_{p,k} = P_{p,k} - P_{k,p}$, $k > p$ are algebraically independent in $\mathcal{P}_{\mathfrak{S}}(\ell_p)$.*

Proof. Let $\lambda \neq 0$. Then

$$(\psi_\lambda \triangleleft \delta_{\tilde{x} \triangleleft x})(A_{p,k}) = \delta_{\tilde{x} \triangleleft x}(A_{p,k}) + \lambda^p \delta_{\tilde{x} \triangleleft x}(P_k) = \lambda^p 2P_k(x),$$

because $\delta_{\tilde{x} \triangleleft x}(A_{p,k}) = 0$. Since the map $P \mapsto \delta_{\tilde{x} \triangleleft x}(P)$ is a homomorphism and polynomials $\{\lambda^p 2P_k\}$, $k > p$ are algebraically independent, we can apply Theorem 3 and get that polynomials $\{A_{p,k}\}$ are algebraically independent as polynomials on c_{00} . So they are algebraically independent as polynomials on ℓ_p as well. $\square$

5.5. Subsymmetric Polynomials on Abstract Hilbert Spaces

Let $\mathcal{H}$ be an abstract separable Hilbert space with the inner product $\langle \cdot | \cdot \rangle$ and an orthonormal basis (e_n) . Then $\mathcal{H}$ is isomorphic to ℓ_2 with respect to the isomorphism J define on the basis vectors by

$$J: e_n \mapsto (\underbrace{0, \dots, 0}_n, 1, 0, \dots)$$

and so each subsymmetric polynomial P on ℓ_2 is well-defined on $\mathcal{H}$ as $P(h) := P(J(h))$. However, some specific structures of $\mathcal{H}$ can provide additional properties of subsymmetric polynomials.

Example 6. Let $\mathcal{H}$ be the Hardy space $H^2(\mathbb{D})$ of analytic functions

$$h(z) = \sum_{k=0}^{\infty} h_k z^k$$

on the open unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ such that

$$\|h\|^2 = \langle h | h \rangle = \int_{|z|=1} h(z) \overline{h(z)} dz = \sum_{k=0}^{\infty} |h_k|^2 < \infty.$$

Thus, polynomial $e_k(z) = z^k, k = 0, 1, 2, \dots$ form an orthonormal basis in $H^2(\mathbb{D})$ and so each standard subsymmetric polynomial on $H^2(\mathbb{D})$ is of the form

$$P_{\alpha_1, \dots, \alpha_n}(h) = \sum_{i_1 < \dots < i_n} h_{i_1}^{\alpha_1} \dots h_{i_n}^{\alpha_n}, \quad i_m \geq 2,$$

where coefficients h_k can be computed as

$$h_k = \int_{|z|=1} h(z) \overline{z^k} dz.$$

The operators

$$\mathcal{C}_j: (h_0, \dots, h_n, \dots) \mapsto (h_0, \dots, h_{j-1}, 0, h_j, h_{j+1}, \dots), \quad j = 0, 1, 2, \dots$$

can be represented as operators of multiplication by the independent variable z :

$$\mathcal{C}_j(h) = \mathcal{C}_j\left(\sum_{k=0}^{\infty} h_k z^k\right) = \sum_{k=0}^{j-1} h_k z^k + z \sum_{k=j}^{\infty} h_k z^k.$$

In particular, $\mathcal{C}_0(h) = zh$ is the classical shift operator on $H^2(\mathbb{D})$ (see [38] for details). Note that in [38] invariant subspaces on $H^2(\mathbb{D})$ with respect to the shift $\mathcal{C}_0$ were investigated.

The operators $\mathcal{C}_j$ naturally act on the space $H_b(H^2(\mathbb{D}))$ of analytic functions of bounded type on $H^2(\mathbb{D})$ as

$$\widehat{\mathcal{C}}_j(f)(h) = f(\mathcal{C}_j(h)).$$

Since $\widehat{\mathcal{C}}_j$ is the operator of composition with the continuous linear operator $\mathcal{C}_j$, it is a continuous algebra homomorphism from $H_b(H^2(\mathbb{D}))$ to itself (see e.g., [20]).

Proposition 13. The space $H_{b\mathfrak{S}}(H^2(\mathbb{D}))$ of subsymmetric analytic functions of bounded type on $H^2(\mathbb{D})$ is a maximal subspace of $H_b(H^2(\mathbb{D}))$ consisting of invariant elements for all operators $\widehat{\mathcal{C}}_j$.

Proof. According to the definition of subsymmetric functions, $f \in H_b(H^2(\mathbb{D}))$ is subsymmetric if and only if it is invariant about every operator $\widehat{\mathcal{C}}_j$. $\square$

Note that not every $\widehat{\mathcal{C}}_0$ -invariant polynomial is subsymmetric.

Example 7. Let

$$P(h) = \sum_{k=0}^{\infty} h_k h_{k+1}.$$

Then P is $\widehat{\mathcal{C}}_0$ -invariant but not subsymmetric. Indeed,

$$\widehat{\mathcal{C}}_0(P)(h) = P(\mathcal{C}_0(h)) = P(0, h_0, h_1, h_2, \dots) = 0h_0 + h_0h_1 + h_1h_2 + \dots = P(h),$$

while

$$\widehat{\mathcal{C}}_1(P)(h) = P(\mathcal{C}_1(h)) = P(h_0, 0, h_1, h_2, \dots) = 0h_0 + 0h_1 + h_1h_2 + \dots \neq P(h).$$

Let us consider another example of the Hilbert space $\mathcal{H}$.

Example 8. Let $\mathcal{H}$ be the real Hilbert space with the orthogonal basis consisting of Chebyshev polynomials of the first kind T_n , $n \in \mathbb{N}$, and inner product

$$\langle u|v \rangle = \int_{-1}^1 u(t)v(t) \frac{dt}{1-t^2}.$$

Let us recall that Chebyshev polynomials of the first kind can be determined by the equality

$$T_n(\cos \vartheta) = \cos n\vartheta, \quad \vartheta \in \mathbb{R}. \quad (14)$$

The Hilbert space $\mathcal{H}$ can be considered as the weighted space $L_2[-1, 1]$, namely, $\mathcal{H} = L_2([-1, 1], \frac{1}{1-t^2})$. The sequence of polynomials

$$\frac{1}{\pi}, \quad \frac{2}{\pi}T_1, \quad \frac{2}{\pi}T_2, \quad \dots$$

form an orthonormal basis in $L_2([-1, 1], \frac{1}{1-t^2})$. More information about Chebyshev polynomials can be found in [39]. Thus, standard subsymmetric polynomials on $L_2([-1, 1], \frac{1}{1-t^2})$ are of the form

$$P_{\alpha_1, \dots, \alpha_n}(g) = \sum_{i_1 < \dots < i_n} h_{i_1}^{\alpha_1} \cdots g_{i_n}^{\alpha_n}, \quad i_m \geq 2,$$

where $g \in L_2([-1, 1], \frac{1}{1-t^2})$,

$$g_0 = \frac{1}{\pi} \int_{-1}^1 g(t) \frac{dt}{1-t^2}, \quad \text{and} \quad g_n = \frac{2}{\pi} \int_{-1}^1 T_n(t)g(t) \frac{dt}{1-t^2}.$$

From (14) it follows that for the composition $T_k \circ T_m$ of polynomials T_k and T_m we have $T_k \circ T_m(t) = T_k(T_m(t)) = T_{km}(t)$, $k, m \in \mathbb{N}$. Hence, each polynomial T_k generates a continuous linear operator $g \mapsto T_k \circ g$ on $L_2([-1, 1], \frac{1}{1-t^2})$ so that

$$T_k \circ g(t) = \frac{g_0}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} g_n T_{kn}.$$

In other words, the operator of composition with T_k maps $(g_0, g_1, g_2, \dots)$ to

$$(g_0, \underbrace{0, \dots, 0}_k, \underbrace{g_1, 0, \dots, 0}_k, \underbrace{g_2, 0, \dots, 0}_k, g_3, \dots).$$

Clearly, if f is a subsymmetric function, then $f(T_k \circ g) = f(g)$. Thus, if we set $\hat{T}_k(f) = f(T_k \circ g)$, then $\hat{T}_k$ is a continuous operator on $H_b(L_2([-1, 1], \frac{1}{1-t^2}))$, and each function in $H_b(L_2([-1, 1], \frac{1}{1-t^2}))$ is invariant with respect to every $\hat{T}_k$. Note that the space of T_k -invariant analytic functions of bounded type is larger than $H_b(L_2([-1, 1], \frac{1}{1-t^2}))$. Indeed, the polynomial $Q(g) = g_0^2$ is invariant with respect to $\hat{T}_k$ for every $k \in \mathbb{N}$ but it is not subsymmetric.

6. Zeros of Subsymmetric Polynomials and Subspaces of Symmetry

Let P be an n -homogeneous polynomial on a complex infinite-dimensional linear space. According to known result in [25], for every $x_0 \in \ker P$ there is an infinite-dimensional subspace in $\ker P$ containing x_0 . Actually, the main idea of the proof of the result in [25] can be formulated as the following proposition.

Proposition 14. *Let X be an infinite dimensional complex linear space and P be an n -homogeneous $\mathbb{C}$ -valued polynomial. For every $x_0 \in \ker P$ there exists an infinite linearly independent sequence $(x^{(k)}) \subset X$ such that for every $m \in \mathbb{N}$,*

$$P(t_1 x^{(1)} + t_2 x^{(2)} + \dots + t_m x^{(m)}) = t_1^n P(x^{(1)}) + t_2^n P(x^{(2)}) + \dots + t_m^n P(x^{(m)}), \quad (15)$$

and x_0 belongs to the linear span of $(x^{(k)})$.

The diagonal representation (15) suggests that a restriction of P to a subspace is symmetric with respect to an appropriate basis, i.e., $g_k = x^{(k)} / P(x^{(k)})$. However, in the general case, the values $P(x_m)$ can be nonzero only for a finite number of vectors and so we cannot claim that any polynomial on X is symmetric with respect to a basis on an infinite-dimensional subspace. In this section, we consider zeros of subsymmetric polynomials and a linear subspace V for a given subsymmetric polynomial P such the restriction of P to V is a nontrivial symmetric polynomial in some basis.

For a given natural number N we define by Ξ_N the following homomorphism of $\mathcal{P}_{\mathfrak{S}}(\ell_p)$,

$$\Xi_N(P)(x) = P(x \triangleleft (-1)^{1/N} x),$$

where $(-1)^{1/N}$; we understand the principal value of the N^{th} root of -1 .

Proposition 15. *Let $P_{\alpha_1, \dots, \alpha_n}$ be an M -homogeneous standard polynomial in $\mathcal{P}_{\mathfrak{S}}(\ell_p)$. Then $\Xi_N(P_{\alpha_1, \dots, \alpha_n})$ is a homogeneous polynomial, $\deg \Xi_N(P_{\alpha_1, \dots, \alpha_n}) = M$ or $\Xi_N(P_{\alpha_1, \dots, \alpha_n}) = 0$. If $M = N$, then $\Xi_N(P_{\alpha_1, \dots, \alpha_n}) \in \mathcal{P}_{\mathfrak{S}}^{(n-1)}(\ell_p)$.*

Proof. According to (7) we have

$$\begin{aligned} \Xi_N(P_{\alpha_1, \dots, \alpha_n})(x) &= P_{\alpha_1, \dots, \alpha_n}(x) \\ &+ \sum_{k=1}^n (-1)^{\frac{\alpha_{k+1} + \dots + \alpha_n}{N}} P_{\alpha_1, \dots, \alpha_k}(x) P_{\alpha_{k+1}, \dots, \alpha_n}(x) + (-1)^{M/N} P_{\alpha_1, \dots, \alpha_n}(x). \end{aligned}$$

Thus, $\deg \Xi_N(P_{\alpha_1, \dots, \alpha_n}) = \alpha_1 + \dots + \alpha_n = M$ if $\Xi_N(P_{\alpha_1, \dots, \alpha_n}) \neq 0$. If $M = N$, then

$$\Xi_N(P_{\alpha_1, \dots, \alpha_n})(x) = \sum_{k=1}^n (-1)^{\frac{\alpha_{k+1} + \dots + \alpha_n}{N}} P_{\alpha_1, \dots, \alpha_k}(x) P_{\alpha_{k+1}, \dots, \alpha_n}(x).$$

Hence, for this case, $\Xi_N(P_{\alpha_1, \dots, \alpha_n})$ is an algebraic combination of polynomials in $\mathcal{P}_{\mathfrak{S}}^{(n-1)}(\ell_p)$. $\square$

Note that if $P_{\alpha_1, \dots, \alpha_n} = P_N$, then $\Xi_N(P_N) = 0$.

Corollary 4. Let P be a homogeneous subsymmetric polynomial on ℓ_p . Then there is a finite sequence of natural numbers $N_1, N_2, \dots, N_m$ such that

$$\Xi_{N_m} \circ \dots \circ \Xi_{N_1}(P) = 0,$$

where “ $\circ$ ” is the composition of mappings.

Proof. Suppose that $P \in \mathcal{P}_{\mathfrak{S}}^{(n)}(\ell_p)$ and $P \notin \mathcal{P}_{\mathfrak{S}}^{(n-1)}(\ell_p)$ for some $n \in \mathbb{N}$. Then P is a finite algebraic combination of standard polynomials $Q_1, Q_2, \dots$ in $\mathcal{P}_{\mathfrak{S}}^{(n)}(\ell_p)$. Some of them, say $Q_{i_1}, Q_{i_2}, \dots, Q_{i_s}$, are in $\mathcal{P}_{\mathfrak{S}}^{(n)}(\ell_p) \setminus \mathcal{P}_{\mathfrak{S}}^{(n-1)}(\ell_p)$. Let $N_1 = \deg Q_{i_1}$, $N_2 = \deg Q_{i_2}$ and so on. Then, by Proposition 15,

$$\Xi_{N_s} \circ \dots \circ \Xi_{N_1}(P) \in \mathcal{P}_{\mathfrak{S}}^{(n-1)}(\ell_p),$$

and either $\deg \Xi_{N_s} \circ \dots \circ \Xi_{N_1}(P) = \deg P$ or $P = 0$. If $P \neq 0$, then we can apply this process to $\Xi_{N_s} \circ \dots \circ \Xi_{N_1}(P)$. Since, $\Xi_N(P_N) = 0$, after a finite number m of steps, we will have $\Xi_{N_m} \circ \dots \circ \Xi_{N_1}(P) = 0$. $\square$

Let us recall that a linearly independent sequence e_n in a Banach space X is a Schauder basis of X (see [40] for detailed) if every $x \in X$ has a unique representation $x = \sum_{n=1}^{\infty} x_n e_n$ for some numbers x_n such that

$$\lim_{m \rightarrow \infty} \left\| x - \sum_{n=1}^m x_n e_n \right\| = 0.$$

Theorem 8.

- (i) For every subsymmetric homogeneous polynomial P on ℓ_p and a natural number d , there is a d -dimensional subspace V_d in ℓ_p and a linear basis (g_i) in V_d such that P is symmetric on V_d with respect to the basis.
- (ii) For every subsymmetric homogeneous polynomial P on ℓ_p there exists a well-ordered set $\mathfrak{A}$ an infinite-dimensional subspace V in $\ell_p(\mathfrak{A})$, and a Schauder basis (g_i) in V such that $P^{\mathfrak{A}}$ is symmetric on V with respect to the basis.

Proof. (i) For given d and $N_1 \in \mathbb{N}$ we consider the following subspace

$$V_d^{(1)} = \{(x_1, \dots, x_d, (-1)^{1/N_1} x_1, \dots, (-1)^{1/N_1} x_d, 0, 0, \dots)\} \subset \ell_p.$$

The space $V_d^{(1)}$ is a d -dimensional subspace in ℓ_p with basis

$$g_k^{(1)} = \underbrace{(0, \dots, 0, 1, 0, \dots, 0, (-1)^{1/N_1}, 0, \dots)}_{k}, \quad k = 1, \dots, d.$$

If

$$y = (x_1, \dots, x_d, (-1)^{1/N_1} x_1, \dots, (-1)^{1/N_1} x_d, 0, 0, \dots),$$

then $P(y) = \Xi_{N_1}(P)(x_1, \dots, x_d, 0, \dots)$.

For another number N_2 , we can construct a space $V_d^{(2)}$ such that for every $z \in V_d^{(2)}$,

$$P(z) = \Xi_{N_2} \circ \Xi_{N_1}(P)(x_1, \dots, x_d, 0, \dots).$$

Indeed, let $V_d^{(2)}$ consist of the vectors of the form

$$z = (x_1, \dots, x_d, (-1)^{1/N_1} x_1, \dots, (-1)^{1/N_1} x_d, (-1)^{1/N_2} x_1, \dots, (-1)^{1/N_2} x_d, (-1)^{1/(N_1 N_2)} x_1, \dots, (-1)^{1/(N_1 N_2)} x_d, 0, \dots).$$

Then,

$$\Xi_{N_2} \circ \Xi_{N_1}(P)(x_1, \dots, x_d, 0, \dots) = \Xi_{N_2}(P)(x_1, \dots, x_d, (-1)^{1/N_1} x_1, \dots, (-1)^{1/N_1} x_d, 0, \dots) = P(z).$$

Clearly, $V_d^{(2)}$ is a d -dimensional subspace and

$$g_k^{(2)} = (\underbrace{0, \dots, 0, 1, 0, \dots, 0}_{k}, \underbrace{0, \dots, 0, (-1)^{1/N_1}}_{d+k}, \underbrace{0, \dots, 0, (-1)^{1/N_2}}_{k}, \underbrace{0, \dots, 0, (-1)^{1/(N_1 N_2)}}_{d+k}, 0, \dots),$$

$k = 1, \dots, d$, is a basis. By continuing this process, we can construct for a given finite sequence $N_1, \dots, N_m$ such that for every $1 \leq j \leq m$ there exists a d -dimensional subspace $V_d^{(j)}$ with a basis $g_1^{(j)}, \dots, g_d^{(j)}$ such that

$$P\left(\sum_{i=1}^j x_i g_i^{(j)}\right) = \Xi_{N_j} \circ \dots \circ \Xi_{N_1}(P)(x_1, \dots, x_d, 0, \dots).$$

From here, in particular, it follows that the restriction of P to $V_d^{(j)}$ is subsymmetric with respect to the basis $(g_i^{(j)})$. Let us take the sequence $N_1, \dots, N_m$ as in Corollary 4. Then for some $j \leq m$ the polynomial $\Xi_{N_j} \circ \dots \circ \Xi_{N_1}(P)$ is in $\mathcal{P}_{\mathfrak{S}}^{(1)}(\ell_p)$ while $\Xi_{N_{j-1}} \circ \dots \circ \Xi_{N_1}(P)$ is not. Thus, the space $V_d = V_d^{(j)}$ and its basis $(g_i) = (g_i^{(j)})$ are such that the restriction of P to $V_d = V_d^{(j)}$ is symmetric with respect to (g_i) .

(ii) As we observed above, the functional $\delta_{x \prec y}$ is not a point evaluation but it can be interpreted as a point evaluation functional on $\mathcal{P}_{\mathfrak{S}}^{\mathfrak{A}}(\ell_p(\mathfrak{A}))$ for some appropriate $\mathfrak{A}$. Let $(\mathfrak{A}, \preceq)$ be the following well-ordered set

$$\mathfrak{A} = \bigcup_{k=1}^{\infty} \mathbb{N}_k,$$

where each $\mathbb{N}_k$ is a copy of $\mathbb{N}$ with the natural order, that is, $n \preceq m$ if and only if $n \leq m$, $n, m \in \mathbb{N}_k$. If $n \in \mathbb{N}_k$ and $m \in \mathbb{N}_j$, $k \neq j$, then $n \prec m$ if and only if $k < j$. Let (e_i^k) be the standard basis in $\ell_p(\mathbb{N}_k)$, that is,

$$e_i^k = (\underbrace{0, \dots, 0}_i, 1, 0, \dots), \quad i \in \mathbb{N}_k.$$

Every vector $y \in \ell_p(\mathfrak{A})$ can be represented as

$$y = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} y_i^k e_i^k$$

for some complex numbers y_i^k such that

$$\|y\|^p = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |y_i^k|^p < \infty.$$

For a given $N_1 \in \mathbb{N}$ we consider the following subspace $V^{(1)}$ of $\ell_p(\mathfrak{A})$,

$$V^{(1)} = \left\{ \sum_{i=1}^{\infty} x_i e_i^1 + (-1)^{1/N_1} \sum_{i=1}^{\infty} x_i e_i^2 \right\},$$

where $x = (x_1, x_2, \dots)$ goes over ℓ_p . We can see that

$$\Xi_{N_1}(P)(x) = P^{\mathfrak{A}} \left(\sum_{i=1}^{\infty} x_i e_i^1 + (-1)^{1/N_1} \sum_{i=1}^{\infty} x_i e_i^2 \right).$$

Also, it is easy to check that vectors $g_i^{(1)} = e_i^1 + (-1)^{1/N_1} e_i^2$ form a Schauder basis in $V^{(1)}$. Suppose that for given natural numbers $N_1, \dots, N_m$, we already constructed a subspace $V^{(m-1)} \subset \ell_p(\mathfrak{A})$ of the form

$$\left\{ \sum_{i=1}^{\infty} y_i^1 e_i^1 + \sum_{i=1}^{\infty} y_i^2 e_i^2 + \dots + \sum_{i=1}^{\infty} y_i^r e_i^r \right\}$$

for some finite number r and a Schauder basis $(g_i^{(m-1)})$ in $V^{(m-1)}$ such that

$$\Xi_{N_{m-1}} \circ \dots \circ \Xi_{N_1}(P)(x) = P \left(\sum_{i=1}^{\infty} x_i g_i^{(m-1)} \right).$$

Then the space $V^{(m)}$ is defined as

$$V^{(m)} = \left\{ \sum_{i=1}^{\infty} y_i^1 e_i^1 + \sum_{i=1}^{\infty} y_i^2 e_i^2 + \dots + \sum_{i=1}^{\infty} y_i^r e_i^r + (-1)^{1/N_m} \left(\sum_{i=1}^{\infty} y_i^1 e_i^{r+1} + \sum_{i=1}^{\infty} y_i^2 e_i^{r+2} + \dots + \sum_{i=1}^{\infty} y_i^r e_i^{2r} \right) \right\},$$

and the basis $(g_i^{(m)})$ as

$$g_i^{(m)} = g_i^{(m-1)} + (-1)^{1/N_m} g_i^{(m-1)}.$$

Taking the sequence $N_1, \dots, N_m$ as in Corollary 4, we have that for some $j \leq m$ the polynomial $\Xi_{N_j} \circ \dots \circ \Xi_{N_1}(P)$ is symmetric with respect to the basis $(g_i^{(j)})$. Thus, the space $V_d = V_d^{(j)}$. $\square$

Let us show that there are some finitely dimensional subspaces such that the restriction of any subsymmetric polynomial of some special form to these subspaces are symmetric with respect to some basis.

Proposition 16. For a number $d \in \mathbb{N}$ there is a d -dimensional subspace V_d in ℓ_p such that every polynomial $P_{k,m}$ is nontrivial and symmetric on V_d with respect to a basis.

Proof. Indeed, we know that $\Theta(P_{k,m})(x) = \delta_{\tilde{x} \sim x}(P_{k,m})$ is a nontrivial symmetric polynomial. Thus, restriction of $P_{k,m}$ to the d -dimensional subspace spanned on the vectors

$$g_k = (\underbrace{0, \dots, 0}_k, 1, \underbrace{0, \dots, 0}_k, 1, \underbrace{0, \dots, 0}_k), \quad 1 \leq k \leq d$$

is symmetric with respect to (g_k) . $\square$

Proposition 17. For a number $d \in \mathbb{N}$ there is a d -dimensional subspace V_d in ℓ_p such that every polynomial of the form $A_{k,m} = P_{k,m} - P_{m,k}$, $0 < k < m$ vanishes on V_d .

Proof. It is easy to check that $\Theta(A_{k,m}) = 0$ for every $A_{k,m}$. Thus each $A_{k,m}$ vanishes on the subspace V_d as in Proposition 16. $\square$

Let us recall that a polynomial map

$$\mathcal{Q}: (z_1, \dots, z_n) \mapsto (Q_1(z_1, \dots, z_n), \dots, Q_n(z_1, \dots, z_n)), \quad (z_1, \dots, z_n) \in \mathbb{C}^n$$

is *proper* if for every compact subset $K \subset \mathbb{C}^n$, the preimage $(\mathcal{Q})^{-1}(K)$ is compact in $\mathbb{C}^n$. It is known (see, e.g., [41], Chapter 15) that a proper map is always surjective and open. Moreover, in [42], it is shown that if $Q_1, \dots, Q_n$ are homogeneous polynomials, then $\mathcal{Q}$ is proper if and only if $\mathcal{Q}^{-1}(0) = \{0\}$. Note that the surjectivity of a proper map implies that polynomials $Q_1, \dots, Q_n$ are algebraically independent. Note that according to Proposition 12, polynomials $A_{1,k}$, $k > 1$ on ℓ_1 are algebraically independent but the restriction of $A_{1,k}$ to any M -dimensional subspace spanned on the basis vectors e_i , $i = 1, \dots, M$, $M \geq 2$ generates an improper map $(A_{1,2}(x_1, \dots, x_M), \dots, A_{1,M+1}(x_1, \dots, x_M))$ from $\mathbb{C}^M$ to $\mathbb{C}^M$, because each $A_{1,k}$ vanishes on the d -dimensional subspace V_d , $d = \lfloor M/2 \rfloor$ as in Proposition 17.

7. Applications

7.1. Polynomial Hash Function

An important question in cryptography is about the construction of a hash function h that is assigned to a given sequence of numbers $x = (x_1, \dots, x_N)$ of some sequence $h(x) = w = (w_1, \dots, w_M)$ so that h is injective (i.e., h is a collision-free hash function) and the inverse map $h^{-1}: w \mapsto x$ is difficult for computation.

Our approach allows us to work with finite sequences of nonzero numbers. From Theorem 2 it follows that we can take $M = \frac{N(N+1)}{2}$, and the sequence $w = h(x)$ is defined as

$$\begin{aligned} w_1 &= P_1(x), \\ w_2 &= P_2(x), \\ &\dots \\ w_N &= P_N(x), \\ w_{N+1} &= P_{1,2}(x), \\ w_{N+2} &= P_{2,4}(x), \\ &\dots \\ w_{N(N+1)/2} &= P_{N(N-1)/2, N(N-1)}. \end{aligned} \tag{16}$$

Indeed, according to Theorem 2, $x \simeq y$ if and only if $P(x) = P(y)$ for every symmetric polynomial (that is, $P_i(x) = P_i(y)$, $i = 1, \dots, N$) and $P_{k,2k}(x) = P_{k,2k}(y)$, $k = 1, N(N-1)/2$. Thus, together we have $N + N(N-1)/2 = N(N+1)/2 = M$ polynomials and so the

mapping $w = h(x)$ (16) is injective on the sequences of nonzero complex numbers. Note that if x contains a zero element, then it will be ignored by (16) and the value $h(x)$ will be the same as when we remove this element. To find the inverse mapping h^{-1} , we need to solve the system of high-degree polynomial Equation (16), which is a hard computational problem. More precisely, the problem of solving a polynomial system is *NP*-hard. In other words, the answer can be verified in polynomial time [43,44]. In [43], it is conjectured that the problem of solving a polynomial system over a finite field is exponentially complex.

The proposed construction has the advantage over general methods for polynomial systems that the mapping $x \mapsto h(x)$ is always injective. So h is a collision-free hash function, that is, every value of h has a unique preimage.

A hash function based on symmetric polynomials on ℓ_1 was considered in [45]. In this case, we have a system of symmetric polynomials that can be reduced to a high-degree polynomial of a single variable and solved by approximation methods. The proposed hash function h is subsymmetric and cannot be reduced to a single polynomial.

Example 9. Let $x = (5, 3, 7)$. Then $N = 3$, and $h(x) = (w_1, \dots, w_6)$, where

$$\begin{aligned} w_1 &= P_1(x) = x_1 + x_2 + x_3 = 15, \\ w_2 &= P_2(x) = x_1^2 + x_2^2 + x_3^2 = 83, \\ w_3 &= P_3(x) = x_1^3 + x_2^3 + x_3^3 = 495, \\ w_4 &= P_{1,2}(x) = x_1x_2^2 + x_1x_3^2 + x_2x_3^2 = 437, \\ w_5 &= P_{2,4}(x) = x_1^2x_2^4 + x_1^2x_3^4 + x_2^2x_3^4 = 83659, \\ w_6 &= P_{4,8} = x_1^4x_2^8 + x_1^4x_3^8 + x_2^4x_3^8 = 4074050131. \end{aligned}$$

Example 9 shows that evaluations of $P_{k,mk}$ rapidly grow, and so the hash function h may have big values that depend on the size of x . This disadvantage can be reduced using multiplication in a finite field and other known methods.

Some different approaches to the construction of polynomial hash functions can be found in [46,47].

7.2. Spectra of Operators

Let E be a separable Hilbert space and $A: E \rightarrow E$ be a normal operator with a point spectrum such that the eigenvalues x_i of A are nonzero and $\sum_{i=1}^{\infty} |x_i|^p < \infty$. In other words, A is a *p*-nuclear (or *p*-trace) operator (see [48]). Let (e_i) be the orthonormal basis of eigenvectors of A . Such operators (with real eigenvalues) naturally appear in quantum physics as $A = e^{-\beta\mathcal{H}}$, where $\mathcal{H}$ is the Hamiltonian of a quantum system, and $\beta > 0$ is a physical parameter (see for details e.g., [49–51]). It is well known that the operator A cannot be determined by its spectrum. Establishing additional conditions or some additional spectral data allowing us to determine A , is an important problem of the inverse spectral theory (see e.g., [52] for some particular operators).

All eigenvalues x_i of A , counting with their multiplicities, form a multiset $[x] \in \mathcal{M}_p^S$, where $x = (x_1, x_2, \dots) \in \ell_p$. For every $k \geq \lceil p \rceil$,

$$P_k(x) = \sum_{i=1}^{\infty} x_i^k = \text{tr } A^k,$$

where tr is the trace of an operator. More information about the relationship between traces of *p*-nuclear operators and symmetric analytic functions on ℓ_p can be found in [53]. From Proposition 5 we have the following corollary.

Corollary 5. Let A and B be normal p -nuclear operators with point spectra and the same eigenvectors such that the eigenvalues x_i of A and the eigenvalues y_i of B are nonzero. Then $[x] = [y]$ if and only if there exists a natural number $m \geq [p]$ such that $\operatorname{tr} A^n = \operatorname{tr} B^n$ for every $n \geq m$.

Since we suppose that all eigenvalues x_i of A are nonzero, the ordered multiset $[[x]]$ completely defines the operator A .

Let $E \wedge E$ be the skew-symmetric tensor product of E . That is, $E \wedge E$ is the closed subspace of the Hilbert tensor product $E \otimes E$ spanned on the orthogonal vectors $e_i \wedge e_j = \frac{e_i \otimes e_j - e_j \otimes e_i}{2}$. In other words, the vectors $e_i \wedge e_j$, $i < j$ form an orthogonal basis in $E \wedge E$. For given operators A and B on E we denote by $A \wedge B$ the restriction of $A \otimes B$ to $E \wedge E$. That is, $(A \wedge B)(e_i \wedge e_j) = A(e_i) \wedge B(e_j)$. Then, for any natural numbers k and m ,

$$(A^k \wedge A^m)(e_i \wedge e_j) = x_i^k x_j^m,$$

and if A is p -nuclear, and $k \geq p$ and $m \geq p$, then

$$\operatorname{tr}(A^k \wedge A^m) = \sum_{i < j} x_i^k x_j^m = P_{k,m}(x).$$

Thus, applying Theorem 4, we have the following result.

Theorem 9. Let A and B be normal p -nuclear operators with point spectra and the same eigenvectors such that the eigenvalues x_i of A and the eigenvalues y_i of B are nonzero. Then $A = B$ if and only if there exists a natural number $m \geq [p]$ such that $\operatorname{tr} A^n = \operatorname{tr} B^n$ and $\operatorname{tr} [(A \wedge A^2)^n] = \operatorname{tr} [(B \wedge B^2)^n]$ for every $n \geq m$.

Proof. As we mentioned above, it is enough to show that $[[x]] = [[y]]$. This equality follows from Theorem 4 because

$$\operatorname{tr} [(A \wedge A^2)^n] = \operatorname{tr}(A^n \wedge A^{2n}) = P_{n,2n}(x),$$

and

$$\operatorname{tr} [(B \wedge B^2)^n] = \operatorname{tr}(B^n \wedge B^{2n}) = P_{n,2n}(y).$$

□

8. Conclusions

In the paper, we establish algebraic properties of subsymmetric polynomials on c_{00} and ℓ_p and describe some nontrivial sequences of algebraically independent subsymmetric polynomials. However, the problem of the existence of an algebraic basis in the algebra of subsymmetric polynomials of an infinite number of variables remains open. We proved that an ordered multiset $[[x]]$ is completely determined by evaluations $P_{n,2n}(x)$ and used this fact to propose some applications to cryptography and operator theory. We considered, probably, the most general case when we can extend subsymmetric polynomials to subsymmetric polynomials on a larger space such that the operator of extension is a continuous homomorphism of algebras. We applied these results to the algebra $H_{b\in}(\ell_p)$ of subsymmetric analytic functions of bounded type and constructed some nontrivial characters of this algebra. In the paper, we introduced a subsymmetric translation operator on $H_{b\in}(\ell_p)$ and extended it to a convolution on the set of characters of this algebra. Additionally, using a constructed homomorphism, we considered certain subspaces of sets of zeros of subsymmetric polynomials and subspaces where a subsymmetric polynomial is symmetric with respect to a basis.

We proposed some applications of the obtained results to cryptography and operator theory. In particular, we constructed a collision-free hash function based on subsymmetric polynomials.

Further investigations will focus on the problem of the existence of an algebraic basis in the algebra of all subsymmetric polynomials and on the description of the spectrum of $H_{b\mathfrak{S}}(\ell_p)$ as it was done in [7] for the algebra $H_{bs}(\ell_1)$. In addition, it would be interesting to consider an inner product on $\mathcal{P}_{\mathfrak{S}}(c_{00})$ and its corresponding Hilbert space—the completion of $\mathcal{P}_{\mathfrak{S}}(c_{00})$ with respect to the inner product norm. These kinds of Hilbert spaces in the case of symmetric polynomials were considered in [54].

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References

1. Macdonald, I.G. *Symmetric Functions and Orthogonal Polynomials*; University Lecture Series; AMS: Providence, RI, USA, 1997; Volume 12.
2. Nemirovskii, A.; Semenov, S. On polynomial approximation of functions on Hilbert space. *Mat. USSR-Sb.* **1973**, *21*, 255–277. [\[CrossRef\]](#)
3. González, M.; Gonzalo, R.; Jaramillo, J.A. Symmetric polynomials on rearrangement-invariant function spaces. *J. Lond. Math. Soc.* **1999**, *59*, 681–697. [\[CrossRef\]](#)
4. Alencar, R.; Aron, R.; Galindo, P.; Zagorodnyuk, A. Algebra of symmetric holomorphic functions on ℓ_p . *Bull. Lond. Math. Soc.* **2003**, *35*, 55–64. [\[CrossRef\]](#)
5. Aron, R.; Galindo, P.; Pinasco, D.; Zalduendo, I. Group-symmetric holomorphic functions on a Banach space. *Bull. Lond. Math. Soc.* **2016**, *48*, 779–796. [\[CrossRef\]](#)
6. Chernega, I.; Galindo, P.; Zagorodnyuk, A. Some algebras of symmetric analytic functions and their spectra. *Proc. Edinb. Math. Soc.* **2012**, *55*, 125–142. [\[CrossRef\]](#)
7. Chernega, I.; Galindo, P.; Zagorodnyuk, A. On the spectrum of the algebra of bounded-type symmetric analytic functions on l_1 . *Math. Nachr.* **2024**, *297*, 3835–3846. [\[CrossRef\]](#)
8. Chernega, I.; Baziv, N. Symmetric Polynomials on the tensor product of the space of absolutely summable sequences. *AIP Conf. Proc.* **2022**, *2483*, 030004. [\[CrossRef\]](#)
9. Falcó, J.; García, D.; Jung, M.; Maestre, M. Group-invariant separating polynomials on a Banach space. *Publ. Mat.* **2022**, *66*, 207–233. [\[CrossRef\]](#)
10. Halushchak, S. Spectra of some algebras of entire functions of bounded type, generated by a sequence of polynomials. *Carpathian Math. Publ.* **2019**, *11*, 311–320. [\[CrossRef\]](#)
11. Vasylyshyn, S. Spectra of algebras of analytic functions, generated by sequences of polynomials on Banach spaces, and operations on spectra. *Carpathian Math. Publ.* **2023**, *15*, 104–119. [\[CrossRef\]](#)
12. Vasylyshyn, T. Algebras of symmetric and block-symmetric functions on spaces of Lebesgue measurable functions. *Carpathian Math. Publ.* **2024**, *16*, 174–189. [\[CrossRef\]](#)
13. Vasylyshyn, T.V. Isomorphisms of algebras of symmetric functions on spaces. *Mat. Stud.* **2025**, *63*, 77–87. [\[CrossRef\]](#)
14. Vasylyshyn, T. Symmetric analytic functions on the Cartesian power of the complex Banach space of Lebesgue measurable essentially bounded functions on $[0, 1]$. *J. Math. Anal. Appl.* **2022**, *509*, 125977. [\[CrossRef\]](#)
15. Ponomarov, R.V.; Vasylyshyn, T.V. Symmetric polynomials on Cartesian products of Banach spaces of Lebesgue integrable functions. *Carpathian Math. Publ.* **2025**, *17*, 483–515. [\[CrossRef\]](#)
16. D’Alessandro, S.; Hájek, P. Polynomial algebras and smooth functions in Banach spaces. *J. Funct. Anal.* **2014**, *266*, 1627–1646. [\[CrossRef\]](#)
17. Hájek, P. Polynomial algebras on classical Banach Spaces. *Isr. J. Math.* **1998**, *106*, 209–220. [\[CrossRef\]](#)

18. D'Alessandro, S.; Hájek, P.; Johannis, M. Erratum to: "Polynomial algebras on classical Banach spaces". *Isr. J. Math.* **2015**, *207*, 1003–1012. [CrossRef]
19. Gonzalo, R. Multilinear forms, subsymmetric polynomials, and spreading models on Banach spaces. *J. Math. Anal. Appl.* **1996**, *202*, 379–397. [CrossRef]
20. Aron, R.M.; Cole, B.J.; Gamelin, T.W. Spectra of algebras of analytic functions on a Banach space. *J. Reine Angew. Math.* **1991**, *415*, 51–93.
21. Aron, R.M.; Cole, B.J.; Gamelin, T.W. Weak-star continuous analytic functions. *Can. J. Math.* **1995**, *47*, 673–683. [CrossRef]
22. Aron, R.M.; Galindo, P.; García, D.; Maestre, M. Regularity and algebras of analytic functions in infinite dimensions. *Trans. Am. Math. Soc.* **1996**, *348*, 543–559. [CrossRef]
23. Carando, D.; García, D.; Maestre, M. Homomorphisms and composition operators on algebras of analytic functions of bounded type. *Adv. Math.* **2005**, *197*, 607–629. [CrossRef]
24. Carando, D.; Dimant, V.; Rodríguez, J.T. Homomorphisms on algebras of analytic functions on non-symmetrically regular spaces. *Math. Z.* **2023**, *304*, 17. [CrossRef]
25. Plichko, A.; Zagorodnyuk, A. On automatic continuity and three problems of The Scottish book concerning the boundedness of polynomial functionals. *J. Math. Anal. Appl.* **1998**, *220*, 477–494. [CrossRef]
26. Aron, R.M.; Rueda, M.P. A problem concerning zero-subspaces of homogeneous polynomial. *Linear Topol. Spaces Complex Anal.* **1997**, *3*, 20–23.
27. Aron, R.; Seoane-Sepúlveda, J.B. Some comments about lineability and non-lineability. *J. Math. Sci.* **2025**. [CrossRef]
28. Aires, M.; Botelho, G. Zero sets of homogeneous polynomials containing infinite dimensional spaces. *Rev. Mat. Complut.* **2025**. [CrossRef]
29. Banach, T.; Plichko, A.; Zagorodnyuk, A. Zeros of quadratic functionals on non-separable spaces. *Colloq. Math.* **2004**, *100*, 141–147. [CrossRef]
30. Botelho, G.; Matos, M.C.; Pellegrino, D. Lineability of summing sets of homogeneous polynomials. *Linear Multilinear Algebra* **2010**, *58*, 61–74. [CrossRef]
31. Ferrer, J.; García, D.; Maestre, M.; Seoane-Sepúlveda, J.B. On the zero-set of 2-homogeneous polynomials in Banach spaces. *Linear Multilinear Algebra* **2019**, *67*, 1958–1970. [CrossRef]
32. Aron, R.M.; Berner, P.D. A Hahn-Banach extension theorem for analytic mappings. *Bull. Soc. Math. Fr.* **1978**, *106*, 3–24. [CrossRef]
33. Dineen, S. *Complex Analysis on Infinite Dimensional Spaces*; Springer: Berlin/Heidelberg, Germany, 1999. [CrossRef]
34. Mujica, J. *Complex Analysis in Banach Spaces*; North-Holland: Amsterdam, The Netherlands; New York, NY, USA; Oxford, UK, 1986.
35. Weyl, H. *The Classical Groups: Their Invariants and Representations*; Princeton University Press: Princeton, NJ, USA, 1973. [CrossRef]
36. Zagorodnyuk, A.V. Groups of symmetries of the set of zeros of polynomial functionals on complex Banach spaces. *J. Math. Sci.* **2001**, *104*, 1428–1431. [CrossRef]
37. Zelenyuk, Y.G. *Ultrafilters and Topologies on Groups*; De Gruyter: Berlin, Germany, 2011.
38. Nikolski, N.K. *Treatise on the Shift Operator: Spectral Function Theory*; Springer: Berlin/Heidelberg, Germany, 1986.
39. Mason, J. Some properties and applications of Chebyshev polynomial and rational approximation. In *Rational Approximation and Interpolation*; Graves-Morris, P.R., Saff, E.B., Varga, R.S., Eds.; Lecture Notes in Mathematics, vol 1105; Springer: Berlin/Heidelberg, Germany, 1984; pp. 27–48. [CrossRef]
40. Lindenstrauss, J.; Tzafriri, L. *Classical Banach Spaces, Vol. I, Sequence Spaces*; Springer: Berlin/Heidelberg, Germany, 1977.
41. Rudin, W. *Function Theory in the Unit Ball of $\mathbb{C}^n$* ; Reprint of the 1980 Edition; Classics in Mathematics; Springer: Berlin/Heidelberg, Germany, 2008.
42. Rudin, W. Proper holomorphic maps and finite reflection groups. *Indiana Univ. Math. J.* **1982**, *31*, 701–720. [CrossRef]
43. Ding, J.; Schmidt, D.J. Multivariable Public Key Crypto-Systems, *Cryptology ePrint Archive*, 2004, Report 2004/350. Available online: <https://eprint.iacr.org/2004/350.pdf> (accessed on 7 November 2025).
44. Ogura, N. On Multivariate Public-Key Cryptosystems. Ph.D. thesis, Tokyo Metropolitan University, Tokyo, Japan, March 2012. Available online: <https://tokyo-metro-u.repo.nii.ac.jp/records/2120> (accessed on 7 November 2025).
45. Novosad, Z.; Pylypiv, V.M.; Sharyn, S.; Zagorodnyuk, A.V. Dynamics of an infinite-dimensional symmetric logistic mapping. *Res. Math.* **2025**, *33*, 54–74. [CrossRef] [PubMed]
46. Lemire, D.; Kaser, O. Recursive n -gram hashing is pairwise independent, at best. *Comput. Speech Lang.* **2010**, *24*, 698–710. [CrossRef]
47. Lemire, D.; Kaser, O. Strongly universal string hashing is fast. *Comput. J.* **2014**, *57*, 1624–1638. [CrossRef]
48. Chen, D.; Belacel, A.; Chávez-Domínguez, J.A. Positively p -nuclear operators, positively p -integral operators and approximation properties. *Positivity* **2022**, *26*, 9. [CrossRef]
49. Schmidt, H.J.; Schnack, J. Thermodynamic fermion-boson symmetry in harmonic oscillator potentials. *Phys. A Stat. Mech. Appl.* **1999**, *265*, 584–589. [CrossRef]

50. Chernega, I.; Martsinkiv, M.; Vasylyshyn, T.; Zagorodnyuk, A. Applications of supersymmetric polynomials in statistical quantum physics. *Quantum Rep.* **2023**, *5*, 683–697. [[CrossRef](#)]
51. Hryniv, R.; Kravtsiv, V.; Vasylyshyn, T.; Zagorodnyuk, A. Symmetric and supersymmetric polynomials on ℓ_p and partition functions in quantum statistical physics. *Phys. Scr.* **2025**, *100*, 075208. [[CrossRef](#)]
52. Gesztesy, F.; Simon, B. Inverse spectral analysis with partial information on the potential, II. The case of discrete spectrum. *Trans. Amer. Math. Soc.* **2000**, *352*, 2765–2787. [[CrossRef](#)]
53. Burtnyak, I.; Chernega, I.; Hladkyi, V.; Labachuk, O.; Novosad, Z. Application of symmetric analytic functions to spectra of linear operators. *Carpathian Math. Publ.* **2021**, *13*, 701–710. [[CrossRef](#)]
54. Holubchak, O.M. Hilbert space of symmetric functions on ℓ_1 . *J. Math. Sci.* **2012**, *185*, 809–814. [[CrossRef](#)]

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