

Article

Several Subordination Features Using Bessel-Type Operator

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Abstract

For the function solution to the well-known homogeneous Bessel differential equation, we utilized a normalized form of this function to define a certain operator on a subclass of analytic functions. Using this operator, we introduced various subordination properties. We also examined the sufficient starlikeness conditions and provided some estimates for a specific subclass of univalent functions defined in the unit disc.

Keywords: analytic; starlike; univalent; subordination; Bessel

MSC: 30C45

1. Introduction

Consider $\mathcal{F}$ to represent the family of analytic and univalent functions in the open unit disc $\nabla := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, with the following series representation:

$$f(\zeta) = \zeta + \sum_{n=1}^{\infty} a_{n+1} \zeta^{n+1}. \quad (1)$$

The solution for the differential equation ([1])

$$\zeta^2 w''(\zeta) + b\zeta w'(\zeta) + [c\zeta^2 - p^2 + (1-b)p]w(\zeta) = 0 \quad (b, c, p \in \mathbb{C}), \quad (2)$$

is $w_{b,c,p}(\zeta)$, which is the generalized Bessel function, and has the following representation:

$$w_{b,c,p}(\zeta) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n! \Gamma\left(n + p + \frac{b+1}{2}\right)} \left(\frac{\zeta}{2}\right)^{2n+p}, \quad (3)$$

for $b, c, p, \zeta \in \mathbb{C}$, and $p + \frac{b+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$. Classical, modified, and spherical Bessel functions can all be studied in a unified way through series (3), as follows:

- (i) Acquiring $c = b = 1$ within (3) gives the familiar Bessel function of the first kind of order p defined by Watson [2] (Baricz [1]):

$$J_p(\zeta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(p + n + 1)} \left(\frac{\zeta}{2}\right)^{2n+p}; \quad \zeta \in \mathbb{C}.$$



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- (ii) Acquiring $c = -1$ and $b = 1$ within (3) gives the modified Bessel function of the first kind of order p given as ([1,2])

$$I_p(\zeta) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(p+n+1)} \left(\frac{\zeta}{2}\right)^{2n+p}; \quad \zeta \in \mathbb{C}.$$

- (iii) Acquiring $c = 1$ and $b = 2$ within (3) gives the spherical Bessel function of the first kind of order p expressed as ([1])

$$j_p(\zeta) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(p+n+\frac{3}{2})} \left(\frac{\zeta}{2}\right)^{2n+p}; \quad \zeta \in \mathbb{C}.$$

Now, we employ the function $\varphi_{b,c,p}(\zeta)$, which is a normalization for $w_{b,c,p}(\zeta)$, with the following ([3,4]):

$$\varphi_{b,c,p}(\zeta) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) \zeta^{1-\frac{p}{2}} w_{b,c,p}(\sqrt{\zeta}) \quad \left(b, c, p \in \mathbb{C}, p + \frac{b+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \quad (4)$$

Thus, $\varphi_{b,c,p}(\zeta)$ takes the following series form:

$$\varphi_{\kappa,c}(\zeta) = \zeta + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n (\kappa)_n} \frac{\zeta^{n+1}}{n!} \quad \left(\kappa = p + \frac{b+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-, b, c, p \in \mathbb{C}\right). \quad (5)$$

Now, the convolution-type operator

$$\mathcal{B}_{\kappa}^c : \mathcal{F} \longrightarrow \mathcal{F},$$

is defined as follows:

$$\begin{aligned} \mathcal{B}_{\kappa}^c f(\zeta) &:= \varphi_{\kappa,c}(\zeta) * f(\zeta) \\ &= \zeta + \sum_{n=1}^{\infty} \frac{(-c)^n a_{n+1}}{4^n (\kappa)_n} \frac{\zeta^{n+1}}{n!}, \end{aligned} \quad (6)$$

where $(\kappa)_n$ refers to the Pochhammer (shifted factorial), which is expressed as

$$(\kappa)_n = \frac{\Gamma(\kappa+n)}{\Gamma(\kappa)} = \begin{cases} 1, & (n=0, \kappa \in \mathbb{C} \setminus \{0\}), \\ \kappa(\kappa+1)\dots(\kappa+n-1), & (n \in \mathbb{N}, \kappa \in \mathbb{C}). \end{cases}$$

Applying (6),

$$\zeta(\mathcal{B}_{\kappa}^c f(\zeta))' = (\kappa-1)\mathcal{B}_{\kappa-1}^c f(\zeta) - (\kappa-2)\mathcal{B}_{\kappa}^c f(\zeta). \quad (7)$$

We list some operators as special cases:

- (i) By acquiring $c = -4$ in (6), we obtain the operator $\mathcal{N}_{\kappa} : \mathcal{F} \longrightarrow \mathcal{F}$, defined by

$$\mathcal{N}_{\kappa} f(\zeta) = \zeta + \sum_{n=1}^{\infty} \frac{a_{n+1}}{(\kappa)_n} \frac{\zeta^{n+1}}{n!}; \quad \kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad (8)$$

- (ii) By acquiring $c = b = 1$ in (6), we arrive to the operator $\mathcal{J}_p : \mathcal{F} \longrightarrow \mathcal{F}$, defined by

$$\mathcal{J}_p f(\zeta) = \zeta + \sum_{n=1}^{\infty} \frac{(-1)^n a_{n+1}}{4^n (p+1)_n} \frac{\zeta^{n+1}}{n!}; \quad p+1 \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad (9)$$

(iii) By acquiring $c = -1$ and $b = 1$ in (6), we have the operator $\mathcal{I}_p : \mathcal{F} \rightarrow \mathcal{F}$, defined by

$$\mathcal{I}_p f(\zeta) = \zeta + \sum_{n=1}^{\infty} \frac{a_{n+1}}{4^n (p+1)_n} \frac{\zeta^{n+1}}{n!}; \quad p+1 \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad (10)$$

(iv) By acquiring $c = 1$ and $b = 2$ in (6), we have the operator $\mathcal{S}_p : \mathcal{F} \rightarrow \mathcal{F}$, defined by

$$\mathcal{S}_p f(\zeta) = \zeta + \sum_{n=1}^{\infty} \frac{(-1)^n a_{n+1}}{4^n (p + \frac{3}{2})_n} \frac{\zeta^{n+1}}{n!}; \quad p + \frac{3}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-. \quad (11)$$

Now, and with the help of the operator $\mathcal{B}_\kappa^c$, in the following definition, we define a subclass of functions $S_\kappa^c(\alpha; C, D) \subset \mathcal{F}$. The investigations within this paper are focused on studying the properties of functions belonging to this subclass.

Definition 1. For real $-1 \leq D < C \leq 1$, $0 \leq \alpha < 1$ and $f \in \mathcal{F}$, $f \in S_\kappa^c(\alpha; C, D)$ if it fulfills the subordination:

$$\frac{1}{1-\alpha} \left(\frac{\zeta (\mathcal{B}_\kappa^c f(\zeta))'}{\mathcal{B}_\kappa^c f(\zeta)} - \alpha \right) \prec \frac{1+C\zeta}{1+D\zeta}. \quad (12)$$

Alternatively, in a similar way,

$$S_\kappa^c(\alpha; C, D) = \left\{ f \in \mathcal{F} : \left| \frac{\frac{\zeta (\mathcal{B}_\kappa^c f(\zeta))'}{\mathcal{B}_\kappa^c f(\zeta)} - \alpha}{D \frac{\zeta (\mathcal{B}_\kappa^c f(\zeta))'}{\mathcal{B}_\kappa^c f(\zeta)} - [D+(C-D)(1-\alpha)]} \right| < 1; \zeta \in \nabla \right\}.$$

$$\left(-1 \leq D < C \leq 1; 0 \leq \alpha < 1; \kappa = p + \frac{b+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-; b, c, p \in \mathbb{C} \right)$$

Additionally, for $C = -D = 1$, we write $S_\kappa^c(\alpha; 1, -1) = S_\kappa^c(\alpha)$, where

$$S_\kappa^c(\alpha) = \left\{ f \in \mathcal{F} : \Re \left\{ \frac{\zeta (\mathcal{B}_\kappa^c f(\zeta))'}{\mathcal{B}_\kappa^c f(\zeta)} \right\} > \alpha; \zeta \in \nabla \right\}. \quad (13)$$

We recommend the papers [1,4–8] for additional results on the modification of the generalized Bessel function. These papers established a variety of functional inequalities, integral representations, extensions of some known trigonometric inequalities, starlikeness, convexity, and univalence. Using the normalized form of the first-kind ordinary Bessel function and the normalized form of the first-kind generalized Bessel functions, respectively, Baricz and Frasin [7] and Deniz et al. [4] were interested in the univalence of certain integral operators. Several sufficient circumstances for the convexity and strong convexity of the integral operators formed by the normalized expression of the ordinary Bessel function of the first kind were obtained by Frasin [9]. Additionally, numerous authors have addressed the issue of some generalized integral operators' geometric features (such as convexity, starlikeness, and univalence). In addition, we recommend [10–18]. For more recent works about Bessel functions, we refer to [19–23].

The primary objective of this work is to introduce a variety of subordination properties involving the linear operator $\mathcal{B}_\kappa^c$ associated with the generalized Bessel functions defined by (6). Also, we investigated some estimates and sufficient starlike conditions of certain subclass of univalent functions defined in ∇ .

2. Preliminaries

We start by going over each of the next lemmas, which are necessary for our current study.

Lemma 1 ([24,25]). Let h be a convex (univalent) analytic function in ∇ such that $h(0) = 1$. Assume, further, that the function ϕ is analytic and denoted by

$$\phi(\zeta) = 1 + c_1\zeta + c_2\zeta^2 + \dots \quad (14)$$

If

$$\phi(\zeta) + \frac{\zeta\phi'(\zeta)}{\delta} \prec h(\zeta) \quad (\delta \neq 0, \Re\{\delta\} \geq 0, \zeta \in \nabla),$$

then

$$\phi(\zeta) \prec \psi(\zeta) = \frac{\delta}{\zeta^\delta} \int_0^\zeta t^{\delta-1} h(t) dt \prec h(\zeta) \quad (\delta \neq 0, \Re\{\delta\} \geq 0, \zeta \in \nabla), \quad (15)$$

and $\psi(\zeta)$ is the best dominant of (15).

Lemma 2 ([25]). For $\Psi : \mathbb{C}^2 \times \nabla \longrightarrow \mathbb{C}$ fulfills the subsequent requirement:

$$\Re\{\Psi(ix, y; \zeta)\} \leq \varepsilon,$$

for all $x \in \mathbb{R}$ and $y \leq -\frac{1}{2}(1+x^2)$, and for all $\zeta \in \nabla$. If the function ϕ of the form (14) is analytic in ∇ and

$$\Re\{\Psi(\phi(\zeta), \zeta\phi'(\zeta); \zeta)\} > \varepsilon \quad (\zeta \in \nabla),$$

this leads to

$$\Re\{\phi(\zeta)\} > 0 \quad (\zeta \in \nabla).$$

Lemma 3 ([26]). Let $\lambda \in \mathbb{R} \setminus \{0\}$, $0 \leq \beta < 1$ and $a/\lambda > 0$. Additionally, assume that $\Psi(\zeta) = 1 + c_n\zeta^n + c_{n+1}\zeta^{n+1} + \dots$ is analytic in ∇ , and

$$\Psi(\zeta) \prec 1 + \frac{aM}{n\lambda + a}\zeta \quad (n \in \mathbb{N}),$$

where

$$M = \frac{(1-\beta)|\lambda| \left(1 + \frac{n\lambda}{a}\right)}{|1 - \lambda + \lambda\beta| + \sqrt{1 + \left(1 + \frac{n\lambda}{a}\right)^2}}.$$

If the function $\theta(\zeta) = 1 + d_n\zeta^n + d_{n+1}\zeta^{n+1} + \dots$ is analytic in ∇ and satisfies

$$\Psi(\zeta)\{1 - \lambda + \lambda[(1-\beta)\theta(\zeta) + \beta]\} \prec 1 + M\zeta,$$

this gives that

$$\Re\{\theta(\zeta)\} > 0 \quad (\zeta \in \nabla).$$

Let $P(\gamma)$ be the family of all functions ϕ expressed with (14), analytic in ∇ , and that satisfies $\Re\{\phi(\zeta)\} > \gamma$ ($0 \leq \gamma < 1, \zeta \in \nabla$).

Lemma 4 ([27]). If $\phi \in P(\gamma)$, then

$$\Re\{\phi(\zeta)\} > 2\gamma - 1 + \frac{2(1-\gamma)}{1+|\zeta|} \quad (0 \leq \gamma < 1, \zeta \in \nabla).$$

Lemma 5 ([28]). Let the functions $\phi \in P(\gamma_1)$ and $\psi \in P(\gamma_2)$ be such that $0 \leq \gamma_1 < \gamma_2 < 1$. Then,

$$\phi * \psi \in P(\gamma_3),$$

where $\gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2)$.

For complex numbers α_1, α_2 , and β_1 ($\beta_1 \notin \mathbb{Z}_0^-$), the Gauss hypergeometric function is given in [29] (Chapter 14):

$${}_2F_1(\alpha_1, \alpha_2; \beta_1; \zeta) = 1 + \frac{\alpha_1 \alpha_2}{\beta_1} \frac{\zeta}{1!} + \frac{\alpha_1(\alpha_1 + 1)\alpha_2(\alpha_2 + 1)}{\beta_1(\beta_1 + 1)} \frac{\zeta^2}{2!} + \dots$$

Lemma 6 ([29]). For any complex parameters α_1, α_2 , and β_1 ($\beta_1 \notin \mathbb{Z}_0^-$), the following equalities hold:

$$\int_0^1 t^{\alpha_2-1} (1-t)^{\beta_1-\alpha_2-1} (1-\zeta t)^{-\alpha_1} dt = \frac{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}{\Gamma(\beta_1)} {}_2F_1(\alpha_1, \alpha_2; \beta_1; \zeta), \quad (16)$$

where $\Re\{\beta_1\} > \Re\{\alpha_2\} > 0$ and $\zeta \in \nabla$. Also,

$${}_2F_1(\alpha_1, \alpha_2; \beta_1; \zeta) = {}_2F_1(\alpha_2, \alpha_1; \beta_1; \zeta), \quad (17)$$

and

$${}_2F_1(\alpha_1, \alpha_2; \beta_1; \zeta) = (1-\zeta)^{-\alpha_1} {}_2F_1\left(\alpha_1, \beta_1 - \alpha_2; \beta_1; \frac{\zeta}{1-\zeta}\right). \quad (18)$$

3. Subordination Properties Involving $\mathcal{B}_\kappa^c$

Throughout this article, unless specified otherwise, $-1 \leq D < C \leq 1$; $0 \leq \alpha < 1$; $\kappa = p + \frac{b+1}{2} \in \mathbb{R} \setminus \mathbb{Z}_0^-$; $b, c, p \in \mathbb{R}$; and $\zeta \in \nabla$. We use the operator $\mathcal{B}_\kappa^c$ to introduce certain convolution and subordination features.

Theorem 1. Assume $\lambda > 0$ and $-1 \leq D_j < C_j \leq 1$. If $f_j \in \mathcal{F}$ fulfill the subordination,

$$(1-\lambda) \frac{\mathcal{B}_\kappa^c f_j(\zeta)}{\zeta} + \lambda \frac{\mathcal{B}_{\kappa-1}^c f_j(\zeta)}{\zeta} \prec \frac{1+C_j\zeta}{1+D_j\zeta} \text{ (for } j=1,2), \quad (19)$$

Then,

$$(1-\lambda) \frac{\mathcal{B}_\kappa^c F(\zeta)}{\zeta} + \lambda \frac{\mathcal{B}_{\kappa-1}^c F(\zeta)}{\zeta} \prec \frac{1+(1-2\eta_0)\zeta}{1-\zeta},$$

where $F(\zeta) = \mathcal{B}_\kappa^c(f_1 * f_2)(\zeta)$. Also,

$$\eta_0 = 1 - \frac{4(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)} \left[1 - \frac{1}{2} {}_2F_1\left(1, 1; \frac{\kappa-1}{\lambda} + 1; \frac{1}{2}\right) \right]. \quad (20)$$

Also, in the case of $D_j = -1$, we obtain the best possible subordinating consequence.

Proof. Let $f_j \in \mathcal{F}$ satisfy (19). Also,

$$\varphi_j(\zeta) = (1-\lambda) \frac{\mathcal{B}_\kappa^c f_j(\zeta)}{\zeta} + \lambda \frac{\mathcal{B}_{\kappa-1}^c f_j(\zeta)}{\zeta} \quad (j=1,2), \quad (21)$$

gives

$$\varphi_j \in P(\gamma_j) \quad (\gamma_j = \frac{1-C_j}{1-D_j}).$$

Using (7) and (21),

$$\mathcal{B}_\kappa^c f_j(\zeta) = \frac{\kappa-1}{\lambda} \zeta^{1-\frac{\kappa-1}{\lambda}} \int_0^\zeta t^{\frac{\kappa-1}{\lambda}-1} \varphi_j(t) dt, \quad (22)$$

if $F(\zeta) = \mathcal{B}_\kappa^c(f_1 * f_2)(\zeta)$, applying (22) and making use of

$$\mathcal{B}_\kappa^c F(\zeta) = \mathcal{B}_\kappa^c(\mathcal{B}_\kappa^c(f_1 * f_2)(\zeta)) = \mathcal{B}_\kappa^c(f_1)(\zeta) * \mathcal{B}_\kappa^c(f_2)(\zeta),$$

Simple computations show that

$$\mathcal{B}_\kappa^c F(\zeta) = \frac{\kappa-1}{\lambda} \zeta^{1-\frac{\kappa-1}{\lambda}} \int_0^\zeta t^{\frac{\kappa-1}{\lambda}-1} \varphi_0(t) dt,$$

where

$$\varphi_0(\zeta) = (1-\lambda) \frac{\mathcal{B}_\kappa^c F(\zeta)}{\zeta} + \lambda \frac{\mathcal{B}_{\kappa-1}^c F(\zeta)}{\zeta} = \frac{\kappa-1}{\lambda} \zeta^{-\frac{\kappa-1}{\lambda}} \int_0^\zeta t^{\frac{\kappa-1}{\lambda}-1} (\varphi_1 * \varphi_2)(t) dt. \quad (23)$$

Given that $\varphi_1 \in P(\gamma_1)$ and $\varphi_2 \in P(\gamma_2)$, Lemma 5 implies that

$$\varphi_1 * \varphi_2 \in P(\gamma_3) \text{ with } \gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2),$$

and the bound γ_3 is the best. Applying Lemma 4 to (23),

$$\begin{aligned} \Re\{\varphi_0(\zeta)\} &= \frac{\kappa-1}{\lambda} \int_0^1 u^{\frac{\kappa-1}{\lambda}-1} \Re\{(p_1 * p_2)(u\zeta)\} du \\ &\geq \frac{\kappa-1}{\lambda} \int_0^1 u^{\frac{\kappa-1}{\lambda}-1} \left[2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u|\zeta|} \right] du \\ &> \frac{\kappa-1}{\lambda} \int_0^1 u^{\frac{\kappa-1}{\lambda}-1} \left[2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u} \right] du \\ &= 1 - \frac{4(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)} \left(1 - \frac{\kappa-1}{\lambda} \int_0^1 \frac{u^{\frac{\kappa-1}{\lambda}-1}}{1+u} du \right) \\ &= \eta_0, \end{aligned}$$

where η_0 is expressed in (20). If $D_1 = D_2 = -1$, $f_j \in \mathcal{F}$ fulfills (19) and

$$\mathcal{B}_\kappa^c f_j(\zeta) = \frac{\kappa-1}{\lambda} \zeta^{1-\frac{\kappa-1}{\lambda}} \int_0^\zeta t^{\frac{\kappa-1}{\lambda}-1} \frac{1+C_j t}{1-t} dt,$$

Then, Lemmas 4 and 6 lead to

$$\begin{aligned} \varphi_0(\zeta) &= \frac{\kappa-1}{\lambda} \int_0^1 u^{\frac{\kappa-1}{\lambda}-1} \left(1 - (1+C_1)(1+C_2) + \frac{(1+C_1)(1+C_2)}{1-u\zeta} \right) du \\ &= 1 - (1+C_1)(1+C_2) + (1+C_1)(1+C_2)(1-\zeta)^{-1} {}_2F_1\left(1, 1; \frac{\kappa-1}{\lambda} + 1; \frac{\zeta}{\zeta-1}\right) \\ &\rightarrow 1 - (1+C_1)(1+C_2) + \frac{1}{2}(1+C_1)(1+C_2) {}_2F_1\left(1, 1; \frac{\kappa-1}{\lambda} + 1; \frac{1}{2}\right) \text{ as } \zeta \rightarrow 1^-, \end{aligned}$$

which concludes Theorem 1 exactly. $\square$

Let $C_j = 1 - 2\eta_j$, $D_j = -1$ ($0 \leq \eta_j < 1; j \in \{1, 2\}$), and $\lambda = 1$ in Theorem 1. Then, we have the subsequent corollary.

Corollary 1. Let $f_j \in \mathcal{F}$, ($j = 1, 2$), fulfill

$$\Re\left\{\frac{\mathcal{B}_{\kappa-1}^c f_j(\zeta)}{\zeta}\right\} > \eta_j \quad (0 \leq \eta_j < 1, j = 1, 2),$$

Then,

$$\Re \left\{ \frac{\mathcal{B}_{\kappa-1}^c F(\zeta)}{\zeta} \right\} > \eta_0^*,$$

where $F(\zeta) = \mathcal{B}_{\kappa}^c(f_1 * f_2)(\zeta)$ and $\eta_0^* = 1 - 4(1 - \eta_1)(1 - \eta_2) \left[1 - \frac{1}{2} {}_2F_1 \left(1, 1; \kappa; \frac{1}{2} \right) \right]$.

Theorem 2. Let $\lambda > 0$ and the function $f \in \mathcal{F}$ satisfy the following subordination condition:

$$(1 - \lambda) \frac{\mathcal{B}_{\kappa}^c f(\zeta)}{\zeta} + \lambda \frac{\mathcal{B}_{\kappa-1}^c f(\zeta)}{\zeta} \prec \frac{1 + C\zeta}{1 + D\zeta}. \quad (24)$$

Then,

$$\Re \left\{ \left(\frac{\mathcal{B}_{\kappa}^c f(\zeta)}{\zeta} \right)^{\frac{1}{\sigma}} \right\} > \tau^{\frac{1}{\sigma}} \quad (\sigma \in \mathbb{N}), \quad (25)$$

where

$$\tau = \begin{cases} \frac{C}{D} + (1 - \frac{C}{D})(1 - D)^{-1} {}_2F_1 \left(1, 1; \frac{\kappa-1}{\lambda} + 1; \frac{D}{D-1} \right), & (D \neq 0), \\ 1 - \frac{\kappa-1}{\kappa-1+\lambda} C, & (D = 0). \end{cases}$$

The subordination is the best possible solution.

Proof. Let

$$\phi(\zeta) = \frac{\mathcal{B}_{\kappa}^c f(\zeta)}{\zeta} \quad (f \in \mathcal{F}), \quad (26)$$

Consequently, ϕ is analytic in ∇ and takes the representation in (14). Differentiating (26) and applying (7), we arrive to

$$\frac{\mathcal{B}_{\kappa-1}^c f(\zeta)}{\zeta} = \phi(\zeta) + \frac{\zeta \phi'(\zeta)}{\kappa - 1}. \quad (27)$$

From (24), (26), and (27), we get

$$\phi(\zeta) + \frac{\lambda}{\kappa - 1} \zeta \phi'(\zeta) \prec \frac{1 + C\zeta}{1 + D\zeta}.$$

Using Lemma 1,

$$\begin{aligned} \frac{\mathcal{B}_{\kappa}^c f(\zeta)}{\zeta} \prec Q(\zeta) &= \frac{\kappa - 1}{\lambda} \zeta^{-\frac{\kappa-1}{\lambda}} \int_0^{\zeta} t^{\frac{\kappa-1}{\lambda}-1} \left(\frac{1 + Ct}{1 + Dt} \right) dt \\ &= \begin{cases} \frac{C}{D} + (1 - \frac{C}{D})(1 - D\zeta)^{-1} {}_2F_1 \left(1, 1; \frac{\kappa-1}{\lambda} + 1; \frac{D\zeta}{1 + D\zeta} \right), & (D \neq 0), \\ 1 + \frac{\kappa-1}{\kappa-1+\lambda} C\zeta, & (D = 0). \end{cases} \end{aligned}$$

After that, we need to clarify that

$$\inf_{|\zeta| < 1} \{ \Re \{ Q(\zeta) \} \} = Q(-1). \quad (28)$$

Actually,

$$\Re \left\{ \frac{1 + C\zeta}{1 + D\zeta} \right\} \geq \frac{1 - Cr}{1 - Dr}; \quad |\zeta| \leq r < 1,$$

Let

$$G(s, \zeta) = \frac{1 + C\zeta s}{1 + D\zeta s} \quad (0 \leq s \leq 1) \quad \text{and} \quad d\nu(s) = \frac{(\kappa-1)s^{\frac{\kappa-1}{\lambda}-1}}{\lambda} ds,$$

be a positive measure, and we have

$$Q(\zeta) = \int_0^1 G(s, \zeta) dv(s),$$

Thus,

$$\Re\{Q(\zeta)\} \geq \int_0^1 \left(\frac{1 - Csr}{1 - Dsr} \right) dv(s) = Q(-r) \quad (|\zeta| \leq r < 1).$$

Choosing $r \rightarrow 1^-$, we have (28). Now, applying

$$\Re\{w^{\frac{1}{m}}\} \geq (\Re\{w\})^{\frac{1}{m}} \quad (\Re\{w\} > 0, 0 < m \leq 1),$$

and (28) directly leads to (25). To demonstrate that (25) is the best, we take into consideration the function $f \in \mathcal{F}$ expressed by

$$\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} = \frac{\kappa-1}{\lambda} \int_0^1 u^{\frac{\kappa-1}{\lambda}-1} \frac{1 + Cu\zeta}{1 + Du\zeta} du,$$

from which it is easily seen that

$$(1 - \lambda) \frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} + \lambda \frac{\mathcal{B}_{\kappa-1}^c f(\zeta)}{\zeta} = \frac{1 + C\zeta}{1 + D\zeta},$$

and that

$$\begin{aligned} \frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} &\rightarrow \frac{\kappa-1}{\lambda} \int_0^1 u^{\frac{\kappa-1}{\lambda}-1} \frac{1 - Cu}{1 - Du} du \\ &= \begin{cases} \frac{C}{D} + (1 - \frac{C}{D})(1 - D)^{-1} {}_2F_1\left(1, 1, \frac{\kappa-1}{\lambda} + 1; \frac{D}{D-1}\right), & (D \neq 0), \\ 1 - \frac{\kappa-1}{\kappa-1+\lambda} C, & (D = 0), \end{cases} \end{aligned}$$

as $\zeta \rightarrow -1$ and using Lemma 6, which concludes Theorem 2 exactly. $\square$

In Theorem 2, take $\lambda = \sigma = 1$, $C = 1 - 2\eta$ ($0 \leq \eta < 1$), and $D = -1$. Consequently, we derive the subsequent corollary:

Corollary 2. Let $f \in \mathcal{F}$, and

$$\Re\left(\frac{\mathcal{B}_{\kappa-1}^c f(\zeta)}{\zeta}\right) > \eta,$$

Then,

$$\Re\left\{\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta}\right\} > \eta + (1 - \eta) \left[{}_2F_1\left(1, 1; \kappa; \frac{1}{2}\right) - 1 \right].$$

For $\nu > -1$, the Bernardi–Libera–Livingston integral operator $F_\nu : \mathcal{F} \rightarrow \mathcal{F}$ is given as follows ([30]):

$$\begin{aligned} F_\nu(f)(\zeta) &= \frac{\nu+1}{\zeta^\nu} \int_0^\zeta t^{\nu-1} f(t) dt \\ &= \zeta + \sum_{n=1}^{\infty} \left(\frac{\nu+1}{\nu+n} \right) a_{n+1} \zeta^{n+1}. \end{aligned} \quad (29)$$

Theorem 3. Assuming that $\nu > -1$, $\lambda > 0$, $f \in \mathcal{F}$, and F_ν are defined in (29). If

$$(1-\lambda) \frac{\mathcal{B}_\kappa^c F_\nu(f)(\zeta)}{\zeta} + \lambda \frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} \prec \frac{1+C\zeta}{1+D\zeta}, \quad (30)$$

then

$$\Re \left\{ \frac{\mathcal{B}_\kappa^c F_\nu(f)(\zeta)}{\zeta} \right\} > \rho_0,$$

where

$$\rho_0 = \begin{cases} \frac{C}{D} + (1 - \frac{C}{D})(1-D)^{-1} {}_2F_1\left(1, 1, \frac{\nu+1}{\lambda} + 1; \frac{D}{D-1}\right), & (D \neq 0), \\ 1 - \frac{\nu+1}{\lambda+\nu+1} C, & (D = 0). \end{cases}$$

Proof. It follows from (29) that

$$\zeta(\mathcal{B}_\kappa^c F_\nu(f)(\zeta))' = (\nu+1)\mathcal{B}_\kappa^c f(\zeta) - \nu\mathcal{B}_\kappa^c F_\nu(f)(\zeta). \quad (31)$$

If we let

$$G(\zeta) = \frac{\mathcal{B}_\kappa^c F_\nu(f)(\zeta)}{\zeta}, \quad (32)$$

then the application of (30) gives that

$$(1-\lambda) \frac{\mathcal{B}_\kappa^c F_\nu(f)(\zeta)}{\zeta} + \lambda \frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} = G(\zeta) + \frac{\lambda}{\nu+1} zG'(\zeta) \prec \frac{1+C\zeta}{1+D\zeta}.$$

The rest of Theorem 3 is comparable to that of Theorem 2, so we leave out all of the details. $\square$

4. Certain Estimates About $S_\kappa^c(\alpha; C, D)$

In this part of paper, we introduce subordination relations, distortion, and argument estimations of functions belonging to the class $S_\kappa^c(\alpha; C, D)$.

Theorem 4. If $f \in S_\kappa^c(\alpha; C, D)$, then for all $|s| \leq 1$ and $|t| \leq 1$ ($s \neq t$), we have

$$\frac{t\mathcal{B}_\kappa^c f(s\zeta)}{s\mathcal{B}_\kappa^c f(t\zeta)} \prec \begin{cases} \left(\frac{1+B_\kappa^c s}{1+B_\kappa^c t} \right)^{(1-\alpha)(C-D)/D} & (D \neq 0), \\ \exp(C(1-\alpha)(s-t)\zeta) & (D = 0). \end{cases} \quad (33)$$

Proof. From (12), we get

$$\frac{\zeta(\mathcal{B}_\kappa^c f(\zeta))'}{\mathcal{B}_\kappa^c f(\zeta)} \prec \frac{1+[D+(C-D)(1-\alpha)]\zeta}{1+D\zeta} := g(\zeta). \quad (34)$$

where g is defined in (34) and h

$$h(\zeta) = h(\zeta; s; t) = \int_0^{\zeta} \left(\frac{s}{1-su} - \frac{t}{1-tu} \right) du,$$

is univalent and convex in ∇ . Applying [31] (Theorem 4.1) and (34), we arrive to

$$\left(\frac{\zeta (\mathcal{B}_\kappa^c f(\zeta))'}{\mathcal{B}_\kappa^c f(\zeta)} - 1 \right) * h(\zeta) \prec \frac{(1-\alpha)(C-D)\zeta}{1+D\zeta} * h(\zeta). \quad (35)$$

With $\vartheta(0) = 0$ for each analytic function $\vartheta(\zeta)$ in ∇ , we obtain

$$\vartheta(\zeta) * h(\zeta) = \int_{\zeta t}^{\zeta s} \frac{\vartheta(u)}{u} du. \quad (36)$$

Consequently, using (35) and (36), we get

$$\int_{\zeta t}^{\zeta s} \left(\frac{(\mathcal{B}_\kappa^c f(u))'}{\mathcal{B}_\kappa^c f(u)} - \frac{1}{u} \right) du \prec (1-\alpha)(C-D) \int_{\zeta t}^{\zeta s} \frac{du}{1+Du}.$$

This implies that

$$\exp \left(\int_{\zeta t}^{\zeta s} \left(\frac{(\mathcal{B}_\kappa^c f(u))'}{\mathcal{B}_\kappa^c f(u)} - \frac{1}{u} \right) du \right) \prec \exp \left((1-\alpha)(C-D) \int_{\zeta t}^{\zeta s} \frac{du}{1+Du} \right). \quad (37)$$

By some simplifications on (37), we arrive to (33). $\square$

Theorem 5. Let $f \in S_\kappa^c(\alpha; C, D)$. Then, for $|\zeta| = r < 1$, we have

$$|\mathcal{B}_\kappa^c f(\zeta)| \leq \begin{cases} r(1+Dr)^{(1-\alpha)(C-D)/D} & (D \neq 0), \\ r \exp((1-\alpha)Cr) & (D = 0), \end{cases} \quad (38)$$

$$|\mathcal{B}_\kappa^c f(\zeta)| \geq \begin{cases} r(1-Dr)^{(C-D)(1-\alpha)/D} & (D \neq 0), \\ r \exp(-(1-\alpha)Cr) & (D = 0), \end{cases} \quad (39)$$

and

$$\left| \arg \left(\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} \right) \right| \leq \begin{cases} \frac{(1-\alpha)(C-D)}{|D|} \sin^{-1}(|D|r) & (D \neq 0), \\ (1-\alpha)Cr & (D = 0). \end{cases} \quad (40)$$

The assertions are sharp.

Proof. By setting $t = 0$ and $s = 1$ in (33),

$$\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} = \begin{cases} (1+Dw(\zeta))^{(1-\alpha)(C-D)/D} & (D \neq 0), \\ \exp((1-\alpha)Cw(\zeta)) & (D = 0). \end{cases} \quad (41)$$

Applying Shawar's lemma ([32]), we have $|w(\zeta)| \leq |\zeta| = r$ ($\zeta \in \nabla$).

(i) Whenever $D > 0$, (41) indicates that

$$\begin{aligned} \left| \frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} \right| &= \left| (1 + Dw(\zeta))^{(1-\alpha)(C-D)/D} \right| \\ &= \left| \exp \left(\frac{(1-\alpha)(C-D)}{D} \log(1 + Dw(\zeta)) \right) \right| \\ &= \exp \left[\Re \left\{ \frac{(1-\alpha)(C-D)}{D} \log(1 + Dw(\zeta)) \right\} \right] \\ &= \exp \left(\frac{(1-\alpha)(C-D)}{D} \log|1 + Dw(\zeta)| \right) \\ &= |1 + Dw(\zeta)|^{(1-\alpha)(C-D)/D} \\ &\leq (1 + Dr)^{(1-\alpha)(C-D)/D}. \end{aligned}$$

(ii) Let $D < 0$. Also, we set $Y = -D > 0$. Then,

$$\begin{aligned} |1 + Dw(\zeta)|^{(1-\alpha)(C-D)/D} &= \left| (1 - Yw(\zeta))^{-1} \right|^{(1-\alpha)(C-D)/Y} \\ &\leq (1 - Yr)^{-(1-\alpha)(C-D)/Y} \\ &= (1 + Dr)^{(1-\alpha)(C-D)/D}. \end{aligned}$$

For $D \neq 0$, this establishes inequality (38). The additional inequalities in (38) and (39) can also be demonstrated. Let $|\zeta| = r$ and $D \neq 0$. From (41), we see that

$$\begin{aligned} \left| \arg \left(\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} \right) \right| &= \frac{(1-\alpha)(C-D)}{|D|} |\arg(1 + Dw(\zeta))| \\ &\leq \frac{(1-\alpha)(C-D)}{|D|} \sin^{-1}(|D|r). \end{aligned}$$

Additionally, (40) is a direct result of (41) for the case $D = 0$. It is evident that every estimate is sharp by applying f_0 , where

$$\mathcal{B}_\kappa^c f_0(\zeta) = \begin{cases} \zeta(1 + D\varepsilon\zeta)^{(1-\alpha)(C-D)/D} & (D \neq 0, |\varepsilon| = 1), \\ \zeta \exp((1-\alpha)C\varepsilon\zeta) & (D = 0, |\varepsilon| = 1). \end{cases} \quad (42)$$

This concludes Theorem 5's proof. $\square$

Theorem 6. Assume that $f \in S_\kappa^c(\alpha; C, D)$. Therefore, if $|\zeta| = r < 1$, then

$$\left| \left(\mathcal{B}_\kappa^c f(\zeta) \right)' \right| \leq \begin{cases} [1 + (D\alpha + (1-\alpha)C)r](1 + Dr)^{[(1-\alpha)C - (2-\alpha)D]/D} & (D \neq 0), \\ [1 + (D\alpha + (1-\alpha)C)r] \exp((1-\alpha)Cr) & (D = 0), \end{cases} \quad (43)$$

$$\left| \left(\mathcal{B}_\kappa^c f(\zeta) \right)' \right| \geq \begin{cases} [1 - (D\alpha + (1-\alpha)C)r](1 - Dr)^{[(1-\alpha)C - (2-\alpha)D]/D} & (D \neq 0), \\ [1 - (D\alpha + (1-\alpha)C)r] \exp(-(1-\alpha)Cr) & (D = 0), \end{cases} \quad (44)$$

and

$$\left| \arg \left(\left(\mathcal{B}_\kappa^c f(\zeta) \right)' \right) \right| \leq \begin{cases} \frac{(1-\alpha)(C-D)}{|D|} \sin^{-1}(|D|r) + \sin^{-1} \left(\frac{(1-\alpha)(C-D)r}{1 - D[\alpha D + (1-\alpha)C]r^2} \right) & (D \neq 0), \\ (1-\alpha)Cr + \sin^{-1}((\alpha D + (1-\alpha)C)r) & (D = 0). \end{cases} \quad (45)$$

Every estimate made here is sharp.

Proof. Let

$$\phi(\zeta) = \frac{\zeta(\mathcal{B}_\kappa^c f(\zeta))'}{\mathcal{B}_\kappa^c f(\zeta)}, \quad (46)$$

Then, $\phi(0) = 1$, ϕ is analytic, and

$$\phi(\zeta) \prec \frac{1 + C_0 \zeta}{1 + D \zeta} \quad (C_0 := \alpha D + (1 - \alpha)C).$$

Function ϕ is known to satisfy the following sharp estimates, according to [33]:

$$\frac{1 - C_0 r}{1 - D r} \leq |\phi(\zeta)| \leq \frac{1 + C_0 r}{1 + D r} \quad (|\zeta| = r < 1), \quad (47)$$

and

$$\left| \phi(\zeta) - \frac{1 - C_0 D r^2}{1 - D^2 r^2} \right| \leq \frac{(C_0 - D)r}{1 - D^2 r^2} \quad (|\zeta| = r < 1), \quad (48)$$

as well as

$$\left| \arg(\phi(\zeta)) \right| \leq \sin^{-1} \left(\frac{(C_0 - D)r}{1 - C_0 D r^2} \right) \quad (|\zeta| = r < 1). \quad (49)$$

We derive estimates (43), (44), and (45) of Theorem 6 using (47), (48), and (49), in addition to the estimates provided by Theorem 5 in (46). Furthermore, for the function f_0 described by (42), all of the estimates are sharp. $\square$

5. Some Sufficient Starlike Conditions

A function $f \in \mathcal{F}$ is said to be a starlike function of order α if it satisfies the inequality $\Re\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right) > \alpha$ ($0 \leq \alpha < 1$, $\zeta \in \nabla$). Here, we present several conditions for functions requiring specific subordination attributes in order to belong to the class $S_\kappa^c(\alpha)$, which is defined by (13).

Theorem 7. If $\lambda > 0$, $\kappa \neq 1$ and $f \in \mathcal{F}$ satisfies

$$(1 - \lambda) \frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} + \lambda \frac{\mathcal{B}_{\kappa-1}^c f(\zeta)}{\zeta} \prec 1 + M_1 \zeta, \quad (50)$$

where

$$M_1 = \frac{\zeta \eta}{|\eta - 1| + \sqrt{1 + \zeta^2}},$$

with $\zeta = \frac{\kappa-1+\lambda}{\kappa-1}$ and $\eta = \frac{\lambda(1-\alpha)}{\kappa-1}$, then $f \in S_\kappa^c(\alpha)$.

Proof. Set

$$\phi(\zeta) = \frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta}, \quad (51)$$

Then, ϕ takes representation (14), analytic on ∇ . Applying Theorem 2 with $C = M_1$, $D = 0$, $\sigma = 1$, we arrive at

$$\phi(\zeta) \prec 1 + \frac{\kappa - 1}{\kappa - 1 + \lambda} M_1 \zeta, \quad (52)$$

or

$$|\phi(\zeta) - 1| < \frac{M_1}{\zeta} = N < 1; \quad \zeta = \frac{\kappa-1+\lambda}{\kappa-1}. \quad (53)$$

If we set

$$p(\zeta) = \frac{1}{1 - \alpha} \left(\frac{\zeta(\mathcal{B}_\kappa^c f(\zeta))'}{\mathcal{B}_\kappa^c f(\zeta)} - \alpha \right), \quad (54)$$

and then apply identities (7) and (51) together, we get

$$\frac{\mathcal{B}_{\kappa-1}^c f(\zeta)}{\zeta} = \left[\left(1 - \frac{1-\alpha}{\kappa-1} + \frac{1-\alpha}{\kappa-1} p(\zeta) \right) \right] \phi(\zeta). \quad (55)$$

Relation (50) can be expressed as follows in light of (55):

$$|(1-\eta)\phi(\zeta) + \eta p(\zeta)\phi(\zeta) - 1| < M_1 = \zeta N. \quad (56)$$

We have to prove that (56) produces

$$\Re\{p(\zeta)\} > 0. \quad (57)$$

If

$$\Re\{p(\zeta)\} \not> 0,$$

then for some $x \in \mathbb{R}$, $\exists z_0 \in \nabla$ with $p(z_0) = ix$. It is sufficient to derive a contradiction from the following inequality in order to demonstrate (57):

$$W = |(1-\eta)\phi(z_0) + \eta p(z_0)\phi(z_0) - 1| \geq M_1.$$

Take $u + iv = \phi(\zeta_0)$. Next, we use (53) to determine that

$$\begin{aligned} W^2 &= |(1-\eta)\phi(\zeta_0) + \eta p(\zeta_0)\phi(\zeta_0) - 1|^2 \\ &= (u^2 + v^2)\eta^2 x^2 + 2\eta vx + |(1-\eta)\phi(\zeta_0) - 1|^2 \\ &\geq (u^2 + v^2)\eta^2 x^2 + 2\eta vx + (\eta - |1-\eta|N)^2, \end{aligned}$$

Then,

$$W^2 - M_1^2 \geq (u^2 + v^2)\eta^2 x^2 + 2\eta vx + (\eta - |1-\eta|N)^2 - \zeta^2 N^2.$$

Setting

$$\Phi(x) = W^2 - M_1^2 = (u^2 + v^2)\eta^2 x^2 + 2\eta vx + (\eta - |1-\eta|N)^2 - \zeta^2 N^2,$$

we observe that if $\Phi(x) \geq 0$ for any real x , then (56) becomes true.

Given that

$$(u^2 + v^2)\eta^2 > 0,$$

if the discriminant $\Delta \leq 0$, then the inequality $\Phi(x) \geq 0$ is true. Thus,

$$\Delta = 4 \left\{ \eta^2 v^2 - \eta^2 (u^2 + v^2) \left[(\eta - |1-\eta|N)^2 - \zeta^2 N^2 \right] \right\} \leq 0,$$

is identical to

$$v^2 \left[1 - (\eta - |1-\eta|N)^2 + \zeta^2 N^2 \right] \leq u^2 \left[(\eta - |1-\eta|N)^2 - \zeta^2 N^2 \right].$$

By setting $\phi(\zeta_0) - 1 = \rho e^{i\theta}$ such that $\theta \in \mathbb{R}$,

$$\frac{v^2}{u^2} = \frac{\rho^2 \sin^2 \theta}{(1 + \rho \cos \theta)^2}.$$

Since $\cos \theta = -\rho$ is where the above expression reaches the greatest value, using (53), we get

$$\frac{v^2}{u^2} \leq \frac{\rho^2}{1-\rho^2} \leq \frac{N^2}{1-N^2} = \frac{(\eta - |1-\eta|N)^2 - \xi^2 N^2}{1 - (\eta - |1-\eta|N)^2 + \xi^2 N^2}.$$

This results in $\Delta \leq 0$. Consequently, $W \geq M_1$, which is in opposition to (56). Therefore,

$$\Re\{p(\zeta)\} > 0.$$

By demonstrating that $f \in S_\kappa^c(\alpha)$, the proof of Theorem 7 is finished. $\square$

Taking $\alpha = 0, \lambda = 1, \kappa = 2$, and $c = 8$ in Theorem 7, we can introduce the following illustrative example.

Example 1. For function $f_1(\zeta) = \frac{\zeta}{1-\zeta^2} = \zeta + \zeta^3 + \zeta^5 + \dots$, it is clear that f_1 is analytic and univalent and $f_1 \in \mathcal{F}$. A straightforward application of Theorem 7 reports that

$$\Re\left\{\frac{\zeta(\mathcal{B}_2^8 f_1(\zeta))'}{\mathcal{B}_2^8 f_1(\zeta)}\right\} > 0, \quad (58)$$

since

$$\frac{\mathcal{B}_1^8 f_1(\zeta)}{\zeta} \prec 1 + \frac{2}{\sqrt{5}}\zeta. \quad (59)$$

where

$$\mathcal{B}_2^8 f_1(\zeta) = \zeta + \sum_{n=1}^{\infty} \left(\frac{2^n}{(2n)!}\right)^2 \zeta^{2n+1}, \quad (60)$$

For inequality (58), see Figure 1b. For subordination (59), see Figure 1a.

Taking the restriction $\lambda = 1$ in Theorem 7, we state the following corollary:

Corollary 3. For $\kappa \neq 1, f \in \mathcal{F}$ satisfies

$$\left|\frac{\mathcal{B}_{\kappa-1}^c f(\zeta)}{\zeta} - 1\right| < M_1,$$

where

$$M_1 = \frac{\xi\eta}{\left|\frac{1-\alpha}{\kappa-1} - 1\right| + \sqrt{1 + \left(\frac{\kappa}{\kappa-1}\right)^2}}.$$

Then, $f \in S_\kappa^c(\alpha)$.

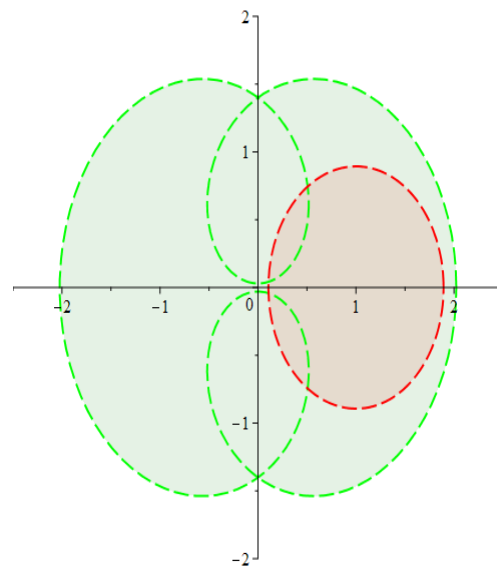
Theorem 8. Let $\lambda > 0, \mu \geq 0$, and $f \in \mathcal{F}$ such that $\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} \neq 0$, fulfilling the following subordination:

$$(1-\lambda)\left(\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta}\right)^\mu + \lambda(\mathcal{B}_\kappa^c f(\zeta))'\left(\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta}\right)^{\mu-1} \prec 1 + M_2\zeta, \quad (61)$$

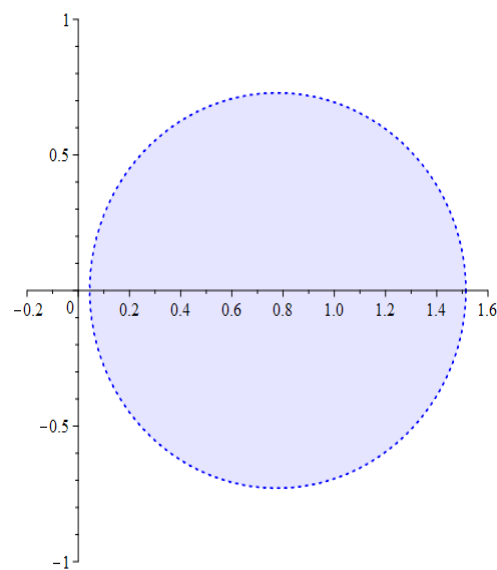
where principal values of powers were taken, and

$$M_2 = \begin{cases} \frac{(1-\alpha)\lambda\left(1+\frac{\lambda}{\mu}\right)}{|1-(1-\alpha)\lambda| + \sqrt{1+\left(1+\frac{\lambda}{\mu}\right)^2}} & (\mu > 0), \\ (1-\alpha)\lambda & (\mu = 0). \end{cases}$$

Then, $f \in S_{\kappa}^c(\alpha)$.



(a)



(b)

Figure 1. (a) The range of the function $1 + \frac{2}{\sqrt{5}}\zeta$ is given in green color. The range of the function $\frac{\mathcal{B}_1^s f_1(\zeta)}{\zeta}$ is given in red color. (b) The range of the function $\frac{\zeta(\mathcal{B}_2^s f_1(\zeta))'}{\mathcal{B}_2^s f_1(\zeta)}$ is given in blue color.

Proof. In the case of $\mu = 0$, relation (61) is equivalent to

$$\left| \frac{\zeta(\mathcal{B}_{\kappa}^c f(\zeta))'}{\mathcal{B}_{\kappa}^c f(\zeta)} - 1 \right| < 1 - \alpha.$$

Consequently, $f \in S_{\kappa}^c(\alpha)$ is implied.

In the case of $\mu > 0$, let

$$\phi(\zeta) = \left(\frac{\mathcal{B}_{\kappa}^c f(\zeta)}{\zeta} \right)^{\mu}. \quad (62)$$

In (62), select the principle value. Then, we observe that ϕ is analytic in $|\zeta| < 1$. Moreover, ϕ is of the type (14). Additionally, differentiating (62) leads to

$$(1 - \lambda) \left(\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} \right)^\mu + \lambda (\mathcal{B}_\kappa^c f(\zeta))' \left(\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} \right)^{\mu-1} = \phi(\zeta) + \frac{\lambda}{\mu} \zeta \phi'(\zeta).$$

Considering Lemma 1 ($c = \mu/\lambda$), this results in

$$\phi(\zeta) \prec 1 + \frac{\mu}{\mu + \lambda} M_2 \zeta.$$

Furthermore, using (61) and (62), we have

$$\phi(\zeta) [1 - \lambda + \lambda((1 - \alpha)p(\zeta) + \alpha)] \prec 1 + M_2 \zeta,$$

such that $p(\zeta)$ is expressed in (54). Then, making use of Lemma 3, we arrive at

$$\Re\{p(\zeta)\} > 0,$$

that is,

$$\Re \left\{ \frac{\zeta (\mathcal{B}_\kappa^c f(\zeta))'}{\mathcal{B}_\kappa^c f(\zeta)} \right\} > \alpha.$$

The proof of Theorem 8 is therefore finished. $\square$

Taking $\lambda = \mu = 1$ within Theorem 8 forms the corollary below.

Corollary 4. For $f \in \mathcal{F}$ such that

$$\left| (\mathcal{B}_\kappa^c f(\zeta))' - 1 \right| < \frac{2(1 - \alpha)}{\alpha + \sqrt{5}}, \quad (63)$$

we have $f \in S_\kappa^c(\alpha)$.

Theorem 9. Suppose that $\mu > 0$, $\lambda > 0$, and $\nu > -\mu$. If $f \in \mathcal{F}$ holds, the subordination

$$(1 - \lambda) \left(\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} \right)^\mu + \lambda (\mathcal{B}_\kappa^c f(\zeta))' \left(\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} \right)^{\mu-1} \prec 1 + M_3 \zeta, \quad (64)$$

such that the powers are the principal ones, and

$$M_3 = \frac{(1 - \alpha)\lambda \left(1 + \frac{\lambda}{\mu}\right) (\nu + \mu + 1)}{\left[|1 - (1 - \alpha)\lambda| + \sqrt{1 + \left(1 + \frac{\lambda}{\mu}\right)^2}\right] (\nu + \mu)}, \quad (65)$$

Then, F_ν is expressed by

$$F_\nu(\zeta) := \left(\frac{\nu + \mu}{\zeta^\nu} \int_0^\zeta t^{\nu-1} (f(t))^\mu dt \right)^{1/\mu}, \quad (66)$$

which belongs to $S_\kappa^c(\alpha)$.

Proof. Clearly, $F_\nu \in \mathcal{F}$. Moreover,

$$\zeta^\nu (\mathcal{B}_\kappa^c F_\nu(\zeta))^\mu = (\mu + \nu) \int_0^\zeta t^{\nu-1} (\mathcal{B}_\kappa^c f(t))^\mu dt. \quad (67)$$

When we differentiate (67), we get

$$(\mu + \nu) \left(\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} \right)^\mu = \nu \left(\frac{\mathcal{B}_\kappa^c F_\nu(\zeta)}{\zeta} \right)^\mu + \mu (\mathcal{B}_\kappa^c F_\nu(\zeta))' \left(\frac{\mathcal{B}_\kappa^c F_\nu(\zeta)}{\zeta} \right)^{\mu-1}. \quad (68)$$

Let

$$\mathcal{G}(\zeta) = (1 - \lambda) \left(\frac{\mathcal{B}_\kappa^c F_\nu(\zeta)}{\zeta} \right)^\mu + \lambda (\mathcal{B}_\kappa^c F_\nu(\zeta))' \left(\frac{\mathcal{B}_\kappa^c F_\nu(\zeta)}{\zeta} \right)^{\mu-1}, \quad (69)$$

Then, $\mathcal{G}$ is analytic in $|\zeta| < 1$; also, it takes the type (14). Again, by differentiating (69) and applying (68), we arrive to

$$\mathcal{G}(\zeta) + \frac{\zeta \mathcal{G}'(\zeta)}{\mu + \nu} = (1 - \lambda) \left(\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} \right)^\mu + \lambda (\mathcal{B}_\kappa^c f(\zeta))' \left(\frac{\mathcal{B}_\kappa^c f(\zeta)}{\zeta} \right)^{\mu-1}.$$

The aforementioned formula and assumption (64) now indicate that

$$\mathcal{G}(\zeta) + \frac{\zeta \mathcal{G}'(\zeta)}{\mu + \nu} \prec 1 + M_3 \zeta,$$

where M_3 is given by (65). Now, Lemma 1 gives

$$\mathcal{G}(\zeta) \prec 1 + \frac{\mu + \nu}{\mu + \nu + 1} M_3 \zeta. \quad (70)$$

Finally, in applying Theorem 8, if f has been replaced with F_ν , then $F_\nu \in S_\kappa^c(\alpha)$ follows from (70). This finishes the proof. $\square$

Taking $\lambda = \mu = 1$ and $\nu = 0$ within Theorem 9, the subsequent corollary is obtained.

Corollary 5. Let $f \in \mathcal{F}$ fulfill the subordination

$$\left| (\mathcal{B}_\kappa^c f(\zeta))' - 1 \right| < \frac{4(1 - \alpha)}{\alpha + \sqrt{5}}, \quad (71)$$

Then,

$$\int_0^\zeta \frac{f(t)}{t} dt \in S_\kappa^c(\alpha).$$

Theorem 10. If the functions $\frac{\mathcal{B}_{\kappa-1}^c f_j(\zeta)}{\zeta} \in P(\gamma_j)$ ($\gamma_j = \frac{1-C_j}{1-D_j}$, $-1 \leq D_j < C_j \leq 1$, $f_j \in \mathcal{F}$; $1j = 1, 2$), then $\mathfrak{h}$, defined by

$$\mathfrak{h}(\zeta) = \mathcal{B}_\kappa^c(f_1 * f_2)(\zeta), \quad (72)$$

satisfies

$$\Re \left\{ \frac{\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta)}{\mathcal{B}_\kappa^c \mathfrak{h}(\zeta)} \right\} > 0 \quad (\zeta \in \nabla), \quad (73)$$

provided that

$$\frac{(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)} \leq \frac{2\kappa - 1}{2 \left[\left({}_2F_1 \left(1, 1; \kappa; \frac{1}{2} \right) - 2 \right)^2 + 2(\kappa - 1) \right]}. \quad (74)$$

Proof. We have $\frac{\mathcal{B}_{\kappa-1}^c f_j(\zeta)}{\zeta} \in P(\gamma_j)(\gamma_j = \frac{1-C_j}{1-D_j}; j = 1, 2)$.

Therefore, we can determine from Lemma 5 that

$$\Re \left\{ \frac{\mathcal{B}_{\kappa-1}^c f_1(\zeta)}{\zeta} * \frac{\mathcal{B}_{\kappa-1}^c f_2(\zeta)}{\zeta} \right\} > 1 - 2 \frac{(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)}.$$

Using $\mathfrak{h}(\zeta) = \mathcal{B}_{\kappa}^c f_0(\zeta)$; $f_0(\zeta) = (f_1 * f_2)(\zeta)$, (7) makes it simple to demonstrate that

$$\begin{aligned} \frac{\mathcal{B}_{\kappa-1}^c f_1(\zeta)}{\zeta} * \frac{\mathcal{B}_{\kappa-1}^c f_2(\zeta)}{\zeta} &= \frac{1}{\zeta} \mathcal{B}_{\kappa-1}^c (\mathcal{B}_{\kappa-1}^c f_0(\zeta)) \\ &= \frac{1}{\zeta} \mathcal{B}_{\kappa-1}^c \left(\frac{1}{\kappa-1} \left[(\kappa-2) \mathcal{B}_{\kappa}^c f_0(\zeta) + \zeta (\mathcal{B}_{\kappa}^c f_0(\zeta))' \right] \right) \\ &= \frac{1}{\zeta} \frac{1}{\kappa-1} \left[(\kappa-2) \mathcal{B}_{\kappa-1}^c (\mathcal{B}_{\kappa}^c f_0(\zeta)) + \zeta (\mathcal{B}_{\kappa-1}^c (\mathcal{B}_{\kappa}^c f_0(\zeta)))' \right] \\ &= \frac{\kappa-2}{\kappa-1} \frac{\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta)}{\zeta} + \frac{1}{\kappa-1} (\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta))' \\ &= \frac{\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta)}{\zeta} + \frac{\zeta}{\kappa-1} \left(\frac{\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta)}{\zeta} \right)'. \end{aligned}$$

Thus,

$$\Re \left\{ \frac{\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta)}{\zeta} + \frac{\zeta}{\kappa-1} \left(\frac{\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta)}{\zeta} \right)' \right\} > 1 - 2 \frac{(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)}, \quad (75)$$

Applying Lemma 1 to the case of

$$\delta = \kappa - 1, \quad C = -1 + 4 \frac{(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)} \quad \text{and} \quad D = -1,$$

leads to

$$\Re \left\{ \frac{\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta)}{\zeta} \right\} > 1 + \frac{(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)} \left[{}_2F_1 \left(1, 1; \kappa; \frac{1}{2} \right) - 2 \right] \quad (\zeta \in \nabla). \quad (76)$$

Using inequality (76) together with Theorem 2 for $\lambda = \sigma = 1$,

$$C = -1 - 4 \frac{(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)} \left[{}_2F_1 \left(1, 1; \kappa; \frac{1}{2} \right) - 2 \right],$$

and $D = -1$, we conclude that

$$\Re \{ \theta(\zeta) \} > 1 - 2 \frac{(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)} \left[{}_2F_1 \left(1, 1; \kappa; \frac{1}{2} \right) - 2 \right]^2 \quad (\zeta \in \nabla), \quad (77)$$

where $\theta(\zeta) = \mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta) / \zeta$. If we set

$$\Omega(\zeta) = \frac{\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta)}{\mathcal{B}_{\kappa}^c \mathfrak{h}(\zeta)} \quad (\zeta \in \nabla),$$

then Ω is analytic; also, it takes formula (14). Some calculations can give that

$$\begin{aligned} \frac{\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta)}{\zeta} + \frac{\zeta}{\kappa-1} \left(\frac{\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta)}{\zeta} \right)' &= \theta(\zeta) \left[(\Omega(\zeta))^2 + \frac{\zeta}{\kappa-1} \Omega'(\zeta) \right] \\ &= \Psi(\Omega(\zeta), \zeta \Omega'(\zeta); \zeta), \end{aligned} \quad (78)$$

where $\Psi(u, v; \zeta) = \theta(\zeta)(u^2 + \frac{v}{\kappa-1})$. Thus, using (75) in (78), we deduce that

$$\Re\{\Psi(\Omega(\zeta), \zeta\Omega'(\zeta); \zeta)\} > 1 - 2\frac{(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)} \quad (\zeta \in \nabla).$$

The real $x, y \leq -(x^2 + 1)/2$ give

$$\begin{aligned} \Re\{\Psi(ix, y; \zeta)\} &= \left(\frac{y}{\kappa-1} - x^2\right) \Re\{\theta(\zeta)\} \\ &\leq -\frac{1}{2(\kappa-1)} \left[1 + (2(\kappa-1) + 1)x^2\right] \Re\{\theta(\zeta)\} \\ &\leq \frac{\Re\{\theta(\zeta)\}}{2(\kappa-1)} \leq 1 - 2\frac{(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)} \quad (\zeta \in \nabla), \end{aligned}$$

by (74) and (77). Consequently, we obtain $\Re\{\Omega(\zeta)\} > 0$ ($\zeta \in \nabla$) by applying Lemma 2. The proof is complete. $\square$

Theorem 11. If $\frac{\mathcal{B}_{\kappa-1}^c f_j(\zeta)}{\zeta} \in P(\gamma_j)$ ($\gamma_j = \frac{1-C_j}{1-D_j}$, $-1 \leq D_j < C_j \leq 1$, $f_j \in \mathcal{F}$; $j = 1, 2, 3$), then $\mathfrak{H}(\zeta) = \mathcal{B}_{\kappa}^c(f_1 * f_2 * f_3)(\zeta)$ satisfies

$$\Re\left\{\frac{\mathcal{B}_{2\kappa-2}^c \mathfrak{H}(\zeta)}{\mathcal{B}_{2\kappa-1}^c \mathfrak{H}(\zeta)}\right\} > 0 \quad (\zeta \in \nabla), \quad (79)$$

provided that

$$\frac{(C_1 - D_1)(C_2 - D_2)(C_3 - D_3)}{(1 - D_1)(1 - D_2)(1 - D_3)} < \frac{2\kappa - 1}{4\left[\left({}_2F_1\left(1, 1; \kappa; \frac{1}{2}\right) - 2\right)^2 + 2(\kappa - 1)\right]}. \quad (80)$$

Proof. Using (72), it is easy to verify that

$$\begin{aligned} \Re\left\{\frac{\mathcal{B}_{2\kappa-2}^c \mathfrak{H}(\zeta)}{\zeta} + \frac{\zeta}{\kappa-1} \left(\frac{\mathcal{B}_{2\kappa-2}^c \mathfrak{H}(\zeta)}{\zeta}\right)'\right\} &= \Re\left\{\frac{\mathcal{B}_{\kappa-1}^c f_1(\zeta)}{\zeta} * \frac{\mathcal{B}_{\kappa-1}^c f_2(\zeta)}{\zeta} * \frac{\mathcal{B}_{\kappa-1}^c f_3(\zeta)}{\zeta}\right\} \\ &> \frac{2\kappa - 1}{4\left[\left({}_2F_1\left(1, 1; \kappa; \frac{1}{2}\right) - 2\right)^2 + 2(\kappa - 1)\right]} \quad (\zeta \in \nabla), \end{aligned}$$

and we skip the details since the proof of Theorem 11 can be finished in a way similar to that of Theorem 10. $\square$

Theorem 12. For $j \in \{1, 2\}$, let $f_j \in \mathcal{F}$. Additionally, for $\mathfrak{h}$ expressed in (72), it satisfies

$$\Re\left\{\frac{\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta)}{\zeta}\right\} > 1 - \frac{2\kappa - 1}{2\left[\left({}_2F_1\left(1, 1; \kappa; \frac{1}{2}\right) - 2\right)^2 + 2(\kappa - 1)\right]} \quad (\zeta \in \nabla),$$

Then,

$$\Re\left\{\frac{\mathcal{B}_{\kappa-1}^c \mathfrak{G}(\zeta)}{\mathcal{B}_{\kappa}^c \mathfrak{G}(\zeta)}\right\} > 0 \quad (\zeta \in \nabla),$$

where

$$\mathfrak{G}(\zeta) = \frac{\kappa-1}{\zeta^{\kappa-2}} \int_0^{\zeta} t^{\kappa-3} \mathfrak{h}(t) dt \quad (\zeta \in \nabla).$$

Proof. Use the following inequality:

$$\Re \left\{ \frac{\mathcal{B}_{\kappa-1}^c \mathfrak{G}(\zeta)}{\zeta} + \frac{\zeta}{\kappa-1} \left(\frac{\mathcal{B}_{\kappa-1}^c \mathfrak{G}(\zeta)}{\zeta} \right)' \right\} = \Re \left\{ \frac{\mathcal{B}_{\kappa-1}^c \mathfrak{h}(\zeta)}{\zeta} \right\} \\ > 1 - \frac{2\kappa-1}{2 \left[\left({}_2F_1 \left(1, 1; \kappa; \frac{1}{2} \right) - 2 \right)^2 + 2(\kappa-1) \right]} \quad (\zeta \in \nabla).$$

At this stage, the proof can be completed, and the details are left out. $\square$

6. Conclusions

The Bessel function is one of the most significant special functions. A generalized version of this function was utilized in this article. This extended Bessel function yields new results as well as direct implications for some modified versions of the ordinary Bessel function, of which there are three forms (modified, spherical, and ordinary). Based on this role of the generalized function, we drew upon previous studies and applied them to obtain the operator $\mathcal{B}_{\kappa}^c$ on a class of analytic functions defined on the unit disk. This class also generalizes well-known classes of functions in the field of Geometric Function Theory of complex variables by employing the principle of differential subordination and the concept of convolution of functions. The operator $\mathcal{B}_{\kappa}^c$ generalizes the other existing operator by specializing parameters κ and c . Additionally, we looked into some sufficient starlike requirements and assessments for a particular subset of univalent functions in ∇ . By specializing the parameters in all theorems obtained here, we will be able to determine the consequences of $\mathcal{N}_{\kappa}$, $\mathcal{J}_p$, $\mathcal{I}_p$, and $\mathcal{S}_p$ as special cases of $\mathcal{B}_{\kappa}^c$. For further studies, we recommend investigations to be conducted on the Mittag-Leffler function [34], Hurwitz–Lerch Zeta function [35], Dini [36] and Einstein [37,38] functions, etc.

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