

Article

Optimal GQPC Codes over the Finite Field F_q

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Abstract

This paper presents the algebraic structure of generalized quasi-polycyclic (GQPC) codes, which is a generalization of the right quasi-polycyclic (QPC) and generalized quasi-cyclic (GQC) codes over a finite field F_q . Here, we mainly study the multi-generator polynomial of the right GQPC codes of index l . In this regard, we use the Chinese Remainder Theorem to decompose the right GQPC codes into their constituent codes. Further, we determine the dimension of a right GQPC code and provide a method for finding a normalized generating set for a multi-generator right GQPC code. As a by-product, we provide some examples of GQPC codes and obtain several optimal and near-optimal 2-generator right GQPC codes of index 2 over F_2 .

Keywords: right polycyclic codes; quasi-polycyclic codes; normalized generating set of polynomials; optimal codes

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1. Introduction

The class of polycyclic (Pseudocyclic) codes emerges as the generalized class of linear codes from the study of cyclic, negacyclic, and constacyclic codes. In 1990, Krishna and Sarwate [1] introduced the term pseudocyclic code while studying constacyclic code. Later, Lopez-Permouth et al. [2] renamed pseudocyclic codes as polycyclic codes. Recall that a right polycyclic (RPC) code on a finite field F_q of length n is the ideal of the quotient ring $F_q[z] / \langle f(z) \rangle$ where $f(z)$ is a monic polynomial of degree n . Many researchers have studied the algebraic structure of polycyclic codes in various ways over different code alphabets, considering both finite fields and rings [3–8]. Further, some of the studies [9,10] have explored quantum MDS codes as applications of RPC codes. Later, Lopez-Permouth et al. [2] established the duality relationship between polycyclic codes and sequential codes.

On the other hand, we have another generalization of the family of cyclic codes, known as quasi-cyclic (QC) codes. A QC code of length n and index l is defined as a nonempty subspace C of F_q^{nl} with the property that the cyclic shift of any arbitrary codeword of C by l components is again a codeword of C . Note that for $l = 1$, the family of QC codes is in fact the cyclic codes. Further, an important subclass of QC codes is the 1-generator QC codes. Similar to the cyclic codes [11,12], these codes employ a 1-generator polynomial to achieve a compact representation and facilitate efficient encoding. For more details, refer to [13,14]. Recently, Hassan et al. [15] extended the QC and quasi-twisted frameworks to the 1-generator right (or left) quasi-polycyclic (QPC) codes by replacing the cyclic shift with the right (or left) polycyclic shift. However, from the structural point of view, these codes can

be realized as an $F_q[z]$ -submodule of $(F_q[z]/\langle f(z) \rangle)^l$. This approach has also facilitated the construction of many best-known linear and quantum codes over finite fields and rings, as reported in [15–17]. However, their work primarily considered on 1-generator QPC codes, providing a valuable tabulation of optimal codes for short lengths and dimensions. In addition to this framework, generalized quasi-cyclic (GQC) and generalized quasi-twisted (multi-twisted) codes have further extensions and remain an active area of research. In particular, many researchers extended the concept of QC codes to the GQC setting in both commutative and noncommutative polynomial rings [18–26]. These generalizations unify several well-known code families under a single algebraic structure. It has led to the explicit construction of numerous new linear codes, including many that meet or improve the previously best-known parameters.

Ou-Azzou et al. [27] studied GQPC codes in skew polynomial rings with derivations over finite fields. Subsequently, Kabbouch et al. [28] extended the notions of GQC and QPC codes to define right GQPC codes over F_q . Both works focused on 1-generator right GQPC codes and reported several good linear codes, with Ou-Azzou's constructions corresponding to short code lengths and small dimensions. In this research paper, we extend the concept of right QPC to GQPC codes over the finite fields F_q with l induced vectors $a_1^{t_1}, a_2^{t_2}, \dots, a_l^{t_l}$ in $F_q^{t_i}$ for all $i = 1, 2, \dots, l$, respectively. That is, we do not restrict our work to a 1-generator right GQPC code only. Our main contributions in this paper are as follows:

1. Establish the decomposition theorem for the right GQPC codes of index l .
2. Discuss a method to obtain a normalized generating set for the ρ -generator right GQPC codes of index l .
3. Using the MAGMA software [29], constructing many optimal and near-optimal 2-generator right GQPC codes whose parameters achieve the bounds given in the Codes Table [30].

This study proceeds as follows: Section 2 recalls the basic concepts of RPC codes over the finite field F_q . In Section 3, we describe the structure of the right GQPC code and demonstrate the code decomposition theorem with examples. Section 4 contains the normalized generating set of the right GQPC code as the ρ -generator polynomial of the index l . Additionally, this section presents a table listing optimal and near-optimal GQPC codes of index 2 derived from the 2-generator polynomial set. Section 5 concludes the work.

2. Background

In this section, we recall the basic concepts of right polycyclic (RPC) codes over the finite field F_q that we will use throughout this paper. Let F_q be a finite field with q elements, where $q = p^d$ for some prime number p and an integer $d \geq 1$. A subspace C of F_q^n (n -dimensional vector space) is called a linear code of length n over F_q , and elements of C are called codewords. Let $c = (c_0, c_1, \dots, c_{n-1})$ be a codeword in C . Then the number of nonzero elements in c is called its (Hamming) weight. It is denoted by $w_H(c)$. The weight $w_H(C)$ of C is the minimum among the weights of all its nonzero codewords. Let $a = (a_0, a_1, \dots, a_{n-1})$, $b = (b_0, b_1, \dots, b_{n-1}) \in F_q^n$. Then the (Hamming) distance between a and b is defined as $d(a, b) = |\{i : a_i \neq b_i\}|$, equivalently $d(a, b) = w_H(a - b)$. The distance of code C , denoted by $d_H(C)$, is defined as $d_H(C) = \min\{d(a, b) : a, b \in C, a \neq b\}$. For a linear code $C \subseteq F_q^n$, $d_H(C) = w_H(C)$. Note that a k -dimensional linear code C of length n over the finite field F_q is represented by $[n, k, d_H]$, where d_H is the distance of C .

Definition 1. Let $(y_0, y_1, \dots, y_{n-1})$ and $\mathbf{a} = (a_0, a_1, \dots, a_{n-1})$ be two vectors in F_q^n with $a_0 \neq 0$. The right polyshift operator $P_r^{\mathbf{a}} : F_q^n \mapsto F_q^n$ is given by

$$(y_0, y_1, \dots, y_{n-1}) \mapsto y_{n-1}\mathbf{a} + (0, y_0, \dots, y_{n-2}).$$

Definition 2. Let $C \subseteq F_q^n$ be a linear code over F_q and $\mathbf{a} = (a_0, a_1, \dots, a_{n-1}) \in F_q^n$. Then C is called an RPC code induced by a vector $\mathbf{a}$ if for every codeword $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in C$, the right polyshift of $\mathbf{c}$ is $P_r^{\mathbf{a}}(\mathbf{c}) = c_{n-1}\mathbf{a} + (0, c_0, \dots, c_{n-2}) \in C$. For a linear code C , $P_r^{\mathbf{a}}$ is called the RPC shift induced by $\mathbf{a}$.

In light of the above definition, we can express $P_r^{\mathbf{a}}(\mathbf{c})$ in terms of the following matrix:

$$P_r^{\mathbf{a}}(\mathbf{c}) = \mathbf{c}A_r, \text{ where } A_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_0 & a_1 & a_2 & \cdots & 0 & a_{n-1} \end{pmatrix}.$$

Hence, a linear code C is a right polycyclic code if it is an invariant subspace under right multiplication by A_r .

Remark 1. In Definition 2 and throughout the paper, we use the term right polycyclic codes corresponding to the right polyshift, and it has no connection with the concept of right modules.

RPC codes are the generalization of cyclic and constacyclic codes over the field F_q . Recall that cyclic and constacyclic codes of length n over F_q can be described as ideals of rings $F_q[z]/\langle z^n - \lambda \rangle$ and $F_q[z]/\langle z^n - 1 \rangle$, respectively, where λ is a nonzero element of F_q . Extending this concept, an RPC code of length n is also an ideal of the ring $F_q[z]/\langle z^n - \mathbf{a}(z) \rangle$, where $\mathbf{a}(z)$ is a polynomial over F_q of degree strictly less than n (see [2,16]). For convenience, we recall their structural properties and restate them in our notation.

Let $R = F_q[z]$ be a polynomial ring over the field F_q . Consider a vector $\mathbf{a} = (a_0, a_1, \dots, a_{n-1})$ in F_q^n , and associate this with the polynomial $\mathbf{a}(z) = a_0 + a_1z + \dots + a_{n-1}z^{n-1}$ in the polynomial ring R . Let $P_{n,\mathbf{a}}(z) = z^n - \mathbf{a}(z)$ be a polynomial of degree n with $P_{n,\mathbf{a}}(0) \neq 0$. From [31] (Page No. 84), there always exists a positive integer m less than or equal to $q^n - 1$ such that $P_{n,\mathbf{a}}(z)$ divides $z^m - 1$. The lowest positive integer m for which this property holds is called the order of the polynomial $P_{n,\mathbf{a}}(z)$. In the case of $P_{n,\mathbf{a}}(0) = 0$, we can write $P_{n,\mathbf{a}}(z) = z^k Q(z)$ for some polynomial $Q(z) \in R$ with $Q(0) \neq 0$ and a positive integer $k \geq 1$. In addition, the order of $P_{n,\mathbf{a}}(z)$ is defined as the order of $Q(z)$. Consider a principal ideal $\langle P_{n,\mathbf{a}}(z) \rangle$ in the polynomial ring R and denote the quotient ring $R/\langle P_{n,\mathbf{a}}(z) \rangle$ by $R_n^{\mathbf{a}}$. Define a F_q -linear map

$$\Phi : F_q^n \mapsto R_n^{\mathbf{a}} \text{ given by}$$

$$(r_0, r_1, \dots, r_{n-1}) \mapsto r_0 + r_1z + \dots + r_{n-1}z^{n-1},$$

in the polynomial ring R . Let $\mathbf{c} = (c_0, c_1, \dots, c_{n-1})$ be an element of the linear code C . Then $\phi(\mathbf{c}) = c_0 + c_1z + \dots + c_{n-1}z^{n-1}$ is the polynomial representation of the codeword $\mathbf{c}$. Observe that $z\phi(\mathbf{c}) = c_0z + c_1z^2 + \dots + c_{n-1}z^n$. Since $z^n = \mathbf{a}(z)$ in $R_n^{\mathbf{a}}$, it follows that $z\phi(\mathbf{c}) = c_0z + c_1z^2 + \dots + c_{n-1}\mathbf{a}(z)$ which corresponds to the RPC shift $P_r^{\mathbf{a}}(\mathbf{c})$ of a codeword $\mathbf{c}$, i.e., $z\phi(\mathbf{c}) \in C$. Based on the above discussion, the following proposition presents the ideal structure of an RPC code over the field F_q .

Proposition 1 ([16] Section 2). Let C be a linear code over F_q of length n . Then C is an RPC code induced by a vector $\mathbf{a} = (a_0, a_1, \dots, a_{n-1})$ if and only if C is an ideal of the quotient ring $R_n^{\mathbf{a}}$.

Let C be an RPC code of length n over the field F_q . Then the following results [5] hold:

1. C is principally generated by a least degree monic polynomial $f(z)$ such that $f(z) \mid P_{n,\mathbf{a}}(z)$.
2. C has a basis set $\{f(z), zf(z), \dots, z^{k-1}f(z)\}$ and the corresponding generating matrix for C is

$$M = \begin{pmatrix} \Phi^{-1}(f(z)) \\ \Phi^{-1}(zf(z)) \\ \vdots \\ \Phi^{-1}(z^{k-1}f(z)) \end{pmatrix}, \text{ where } k = n - \deg(f(z)).$$

3. $\dim C$, the dimension of C is $k = n - \deg(f(z))$.

Lemma 1. Let $P_{n,\mathbf{a}}(z) = z^n - \mathbf{a}(z) \in R$, and it factors into monic polynomials as $P_{n,\mathbf{a}}(z) = b(z)d(z)$. Suppose C is an RPC code generated by $d(z)$ with $b(z)$ as its corresponding parity-check polynomial. Then a polynomial $r(z) \in R_n^{\mathbf{a}}$ is a generator of the code C if and only if there exists a polynomial $s(z)$ in R such that $r(z) = s(z)d(z)$ and $\gcd(b(z), s(z)) = 1$.

Proof. Consider $r(z) = s(z)d(z)$, where $\gcd(b(z), s(z)) = 1$. Then, there exist some polynomials $\mu(z)$ and $\nu(z)$ in R such that

$$\mu(z)s(z) + \nu(z)b(z) = 1. \quad (1)$$

If both sides of the Equation (1) are multiplied by $d(z)$, then we get $\mu(z)s(z)d(z) + \nu(z)b(z)d(z) = d(z)$. This expression leads to $\mu(z)r(z) = d(z) \pmod{(z^n - \mathbf{a}(z))}$. Hence, $d(z) \in Rr(z)$ where $Rr(z)$ is the ideal of R . Consequently, $Rd(z) \subseteq Rr(z)$. Otherwise, since $r(z) = s(z)d(z)$, it is clear that $Rr(z) \subseteq Rd(z)$, and hence $Rr(z) = Rd(z) = C$. Thus, it follows that $r(z) = s(z)d(z)$ is a generator of C . $\square$

3. Right Generalized Quasi-Polycyclic (GQPC) Codes

Here, we study the structural properties of right GQPC codes over the finite field F_q , mainly focusing on the algebraic structure of ρ -generator polynomial right GQPC codes. The definition of right QPC codes of length lm and index l given in [16] (Definition 3.2) extends naturally to right GQPC codes over F_q by considering l different values of m . Similarly, a GQC code of block length $(t_1, t_2, \dots, t_l)$ and index l extends to a right GQPC code by replacing the cyclic shift with the RPC shift induced by l distinct vectors. We begin with the definition of a GQC code.

Definition 3. Let $t_1, t_2, \dots, t_l \in \mathbb{Z}^+$ (set of positive integers). Suppose $C \subseteq F_q^{t_1} \times F_q^{t_2} \times \dots \times F_q^{t_l}$ is a linear code of length $N = t_1 + t_2 + \dots + t_l$ and index l over the field F_q . Consider a codeword $\mathbf{c} = (c_{t_1}, c_{t_2}, \dots, c_{t_l})$ in C where $c_{t_i} = (c_{i,0}, c_{i,1}, \dots, c_{i,t_i-1}) \in F_q^{t_i}$ for all $i = 1, 2, \dots, l$. Then C is said to be a GQC code, if for any $\mathbf{c} \in C$,

$$T(\mathbf{c}) = (T(c_{t_1}), T(c_{t_2}), \dots, T(c_{t_l})) \in C,$$

where T is the cyclic shift operator.

By replacing the cyclic shift T in each component of the above definition of GQC codes with the RPC shifts $P_r^{a_i}$ induced by the vector $a_i \in F_q^{t_i}$ for $i = 1, 2, \dots, l$, we obtain the class of GQPC codes, which naturally includes the right QPC codes as a special case.

Definition 4. A linear code $C \subseteq F_q^{t_1} \times F_q^{t_2} \times \dots \times F_q^{t_l}$ is a right GQPC code of length $N = t_1 + t_2 + \dots + t_l$ and index l if for any codeword $c \in C$, $(P_r^{a_1}(c_{t_1}), P_r^{a_2}(c_{t_2}), \dots, P_r^{a_l}(c_{t_l})) \in C$, where

$$P_r^{a_i}(c_{t_i}) = (0, c_{i,0}, c_{i,1}, \dots, c_{i,t_i-2}) + c_{i,t_i-1}(a_{i,0}, a_{i,1}, \dots, a_{i,t_i-1}) \\ = \begin{pmatrix} 0 + c_{i,t_i-1}a_{i,0} \\ c_{i,1} + c_{i,t_i-1}a_{i,1} \\ \vdots \\ c_{i,t_i-2} + c_{i,t_i-1}a_{i,t_i-1} \end{pmatrix}^{T'},$$

with induced vector $a_i = (a_{i,0}, a_{i,1}, \dots, a_{i,t_i-1}) \in F_q^{t_i}$, for all $i = 1, 2, \dots, l$. Here, T' denotes the transpose of a matrix.

If each $a_i = (1, 0, \dots, 0)$, then the right GQPC code is just the GQC code, and if all $t_1 = t_2 = \dots = t_l$, then the right GQPC code is the right QPC code.

Consider the quotient ring $R_{t_i}^{a_i} = R / \langle P_{t_i, a_i}(z) \rangle$, where $P_{t_i, a_i}(z) = z^{t_i} - a_i(z)$ and $a_i(z)$ is the polynomial representation of the corresponding induced vector a_i for all $i = 1, 2, \dots, l$. Denote $\mathbf{S} = R_{t_1}^{a_1} \times R_{t_2}^{a_2} \times \dots \times R_{t_l}^{a_l}$ (throughout the paper). Let $r(z) = r_0 + r_1z + \dots + r_mz^m \in R$, and $s(z) = (s_1(z), s_2(z), \dots, s_l(z))$ be an element in the set $\mathbf{S}$. We introduce the multiplication rule over $\mathbf{S}$ by

$$v(z) \cdot s(z) = (v(z) \cdot s_1(z), v(z) \cdot s_2(z), \dots, v(z) \cdot s_l(z)), \quad (2)$$

where $v(z) \cdot s_i(z)$ belonging to $R_{t_i}^{a_i}$ for all $i = 1, 2, \dots, l$. From Equation (2), it can easily be checked that the ring $\mathbf{S}$ is an R -module. We define a map

$$M : F_q^{t_1} \times F_q^{t_2} \times \dots \times F_q^{t_l} \mapsto \mathbf{S} \text{ given by } (c_{t_1}, c_{t_2}, \dots, c_{t_l}) \mapsto (c_{t_1}(z), c_{t_2}(z), \dots, c_{t_l}(z)), \quad (3)$$

where $c_{t_i}(z) = c_{i,0} + c_{i,1}(z) + c_{i,2}(z^2) + \dots + c_{i,t_i-1}z^{t_i-1} \in R_{t_i}^{a_i}$. Then, we get the following result, which defines the correspondence between the linear and the right GQPC codes.

Proposition 2 ([28]). The map M defined in Equation (3) gives rise to a bijective map between the set of all right GQPC codes of block length $(t_1, t_2, \dots, t_l)$ and index l over the field F_q , and the set of all R -submodules of the R -module $\mathbf{S}$.

Let C be a right GQPC code of block length $(t_1, t_2, \dots, t_l)$ and index l given by $C = (C_{t_1}, C_{t_2}, \dots, C_{t_l})$, where each C_{t_i} is an RPC code of length t_i and generated by $g_{t_i}(z) \in R_{t_i}$ for $i = 1, 2, \dots, l$. Then a 1-generator polynomial of C is an l -tuple $(e_1(z), e_2(z), \dots, e_l(z))$, where each $e_i(z) = h_i(z)g_{t_i}(z)$, $h_i(z)$ is relatively prime to the check polynomial of C_{t_i} and $g_{t_i}(z) \mid (z^{t_i} - a_i(z))$ for all $i = 1, 2, \dots, l$. Consider a 1-generator right GQPC code C of block length $(t_1, t_2, \dots, t_l)$ and index l over R generated by $e(z) = (e_1(z), e_2(z), \dots, e_l(z)) \in \mathbf{S}$. Then C can be written as

$$C = \{r(z)(e_1(z), e_2(z), \dots, e_l(z)) : r(z) \in R\} \\ = \{(r(z)e_1(z), r(z)e_2(z), \dots, r(z)e_l(z)) : r(z) \in R\}.$$

A right GQPC code C of block length $(t_1, t_2, \dots, t_l)$ and index l is called a ρ -generator (multi-generator) polynomial over F_q if ρ is the smallest positive integer greater than 1 for which there are codewords $b_i(z) = (b_{i,1}(z), b_{i,2}(z), \dots, b_{i,l}(z)) \in \mathbf{S}$ for $1 \leq i \leq \rho$ in C such that $C = Rb_1(z) + Rb_2(z) + \dots + Rb_\rho(z)$. To further investigate the algebraic structure of ρ -generator right GQPC codes, particularly through the factorization of their defining polynomials, we recall the following fundamental result, the Chinese Remainder Theorem (CRT).

Definition 5 ([32]). Consider a set of polynomials $\{p_1(z), p_2(z), \dots, p_L(z)\}$ in R , which are pairwise coprime over F_q . Let $P(z) = p_1(z)p_2(z) \cdots p_L(z)$. If $p'_1(z), p'_2(z), \dots, p'_L(z)$ are any set of polynomials in R , then there is exactly one polynomial $e(z)$ having $\deg(e(z)) < \deg(P(z))$ such that

$$e(z) \equiv p'_i(z) \pmod{p_i(z)}, \text{ for every } 0 \leq i \leq L. \quad (4)$$

Also, let $r_i(z) \in R$ be such that

$$\frac{P(z)}{p_i(z)} r_i(z) \equiv 1 \pmod{p_i(z)}, 0 \leq i \leq L.$$

Then the solution for Equation (4) is $e(z) = \sum_{i=1}^L \frac{P(z)}{p_i(z)} r_i(z) p'_i(z)$ reduced $(\pmod{P(z)})$.

Code Decomposition

Esmaili and Yari [23] presented the decomposition theorem of GQC codes that decompose into their shorter constituent codes using the CRT. According to the CRT, the ring $R/\langle z^n - 1 \rangle$ decomposes into a direct product of finite fields if n is coprime with the prime p . Here, we introduce the decomposition theorem for the right GQPC codes over the field F_q .

Let $P_{t_i, a_i}(z) \in R$, such that $P_{t_i, a_i}(0) \neq 0$ be a monic polynomial of degree t_i and order n_i with the condition $\gcd(n_i, q) = 1$ for $i = 1, 2, \dots, l$. As $\gcd(n_i, q) = 1$, $z^{n_i} - 1$ reduces into irreducible polynomials in R and since $P_{t_i, a_i}(z)$ divides $z^{n_i} - 1$, then $P_{t_i, a_i}(z)$ factors into completely distinct irreducible polynomials for $i = 1, 2, \dots, l$. Assume that the number of irreducible factors of all the $P_{t_i, a_i}(z)$ decompositions is t for all $i = 1, 2, \dots, l$. Hence, we have

$$P_{t_i, a_i}(z) = z^{t_i} - a_i(z) \quad (5)$$

$$= p_{t,1}(z)^{u_{i,1}} p_{t,2}(z)^{u_{i,2}} \dots p_{t,t}(z)^{u_{i,t}}, \quad (6)$$

where $p_{t,i,j}(z)$ are distinct irreducible polynomials for all $1 \leq j \leq t$ and $u_{i,j} \in \{0, 1\}$. Now, $u_{i,j} = 1$ for some $1 \leq j \leq t$, then $U_{t,j} = R/\langle p_{t,j}(z) \rangle$ is the finite field extension of F_q . For $1 \leq j \leq t$ and $i = 1, 2, \dots, l$, we define

$$U_{t,i,j} = \begin{cases} U_{t,j}, & \text{if } u_{i,j} = 1, \\ \{0\}, & \text{if } u_{i,j} = 0. \end{cases} \quad (7)$$

Consider a root β_j for each $P_{t_i, a_i}(z)$. Now, for $d_{t_i}(z) \in R_{t_i}^{a_i} = R/\langle z^{t_i} - a_i(z) \rangle$ and $1 \leq j \leq t$, define

$$d_{t,i,j} = \begin{cases} d_{t_i}(\beta_j), & \text{if } U_{t,i,j} = U_{t,j}, \\ 0, & \text{if } U_{t,i,j} = \{0\}. \end{cases} \quad (8)$$

Using the above Equations (7) and (8) and CRT, for each $i = 1, 2, \dots, l$, we get the following ring isomorphism

$$R_{t_i}^{a_i} \cong \bigoplus_{j=1}^t U_{t_i,j}, \quad (9)$$

where the isomorphism is given by $d_{t_i}(z)$ to $(d_{t_i,1} + d_{t_i,2} + \dots + d_{t_i,t})$. Then

$$\mathbf{S} = R_{t_1}^{a_1} \times R_{t_2}^{a_2} \times \dots \times R_{t_l}^{a_l} \cong \left(\bigoplus_{j=1}^t U_{t_1,j} \right) \times \dots \times \left(\bigoplus_{j=1}^t U_{t_l,j} \right) \quad (10)$$

$$\cong \bigoplus_{j=1}^t (U_{t_1,j} \times \dots \times U_{t_l,j}), \quad (11)$$

where $(d_{t_1}(z), d_{t_2}(z), \dots, d_{t_l}(z)) \in \mathbf{S}$ maps to $\sum_{j=1}^t (d_{t_1,j}, d_{t_2,j}, \dots, d_{t_l,j})$. Since some $U_{t_i,j}$ will be $\{0\}$, the corresponding coordinates of the codewords in that particular constituent will be 0. Hence, from Proposition 2, a right GQPC code can be observed as a submodule of $\bigoplus_{j=1}^t U_{t_j}^l$ as for each i , $U_{t_i,j}$ is either U_{t_j} or $\{0\} \subset U_{t_j}$.

Proposition 3. Let the right GQPC code $C \subset \mathbf{S}$ of block length $(t_1, t_2, \dots, t_l)$ and index l be generated as an R -submodule by

$$\{(d_{t_1}^1(z), \dots, d_{t_1}^1(z)), (d_{t_1}^2(z), \dots, d_{t_1}^2(z)), \dots, (d_{t_1}^s(z), \dots, d_{t_1}^s(z))\} \subset \mathbf{S}.$$

Then as a submodule of $\bigoplus_{j=1}^t U_{t_j}^l$, C takes the form inside $\bigoplus_{j=1}^t U_{t_j}^l$, as $C = \bigoplus_{j=1}^t C_j$, where every C_j is a l length U_{t_j} -linear code and written as

$$C_j = \text{Span}_{U_{t_j}} \{(d_{j,1}^{b,1}, \dots, d_{j,l}^{b,l}) : 1 \leq b \leq s\}, \text{ for } 1 \leq j \leq t. \quad (12)$$

Proof. Since C is a submodule of $\mathbf{S}$, we have

$$C = \{v_1(z)(d_{t_1}^1(z), \dots, d_{t_1}^1(z)) + v_2(z)(d_{t_1}^2(z), \dots, d_{t_1}^2(z)) + \dots + v_s(z)(d_{t_1}^s(z), \dots, d_{t_1}^s(z)) : v_1, v_2, \dots, v_s \in R\}$$

Now, from Equation (10), $C_j \subset (U_{t_1,j} \times U_{t_2,j} \times \dots \times U_{t_l,j})$ and can be written as

$$C_j = \{v_1(\beta_j)(d_{t_1,j}^1, d_{t_2,j}^1, \dots, d_{t_l,j}^1) + \dots + v_s(\beta_j)(d_{t_1,j}^s, d_{t_2,j}^s, \dots, d_{t_l,j}^s) : v_1, v_2, \dots, v_s \in R\}.$$

As β_j is a root of $P_{t_j}(z)$, we have $U_{t_j} = F_q(\beta_j)$. Also, each $U_{t_1,j}, U_{t_2,j}, \dots, U_{t_l,j}$ is either U_{t_j} or $\{0\} \subset U_{t_j}$, therefore, $v_1(\beta_j), v_2(\beta_j), \dots, v_s(\beta_j)$ take all possible values in U_{t_j} as the polynomials $v_1(z), v_2(z), \dots, v_s(z)$ vary over R . Thus, we get

$$C_j = \text{Span}_{U_{t_j}} \{(d_{j,1}^{b,1}, \dots, d_{j,l}^{b,l}) : 1 \leq b \leq s\}, \text{ for } 1 \leq j \leq t.$$

□

Theorem 1. Let C be a ρ -generator right GQPC code of block length $(t_1, t_2, \dots, t_l)$ and index l over F_q . If $C = \bigoplus_{i=1}^t C_i$, where C_i is of dimension k_i for $i = 1, 2, \dots, t$, and $K = \max\{k_i : 1 \leq i \leq t\}$. Then $\rho = K$. In addition, any right GQPC code C with constituent codes C_i of dimension k_i , satisfying $\rho = \max k_i$, is a ρ -generator right GQPC code.

Proof. Let C be a ρ -generator right GQPC code generated by the elements $e^j(z) = (e_1^j(z), \dots, e_l^j(z)) \in \mathbf{S}$ for $j = 1, 2, \dots, \rho$. Then for each i , C_i is spanned as R -submodule by $\bar{e}^j(z) = (\bar{e}_1^j(z), \dots, \bar{e}_l^j(z))$, where $\bar{e}_r^j(z) = e_r^j(z) \bmod (p_{t,i}(z))$, if $p_{t,i}(z)$ is a factor of $P_{t_i,a_i}(z)$ and $\bar{e}_r^j(z) = 0$ otherwise, for all $r = 1, 2, \dots, l$. Further, let β_i be a root of

irreducible factor $P_{t,i}(z)$ of $P_{t,a_i}(z)$. Then, for each $i = 1, 2, \dots, t$, the generator matrix of C_i contains the j^{th} row as $\vec{e}^j = (e_1^j(\beta_i), e_2^j(\beta_i), \dots, e_l^j(\beta_i))$, $j = 1, 2, \dots, \rho$. Hence, $k_i \leq \rho$ for every i , and thus $K = \max\{k_i : 1 \leq i \leq t\} \leq \rho$.

On the other hand, as $K = \max k_i$, there exists $e_i^j(z) \in R^l$, $1 \leq j \leq K$, such that $e_i^j(z)$ span C_i , for $1 \leq i \leq t$, as a R -submodule. Then by Proposition 3, for every $1 \leq j \leq K$ there exists $e^j(z) \in C$ such that $e_i^j(z) = e^j(z) \pmod{(p_{t,j}(z))}$ and C is generated by $e_i^j(z)$ for $1 \leq j \leq K$. Thus, $\rho \leq K$ which implies that $\rho = K$. $\square$

If C is a 1-generator right GQPC code of block length $(t_1, t_2, \dots, t_l)$ and index l over F_q , then by Theorem 1, every C_i , $i = 1, 2, \dots, t$, is either an $[l, 1]$ linear code or the trivial code over $R/\langle p_{t,j}(z) \rangle$. Conversely, any right GQPC code whose constituent codes C_i are of dimension at most 1 is a 1-generator right GQPC code.

Example 1. Let $q = 2$, $t_1 = 4$, $t_2 = 6$, and $t_3 = 8$. Consider the monic polynomials

$$\begin{aligned} P_{t_1,a_1}(z) &= z^4 + z^2 + z + 1 \text{ of order 7, where } a_1 = (1, 1, 1, 0) \in F_2^4 \\ &= (z+1)(z^3 + z^2 + 1), \\ P_{t_2,a_2}(z) &= z^6 + z^4 + z + 1 \text{ of order 21, where } a_2 = (1, 1, 0, 0, 1, 0, 0) \in F_2^6 \\ &= (z+1)(z^2 + z + 1)(z^3 + z + 1), \text{ and} \\ P_{t_3,a_3}(z) &= z^8 + z^6 + z^4 + z + 1 \text{ of order 217, where } a_3 = (1, 1, 0, 0, 1, 0, 1, 0, 0) \in F_2^8 \\ &= (z^3 + z^2 + 1)(z^5 + z^4 + z^2 + z + 1). \end{aligned}$$

In this case, we have

$$R_{t_1}^{a_1} \times R_{t_2}^{a_2} \times R_{t_3}^{a_3} = F_2[z]/\langle P_{t_1,a_1}(z) \rangle \times F_2[z]/\langle P_{t_2,a_2}(z) \rangle \times F_2[z]/\langle P_{t_3,a_3}(z) \rangle.$$

Suppose $P_{5,1}(z) = (z+1)$, $P_{5,2}(z) = (z^2 + z + 1)$, $P_{5,3}(z) = (z^3 + z^2 + 1)$, $P_{5,4}(z) = (z^3 + z + 1)$ and $P_{5,5}(z) = (z^5 + z^4 + z^2 + z + 1)$. Then, we have $U_{t_1} \cong F_2 \cong \frac{F_2[z]}{\langle z+1 \rangle}$, $U_{t_2} \cong F_4 \cong \frac{F_2[z]}{\langle z^2 + z + 1 \rangle}$, $U_{t_3} \cong F_8 \cong \frac{F_2[z]}{\langle z^3 + z^2 + 1 \rangle}$, $U_{t_4} \cong F_8 \cong \frac{F_2[z]}{\langle z^3 + z + 1 \rangle}$ and $U_{t_5} \cong F_{32} \cong \frac{F_2[z]}{\langle z^5 + z^4 + z^2 + z + 1 \rangle}$. Now, following the same notations as in Equation (7), we have

$$\begin{aligned} U_{t_1,1} &= F_2, & U_{t_1,2} &= \{0\}, & U_{t_1,3} &= F_8, & U_{t_1,4} &= \{0\}, & U_{t_1,5} &= \{0\} \\ U_{t_2,1} &= F_2, & U_{t_2,2} &= F_4, & U_{t_2,3} &= \{0\}, & U_{t_2,4} &= F_8, & U_{t_2,5} &= \{0\} \\ U_{t_3,1} &= \{0\}, & U_{t_3,2} &= \{0\}, & U_{t_3,3} &= F_8, & U_{t_3,4} &= \{0\}, & U_{t_3,5} &= F_{32}. \end{aligned}$$

Thus, we get

$$\begin{aligned} R_{t_1}^{a_1} \times R_{t_2}^{a_2} \times R_{t_3}^{a_3} &\cong (U_{t_1} \times U_{t_1} \times \{0\}) \oplus (\{0\} \times U_{t_2} \times \{0\}) \oplus (U_{t_3} \times \{0\} \times U_{t_3}) \\ &\quad \oplus (\{0\} \times U_{t_4} \times \{0\}) \oplus (\{0\} \times \{0\} \times U_{t_5}). \end{aligned}$$

We take C as a right GQPC code of block length $(4, 6, 8)$ generated by $C = \langle z + z^2, z^3 + z + 1, z^5 + z^4 + z^2 + z + 1 \rangle$. Then the right GQPC code C can be decomposed into $C = \bigoplus_{i=1}^5 C_i$. Let β_i be a root of $P_{5,i}(z)$ for $i = 1, 2, \dots, 5$, with $\beta_1 = 1$. Then, C has the following components:

$$\begin{aligned}
C_1 : & \text{ an } [l = 3, 1] \text{ linear code over } \frac{F_2[z]}{\langle z + 1 \rangle} \cong F_2 \text{ with the generator matrix } [0, 1, 1], \\
C_2 : & \text{ an } [l = 3, 1] \text{ linear code over } \frac{F_2[z]}{\langle z^2 + z + 1 \rangle} \cong F_4 \text{ with generator matrix } [0, \beta_2^3 + \beta_2^2, 0], \\
C_4 : & \text{ an } [l = 3, 0] \text{ trivial linear code over } \frac{F_2[z]}{\langle z^3 + z + 1 \rangle} \cong F_8, \\
C_5 : & \text{ an } [l = 3, 0] \text{ trivial linear code over } \frac{F_2[z]}{\langle z^5 + z^4 + z^2 + z + 1 \rangle} \cong F_{32}, \\
C_3 : & \text{ an } [l = 3, 1] \text{ linear code over } \frac{F_2[z]}{\langle z^3 + z^2 + 1 \rangle} \cong F_8 \text{ with generator matrix} \\
& [\beta_3 + \beta_3^2, 0, \beta_3^3 + \beta_3^4 + \beta_3^5].
\end{aligned}$$

Let k_i be the dimension of C_i , $i = 1, 2, 3, 4, 5$. Then $\max\{k_i\} = 1$ which is the number of generator of GQPC code C .

Example 2. Let $q = 3$, $t_1 = 4$ and $t_2 = 6$. Consider the monic polynomials

$$\begin{aligned}
P_{t_1, a_1}(z) &= z^4 + 2z^3 + z^2 + z + 1 \text{ of order 8, where } a_1 = (1, 1, 1, 2) \in F_3^4 \\
&= (z + 1)(z + 2)(z^2 + 2z + 2); \\
P_{t_2, a_2}(z) &= z^6 + 2z^5 + 2z^4 + 2z^2 + z + 1 \text{ of order 8, where } a_2 = (1, 1, 2, 0, 2, 2) \in F_3^6 \\
&= (z + 1)(z + 2)(z^2 + 1)(z^2 + 2z + 2).
\end{aligned}$$

In this case, we have

$$S = R_{t_1}^{a_1} \times R_{t_2}^{a_2} = F_3[z] / \langle P_{t_1, a_1}(z) \rangle \times F_3[z] / \langle P_{t_2, a_2}(z) \rangle.$$

Continuing with the same notations as used in Equation (5), let $P_{4,1}(z) = z + 1$, $P_{4,2}(z) = z + 2$, $P_{4,3}(z) = z^2 + 1$ and $P_{4,4}(z) = z^2 + 2z + 2$. Then, from Equation (8), we have $U_{t_1} \cong F_3 \cong \frac{F_3[z]}{\langle z + 1 \rangle}$, $U_{t_2} \cong F_3 \cong \frac{F_3[z]}{\langle z + 2 \rangle}$, $U_{t_3} \cong F_{3^2} \cong \frac{F_3[z]}{\langle z^2 + 1 \rangle}$ and $U_{t_4} \cong F_{3^2} \cong \frac{F_3[z]}{\langle z^2 + 2z + 2 \rangle}$. Now, following the notation and simple calculation as given in Equation (7), we have

$$\begin{aligned}
U_{t_1,1} &= U_{t_1} = F_3, & U_{t_1,2} &= U_{t_2} = F_3, & U_{t_1,3} &= \{0\}, & U_{t_1,4} &= U_{t_4} = F_{3^2} \\
U_{t_2,1} &= U_{t_1} = F_3, & U_{t_2,2} &= U_{t_2} = F_3, & U_{t_2,3} &= U_{t_3} = F_{3^2}, & U_{t_2,4} &= U_{t_4} = F_{3^2}.
\end{aligned}$$

Using the CRT and following Equations (9) and (10), we get

$$S = R_{t_1}^{a_1} \times R_{t_2}^{a_2} \cong (U_{t_1} \times U_{t_1}) \oplus (U_{t_2} \times U_{t_2}) \oplus (\{0\} \times U_{t_3}) \oplus (U_{t_4} \times U_{t_4}).$$

Let $C \subset S$ be a 2-generator right GQPC code of block length $(4, 6)$ generated by $e_1(z) = (z^2 + z, z^2 + 2z)$ and $e_2(z) = (z^2 + 2, z^3 + z^2 + 1)$. Then, right GQPC code can be decompose into $C = \bigoplus_{j=1}^{t=4} C_j$. Furthermore, for $t = 4$, let β_i be a root of $P_{t,i}(z)$ for $i = 1, 2, \dots, t$, where $\beta_1 = 2$ and $\beta_2 = 1$. Then, constituents of code C are as follows:

$$\begin{aligned}
C_1 : & \text{ a } [2, 1] \text{ linear code with the basis } \{(0, 1)\} \text{ over } U_{t_1}, \\
C_2 : & \text{ a } [2, 1] \text{ linear code with the basis } \{(2, 0)\} \text{ over } U_{t_2}, \\
C_3 : & \text{ a } [2, 1] \text{ linear code with the basis } \{0, \beta_3^2 + 2\beta_3\} \text{ over } U_{t_3}, \\
C_4 : & \text{ a } [2, 2] \text{ linear code with the basis } \{(\beta_4(\beta_4 + 1), 2), ((\beta_4 + 2)(\beta_4 + 1), 0)\} \text{ over } U_{t_4}.
\end{aligned}$$

Let k_j be the dimension of C_j , $j = 1, 2, 3, 4$. Then $\max\{k_j\} = 2$ which is the number of generator of right GQPC code C .

4. Normalized Generating Set for a GQPC Code

This section provides a normalized generating set of the ρ -generator right GQPC code of block length $(t_1, t_2, \dots, t_l)$ and index l over the field F_q , and a method to obtain a normalized generating set for a ρ -generator polynomial set that satisfies certain conditions by following the construction given in [20].

Definition 6. Let $\{A_1, A_2, \dots, A_k\} \subset S$ be a formal generating set of a right GQPC code C . If it can be transformed into the following standard form

$$\{(f_{11}(z), 0, \dots, 0), (f_{21}(z), f_{22}(z), \dots, 0), \dots, (f_{l1}(z), f_{l2}(z), \dots, f_{ll}(z))\},$$

where $f_{ij}(z) \in R_{t_j}^{a_j}$, $f_{ii}(z) \mid (z^{t_i} - a_i(z))$ for all $i = 1, 2, \dots, l$ and $j = 1, 2, \dots, i$. Then this standard form of a formal generating set is termed a normalized generating set.

In the following, we provide a method to construct a normalized generating set from any generating set for a right GQPC code C . Assume that

$$\begin{aligned} \mathbf{A}_1 &= (A_{11}(z), A_{12}(z), \dots, A_{1l}(z)), \\ \mathbf{A}_2 &= (A_{21}(z), A_{22}(z), \dots, A_{2l}(z)), \\ &\vdots \\ \mathbf{A}_k &= (A_{k1}(z), A_{k2}(z), \dots, A_{kl}(z)), \end{aligned}$$

form a generating set for the right GQPC code C of block length $(t_1, t_2, \dots, t_l)$ and index l .

- Step (i): Construct $f_{ll}(z) = \gcd(A_{1l}(z), A_{2l}(z), \dots, A_{kl}(z), z^{t_l} - a_l(z))$. Suppose that

$$\frac{A_{1l}(z)}{f_{ll}(z)}, \frac{A_{2l}(z)}{f_{ll}(z)}, \dots, \frac{A_{kl}(z)}{f_{ll}(z)}$$

are pairwise coprime over F_q . Then from the map defined in the proof of Theorem 2, $M_l(C) = \langle A_{1l}(z), A_{2l}(z), A_{3l}(z), \dots, A_{kl}(z) \rangle$. Since $M_l(C)$ is an RPC code of length t_l , we have $M_l(C) = \langle f_{ll}(z) \rangle$. Now, using the Euclidean algorithm over F_q , there exist $r_1(z), r_2(z), \dots, r_k(z), u(z) \in R$ such that

$$r_1(z)A_{1l}(z) + r_2(z)A_{2l}(z) + \dots + r_k(z)A_{kl}(z) + u(z)(z^{t_l} - a_l(z)) = f_{ll}(z).$$

If $f_{ll}(z) \equiv 0 \pmod{(z^{t_l} - a_l(z))}$, then take $\mathbf{a}_l(z) = (0, 0, \dots, z^{t_l} - a_l(z))$. Otherwise, consider

$$r_1(z)A_{1i}(z) + r_2(z)A_{2i}(z) + \dots + r_k(z)A_{ki}(z) = f_{li}(z) \pmod{(z^{t_i} - a_i(z))} \text{ for } i < l.$$

Hence, $\mathbf{a}_l(z) = (f_{l1}(z), f_{l2}(z), \dots, f_{ll}(z))$.

- Step (ii): Let $\alpha_i = \frac{A_{il}(z)}{f_{ll}(z)}$ for $i = 1, 2, \dots, l$. Then

$$\mathbf{A}_i^1 = \mathbf{A}_i - \alpha_i \mathbf{a}_l(z) = (A_{i1}^1(z), A_{i2}^1(z), \dots, A_{i,l-1}^1(z), 0).$$

Now, consider $f_{l-1,l-1}(z) = \gcd_q(A_{1,l-1}^1(z), A_{2,l-1}^1(z), \dots, A_{k,l-1}^1(z), z^{t_{l-1}} - a_{l-1}(z))$ and as in step (i), assume that $\frac{A_{1,l-1}^1(z)}{f_{l-1,l-1}(z)}, \frac{A_{2,l-1}^1(z)}{f_{l-1,l-1}(z)}, \dots, \frac{A_{k,l-1}^1(z)}{f_{l-1,l-1}(z)}$ are pairwise coprime in R . Then using the same procedure as in step (i), we can get

$$a_{l-1}(z) = (f_{l-1,1}(z), f_{l-1,2}(z), \dots, f_{l-1,l-1}(z), 0).$$

- Step (iii): Let $\alpha_i^1 = \frac{A_{i,l-1}^1(z)}{f_{l-1,l-1}(z)}$ for $i = 1, 2, \dots, l$. Then

$$A_i^2 = A_i^1 - \alpha_i^1 a_{l-1}(z) = (A_{i,1}^2(z), A_{i,2}^2(z), \dots, A_{i,l-2}^2(z), 0, 0).$$

Employing the same approach as in step (i), we have

$$f_{l-2,l-2}(z) = \gcd(A_{1,l-2}^2(z), A_{2,l-2}^2(z), \dots, A_{l,l-2}^2(z), z^{t_{l-2}} - a_{l-2}(z)).$$

Then by step (i) and $A_1^2, A_2^2, \dots, A_l^2$, we can construct

$$a_{l-2}(z) = (f_{l-2,1}(z), f_{l-2,2}(z), \dots, f_{l-2,l-2}(z), 0, 0).$$

Repeating the above procedure, we finally obtain $a_1(z) = (f_{1,1}(z), 0, 0, \dots, 0)$. Hence, above method can yield a normalized generating set $\{a_1(z), a_2(z), \dots, a_l(z)\}$.

Example 3. Let $q = 2$, $t_1 = 6$ and $t_2 = 4$. Consider the monic polynomials

$$\begin{aligned} P_{t_1, a_1}(z) &= z^6 - z^2 + 1, \text{ where } a_1 = (1, 0, 1, 0, 0, 0) \in F_2^6 \\ &= (z^3 + z + 1)^2, \\ P_{t_2, a_2}(z) &= z^4 + z^2 + z + 1, \text{ where } a_2 = (1, 1, 1, 0) \in F_2^4 \\ &= (z + 1)(z^3 + z^2 + 1). \end{aligned}$$

Let $A_1 = \{z^3 + z + 1, z + 1\}$ and $A_2 = \{0, z(z + 1)\}$ be a formal 2-generating set of right GQPC code of index 2 over F_2 . Here, $S = R_{t_1}^{a_1} \times R_{t_2}^{a_2} = F_2[z] / \langle P_{t_1, a_1}(z) \rangle \times F_2[z] / \langle P_{t_2, a_2}(z) \rangle$. As in step (i) of the above method, by a simple calculation

$$f_{22}(z) = \gcd(z + 1, z(z + 1), z^4 + z^2 + z + 1) = z + 1.$$

Now, by the Euclidean algorithm, there exist two polynomials $r_1(z) = 1$ and $r_2(z) = 0$ such that $z + 1 = r_1(z)(z + 1) + r_2(z)z(z + 1)$. Since $f_{22}(z) \neq 0 \pmod{P_{t_2, a_2}(z)}$,

$$\begin{aligned} f_{21}(z) &= r_1(z) \cdot (z^3 + z + 1) + r_2(z) \cdot 0 \pmod{z^6 - z^2 + 1} \\ &= z^3 + z + 1. \end{aligned}$$

Hence, $a_2(z) = (z^3 + z + 1, z + 1)$. Following step (ii) of the above method, $\alpha_1 = 1$ and $\alpha_2 = z$. Then

$$\begin{aligned} A_1^1 &= A_1 - \alpha_1 \cdot a_2(z) \\ &= \{z^3 + z + 1, z + 1\} - \{z^3 + z + 1, z + 1\} = \{0, 0\}, \\ A_2^1 &= A_2 - \alpha_2 \cdot a_2(z) \\ &= \{0, z(z + 1)\} - \{z(z^3 + z + 1), z(z + 1)\} \\ &= \{z(z^3 + z + 1), 0\}. \end{aligned}$$

Again, following step (i), we obtain $f_{11}(z) = \gcd(z^6 - z^2 + 1, z(z^3 + z + 1)) = z^3 + z + 1$, $f_{12}(z) = 0$. Hence, $\mathbf{a}_1(z) = \{f_{11}(z), f_{12}(z)\}$. Thus, a Normalized generating set for a right GQPC code is $\{\mathbf{a}_1(z), \mathbf{a}_2(z)\}$.

Example 4. Let $q = 5$, $t_1 = 4$, $t_2 = t_3 = 6$. Consider monic polynomials

$$\begin{aligned} P_{t_1, \mathbf{a}_1}(z) &= z^4 + z^2 + 3z + 1, \text{ where } \mathbf{a}_1 = (1, 3, 1, 0) \in F_5^4 \\ &= (z + 1)(z + 2)(z^2 + 2z + 3), \\ P_{t_2, \mathbf{a}_2}(z) &= z^6 + 2z^4 + z^2 + z + 2, \text{ where } \mathbf{a}_2 = (2, 1, 1, 0, 2, 0) \in F_5^6 \\ &= (z + 2)(z + 1)^2(z^3 + z^2 + 3z + 1), \\ P_{t_3, \mathbf{a}_3}(z) &= z^6 + 2z^4 + z^2 + z + 2, \text{ where } \mathbf{a}_3 = (2, 1, 1, 0, 2, 0) \in F_5^6 \\ &= (z + 2)(z + 1)^2(z^3 + z^2 + 3z + 1). \end{aligned}$$

Let $A_1 = \{z^3 + 4z^2 + 2z + 1, z^4 + z^2 + 3z + 1, 0\}$, $A_2 = \{0, z^2 + 3z + 2, z^4 + z^3 + 3z^2 + z\}$ and $A_3 = \{z^3 + 4z + 4, 0, z^4 + 4z^3 + z^2 + 3\}$ be a formal 3-generating set of right GQPC code of index 3 over F_5 . Here, $S = R_{t_1}^{\mathbf{a}_1} \times R_{t_2}^{\mathbf{a}_2} \times R_{t_3}^{\mathbf{a}_3} = F_5[z] / \langle P_{t_1, \mathbf{a}_1}(z) \rangle \times F_5[z] / \langle P_{t_2, \mathbf{a}_2}(z) \rangle \times F_5[z] / \langle P_{t_3, \mathbf{a}_3}(z) \rangle$, and $\{A_1, A_2, A_3\} \subset S$. By following the above method and simple calculation,

$$f_{33}(z) = \gcd(z^4 + 4z^3 + z^2 + 3, z^4 + z^3 + 3z^2 + z, P_{t_3, \mathbf{a}_3}(z)) = z^3 + z^2 + 3z + 1.$$

Now, by the Euclidean algorithm, there exist polynomials $r_1(z) = 0$, $r_2(z) = 3$, $r_3(z) = 2$ and $u(z) = 0$ such that

$$f_{33}(z) = r_1(z) \cdot 0 + r_2(z) \cdot (z^4 + z^3 + 3z^2 + z) + r_3(z) \cdot (z^4 + 4z^3 + z^2 + 3) + u(z)P_{t_3, \mathbf{a}_3}(z).$$

Since $f_{33}(z) \neq 0 \pmod{P_{t_3, \mathbf{a}_3}(z)}$,

$$\begin{aligned} f_{32}(z) &= r_1(z)(z^4 + 3z^3 + 2z + 2) + r_2(z)(z^2 + 3z + 2) + r_3(z) \cdot 0 \\ &= 3z^2 + z + 1, \text{ and} \\ f_{31}(z) &= r_1(z)(z^3 + 4z^2 + 2z + 1) + r_2(z) \cdot 0 + r_3(z)(z^3 + 4z + 4) \\ &= 2z^3 + 3z + 3. \end{aligned}$$

Hence, $\mathbf{a}_3(z) = \{f_{31}(z), f_{32}(z), f_{33}(z)\}$. Following step (ii), $\alpha_1 = 0$, $\alpha_2 = z$ and $\alpha_3 = z + 3$. Then

$$\begin{aligned} A_1^1 &= A_1 - \alpha_1 \cdot \mathbf{a}_3(z) = A_1, \text{ (as } \alpha_1 = 0), \\ A_2^1 &= A_2 - \alpha_2 \cdot \mathbf{a}_3(z) = \{3z^4 + 2z^2 + 2z, 3z^2 + 2z + 1, 0\}, \\ A_3^1 &= A_3 - \alpha_3 \cdot \mathbf{a}_3(z) = \{3z^4 + 2z^2 + 2z, 2z^3 + z + 2, 0\}. \end{aligned}$$

By following step (i) and simple calculation, we get $f_{22}(z) = \gcd(z^4 + 3z^3 + 2z + 2, 3z^2 + 2z + 1, 2z^3 + z + 2) = 1$. Now, using Euclidean algorithm, there exist polynomials $r'_1(z) = 0$, $r'_2(z) = 2z^2 + z + 3$ and $r'_3(z) = 2z + 4$ such that

$$1 = 0 \cdot (z^4 + 3z^3 + 2z + 2) + (2z^2 + z + 3) \cdot (3z^2 + 2z + 1) + (2z + 4) \cdot (2z^3 + z + 2).$$

Since $f_{22}(z) \neq 0 \pmod{P_{t_2, \mathbf{a}_2}(z)}$, we have

$$\begin{aligned} f_{21}(z) &= r'_1(z)(z^3 + 4z^2 + 2z + 1) + r'_2(z)(3z^4 + 2z^2 + 2z) + r'_3(z)(3z^4 + 2z^2 + 2z) \\ &= z^6 + 4z^5 + 4z \pmod{P_{t_1, \mathbf{a}_1}(z)} \\ &= 3z^3 + 3z^2 + 3z + 1. \end{aligned}$$

Hence, we have $\mathbf{a}_2(z) = (f_{21}(z), f_{22}(z), 0)$. Further, following step (ii), we have

$$\alpha_1^1 = \frac{z^4 + 3z^3 + 2z + 2}{f_{22}(z) = 1}, \alpha_2^1 = \frac{3z^2 + 2z + 1}{f_{22}(z) = 1}, \alpha_3^1 = \frac{2z^3 + z + 2}{f_{22}(z) = 1}.$$

Then

$$\begin{aligned} A_1^2 &= A_1^1 - \alpha_1^1 \mathbf{a}_2(z) = \{4z^3 + z^2 + 3z + 4, 0, 0\}, \\ A_2^2 &= A_2^1 - \alpha_2^1 \mathbf{a}_2(z) = \{z^3 + 4z^2 + 2z + 1, 0, 0\}, \\ A_3^2 &= A_3^1 - \alpha_3^1 \mathbf{a}_2(z) = \{3z^3 + 2z^2 + z + 3, 0, 0\}. \end{aligned}$$

Again, following step (i), we get $f_{11}(z) = \gcd(4z^3 + z^2 + 3z + 4, z^3 + 4z^2 + 2z + 1, 3z^3 + 2z^2 + z + 3) = z^3 + 4z^2 + 2z + 1$. Further, $f_{12}(z) = 0$ and $f_{13}(z) = 0$. Hence, $\mathbf{a}_1(z) = \{f_{11}(z), f_{12}(z), f_{13}(z)\}$. Thus, a normalized generating set for the right GQPC code is $\{\mathbf{a}_1(z), \mathbf{a}_2(z), \mathbf{a}_3(z)\}$.

Theorem 2. Let C be a right GQPC code of block length $(t_1, t_2, \dots, t_l)$ and index l over the field F_q , where $N = \sum_{i=1}^l t_i$ is the length of C . Then there is a generating set $\{\mathbf{a}_1(z), \mathbf{a}_2(z), \dots, \mathbf{a}_l(z)\}$ such that $C = \langle \mathbf{a}_1(z), \mathbf{a}_2(z), \dots, \mathbf{a}_l(z) \rangle$, where

$$\begin{aligned} \mathbf{a}_1(z) &= (f_{11}(z), 0, \dots, 0), \\ \mathbf{a}_2(z) &= (f_{21}(z), f_{22}(z), \dots, 0), \\ &\vdots \\ \mathbf{a}_l(z) &= (f_{l1}(z), f_{l2}(z), \dots, f_{ll}(z)), \end{aligned}$$

and $f_{ij}(z) \in R_{t_j}^{\mathbf{a}_i}$, $f_{ii}(z) \mid (z^{t_i} - \mathbf{a}_i(z))$ for all $i = 1, 2, \dots, l$ and $j = 1, 2, \dots, i$.

Proof. For index $l = 1$, C is an RPC code of length t_1 . Hence, from Proposition 1, C is an ideal of $R_{t_1}^{\mathbf{a}_1}$. Then, there exists a monic polynomial of the least degree $f_{11}(z) \in R$ such that $f_{11}(z) \mid z^{t_1} - \mathbf{a}_1(z)$ and $C = \langle f_{11}(z) \rangle$ (see [5] (Proposition 1)). Now, assume that the result holds for $k < l$ over F_q . Consider a module homomorphism for all $i = 1, 2, \dots, l$,

$$\begin{aligned} M_i : R_{t_1}^{\mathbf{a}_1} \times R_{t_2}^{\mathbf{a}_2} \times \dots \times R_{t_l}^{\mathbf{a}_l} &\mapsto R_{t_i}^{\mathbf{a}_i}, \text{ given by} \\ (b_1(z), b_2(z), \dots, b_l(z)) &\mapsto b_i(z). \end{aligned}$$

For $i = l$, $M_l(C)$ is an RPC code in $R_{t_l}^{\mathbf{a}_l}$ over F_q of length t_l . Then, there exists a monic polynomial of the least degree $f_{ll}(z) \in R$ such that $f_{ll}(z)$ divides $z^{t_l} - \mathbf{a}_l(z)$ and $M_l(C) = \langle f_{ll}(z) \rangle$. Now, consider any codeword $(c_1(z), c_2(z), \dots, c_l(z)) \in C$. Then, there exists a polynomial $\alpha(z) \in R$ such that $c_l(z) = \alpha(z)f_{ll}(z)$. For the above map M_l , $\ker(M_l(C))$ is the right GQPC code of index $l - 1$. Hence, from our assumption, it can be generated by $\{\mathbf{a}_1(z), \mathbf{a}_2(z), \dots, \mathbf{a}_{l-1}(z)\}$. If $\mathbf{a}_l(z) \in C$, then $c(z) - \alpha(z)\mathbf{a}_l(z) \in \ker(M_l(C))$. Thus, $c(z)$ belongs to $\langle \mathbf{a}_1(z), \mathbf{a}_2(z), \dots, \mathbf{a}_l(z) \rangle$. This completes the proof. $\square$

For ease of computation, we present some relationships between constituents of the generator polynomial set in the next result.

Theorem 3. Assume that $C = \langle \mathbf{a}_1(z), \mathbf{a}_2(z), \dots, \mathbf{a}_l(z) \rangle$ is the right GQPC code of block length $(t_1, t_2, \dots, t_l)$ and index l , where $\mathbf{a}_i(z)$ same as in Theorem 2 and $N = \sum_{i=1}^l t_i$. Then $\deg(f_{i+1i}(z)) < \deg(f_{ii}(z))$ for all $i = 1, 2, \dots, l - 1$.

Proof. For index $l = 2$, let $C = \langle \mathbf{a}_1(z), \mathbf{a}_2(z) \rangle$ as given in Theorem 2. Assume that $\deg(f_{11}(z)) < \deg(f_{21}(z))$. Then, using the division algorithm, there exist $h_{21}(z)$ and $r_{21}(z) \in R$ such that $f_{21}(z) = h_{21}(z)f_{11}(z) + r_{21}(z)$, where $r_{21}(z) = 0$ or $\deg(r_{21}(z)) < \deg(f_{11}(z))$. Then

$$\begin{aligned} \langle \mathbf{a}_1(z), \mathbf{a}_2(z) \rangle &= \langle (f_{11}(z), 0), (f_{21}(z), f_{22}(z)) \rangle, \\ &= \langle (f_{11}(z), 0), (h_{21}(z)f_{11}(z) + r_{21}(z), f_{22}(z)) \rangle, \\ &= \langle (f_{11}(z), 0), (h_{21}(z)(f_{11}(z), 0) + (r_{21}(z), f_{22}(z))) \rangle, \\ &= \langle (f_{11}(z), 0), (r_{21}(z), f_{22}(z)) \rangle. \end{aligned}$$

Similarly, for index $l = 3$, let $C = \langle \mathbf{a}_1(z), \mathbf{a}_2(z), \mathbf{a}_3(z) \rangle$. Assume that $\deg(r_{21}(z)) < \deg(f_{31}(z))$ and $\deg(f_{22}(z)) < \deg(f_{32}(z))$. Then by using the division algorithm there exist $h_{31}(z), h_{22}(z), r_{31}(z), r_{32}(z) \in R$ which satisfy $f_{31}(z) = h_{31}(z)r_{21}(z) + r_{31}(z)$, where $r_{31}(z) = 0$ or $\deg(r_{31}(z)) < \deg(r_{21}(z))$ and $f_{32}(z) = h_{32}(z)f_{22}(z) + r_{32}(z)$, where $r_{32}(z) = 0$ or $\deg(r_{32}(z)) < \deg(f_{22}(z))$. Then

$$\begin{aligned} \langle \mathbf{a}_1(z), \mathbf{a}_2(z), \mathbf{a}_3(z) \rangle &= \langle (f_{11}(z), 0, 0), (f_{21}(z), f_{22}(z), 0), (f_{31}(z), f_{32}(z), f_{33}(z)) \rangle \\ &= \left\langle (f_{11}(z), 0, 0), (r_{21}(z), f_{22}(z), 0), (h_{31}(z)r_{21}(z) + r_{31}(z), \right. \\ &\quad \left. h_{32}(z)f_{22}(z) + r_{32}(z), f_{33}(z)) \right\rangle \\ &= \left\langle (f_{11}(z), 0, 0), (r_{21}(z), f_{22}(z), 0), (h_{31}(z)r_{21}(z), h_{32}(z)f_{22}(z), 0) \right. \\ &\quad \left. + (r_{31}(z), r_{32}(z), f_{33}(z)) \right\rangle \\ &= \langle (f_{11}(z), 0, 0), (r_{21}(z), f_{22}(z), 0), (r_{31}(z), r_{32}(z), f_{33}(z)) \rangle. \end{aligned}$$

Hence, the statement holds for index $l = 3$. Now, assume that the statement holds for index $l - 1$. Then consider

$$\begin{aligned} &\langle \mathbf{a}_1(z), \mathbf{a}_2(z), \dots, \mathbf{a}_{l-1}(z), \mathbf{a}_l(z) \rangle \\ &= \left\langle (f_{11}(z), 0, \dots, 0), (f_{21}(z), f_{22}(z), 0, \dots, 0), \dots, (f_{l-11}(z), f_{l-12}(z), \dots, f_{l-1l-1}(z), 0), \right. \\ &\quad \left. (f_{l1}(z), f_{l2}(z), \dots, f_{ll-1}(z), f_{ll}(z)) \right\rangle \end{aligned}$$

as we have assumed that the statement holds for index less than l , we thus have

$$\begin{aligned} &\langle \mathbf{a}_1(z), \mathbf{a}_2(z), \dots, \mathbf{a}_{l-1}(z), \mathbf{a}_l(z) \rangle \\ &= \left\langle (f_{11}(z), 0, \dots, 0), (r_{21}(z), f_{22}(z), 0, \dots, 0), \dots, (r_{l-11}(z), r_{l-12}(z), \dots, r_{l-1l-1}(z), 0), \right. \\ &\quad \left. (f_{l1}(z), f_{l2}(z), \dots, f_{ll-1}(z), f_{ll}(z)) \right\rangle. \end{aligned}$$

Now, using similar arguments for $f_{l1}(z), f_{l2}(z), \dots, f_{ll-1}(z)$ and applying the division algorithm, we can write

$$\begin{aligned} &\langle \mathbf{a}_1(z), \mathbf{a}_2(z), \dots, \mathbf{a}_{l-1}(z), \mathbf{a}_l(z) \rangle \\ &= \left\langle (f_{11}(z), 0, \dots, 0), (r_{21}(z), f_{22}(z), 0, \dots, 0), \dots, (r_{l-11}(z), r_{l-12}(z), \dots, r_{l-1l-1}(z), 0), \right. \\ &\quad \left. (r_{l1}(z), r_{l2}(z), \dots, r_{ll-1}(z), f_{ll}(z)) \right\rangle, \end{aligned}$$

where $\deg(r_{l1}(z)) < \deg(r_{l-11}(z)), \deg(r_{l2}(z)) < \deg(r_{l-12}(z)), \dots, \deg(r_{ll-1}(z)) < \deg(r_{l-1l-1}(z))$. Hence, we get the desired result. $\square$

Theorems 2 and 3 describe the generator set for the right GQPC codes of block length $(t_1, t_2, \dots, t_l)$ and index l . Based on the uniqueness and the structural properties of this generator set, we now establish the following result.

Theorem 4. Let C be a right GQPC code of block length $(t_1, t_2, \dots, t_l)$ and index l , with a normalized generating set as given in the Theorem 2. Then the dimension of C is $\deg(q_{t_1}(z)) + \deg(q_{t_2}(z)) + \dots + \deg(q_{t_l}(z))$, where

$$\begin{aligned} z^{t_1} - a_1(z) &= q_{t_1}(z)f_{11}(z), \\ z^{t_2} - a_2(z) &= q_{t_2}(z)f_{22}(z), \\ &\vdots \\ z^{t_l} - a_l(z) &= q_{t_l}(z)f_{ll}(z). \end{aligned}$$

Proof. To prove the result, we implement the principle of mathematical induction on l . For index $l = 1$, $C = \langle f_{11}(z) \rangle$. It is just the right cyclic code of length t_1 with induced vector a_1 . From [5] (Proposition 1), the basis of C is denoted by S_1^1 and is given by $S_1^1 = \{f_{11}(z), zf_{11}(z), \dots, z^{t_1 - \deg(f_{11}(z)) - 1}f_{11}(z)\}$ and thus the dimension of C is $t_1 - \deg(f_{11}(z))$. Now, assume that index $l = 2$. Then $C = \langle (f_{11}(z), 0), (r_{21}(z), f_{22}(z)) \rangle$. Consider $S_1^2 = \bigcup_{i=0}^{\deg(q_{t_1}(z)) - 1} \{z^i * (f_{11}(z), 0)\} \cup S_2^2 = \bigcup_{j=0}^{\deg(q_{t_2}(z)) - 1} \{z^j * (r_{21}(z), f_{22}(z))\}$. If a codeword $c(z) \in C$, then there exist $q_1(z), q_2(z)$ in R such that

$$\begin{aligned} c(z) &= q_1(z)(f_{11}(z), 0) + q_2(z)(r_{21}(z), f_{22}(z)) \\ &= (q_1(z)f_{11}(z), 0) + (q_2(z)r_{21}(z), q_2(z)f_{22}(z)). \end{aligned}$$

If $\deg(q_1(z)) \geq \deg(q_{t_1}(z))$, then, by the division algorithm, there exist $Q_1(z)$ and $Q'_1(z)$ in R such that

$$q_1(z) = Q_1(z)q_{t_1}(z) + Q'_1(z) \text{ with } Q'_1(z) = 0 \text{ or } \deg(Q'_1(z)) < \deg(q_{t_1}(z)).$$

Then,

$$\begin{aligned} (q_1(z)f_{11}(z), 0) &= ((Q_1(z)q_{t_1}(z) + Q'_1(z))f_{11}(z), 0) \\ &= Q_1(z)(q_{t_1}(z)f_{11}(z), 0) + Q'_1(z)(f_{11}(z), 0) \\ &= Q'_1(z)(f_{11}(z), 0) \in \text{span}(S_1^2). \end{aligned}$$

Also, if $\deg(q_2(z)) \geq \deg(q_{t_2}(z))$, then by the division algorithm there exist $Q_2(z)$ and $Q'_2(z)$ in R such that

$$q_2(z) = Q_2(z)q_{t_2}(z) + Q'_2(z) \text{ with } Q'_2(z) = 0 \text{ or } \deg(Q'_2(z)) < \deg(q_{t_2}(z)).$$

Then

$$\begin{aligned} (q_2(z)r_{21}(z), f_{22}(z)) &= ((Q_2(z)q_{t_2}(z) + Q'_2(z))r_{21}(z), f_{22}(z)) \\ &= Q_2(z)(q_{t_2}(z)r_{21}(z), f_{22}(z)) + Q'_2(z)(r_{21}(z), f_{22}(z)) \\ &= Q_2(z)(q_{t_2}(z)r_{21}(z), 0) + Q'_2(z)(r_{21}(z), f_{22}(z)). \end{aligned}$$

Assume that $q_{t_2}(z)r_{21}(z) \in \langle f_{11}(z) \rangle$. Then $Q_2(z)(q_{t_2}(z)r_{21}(z), 0) \in \langle (f_{11}(z), 0) \rangle$ and $\deg(Q_2(z)) < \deg(q_{t_1}(z))$. Hence,

$$\begin{aligned} c(z) &= q_1(z)(f_{11}(z), 0) + q_2(z)(r_{21}(z), f_{22}(z)) \\ &= Q'_1(z)(f_{11}(z), 0) + Q''_2(z)(f_{11}(z), 0) + Q'_2(z)(r_{21}(z), f_{22}(z)) \\ &= (Q'_1(z) + Q''_2(z))(f_{11}(z), 0) + Q'_2(z)(r_{21}(z), f_{22}(z)), \end{aligned}$$

where $\deg((Q'_1(z) + Q''_2(z))) < \deg(q_{t_1}(z))$. Therefore, $c(z) \in \text{Span}(S_1^2 \cup S_2^2)$, and clearly, $S_1^2 \cup S_2^2$ is a linearly independent set. Thus, $\dim(C) = \deg(q_{t_1}(z)) + \deg(q_{t_2}(z))$.

Similarly, suppose that the statement holds for the index $l-1$ over F_q . We will also show that this statement holds for the index l . Consider any $c(z) \in C = \langle \mathbf{a}_1(z), \mathbf{a}_2(z), \dots, \mathbf{a}_l(z) \rangle$. Then there exist $q_1(z), q_2(z), \dots, q_l(z)$ such that $c(z) = q_1(z)\mathbf{a}_1(z) + q_2(z)\mathbf{a}_2(z) + \dots + q_l(z)\mathbf{a}_l(z)$. Using the proof of Theorem 2, $\ker(M_l(C)) = \langle \mathbf{a}_1(z), \mathbf{a}_2(z), \dots, \mathbf{a}_{l-1}(z) \rangle$ is a right GQPC code of block length $(t_1, t_2, \dots, t_{l-1})$ of index $l-1$ over F_q . Thus, $\ker(M_l(C)) = \bigcup_{i=1}^{l-1} S_i$, where $S_i = \bigcup_{j=0}^{\deg(q_{t_i}(z))-1} \{z^j * (r_{i,1}(z), r_{i,2}(z), \dots, f_{ii}(z), \dots, 0)\}$. Now, consider $q_l(z)\mathbf{a}_l(z) = q_l(z)(r_{l1}(z), r_{l2}(z), \dots, f_{ll}(z))$ in $c(z)$. If $\deg(q_l(z)) > \deg(q_{t_l}(z))$ then by the division algorithm, there exist $Q_l(z)$ and $Q'_l(z) \in R$ such that $q_l(z) = Q_l(z)q_{t_l}(z) + Q'_l(z)$, where $Q'_l(z) = 0$ or $\deg(Q'_l(z)) < \deg(q_{t_l}(z))$. Hence,

$$\begin{aligned} q_l(z)\mathbf{a}_l(z) &= (Q_l(z)q_{t_l}(z) + Q'_l(z))(r_{l1}(z), r_{l2}(z), \dots, f_{ll}(z)) \\ &= Q_l(z)(q_{t_l}(z)r_{l1}(z), q_{t_l}(z)r_{l2}(z), \dots, 0) + Q'_l(z)(r_{l1}(z), r_{l2}(z), \dots, f_{ll}(z)). \end{aligned}$$

Assume that $q_{t_l}(z)r_{l1}(z) \in \langle f_{l-1,1}(z) \rangle$. Then $(q_{t_l}(z)r_{l1}(z), q_{t_l}(z)r_{l2}(z), \dots, 0) \in \ker(M_l(C))$ and $Q'_l(z) = 0$ or $\deg(Q'_l(z)) < \deg(q_{t_l}(z))$. Then we have

$$q_l(z)\mathbf{a}_l(z) \in \text{Span}\left(\bigcup_{i=1}^{l-1} S_i \cup S_l\right).$$

Therefore, $c(z) \in \text{Span}\left(\bigcup_{i=1}^{l-1} S_i \cup S_l\right)$. Clearly, it is a linearly independent set, as none of its elements can be written as a nonzero linear combination of the remaining elements. Thus, $\dim(C) = \deg(q_{t_1}(z)) + \deg(q_{t_2}(z)) + \dots + \deg(q_{t_l}(z))$. $\square$

The next step in our discussion is to characterize the performance of these codes in terms of optimality. To this end, we define the notions of optimal and near-optimal GQPC codes.

Definition 7. A linear code C of parameters $[n, k, d]$ over F_q is said to be optimal (with fixed length and dimension) if no other $[n, k, d']$ code over F_q exists in code stable [30] with $d' > d$.

Definition 8. A linear code C of parameters $[n, k, d]$ over F_q is said to be near-optimal (with fixed length and dimension) if d is one less than that given in code stable [30].

Remark 2. It is noted that if we do not fix the dimension, then “optimality” will occur in two ways: one regarding the code rate and the other regarding the error-correcting capability. As $[7, 2, 4]$ is not optimal over F_2 , because a better code $[7, 3, 4]$ exists.

Example 5. Let $q = 2$, $t_1 = 36$ and $t_2 = 2$. Consider the monic polynomials $P_{t_1, \mathbf{a}_1}(z) = z^{36} + z^{34} + z^{32} + z^{30} + z^{28} + z^{26} + z^{24} + z^{22} + z^{21} + z^{18} + z^{16} + z^{14} + z^{11} + z^9 + z^8 + z^6 + z^4 + z^2 + z + 1$, where

$$\mathbf{a}_1 = (1, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0) \in F_2^{36}$$

Table 1. 2-Generator right GQPC codes over F_2 .

t_1, t_2, N	Polynomials $z^{t_i} - a_i(z)$, $f_{11}(z)$, $r_{21}(z)$ and $f_{22}(z)$	Parameters	Remarks
20, 2, 22	$P_{t_1, a_1}(z) = 11011111011111111111$, $f_{11} = 101101100111$, $r_{21} = 1001001011$, $P_{t_2, a_2}(z) = 101$, $f_{22} = 1$	[22, 11, 6]	Near-optimal, LCD
29, 2, 31	$P_{t_1, a_1}(z) = 1111011010010110101010101011$, $f_{11} = 1011101111011$, $r_{21} = 10011100111$, $P_{t_2, a_2}(z) = 101$, $f_{22} = 1$	[31, 19, 6]	Optimal
34, 2, 36	$P_{t_1, a_1}(z) = 1111011101010101110111011001010101$, $f_{11} = 11110111101010001111$, $r_{21} = 110001100010000011$, $P_{t_2, a_2}(z) = 101$, $f_{22} = 1$	[36, 17, 8]	Optimal
36, 1, 37	$P_{t_1, a_1}(z) = 11101010110100101010011010101010101$, $f_{11} = 10100110111100010001$, $r_{21} = 1100010010100001111$, $P_{t_2, a_2}(z) = 11$, $f_{22} = 1$	[37, 18, 8]	Optimal
36, 1, 37	$P_{t_1, a_1}(z) = 11101010110100101010011010101010101$, $f_{11} = 10000101111101$, $r_{21} = 1111100101011$, $P_{t_2, a_2}(z) = 11$, $f_{22} = 1$	[37, 24, 6]	Optimal
36, 1, 37	$P_{t_1, a_1}(z) = 11101010110100101010011010101010101$, $f_{11} = 1111101011$, $r_{21} = 101011001$, $P_{t_2, a_2}(z) = 11$, $f_{22} = 1$	[37, 28, 4]	Optimal
36, 2, 38	$P_{t_1, a_1}(z) = 11101010110100101010011010101010101$, $f_{11} = 101110111010001$, $r_{21} = 1111100101011$, $P_{t_2, a_2}(z) = 111$, $f_{22} = 1$	[38, 24, 6]	Optimal
38, 2, 40	$P_{t_1, a_1}(z) = 11110110100101101010101010101010101$, $f_{11} = 111110111$, $r_{21} = 10101101$, $P_{t_2, a_2}(z) = 101$, $f_{22} = 11$	[40, 31, 4]	Optimal
38, 2, 40	$P_{t_1, a_1}(z) = 11110110100101101010101010101010101$, $f_{11} = 10101011101111110111$, $r_{21} = 111000100010010001$, $P_{t_2, a_2}(z) = 101$, $f_{22} = 1$	[40, 21, 7]	Near-optimal
40, 2, 42	$P_{t_1, a_1}(z) = 11110110100101101010101010101010101$, $f_{11} = 10010011$, $r_{21} = 101111$, $P_{t_2, a_2}(z) = 101$, $f_{22} = 1$	[42, 35, 4]	Optimal
40, 2, 42	$P_{t_1, a_1}(z) = 11110110100101101010101010101010101$, $f_{11} = 10111100110110110001$, $r_{21} = 1101011101101101111$, $P_{t_2, a_2}(z) = 101$, $f_{22} = 11$	[42, 22, 8]	Optimal, LCD
41, 1, 42	$P_{t_1, a_1}(z) = 11110111010101001110101010101010101001$, $f_{11} = 1001000101011011$, $r_{21} = 111000011001001$, $P_{t_2, a_2}(z) = 11$, $f_{22} = 1$	[42, 27, 6]	Optimal
47, 1, 48	$P_{t_1, a_1}(z) = 10101111010101101011101100101010101010001001$, $f_{11} = 11111100101$, $r_{21} = 1010100011$, $P_{t_2, a_2}(z) = 11$, $f_{22} = 1$	[48, 38, 4]	Optimal
50, 2, 52	$P_{t_1, a_1}(z) = 1111111101010100111011101010101001101010100110111$, $f_{11} = 100010001111$, $r_{21} = 1101011001$, $P_{t_2, a_2}(z) = 111$, $f_{22} = 1$	[52, 41, 4]	Optimal
50, 1, 51	$P_{t_1, a_1}(z) = 1111111101010100111011101010101001101010100110111$, $f_{11} = 100001110001110100111111$, $r_{21} = 11111010000101100010101$, $P_{t_2, a_2}(z) = 11$, $f_{22} = 1$	[51, 28, 8]	Near-optimal
51, 1, 52	$P_{t_1, a_1}(z) = 1111111101010101110101110110010101010101011101001$, $f_{11} = 11111110111011101$, $r_{21} = 1010101101001011$, $P_{t_2, a_2}(z) = 11$, $f_{22} = 1$	[52, 36, 6]	Optimal

5. Conclusions

In this work, we have generalized the concept of QPC and GQC codes over the finite field F_q . Here, we have decomposed the right GQPC codes into linear codes using the Chinese Remainder Theorem and characterized the ρ -generator right GQPC codes (Section 3). Furthermore, we have discussed the normalized generating set of ρ -generator polynomials for the right GQPC codes and their corresponding dimensions and presented a method for finding a normalized generating set from any ρ -generator polynomial set. This way, we have obtained several optimal right GQPC codes (Table 1).

However, it will be interesting to further work on the concatenated structure of the right GQPC code. Also, finding the annihilator dual of these codes would be a promising direction. Furthermore, the construction of quantum codes from these codes remains an open problem.

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