

Article

On the Extended Adjacency Eigenvalues of Graphs and Applications

Hilal A. Ganie ^{1,*}  and Amal Alsululi ^{2,*} ¹ Department of Mathematics, Government Degree College Uri, Srinagar 190001, India² Department of Mathematics, College of Science, University of Bisha, Bisha 61922, Saudi Arabia

* Correspondence: hilahmad1119kt@gmail.com (H.A.G.); alsuloly@ub.edu.sa (A.A.)

Abstract

Let $A_{ex}(\mathcal{G})$ be the extended adjacency matrix of G . The eigenvalues of $A_{ex}(\mathcal{G})$ are called extended adjacency eigenvalues of $\mathcal{G}$. The sum of the absolute values of eigenvalues of the A_{ex} -matrix is called the extended adjacency energy $\mathcal{E}_{ex}(\mathcal{G})$ of $\mathcal{G}$. In this paper, we obtain the A_{ex} -spectrum of the joined union of regular graphs in terms of their adjacency spectrum and the eigenvalues of an auxiliary matrix. Consequently, we derive the A_{ex} -spectrum of the join of two regular graphs, the lexicographic product of regular graphs, and the A_{ex} -spectrum of various families of graphs. Further, as applications of our results, we construct infinite classes of infinite families of extended adjacency equienergetic graphs. We show that the A_{ex} -energy of the join of two regular graphs is greater than or equal to their energy. We also determine the A_{ex} -eigenvalues of the power graph of finite abelian groups.

Keywords: graphs; eigenvalues; extended adjacency eigenvalues; extended adjacency energy; power graph; equienergetic graphs

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1. Introduction

Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a graph, where $V(\mathcal{G}) = \{v_1, v_2, \dots, v_n\}$ is the set of vertices and $E(\mathcal{G})$ is the set of edges in $\mathcal{G}$. A graph $\mathcal{G}$ is called a simple graph if there are no loops or multiple edges. Two vertices u and v are said to be adjacent if there is an edge between them, and non-adjacent otherwise. A graph $\mathcal{G}$ is called connected if any two vertices $v_i, v_j \in V(\mathcal{G})$ can be connected by paths from v_i to v_j . Throughout this paper, we confine ourselves to connected simple graphs. For notions not defined explicitly in this paper, we refer the reader to the standard book [1].

The degree of the vertex $v_i \in V(\mathcal{G})$ is the number of edges adjacent to v_i and is denoted by $d_i = d(v_i)$. The adjacency matrix of a graph $\mathcal{G}$ is $A(\mathcal{G}) = (a_{ij})$, where $a_{ij} = 1$, if v_i and v_j are adjacent and $a_{ij} = 0$, otherwise. The eigenvalues of $A(\mathcal{G})$ are called the eigenvalues of $\mathcal{G}$, and the largest eigenvalue is the spectral radius of $\mathcal{G}$. The energy of $\mathcal{G}$, denoted by $\mathcal{E}(\mathcal{G})$, is defined as the sum of the absolute values of the eigenvalues of $A(\mathcal{G})$; that is,

$$\mathcal{E}(\mathcal{G}) = \sum_{i=1}^n |\lambda_i|,$$

where $\lambda_i, 1 \leq i \leq n$ are the eigenvalues of $A(\mathcal{G})$. The spectral properties, like the bounds for spectral radius and the energy of graphs, are well-studied, and many papers can be

found in the literature with this theme. For some recent papers in this direction and related results, we refer the reader to [2–4].

Yang et al. [5] in 1994 considered an extended version of the adjacency matrix of a graph $\mathcal{G}$ by defining the extended adjacency matrix $A_{ex}(\mathcal{G})$ as a square matrix of order n , whose (i, j) -entry is equal to $\frac{1}{2} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right)$ if v_i is adjacent to v_j , and zero otherwise. The eigenvalues of $A_{ex}(\mathcal{G})$ are called the *extended adjacency eigenvalues* (shortly, A_{ex} -eigenvalues) of $\mathcal{G}$. The set of all A_{ex} -eigenvalues of $\mathcal{G}$, counted with their multiplicities, is called the *extended adjacency spectrum* (shortly, A_{ex} -spectrum). The largest eigenvalue of $A_{ex}(\mathcal{G})$ is called the *extended adjacency spectral radius* (shortly, A_{ex} -spectral radius) of $\mathcal{G}$. The sum of the absolute values of the A_{ex} -eigenvalues is the *extended adjacency energy* $\mathcal{E}_{ex}(\mathcal{G})$ (shortly, A_{ex} -energy) of $\mathcal{G}$. That is,

$$\mathcal{E}_{ex}(\mathcal{G}) = \sum_{i=1}^n |\eta_i|,$$

where $\eta_1, \eta_2, \dots, \eta_n$ are the A_{ex} -eigenvalues of $\mathcal{G}$. Based on the A_{ex} -matrices of molecules, the influences of heteroatoms and multiple bonds were considered in [5]. The authors in [5] have shown that the A_{ex} -spectral radius and energy possess high discriminating power and correlate well with several physicochemical properties and biological activities of organic compounds. Recently, Gutman et al. [6] investigated the dependence of the A_{ex} -energy on molecular structure and established its basic characteristics. Furthermore, in the same paper, they established a relationship between the A_{ex} -energy and the geometric–arithmetic (GA) topological index, and their main finding was that the difference between the A_{ex} -energy and the total π -electron energy is linearly proportional to the difference between the number of edges and the GA index. The investigation of A_{ex} -spectral radius and energy was further extended, and several interesting results were obtained in [7–14].

In [7], the authors considered the problem of constructing graphs with the same A_{ex} -energy and different A_{ex} -spectrum. They showed the existence of graphs with this property. In the rest of this work, we focus on the study of A_{ex} -eigenvalues of the joined union of graphs. The motivation for this study is to devise a general method to generate graphs with the same A_{ex} -energy and different A_{ex} -spectrum. We also focus on the determination of the A_{ex} -eigenvalues of the power graph of finite abelian groups.

The rest of this paper is organized as follows. In Section 2, we obtain the A_{ex} -spectrum of the joined union of regular graphs in terms of the adjacency spectrum of the component graphs and the eigenvalues of an auxiliary matrix. As a consequence, we obtain the A_{ex} -spectrum of the join of two regular graphs and the lexicographic product of a graph with a regular graph. We derive the A_{ex} -spectrum of some well-known families of graphs. In Section 3, we utilize the results obtained in Section 2 to generate some infinite classes of infinite families of graphs with the same A_{ex} -energy and different A_{ex} -spectra. We show that the A_{ex} -energy of the join of two regular graphs is greater than or equal to their energy. In Section 4, we determine the A_{ex} -eigenvalues of the power graph of finite abelian groups.

2. Extended Adjacency Spectrum of Joined Union of Graphs

In this section, we obtain the A_{ex} -spectrum of the joined union of regular graphs. As a consequence, we obtain the A_{ex} -spectrum of the join of two regular graphs and the lexicographic product of a graph with a regular graph. We derive the A_{ex} -spectrum of some well-known families of graphs.

A block matrix is one that has entries as matrices. Consider the block matrix

$$M = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,s} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,s} \\ \vdots & \vdots & \cdots & \vdots \\ A_{s,1} & A_{s,2} & \cdots & A_{s,s} \end{pmatrix},$$

of order n whose rows and columns are partitioned according to a partition $P = \{P_1, P_2, \dots, P_s\}$ of the set $X = \{1, 2, \dots, n\}$. The quotient matrix $\mathcal{Q}$ (see [1,15]) of M is a matrix of order s whose entries are the average row sums of the blocks $A_{i,j}$ of M . The partition P is said to be *equitable* if each block $A_{i,j}$ of M has a constant row (column) sum, and in this case, $\mathcal{Q}$ is called the *equitable quotient matrix*. A vertex partition $\{V_1, V_2, \dots, V_s\}$ of the vertex set $V(\mathcal{G})$ of graph G is equitable if for each i and for all $u, v \in V_i$, we have $|N(u) \cap V_j| = |N(v) \cap V_j|$ for all j . In general, the eigenvalues of $\mathcal{Q}$ interlace the eigenvalues of M . In case the partition is equitable, the following lemma was obtained in [15].

Lemma 1 ([15]). *If the partition P of X of matrix M is equitable, then each eigenvalue of $\mathcal{Q}$ is an eigenvalue of M .*

Let $\mathcal{G} = \mathcal{G}(V, E)$ be a graph of order n , and let $\mathcal{G}_i = \mathcal{G}_i(V_i, E_i)$ be the graphs of order n_i , for $i = 1, \dots, n$. The *joined union* $\mathcal{G}[\mathcal{G}_1, \dots, \mathcal{G}_n]$ of graphs $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$ is the graph $H(W, F)$ with

$$W = \bigcup_{i=1}^n V_i \quad \text{and} \quad F = \bigcup_{i=1}^n E_i \cup \bigcup_{\{v_i, v_j\} \in E} V_i \times V_j.$$

In other words, the joined union is the union of graphs $\mathcal{G}_1, \dots, \mathcal{G}_n$ together with edges $v_{ik}v_{jl}$, where $v_{ik} \in \mathcal{G}_i$ and $v_{jl} \in \mathcal{G}_j$ whenever v_iv_j is an edge in $\mathcal{G}$. Clearly, the usual join of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is a special case of the joined union $K_2[\mathcal{G}_1, \mathcal{G}_2] = \mathcal{G}_1 \vee \mathcal{G}_2$, where K_2 is the complete graph of order 2.

In the following result, we obtain the A_{ex} -spectrum of the joined union of regular graphs $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$ in terms of the adjacency spectrum of the graphs $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$ and the eigenvalues of an auxiliary matrix.

Theorem 1. *Let $\mathcal{G}$ be a graph of order $n \geq 3$ having m edges. Let $\mathcal{G}_i$ be a r_i -regular graph of order n_i having adjacency eigenvalues $\lambda_{i1} = r_i \geq \lambda_{i2} \geq \dots \geq \lambda_{in_i}$, where $i = 1, 2, \dots, n$. The A_{ex} -spectrum of the joined union graph $\mathcal{G}[\mathcal{G}_1, \dots, \mathcal{G}_n]$ of order $N = \sum_{i=1}^n n_i$ consists of the eigenvalues $\lambda_{ik}(\mathcal{G}_i)$ for $i = 1, \dots, n$ and $k = 2, 3, \dots, n_i$. The remaining n eigenvalues are given by the matrix*

$$M = \begin{pmatrix} r_1 & \tau_{12} & \cdots & \tau_{1n} \\ \tau_{21} & r_2 & \cdots & \tau_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{n1} & \tau_{n2} & \cdots & r_n \end{pmatrix},$$

where for $i \neq j$, we have $\tau_{ij} = \frac{n_j}{2} \left(\frac{r_i + c_i}{r_j + c_j} + \frac{r_j + c_j}{r_i + c_i} \right)$, if $v_i \sim v_j$ and equal to 0 if $v_i \not\sim v_j$. Note that $c_i = \sum_{v_j \in N_G(v_i)} n_j$ is the sum of the cardinalities of the graphs $\mathcal{G}_j, j \neq i$ which corresponds to the neighbors of vertex $v_i \in \mathcal{G}$.

Proof. Let $V(\mathcal{G}) = \{v_1, \dots, v_n\}$ be the vertex set of $\mathcal{G}$, and let $V(\mathcal{G}_i) = \{v_{i1}, \dots, v_{in_i}\}$ be the vertex set of $\mathcal{G}_i$ for $i = 1, 2, \dots, n$. Let $H = \mathcal{G}[\mathcal{G}_1, \dots, \mathcal{G}_n]$ be the joined union of the

graphs $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$. Clearly, the order of H is $N = \sum_{i=1}^n n_i$. We first compute the degree of each vertex in $V(H)$. Evidently, the degree of each vertex $v_{ij} \in V(H)$ for $1 \leq i \leq n$ and $1 \leq j \leq n_i$ is the degree inside $\mathcal{G}_i$ plus the sum of cardinalities of the copies of $\mathcal{G}_j, j \neq i$, which corresponds to the neighbors of the vertex v_i in $\mathcal{G}$. That is, for each $v_{ij} \in V(\mathcal{G}_i)$, we have

$$d_H(v_{ij}) = r_i + \sum_{v_j \in N_G(v_i)} n_j = r_i + c_i, \quad (1)$$

where $c_i = \sum_{v_j \in N_G(v_i)} n_j$. Obviously, $d_H(v_{ij})$ is the same for each vertex in $\mathcal{G}_i, 1 \leq j \leq n_i$.

We label the vertices in H from the vertices in $\mathcal{G}_1$ to the vertices in $\mathcal{G}_n$. With this labeling, the A_{ex} -matrix of H can be written as

$$A_{ex}(H) = \begin{pmatrix} A(\mathcal{G}_1) & h(v_1, v_2) & \dots & h(v_1, v_n) \\ h(v_2, v_1) & A(\mathcal{G}_2) & \dots & h(v_2, v_n) \\ \vdots & \vdots & \ddots & \vdots \\ h(v_n, v_1) & h(v_n, v_2) & \dots & A(\mathcal{G}_n) \end{pmatrix},$$

where $h(v_i, v_j) = \frac{1}{2} \left(\frac{r_i + c_i}{r_j + c_j} + \frac{r_j + c_j}{r_i + c_i} \right) J_{n_i \times n_j}$ if $v_i \sim v_j$ in $\mathcal{G}$ and equal to $0_{n_i \times n_j}$ otherwise. Note that when $A(\mathcal{G}_i)$ is the adjacency matrix of $\mathcal{G}_i$, $J_{n_i \times n_j}$ is the all one-matrix and $0_{n_i \times n_j}$ is the zero matrix of order $n_i \times n_j$.

Since the graph $\mathcal{G}_i$ is a r_i -regular graph of order n_i , for all $i = 1, 2, \dots, n$, the all one-vector $e_{n_i} = (1, 1, \dots, 1)^T$ is the eigenvector of the adjacency matrix $A(\mathcal{G}_i)$ corresponding to the eigenvalue r_i , and all other eigenvectors are orthogonal to e_{n_i} (as the matrix $A(\mathcal{G}_i)$ being symmetric is orthogonally diagonalizable). Let λ_{ik} , where $2 \leq k \leq n_i$, be any eigenvalue of $A(\mathcal{G}_i)$ with the corresponding eigenvector $X = (x_{i1}, x_{i2}, \dots, x_{in_i})^T$ satisfying $e_{n_i}^T X = 0$. Clearly, the column vector X can be regarded as a function defined on $V(\mathcal{G}_i)$ assigning the vertex v_{ij} to x_{ij} ; that is, $X(v_{ij}) = x_{ij}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n_i$. Now, consider the vector $Z = (z_1, z_2, \dots, z_N)^T$, where

$$z_j = \begin{cases} x_{ij} & \text{if } v_{ij} \in V(\mathcal{G}_i) \\ 0 & \text{otherwise.} \end{cases}$$

Since $e_{n_i}^T X = 0$ and the coordinates of the vector Z corresponding to the vertices in $\cup_{j \neq i} V_j$ of H are zero, we have

$$A_{ex}(H)Z = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_{ik}X \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_{ik}Z.$$

This shows that Z is an eigenvector of $A_{ex}(H)$ corresponding to the eigenvalue λ_{ik} for every eigenvalue λ_{ik} , where $2 \leq k \leq n_i$, of $A(\mathcal{G}_i)$. From this, it follows that for $1 \leq i \leq n$ and $2 \leq k \leq n_i$, λ_{ik} is an eigenvalue of $A_{ex}(H)$. In this way, we obtain the $\sum_{i=1}^n n_i - n = N - n$

eigenvalues of $A_{ex}(H)$. The remaining n eigenvalues are the zeros of the characteristic polynomial of the following equitable quotient matrix:

$$M = \begin{pmatrix} r_1 & \tau_{12} & \dots & \tau_{1n} \\ \tau_{21} & r_2 & \dots & \tau_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{n1} & \tau_{n2} & \dots & r_n \end{pmatrix},$$

where for $i \neq j$, we have $\tau_{ij} = \frac{n_j}{2} \left(\frac{r_i + c_i}{r_j + c_j} + \frac{r_j + c_j}{r_i + c_i} \right)$ if $v_i \sim v_j$ and equal to 0 if $v_i \not\sim v_j$. This completes the proof. $\square$

Example 1. Consider the graph $H_1 = P_3[K_3, C_4, P_2]$, as shown in Figure 1. For this graph, $n = 3, n_1 = 3, n_2 = 4, n_3 = 2, \mathcal{G} = P_3, \mathcal{G}_1 = K_3, \mathcal{G}_2 = C_4, \mathcal{G}_3 = P_2, r_1 = 2, r_2 = 2, r_3 = 1, N = 9, c_1 = 4, c_2 = 5, c_3 = 4$. Also, the adjacency eigenvalues of K_3 are $r_1 = 2, \lambda_{12} = -1, \lambda_{13} = -1$; the adjacency eigenvalues of C_4 are $r_2 = 2, \lambda_{22} = 0, \lambda_{23} = 0, \lambda_{24} = -2$; and the adjacency eigenvalues of P_2 are $r_3 = 1, \lambda_{32} = -1$. Further, $\tau_{12} = \frac{n_2}{2} \left(\frac{r_1 + c_1}{r_2 + c_2} + \frac{r_2 + c_2}{r_1 + c_1} \right) = \frac{85}{21}, \tau_{21} = \frac{n_1}{2} \left(\frac{r_1 + c_1}{r_2 + c_2} + \frac{r_2 + c_2}{r_1 + c_1} \right) = \frac{85}{28}, \tau_{13} = 0 = \tau_{31}, \tau_{23} = \frac{n_3}{2} \left(\frac{r_2 + c_2}{r_3 + c_3} + \frac{r_3 + c_3}{r_2 + c_2} \right) = \frac{74}{35}, \tau_{32} = \frac{n_2}{2} \left(\frac{r_2 + c_2}{r_3 + c_3} + \frac{r_3 + c_3}{r_2 + c_2} \right) = \frac{148}{35}$. Therefore, based on Theorem 1, the A_{ex} -spectrum of $H_1 = P_3[K_3, C_4, P_2]$ consists of the eigenvalues $\lambda_{12} = -1, \lambda_{13} = -1, \lambda_{22} = 0, \lambda_{23} = 0, \lambda_{24} = -2, \lambda_{32} = -1$. The remaining three A_{ex} -eigenvalues are the eigenvalues of the matrix $M = \begin{pmatrix} 2 & \frac{85}{21} & 0 \\ \frac{85}{28} & 2 & \frac{74}{35} \\ 0 & \frac{148}{35} & 1 \end{pmatrix}$. Via direct calculation, it can be seen that the eigenvalues of M are 6.42490599, 1.42776319, and -2.85266918 (approximately).

The following consequence of Theorem 1 gives the A_{ex} -spectrum of the join of two regular graphs. Note that Corollary 1 was obtained in [7] as Theorem 3.7.

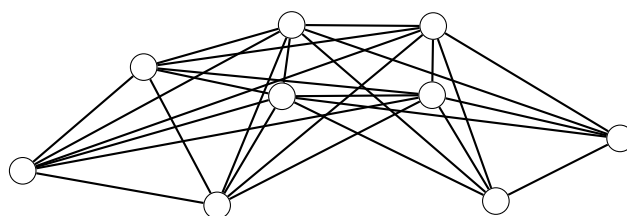


Figure 1. The joined union $P_3[K_3, C_4, P_2]$ of the complete graph K_3 , the cycle C_4 , and the path P_2 with the parent graph P_3 .

Corollary 1. Let $\mathcal{G}_i$ be an r_i -regular graph of order n_i for $i = 1, 2$, and let λ_{ik} , where $2 \leq k \leq n_i$ and $i = 1, 2$ be the adjacency eigenvalues of $\mathcal{G}_i$ other than r_i . Then, the A_{ex} -spectrum of $\mathcal{G} = \mathcal{G}_1 \vee \mathcal{G}_2$ consists of eigenvalues $\lambda_{1k}A(\mathcal{G}_1)$, for $k = 2, \dots, n_1$ and eigenvalues $\lambda_{2k}A(\mathcal{G}_2)$, for $k = 2, \dots, n_2$. The remaining two eigenvalues are given by the matrix

$$\begin{pmatrix} r_1 & \frac{n_2}{2} \left(\frac{r_1 + n_2}{r_2 + n_1} + \frac{r_2 + n_1}{r_1 + n_2} \right) \\ \frac{n_1}{2} \left(\frac{r_1 + n_2}{r_2 + n_1} + \frac{r_2 + n_1}{r_1 + n_2} \right) & r_2 \end{pmatrix},$$

which are $\frac{1}{2} \left(r_1 + r_2 \pm \sqrt{(r_1 - r_2)^2 + n_1 n_2 \left(\frac{r_1 + n_2}{r_2 + n_1} + \frac{r_2 + n_1}{r_1 + n_2} \right)^2} \right)$.

The following observation gives the A_{ex} -spectrum of the join of a regular graph with the union of two regular graphs of distinct degrees.

Corollary 2. Let $\mathcal{G}_i$ be a r_i -regular graph of orders n_i that has adjacency eigenvalues $r_i = \lambda_{i1} \geq \lambda_{i2} \geq \dots \geq \lambda_{in_i}$ for $i = 1, 2, 3$. Then, the A_{ex} -spectrum of $\mathcal{G} = \mathcal{G}_1 \vee (\mathcal{G}_2 \cup \mathcal{G}_3)$ consists of the eigenvalues $\lambda_{1k}(A(\mathcal{G}_1))$ for $k = 2, \dots, n_1$; the eigenvalues $\lambda_{2k}(A(\mathcal{G}_2))$ for $k = 2, \dots, n_2$; and the eigenvalues $\lambda_{3k}(A(\mathcal{G}_3))$ for $k = 2, \dots, n_3$. The remaining three eigenvalues are given by the following matrix:

$$\begin{pmatrix} r_2 & \frac{n_1}{2} \left(\frac{r_2+n_1}{r_1+n_2+n_3} + \frac{r_1+n_2+n_3}{r_2+n_1} \right) & 0 \\ \frac{n_2}{2} \left(\frac{r_1+n_2+n_3}{r_2+n_1} + \frac{r_2+n_1}{r_1+n_2+n_3} \right) & r_1 & \frac{n_3}{2} \left(\frac{r_1+n_2+n_3}{r_3+n_1} + \frac{r_3+n_1}{r_1+n_2+n_3} \right) \\ 0 & \frac{n_1}{2} \left(\frac{r_3+n_1}{r_1+n_2+n_3} + \frac{r_1+n_2+n_3}{r_3+n_1} \right) & r_3 \end{pmatrix}.$$

Proof. Let P_3 be the path on 3 vertices, and let $\mathcal{G} = \mathcal{G}_1 \vee (\mathcal{G}_2 \cup \mathcal{G}_3)$ be the join of the graphs $\mathcal{G}_1$ and $\mathcal{G}_2 \cup \mathcal{G}_3$. It is easy to see that $\mathcal{G} = P_3[\mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_3]$; that is, $\mathcal{G}$ is the joined union of the graphs $\mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_3$ when the parent graph is the path P_3 . Now, using Theorem 1 and noting that $c_2 = n_1, c_1 = n_2 + n_3$, and $c_3 = n_1$, the result follows. $\square$

Example 2. Consider the graph $H_2 = K_3 \vee (C_4 \cup P_2)$ shown in Figure 2. For this graph, $\mathcal{G}_1 = K_3, \mathcal{G}_2 = C_4, \mathcal{G}_3 = P_2, n_1 = 3, n_2 = 4, n_3 = 2, r_1 = 2, r_2 = 2, r_3 = 1$. Also, the adjacency eigenvalues of K_3 are $r_1 = 2, \lambda_{12} = -1, \lambda_{13} = -1$; the adjacency eigenvalues of C_4 are $r_2 = 2, \lambda_{22} = 0, \lambda_{23} = 0, \lambda_{24} = -2$; and the adjacency eigenvalues of P_2 are $r_3 = 1, \lambda_{32} = -1$. Therefore, based on Corollary 2, the A_{ex} -eigenvalues of $H_2 = K_3 \vee (C_4 \cup P_2)$ consist of the eigenvalues $\lambda_{12} = -1, \lambda_{13} = -1, \lambda_{22} = 0, \lambda_{23} = 0, \lambda_{24} = -2, \lambda_{32} = -1$. The remaining three

A_{ex} -eigenvalues of H_2 are the eigenvalues of the following matrix: $\begin{pmatrix} 2 & \frac{111}{35} & 0 \\ \frac{148}{35} & 2 & \frac{5}{2} \\ 0 & \frac{15}{4} & 1 \end{pmatrix}$. Through direct calculation, it can be seen that the eigenvalues of this matrix are 6.59454826, 1.41765991, and -3.01220817 (approximately).

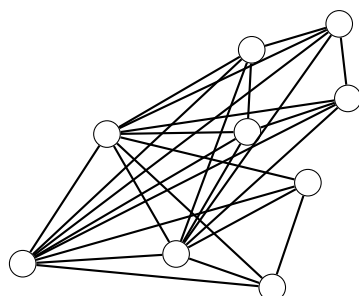


Figure 2. The graph $K_3 \vee (C_4 \cup P_2) = P_3[C_4, K_3, P_2]$.

The lexicographic product $\mathcal{G}[\mathcal{H}]$ of graphs $\mathcal{G}$ and $\mathcal{H}$ is the graph with vertex set $V(\mathcal{G}) \times V(\mathcal{H})$ and edge $(a, x)(b, y) \in E(\mathcal{G}[\mathcal{H}])$ whenever $ab \in E(\mathcal{G})$, or $a = b$ and $xy \in E(\mathcal{H})$. It is interesting to note that the lexicographic product $\mathcal{G}[\mathcal{H}]$ can be constructed by joined union $\mathcal{G}[\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n]$ where $\mathcal{G}_i = \mathcal{H}$ for $1 \leq i \leq n$. Note that in the case of $\mathcal{G}_i = K_1$, we get $\mathcal{G}[K_1, K_1, \dots, K_1] = \mathcal{G}$, which means that every graph $\mathcal{G}$ is the joined union of some classes of graphs. Taking $\mathcal{G}_1 = \mathcal{G}_2 = \dots = \mathcal{G}_n$ in Theorem 1, we obtain the following result, which gives the A_{ex} -spectrum of the lexicographic product $\mathcal{G}[\mathcal{G}_1]$ of a graph $\mathcal{G}$ with a regular graph $\mathcal{G}_1$.

Theorem 2. Let $\mathcal{G}$ be a graph of order $n \geq 3$ having m edges. Let $\mathcal{G}_1$ be a r_1 -regular graph of order n_1 having adjacency eigenvalues $\lambda_1 = r_1 \geq \lambda_2 \geq \dots \geq \lambda_{n_1}$. The A_{ex} -spectrum of $\mathcal{G}[\mathcal{G}_1]$

consists of the eigenvalues $\lambda_k(\mathcal{G}_1)$, where $k = 2, 3, \dots, n_1$, each repeated n times. The remaining n eigenvalues are given by the matrix

$$M = \begin{pmatrix} r_1 & \tau_{12} & \dots & \tau_{1n} \\ \tau_{21} & r_1 & \dots & \tau_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{n1} & \tau_{n2} & \dots & r_1 \end{pmatrix},$$

where for $i \neq j$, we have $\tau_{ij} = \frac{n_1}{2} \left(\frac{r_1 + n_1 d(v_i)}{r_1 + n_1 d(v_j)} + \frac{r_1 + n_1 d(v_j)}{r_1 + n_1 d(v_i)} \right)$ if $v_i \sim v_j$, and equal to 0 if $v_i \not\sim v_j$. Note that $d(v_i)$ is the degree of vertex v_i in $\mathcal{G}$.

Proof. The proof follows directly from Theorem 1 by taking $\mathcal{G}_1 = \mathcal{G}_2 = \dots = \mathcal{G}_n$; $n_1 = n_2 = \dots = n_n$; $r_1 = r_2 = \dots = r_n$ and by using the fact that $c_i = \sum_{v_j \in N_{\mathcal{G}}(v_i)} n_j = n_1 \sum_{v_j \in N_{\mathcal{G}}(v_i)} 1 = n_1 d(v_i)$ together with

$$\tau_{ij} = \frac{n_j}{2} \left(\frac{r_i + c_i}{r_j + c_j} + \frac{r_j + c_j}{r_i + c_i} \right) = \frac{n_1}{2} \left(\frac{r_1 + n_1 d(v_i)}{r_1 + n_1 d(v_j)} + \frac{r_1 + n_1 d(v_j)}{r_1 + n_1 d(v_i)} \right),$$

if $v_i \sim v_j$. $\square$

Example 3. Consider the graph $H_3 = P_3[K_3] = P_3[K_3, K_3, K_3]$, as shown in Figure 3. Let v_1, v_2 and v_3 be the vertices of P_3 with $d(v_1) = 1, d(v_2) = 2$ and $d(v_3) = 1$. For the graph H_3 , we have $\mathcal{G} = P_3, \mathcal{G}_1 = K_3, n = 3, n_1 = 3, r_1 = 2$. Also, the adjacency eigenvalues of K_3 are $r_1 = 2, -1, -1$. Further, $\tau_{12} = \frac{n_1}{2} \left(\frac{r_1 + n_1 d(v_1)}{r_1 + n_1 d(v_2)} + \frac{r_1 + n_1 d(v_2)}{r_1 + n_1 d(v_1)} \right) = \frac{267}{80} = \tau_{21} = \tau_{32} = \tau_{23}$. Therefore, based on Theorem 2, the A_{ex} -eigenvalues of H_3 consist of the eigenvalue -1 repeated six times. The remaining three A_{ex} -eigenvalues are the eigenvalues of the following matrix:

$$\begin{pmatrix} 2 & \frac{267}{80} & 0 \\ \frac{267}{80} & 2 & \frac{267}{80} \\ 0 & \frac{267}{80} & 2 \end{pmatrix}.$$

Through direct calculation, it can be seen that the eigenvalues of this matrix are $2 + \frac{267\sqrt{2}}{80} \approx 6.7199377644, 2 - \frac{267\sqrt{2}}{80} \approx -2.71993776$, and 2.

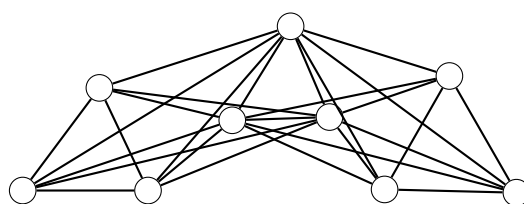


Figure 3. The lexicographic product $P_3[K_3] = P_3[K_3, K_3, K_3]$ of the path P_3 with the regular graph K_3 .

It is clear that the matrix M given in Theorem 2 is completely determined by the graph $\mathcal{G}$. In particular, taking $\mathcal{G}_1 = \overline{K}_{n_1}$, an empty graph on n_1 vertices in Theorem 2, we get the following consequence.

Corollary 3. Let $\mathcal{G}$ be a graph of order $n \geq 3$ having m edges. Let $\overline{K}_{n_1}$ be an empty graph on n_1 vertices. Then, the A_{ex} -spectrum of the $\mathcal{G}[\overline{K}_{n_1}]$ consists of the eigenvalues 0, with multiplicity $n(n_1 - 1)$ times and the n eigenvalues $n_1\eta_1, n_1\eta_2, \dots, n_1\eta_n$, where $\eta_1 \geq \eta_2 \geq \dots \geq \eta_n$ are the A_{ex} -eigenvalues of $\mathcal{G}$.

Proof. The proof follows directly from Theorem 2 by taking $\mathcal{G}_1 = \overline{K}_{n_1}$, $r_1 = 0$, $\lambda_k(\mathcal{G}_1) = 0$, for all k and using the fact that $\tau_{ij} = \frac{n_1}{2} \left(\frac{n_1 d(v_i)}{n_1 d(v_j)} + \frac{n_1 d(v_j)}{n_1 d(v_i)} \right) = \frac{n_1}{2} \left(\frac{d(v_i)}{d(v_j)} + \frac{d(v_j)}{d(v_i)} \right)$ if $v_i \sim v_j$, and equal to 0 if $v_i \not\sim v_j$. $\square$

In particular, by taking $\mathcal{G} = K_n$, the complete graph on n vertices, we obtain the following result as a direct consequence of Theorem 1.

Corollary 4. Let K_n be the complete graph of order $n \geq 3$. Let $\mathcal{G}_i$ be a r_i -regular graph of order n_i having adjacency eigenvalues $\lambda_{i1} = r_i \geq \lambda_{i2} \geq \dots \geq \lambda_{in_i}$, where $i = 1, 2, \dots, n$. The A_{ex} -spectrum of the joined union graph $K_n[\mathcal{G}_1, \dots, \mathcal{G}_n]$ of order $N = \sum_{i=1}^n n_i$ consists of the eigenvalues $\lambda_{ik}(\mathcal{G}_i)$ for $i = 1, \dots, n$ and $k = 2, 3, \dots, n_i$. The remaining n eigenvalues are given by matrix M in Theorem 1 with $\tau_{ij} = \frac{n_j}{2} \left(\frac{N+r_i-n_i}{N+r_j-n_j} + \frac{N+r_j-n_j}{N+r_i-n_i} \right)$, if $v_i \sim v_j$, and equal to 0 if $v_i \not\sim v_j$.

We observe that the complete p -partite graph $K_{n_1, n_2, \dots, n_p}$ is the joined union of graphs $\mathcal{G}_i = \overline{K}_{n_i}$, where the underlying graph is K_p . Therefore, $K_{n_1, n_2, \dots, n_p} \cong K_p[\overline{K}_{n_1}, \overline{K}_{n_2}, \dots, \overline{K}_{n_p}]$. Taking $\mathcal{G}_i = \overline{K}_{n_i}$ in Corollary 4, we get the A_{ex} -spectrum of the complete p -partite graph $K_{n_1, n_2, \dots, n_p}$, which consists of the eigenvalue 0 with multiplicity $N - n$, where $N = \sum_{k=1}^p n_k$. The remaining p eigenvalues are given by matrix M in Theorem 1 with $r_i = 0$, $\tau_{ij} = \frac{n_j}{2} \left(\frac{N-n_i}{N-n_j} + \frac{N-n_j}{N-n_i} \right)$, if $v_i \sim v_j$, and equal to 0 if $v_i \not\sim v_j$.

Let $K_{a,b}$ be the complete bipartite graph with partite sets of cardinality a and b , where $a + b = n$, clearly $K_{a,b} = \overline{K}_a \vee \overline{K}_b$. A complete split graph, denoted by $CS_{\omega, n-\omega}$, is a graph consisting of a clique on ω vertices and an independent set (a subset of vertices of a graph is said to be an independent set if the subgraph induced by them is an empty graph) on the remaining $n - \omega$ vertices such that each vertex of the clique is adjacent to every vertex of the independent set. We note that $CS_{\omega, n-\omega} = K_{\omega} \vee \overline{K}_{n-\omega}$. Similarly, the cone graph is given by $C_{a,b} = C_a \vee \overline{K}_b$, and for $b = 1$, we have wheel graph $W_{n+1} = C_n \vee K_1$ on $n + 1$ vertices. As a special case of Corollary 1, we compute the A_{ex} -spectrum of the complete bipartite graph $K_{a,b}$, the complete split graph $CS_{\omega, n-\omega}$, the cone graph $C_{a,b}$, and the wheel graph W_{n+1} in the following result.

- Corollary 5.** (i) The A_{ex} -spectrum of $K_{a,b}$ consists of the eigenvalue 0 with multiplicity $a + b - 2$ and the eigenvalues $\pm \frac{a^2 + b^2}{2\sqrt{ab}}$.
- (ii) The A_{ex} -spectrum of $CS_{\omega, n-\omega}$ consists of the eigenvalue -1 with multiplicity $\omega - 1$, the eigenvalue 0 with multiplicity $n - \omega - 1$, and the eigenvalues $\frac{1}{2} \left(\omega - 1 \pm \sqrt{(\omega - 1)^2 + \omega(n - \omega) \left(\frac{n-1}{\omega} + \frac{\omega}{n-1} \right)^2} \right)$.
- (iii) The A_{ex} -spectrum of $C_{a,b}$ consists of the eigenvalues $2 \cos \left(\frac{2\pi k}{b} \right)$, where $k = 2, \dots, a - 1$, the eigenvalue 0 with multiplicity $b - 1$, and the eigenvalues $\frac{1}{2} \left(2 \pm \sqrt{4 + ab \left(\frac{b+2}{a} + \frac{a}{b+2} \right)^2} \right)$.
- (iv) The A_{ex} -spectrum of W_{n+1} consists of the eigenvalues $2 \cos \left(\frac{2\pi k}{n} \right)$, where $k = 2, \dots, n - 1$, and the eigenvalues $\frac{1}{2} \left(2 \pm \sqrt{4 + n \left(\frac{3}{n} + \frac{n}{3} \right)^2} \right)$.

Proof. The proofs of (i)–(iv) follow directly from Corollary 1 using the fact that the adjacency spectrum of K_ω consists of eigenvalue $\omega - 1$ and the eigenvalue -1 with multiplicity $\omega - 1$; the adjacency spectrum of $\bar{K}_t$ consists of eigenvalue 0 with multiplicity t , and the adjacency spectrum (see [1]) of C_n is $\left\{2 \cos\left(\frac{2\pi k}{n}\right) : k = 1, 2, \dots, n\right\}$. $\square$

A *friendship* graph F_n is a graph of order $2n + 1$, obtained by joining K_1 with n copies of K_2 ; that is, $F_n = K_1 \vee (nK_2) = K_{1,n}[K_1, \underbrace{K_2, K_2, \dots, K_2}_n]$, where K_1 corresponds to the root vertex (vertex of degree greater than one) in $K_{1,n}$. In particular, by replacing some K_2 s with K_1 s in F_n , we get a *firefly*-type graph denoted by $F_{p,n-p}$, written as follows:

$$F_{p,n-p} = K_{1,n}[K_1, \underbrace{K_1, K_1, \dots, K_1}_p, \underbrace{K_2, K_2, \dots, K_2}_{n-p}].$$

The A_{ex} -spectrum of the *friendship* graph F_n and the *firefly*-type graph $F_{p,n-p}$ are given by the following result.

- Corollary 6.** (i) The A_{ex} -spectrum of F_n consists of the eigenvalue -1 with multiplicity n , the eigenvalue 1 with multiplicity $n - 1$, and the eigenvalues $\frac{1}{2}\left(1 \pm \sqrt{1 + \frac{2(n^2+1)^2}{n}}\right)$.
- (ii) The A_{ex} -spectrum of $F_{p,n-p}$ consists of the eigenvalue 0 with multiplicity $p - 1$, the eigenvalue -1 with multiplicity $n - p$, the eigenvalue 1 with multiplicity $n - p - 1$, and the three eigenvalues of the matrix M given in Corollary 2 with $r_1 = 0, r_2 = 0, r_3 = 1, n_1 = 1, n_2 = p$, and $n_3 = 2(n - p)$.

Proof. (i) follows from Corollary 1 by taking $r_1 = 0, r_2 = 1, n_1 = 1$, and $n_2 = 2n$. Similarly, (ii) follows from Corollary 2 by using the fact that $F_{p,n-p} = K_1 \vee (pK_1 \cup (n - p)K_2)$. $\square$

3. Extended Adjacency Energy

In this section, by using the results obtained in Section 2, we construct some new infinite families of non-cospectral extended adjacency equienergetic graphs.

Two graphs of the same order are said to be equienergetic if they are non-cospectral and have the same energy. The problem of constructing non-cospectral graphs that have the same energy is an active component of present research, and several papers have been published in this direction. For some recent works, we refer the reader to [16,17] and the references therein.

The authors in [7] have extended the concept of equienergetic graphs to the A_{ex} -matrix and defined the extended adjacency equienergetic graphs. Two graphs $\mathcal{G}$ and $\mathcal{H}$ of the same order are said to be extended adjacency equienergetic if they are non-cospectral and have the same A_{ex} -energy, that is, $\mathcal{E}_{ex}(\mathcal{G}) = \mathcal{E}_{ex}(\mathcal{H})$. It is clear that if two graphs are extended adjacency cospectral, they are extended adjacency equienergetic. Therefore, in what follows, we will be interested in finding extended adjacency equienergetic non-cospectral graphs. In fact, we will address the following problem for A_{ex} -eigenvalues.

Problem 1. How can families of non-isomorphic graphs be constructed such that they have equal A_{ex} -energy, but they do not have the same A_{ex} -spectrum?

It is clear that the A_{ex} -energy $\mathcal{E}_{ex}(\mathcal{G})$ of a graph $\mathcal{G}$ represents the trace norm of the matrix $A_{ex}(\mathcal{G})$. Note that the trace norm of a n -square matrix M is defined as $\|M\|_* = \sum_{i=1}^n \sigma_i(M)$, where $\sigma_1(M) \geq \sigma_2(M) \geq \dots \geq \sigma_n(M)$ are the singular values of M (i.e., the square roots of the eigenvalues of MM^* , where M^* is the complex conjugate of M). Therefore, the study of this spectral graph invariant is not only interesting and important

from a spectral graph theory point of view but is also important from a matrix theory point of view.

In the language of matrix theory, the above problem forms a part of the following interesting but hard problem.

Problem 2. *How can matrices of order n with the same trace norm but a different spectrum be characterized?*

For matrices associated with graphs, a large number of families of graphs are known with equal trace norm (energy) but different spectra, see [18]. In general, characterizing the square matrices of order n with equal trace norm is not easy. However, for matrices associated with graphs (called graph matrices), some sufficient conditions are known that guarantee their existence.

Problem 1 was addressed by Adiga et al. in [7], and some infinite families of non-isomorphic non-cospectral extended adjacency equienergetic graphs were constructed. The main tool used by the authors in [7] for the construction of such families was the following operations: join of graphs, double graph of a graph, and the duplication of a graph. In the rest of this section, we aim to construct new infinite families of non-cospectral extended adjacency equienergetic graphs. We will devise a general method to generate infinite classes of infinite families of non-cospectral extended adjacency equienergetic graphs.

The following result gives the A_{ex} -energy of the joined union $\mathcal{G}[\mathcal{G}_1, \dots, \mathcal{G}_n]$ of regular graphs $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$.

Theorem 3. *Let $\mathcal{G}$ be a graph of order $n \geq 2$ with m edges. Let $\mathcal{G}_i$ be a r_i -regular graph of order n_i having energy $\mathcal{E}(\mathcal{G}_i)$ for all $i = 1, 2, \dots, n$. Then,*

$$\mathcal{E}_{ex}(\mathcal{G}[\mathcal{G}_1, \dots, \mathcal{G}_n]) = \sum_{i=1}^n \mathcal{E}(\mathcal{G}_i) - \sum_{i=1}^n r_i + \sum_{i=1}^n |x_i|,$$

where $x_1, x_2, \dots, x_n$ are the eigenvalues of the matrix M given in Theorem 1.

Proof. From Theorem 1, it is clear that if r_i, λ_{ik} , where $k = 2, 3, \dots, n_i$ are the adjacency eigenvalues of $\mathcal{G}_i$, for $1 \leq i \leq n$, then the A_{ex} -eigenvalues of graph $\mathcal{G}[\mathcal{G}_1, \dots, \mathcal{G}_n]$ are λ_{ik} for $k = 2, 3, \dots, n_i$ and $i = 1, 2, \dots, n$. The remaining n eigenvalues are the eigenvalues of the matrix M . Now, using the definition of A_{ex} -energy, the result follows. $\square$

Example 4. Consider the graph $H_1 = P_3[K_3, C_4, P_2]$, as shown in Figure 1. In Example 1, we have found the A_{ex} -eigenvalues of the graph H_1 , which are -1 repeated 3 times, 0 repeated 2 times, -2 repeated once, and the three eigenvalues 6.42490599 , 1.42776319 , and -2.85266918 of the matrix M . Therefore, based on the definition of A_{ex} -energy, we have

$$\begin{aligned} \mathcal{E}_{ex}(H_1) &= \mathcal{E}_{ex}(P_3[K_3, C_4, P_2]) \\ &= 3|-1| + 2|0| + |-2| + |6.42490599| + |1.42776319| + |-2.85266918| \\ &= 15.70533836. \end{aligned}$$

Also, K_3 is a $r_1 = 2$ regular graph with adjacency eigenvalues $2, -1, -1$; C_4 is a $r_2 = 2$ regular graph with adjacency eigenvalues $2, 0, 0, -2$; and P_2 is a $r_3 = 1$ regular graph with adjacency

eigenvalues $1, -1$. Therefore, $\mathcal{E}(K_3) = 4, \mathcal{E}(C_4) = 4, \mathcal{E}(P_2) = 2$ and $\sum_{i=1}^3 r_i = r_1 + r_2 + r_3 = 5$. Therefore, based on Theorem 3, we have

$$\begin{aligned}\mathcal{E}_{ex}(H_1) &= \mathcal{E}_{ex}(P_3[K_3, C_4, P_2]) = \mathcal{E}(K_3) + \mathcal{E}(C_4) + \mathcal{E}(P_2) - \sum_{i=1}^3 r_i + \sum_{i=1}^3 |x_i| \\ &= 5 + \sum_{i=1}^3 |x_i|,\end{aligned}$$

where x_1, x_2, x_3 are the eigenvalues of matrix M . Inputting $x_1 = 6.42490599, x_2 = 1.42776319$, and $x_3 = -2.85266918$, we see that Theorem 3 also gives $\mathcal{E}_{ex}(H_1) = 15.70533836$.

Taking in particular $\mathcal{G} = K_2$ in Theorem 3 and using Theorem 1, we obtain the following result, which gives the A_{ex} -energy of the join of two regular graphs $\mathcal{G}_1$ and $\mathcal{G}_2$.

Corollary 7. For $i = 1, 2$, let $\mathcal{G}_i$ be a r_i -regular graph of order n_i . Then,

$$\mathcal{E}_{ex}(\mathcal{G}_1 \vee \mathcal{G}_2) = \mathcal{E}(\mathcal{G}_1) + \mathcal{E}(\mathcal{G}_2) + (|x_1| + |x_2| - r_1 - r_2),$$

where x_1, x_2 are $\frac{1}{2} \left(r_1 + r_2 \pm \sqrt{(r_1 - r_2)^2 + n_1 n_2 \left(\frac{r_1 + n_2}{r_2 + n_1} + \frac{r_2 + n_1}{r_1 + n_2} \right)^2} \right)$.

Example 5. Consider the graph $H_4 = K_3 \vee C_4$, as shown in Figure 4. For this graph, we have $n_1 = 3, n_2 = 4, r_1 = 2, r_2 = 2$. Also, the adjacency eigenvalues of K_3 are $2, -1, -1$, and the adjacency eigenvalues of C_4 are $2, 0, 0, -2$. Using Corollary 1, it follows that the A_{ex} -eigenvalues of H_4 are $-1, -1, 0, 0, -2$. The remaining two A_{ex} -eigenvalues are the eigenvalues of the matrix $\begin{pmatrix} 2 & \frac{61}{15} \\ \frac{61}{20} & 2 \end{pmatrix}$. Through direct calculation, it can be seen that the eigenvalues of this matrix are $2 + \frac{61\sqrt{3}}{30} \approx 5.52183664$ and $2 - \frac{61\sqrt{3}}{30} \approx -1.52183664$. Therefore, based on the definition of A_{ex} -energy, we have

$$\begin{aligned}\mathcal{E}_{ex}(H_4) &= \mathcal{E}_{ex}(K_3 \vee C_4) \\ &= 2|-1| + 2|0| + |-2| + |5.52183664| + |-1.52183664| \\ &= 11.04367328.\end{aligned}$$

Again, $\mathcal{E}(K_3) = 4, \mathcal{E}(C_4) = 4$, and $r_1 = r_2 = 2$. It follows from Corollary 7 that

$$\begin{aligned}\mathcal{E}_{ex}(H_4) &= \mathcal{E}_{ex}(K_3 \vee C_4) = \mathcal{E}(K_3) + \mathcal{E}(C_4) - (r_1 + r_2) + |x_1| + |x_2| \\ &= 4 + 4 - 4 + |5.52183664| + |-1.52183664| = 11.04367328.\end{aligned}$$

This shows that the value of A_{ex} -energy obtained through direct computation is in agreement with the value given by Corollary 7.

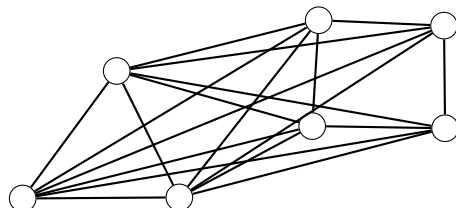


Figure 4. The join of graphs K_3 and C_4 .

Taking in particular $\mathcal{G}_1 = \mathcal{G}_2 = \dots = \mathcal{G}_n$ in Theorem 3, we obtain the following result, which gives the A_{ex} -energy of the lexicographic product $\mathcal{G}[\mathcal{G}_1] = \mathcal{G}[\mathcal{G}_1, \dots, \mathcal{G}_1]$ of $\mathcal{G}$ and $\mathcal{G}_1$, where $\mathcal{G}_1$ is a regular graph.

Corollary 8. Let $\mathcal{G}$ be a graph of order $n \geq 2$ with m edges. Let $\mathcal{G}_1$ be a r_1 -regular graph with energy $\mathcal{E}(\mathcal{G}_1)$. Then,

$$\mathcal{E}_{ex}(\mathcal{G}[\mathcal{G}_1, \dots, \mathcal{G}_1]) = n\mathcal{E}(\mathcal{G}_1) - nr_1 + \sum_{i=1}^n |x_i|,$$

where $x_1, x_2, \dots, x_n$ are the eigenvalues of matrix M given in Theorem 2.

Example 6. Consider the graph $H_3 = P_3[K_3] = P_3[K_3, K_3, K_3]$, as shown in Figure 3. In Example 3, we have shown that the A_{ex} -eigenvalues of H_3 consist of eigenvalue -1 repeated six times and the three eigenvalues $2 + \frac{267\sqrt{2}}{80} \approx 6.71993776$, $2 - \frac{267\sqrt{2}}{80} \approx -2.71993776$, and 2 . Therefore, based on the definition of A_{ex} -energy, we have

$$\begin{aligned} \mathcal{E}_{ex}(H_3) &= \mathcal{E}_{ex}(P_3[K_3]) \\ &= 6|-1| + |6.71993776| + |-2.71993776| + |2| \\ &= 17.43987552. \end{aligned}$$

Again, the adjacency eigenvalues of K_3 are $2, -1, -1$ with degree of regularity $r_1 = 2$. It follows that $\mathcal{E}(K_3) = 4$. Taking $n = 3$ and $\mathcal{G}_1 = K_3$ in Corollary 8, we get

$$\begin{aligned} \mathcal{E}_{ex}(H_3) &= \mathcal{E}_{ex}(P_3[K_3]) \\ &= 3\mathcal{E}(K_3) - 3r_1 + \sum_{i=1}^3 |x_i| \\ &= 3(4) - 3(2) + |6.71993776| + |-2.71993776| + |2| \\ &= 17.43987552, \end{aligned}$$

as $x_1 = 6.71993776$, $x_2 = -2.71993776$ and $x_3 = 2$. This shows that the value of A_{ex} -energy obtained via direct computation is in agreement with the value given by Corollary 8.

Matrix M is determined by the structure of $\mathcal{G}$, the orders n_i , and the degree of regularity r_i of the graphs $\mathcal{G}_i$ for $i = 1, 2, \dots, n$. We have the following observation from Theorem 3.

Corollary 9. Let $\mathcal{G}$ be a graph of order $n \geq 2$ with m edges. Let $\mathcal{G}_i$ and $\mathcal{H}_i$ be r_i -regular non-cospectral graphs of order n_i with $\mathcal{E}(\mathcal{G}_i) = \mathcal{E}(\mathcal{H}_i)$ for all $i = 1, 2, \dots, n$. Then, the graphs $\mathcal{G}[\mathcal{G}_1, \dots, \mathcal{G}_n]$ and $\mathcal{G}[\mathcal{H}_1, \dots, \mathcal{H}_n]$ are non-cospectral with

$$\mathcal{E}_{ex}(\mathcal{G}[\mathcal{G}_1, \dots, \mathcal{G}_n]) = \mathcal{E}_{ex}(\mathcal{G}[\mathcal{H}_1, \dots, \mathcal{H}_n]).$$

Proof. The proof follows from Theorem 3 based on the fact that matrix M is same for the graphs $\mathcal{G}[\mathcal{G}_1, \dots, \mathcal{G}_n]$ and $\mathcal{G}[\mathcal{H}_1, \dots, \mathcal{H}_n]$. $\square$

Example 7. Consider the graphs $H_5 = P_3[K_3, C_4, 2K_2]$ and $H_6 = P_3[C_4, K_3, 2K_2]$, shown in Figure 5. For graphs H_5 and H_6 , we have $\mathcal{G} = P_3$, $\mathcal{G}_1 = K_3$, $\mathcal{G}_2 = C_4$, $\mathcal{G}_3 = 2K_2$, $\mathcal{H}_1 = C_4$, $\mathcal{H}_2 = K_3$, and $\mathcal{H}_3 = 2K_2$. Clearly, graphs $\mathcal{G}_1$ and $\mathcal{H}_1$ are regular of degree 2; graphs $\mathcal{G}_2$ and $\mathcal{H}_2$ are regular of degree 2; and graphs $\mathcal{G}_3$ and $\mathcal{H}_3$ are regular of degree 1. Also, the adjacency eigenvalues of K_3 are $2, -1, -1$, implying that $\mathcal{E}(K_3) = 4$; the adjacency eigenvalues of C_4 are $2, 0, 0, -2$,

implying that $\mathcal{E}(C_4) = 4$; and the adjacency eigenvalues of $2K_2$ are $1, 1, -1, -1$, implying that $\mathcal{E}(2K_2) = 4$. Clearly, $\mathcal{E}(\mathcal{G}_1) = \mathcal{E}(\mathcal{H}_1)$, $\mathcal{E}(\mathcal{G}_2) = \mathcal{E}(\mathcal{H}_2)$, and $\mathcal{E}(\mathcal{G}_3) = \mathcal{E}(\mathcal{H}_3)$. Therefore, based on Corollary 9, we have

$$\mathcal{E}_{ex}(P_3[K_3, C_4, 2K_2]) = \mathcal{E}_{ex}(P_3[C_4, K_3, 2K_2]).$$

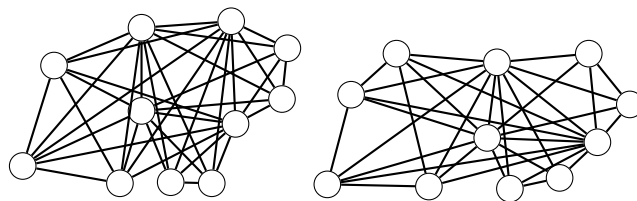


Figure 5. The graphs $P_3[K_3, C_4, 2K_2]$ and $P_3[C_4, K_3, 2K_2]$.

A number of papers can be found in the literature regarding the construction of equienergetic regular graphs; see the book [18] and the references therein. Let H_1 and H_2 be the 4-regular graphs of order 9 shown in Figure 4.1 in [18] (see page number 26). Through direct calculation, we can see that graphs H_1 and H_2 are non-cospectral equienergetic graphs. The double graph $D(\mathcal{G})$ of a graph $\mathcal{G}$ is the graph obtained by taking two copies of $\mathcal{G}$ and joining a vertex in the first copy to a vertex in the second copy whenever they are adjacent in $\mathcal{G}$. It is clear from the definition that if $\mathcal{G}$ is an r -regular graph, then its double graph $D(\mathcal{G})$ is a $2r$ -regular graph. Moreover, it is well-known that $\mathcal{E}(D(\mathcal{G})) = 2\mathcal{E}(\mathcal{G})$, see [18]. Let $L(\mathcal{G}) = L^1(\mathcal{G})$ be the line graph of $\mathcal{G}$, and let $L^k(\mathcal{G}) = L(L^{k-1}(\mathcal{G}))$ for $k = 2, 3, \dots$, be the k -th iterated line graph of $\mathcal{G}$. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two non-cospectral regular graphs of the same order and the same degree $r \geq 3$. Then, it is shown in Theorem 8.4 of [18] that for $k \geq 2$, the iterated line graphs $L^k(\mathcal{G}_1)$ and $L^k(\mathcal{G}_2)$ are non-cospectral equienergetic graphs of the same order and size. Since the graphs H_1 and H_2 are non-cospectral equienergetic 4-regular graphs, it follows from the above discussion that the double graphs $D(H_1)$ and $D(H_2)$ are non-cospectral equienergetic 8-regular graphs. Also, the iterated line graphs $L^k(H_1)$ and $L^k(H_2)$, where $k \geq 2$, are non-cospectral equienergetic regular graphs. Using graphs H_1 and H_2 , we have the following observations from Corollary 9.

Corollary 10. Let $\mathcal{G}$ be a graph of order $n \geq 2$, and let $H_1, H_2, D(H_1), D(H_2), L^k(H_1)$, and $L^k(H_2)$ be the regular graphs defined above.

(i). The graphs

$$\mathcal{G}[\underbrace{H_1, \dots, H_1}_{k_1}, \underbrace{\overline{K}_{n_1}, \dots, \overline{K}_{n_{k_2}}}_{k_2}, \underbrace{K_{t_1}, \dots, K_{t_{k_3}}}_{k_3}, \underbrace{K_{a_1, a_1}, \dots, K_{a_{k_4}, a_{k_4}}}_{k_4}]$$

and

$$\mathcal{G}[\underbrace{H_2, \dots, H_2}_{k_1}, \underbrace{\overline{K}_{n_1}, \dots, \overline{K}_{n_{k_2}}}_{k_2}, \underbrace{K_{t_1}, \dots, K_{t_{k_3}}}_{k_3}, \underbrace{K_{a_1, a_1}, \dots, K_{a_{k_4}, a_{k_4}}}_{k_4}]$$

are non-cospectral extended adjacency equienergetic graphs.

(ii). The graphs

$$\mathcal{G}[\underbrace{D(H_1), \dots, D(H_1)}_{k_1}, \underbrace{\overline{K}_{n_1}, \dots, \overline{K}_{n_{k_2}}}_{k_2}, \underbrace{K_{t_1}, \dots, K_{t_{k_3}}}_{k_3}, \underbrace{K_{a_1, a_1}, \dots, K_{a_{k_4}, a_{k_4}}}_{k_4}]$$

and

$$\mathcal{G}[\underbrace{D(H_2), \dots, D(H_2)}_{k_1}, \underbrace{\bar{K}_{n_1}, \dots, \bar{K}_{n_{k_2}}}_{k_2}, \underbrace{K_{t_1}, \dots, K_{t_{k_3}}}_{k_3}, \underbrace{K_{a_1, a_1}, \dots, K_{a_{k_4}, a_{k_4}}}_{k_4}]$$

are non-cospectral extended adjacency equienergetic graphs.

(iii). The graphs

$$\mathcal{G}[\underbrace{L^k(H_1), \dots, L^k(H_1)}_{k_1}, \underbrace{\bar{K}_{n_1}, \dots, \bar{K}_{n_{k_2}}}_{k_2}, \underbrace{K_{t_1}, \dots, K_{t_{k_3}}}_{k_3}, \underbrace{K_{a_1, a_1}, \dots, K_{a_{k_4}, a_{k_4}}}_{k_4}]$$

and

$$\mathcal{G}[\underbrace{L^k(H_2), \dots, L^k(H_2)}_{k_1}, \underbrace{\bar{K}_{n_1}, \dots, \bar{K}_{n_{k_2}}}_{k_2}, \underbrace{K_{t_1}, \dots, K_{t_{k_3}}}_{k_3}, \underbrace{K_{a_1, a_1}, \dots, K_{a_{k_4}, a_{k_4}}}_{k_4}]$$

are non-cospectral extended adjacency equienergetic graphs, where k_1, k_2, k_3 , and k_4 are whole numbers with $k_1 \geq 2$ or $k_1, k_i \geq 1, i \neq 1$ and $k_1 + k_2 + k_3 + k_4 = n$.

Recently, it is shown that the A_{ex} -energy of a tree T is greater than the energy of T . Here, we show that the A_{ex} -energy of the join of two regular graphs is greater than their energy.

Theorem 4. Let $\mathcal{G}_i$ be a r_i -regular graph of order n_i for $i = 1, 2$. Then,

$$\mathcal{E}_{ex}(\mathcal{G}_1 \vee \mathcal{G}_2) \geq \mathcal{E}(\mathcal{G}_1 \vee \mathcal{G}_2),$$

with equality if and only if $r_1 - r_2 = n_1 - n_2$.

It is well-known (see [18]) that the energy of join $\mathcal{G}_1 \vee \mathcal{G}_2$ of two regular graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is given as follows:

$$\mathcal{E}(\mathcal{G}_1 \vee \mathcal{G}_2) = \mathcal{E}(\mathcal{G}_1) + \mathcal{E}(\mathcal{G}_2) - (r_1 + r_2) + \sqrt{(r_1 - r_2)^2 + 4n_1n_2}. \quad (2)$$

Also, based on Corollary 7, the A_{ex} -energy of join $\mathcal{G}_1 \vee \mathcal{G}_2$ is given as follows:

$$\mathcal{E}_{ex}(\mathcal{G}_1 \vee \mathcal{G}_2) = \mathcal{E}(\mathcal{G}_1) + \mathcal{E}(\mathcal{G}_2) - (r_1 + r_2) + \sqrt{(r_1 - r_2)^2 + n_1n_2 \left(\frac{r_1 + n_2}{r_2 + n_1} + \frac{r_2 + n_1}{r_1 + n_2} \right)^2}. \quad (3)$$

From (2) and (3), we get

$$\mathcal{E}_{ex}(\mathcal{G}_1 \vee \mathcal{G}_2) - \mathcal{E}(\mathcal{G}_1 \vee \mathcal{G}_2) = \sqrt{(r_1 - r_2)^2 + n_1n_2 \left(\frac{r_1 + n_2}{r_2 + n_1} + \frac{r_2 + n_1}{r_1 + n_2} \right)^2} - \sqrt{(r_1 - r_2)^2 + 4n_1n_2}. \quad (4)$$

Now, $\sqrt{(r_1 - r_2)^2 + n_1n_2 \left(\frac{r_1 + n_2}{r_2 + n_1} + \frac{r_2 + n_1}{r_1 + n_2} \right)^2} \geq \sqrt{(r_1 - r_2)^2 + 4n_1n_2}$ gives that $\frac{r_1 + n_2}{r_2 + n_1} + \frac{r_2 + n_1}{r_1 + n_2} \geq 2$, which is always true, as r_1, r_2, n_1, n_2 are positive integers. Using this together with Equation (4), the result follows. Equality occurs if and only if $\frac{r_1 + n_2}{r_2 + n_1} + \frac{r_2 + n_1}{r_1 + n_2} = 2$, that is, if and only if $(r_1 + n_2)^2 + (r_2 + n_1)^2 = 2(r_1 + n_2)(r_2 + n_1)$. This last equality gives

$\left((r_1 + n_2) - (r_2 + n_1)\right)^2 = 0$, which is so if and only if $r_1 - r_2 = n_1 - n_2$. This completes the proof. $\square$

Example 8. Consider the graph $H_4 = K_3 \vee C_4$ shown in Figure 2. For this graph, it is shown in Example 5 that the A_{ex} -eigenvalues of H_4 consist of the eigenvalue -1 repeated two times, the eigenvalue 0 repeated two times, the eigenvalue -2 , and the two eigenvalues 5.52183664 , -1.52183664 . Therefore, based on the definition of A_{ex} -energy, we get

$$\mathcal{E}_{ex}(K_3 \vee C_4) = 11.04367328.$$

Also, K_3 is a $r_1 = 2$ regular graph with $n_1 = 3$ and $\mathcal{E}(K_3) = 4$, and C_4 is a $r_2 = 2$ regular graph with $n_2 = 4$ and $\mathcal{E}(C_4) = 4$. Therefore, based on Theorem 4, the adjacency energy of the graph $K_3 \vee C_4$ is given as follows:

$$\begin{aligned}\mathcal{E}(K_3 \vee C_4) &= \mathcal{E}(K_3) + \mathcal{E}(C_4) - (r_1 + r_2) + \sqrt{(2 - 2)^2 + 4n_1n_2} \\ &= 4 + 4 - 4 + 4\sqrt{3} = 4 + 4\sqrt{3} \approx 10.9282032303.\end{aligned}$$

It is now clear that $\mathcal{E}_{ex}(K_3 \vee C_4) > \mathcal{E}(K_3 \vee C_4)$.

4. Applications to Algebraic Graphs

In this section, we consider the power graph of finite groups. As applications of Theorem 1 and its consequences obtained in Section 2, we determine their A_{ex} -eigenvalues.

If $\mathcal{G}$ is a finite group of order n with identity e , the power graph of $\mathcal{G}$, denoted by $\mathcal{P}(\mathcal{G})$, is the simple graph with vertex set as the elements of group $\mathcal{G}$. Two distinct vertices $x, y \in \mathcal{P}(\mathcal{G})$ are adjacent if and only if one is the positive power of the other, that is, $x^i = y$ or $y^j = x$, for positive integers i, j with $2 \leq i, j \leq n$. Such graphs were introduced in [19], see also [20], and have valuable applications in both algebra and combinatorics.

The divisor d of the positive integer n (written as $d|n$) is a proper divisor of n if $1 < d < n$. Let $\mathcal{D}_n$ be a simple graph with the vertex set as the set of proper divisors $\{d_i : 1, n \neq d_i|n, 1 \leq i \leq t\}$ of n and an edge set $\{d_i d_j : d_i|d_j, 1 \leq i < j \leq t\}$, for $1 \leq i < j \leq t$. If the canonical decomposition of n is $n = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}$, where $r, n_1, n_2, \dots, n_r$ are non-negative integers and $p_1, p_2, \dots, p_r$ are distinct prime numbers, then the order of graph $\mathcal{D}_n$ is $|V(\mathcal{D}_n)| = \prod_{i=1}^r (n_i + 1) - 2$. The structure of the power graph of $\mathbb{Z}_n$ can be written as a joined union of cliques, where cliques correspond to proper divisors of n and are given by the following result.

Theorem 5 ([21]). If $\mathbb{Z}_n$ is a finite cyclic group of order n , then the power graph $\mathcal{P}(\mathbb{Z}_n)$ is given by

$$\mathcal{P}(\mathbb{Z}_n) = K_{\phi(n)+1} \vee \mathcal{D}_n[K_{\phi(d_1)}, K_{\phi(d_2)}, \dots, K_{\phi(d_t)}],$$

where $\phi(n)$ is the Euler's totient function.

The next result gives the A_{ex} -spectrum of $\mathcal{P}(\mathbb{Z}_n)$.

Theorem 6. The A_{ex} -spectrum of $\mathcal{P}(\mathbb{Z}_n)$ consists of the eigenvalue -1 with multiplicity $\phi(n) + \sum_{i=1}^t (\phi(d_i) - 1) = \phi(n) + n - \phi(n) - 1 - t = n - t - 1$, as $\sum_{i=1}^t \phi(d_i) = \sum_{d|n, d \neq 1, n} \phi(d_i) = n - \phi(n) - 1$. The remaining $t + 1$ eigenvalues are the eigenvalues of matrix (5) given below

$$M = \begin{pmatrix} \phi(n) & \tau_{12} & \cdots & \tau_{1(t+1)} \\ \tau_{21} & \phi(d_1) - 1 & \cdots & \tau_{2(t+1)} \\ \tau_{31} & \tau_{32} & \cdots & \tau_{3(t+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{(t+1)1} & \tau_{(t+1)2} & \cdots & \phi(d_t) - 1 \end{pmatrix}, \quad (5)$$

where $\tau_{1j} = \frac{\phi(d_{j-1})}{2} \left(\frac{n-1}{\phi(d_{j-1})+c_j-1} + \frac{\phi(d_{j-1})+c_j-1}{n-1} \right)$ and $\tau_{i1} = \frac{\phi(n)+1}{2} \left(\frac{\phi(d_{i-1})+c_i-1}{n-1} + \frac{n-1}{\phi(d_{i-1})+c_i-1} \right)$ for $2 \leq i, j \leq t+1$. Also, $\tau_{ij} = \frac{\phi(d_{j-1})}{2} \left(\frac{\phi(d_{j-1})-1+c_i}{\phi(d_{j-1})-1+c_j} + \frac{\phi(d_{j-1})-1+c_j}{\phi(d_{i-1})-1+c_i} \right)$, if $v_i \sim v_j$, and equal to 0 if $v_i \not\sim v_j$ for all $2 \leq i, j \leq t+1$ and $c_i = \sum_{v_j \in N_{K_1 \vee \mathcal{D}_n}(v_i)} n_j$.

Proof. Let $\mathcal{P}(\mathbb{Z}_n)$ be the power graph of $\mathbb{Z}_n$. Then, based on Theorem 5, we have

$$\mathcal{P}(\mathbb{Z}_n) = \mathcal{H}[K_{\phi(n)+1}, K_{\phi(d_1)}, K_{\phi(d_2)}, \dots, K_{\phi(d_t)}],$$

where $\mathcal{H} = K_1 \vee \mathcal{D}_n$ is the graph with the vertex set $\{v_1, \dots, v_{t+1}\}$. Let us consider $G_1 = K_{\phi(n)+1}, G_2 = K_{\phi(d_1)}, \dots, G_{t+1} = K_{\phi(d_t)}$ in Theorem 1. Using the fact that graph $K_{\phi(n)+1}$ is $\phi(n)$ regular having adjacency eigenvalues $\phi(n)$ with multiplicity 1 and -1 with multiplicity $\phi(n)$, it follows from Theorem 1 that -1 is the extended adjacency eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity $\phi(n)$. Also, the graph $K_{\phi(d_i)}$ is $\phi_i(d_i) - 1$ regular, having adjacency eigenvalues $\phi(d_i) - 1$ with multiplicity 1 and -1 with multiplicity $\phi(d_i) - 1$ for $i = 1, \dots, t$. It follows from Theorem 1 that -1 is an extended adjacency eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity $\phi(d_i) - 1$, for all $t = 1, 2, \dots, t$. The remaining $t + 1$ A_{ex} -eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ are the eigenvalues of matrix (5). This completes the proof. $\square$

Clearly, from Theorem 6, all the A_{ex} -eigenvalues of the power graph $\mathcal{P}(\mathbb{Z}_n)$ are completely determined except for the $t + 1$ -eigenvalues, which are the eigenvalues of matrix M defined in Theorem 6. Further, it is also clear that matrix M depends upon the structure of graph $\mathcal{D}_n$, which is not known in general. However, if we give some particular values to n , then it may be possible to know the structure of the graph $\mathcal{D}_n$ and hence about the matrix M . This information can be helpful to determine the remaining A_{ex} -eigenvalues of the power graph $\mathcal{P}(\mathbb{Z}_n)$.

If $n = p^s$, where p is a prime and s is a positive integer, then $\mathcal{P}(\mathbb{Z}_n)$ is the complete graph K_n , and its A_{ex} -spectrum is $\{n - 1, -1^{[n-1]}\}$.

Let $n = pq$, where p and q are primes with $p < q$. Then, the power graph $\mathcal{P}(\mathbb{Z}_n)$ can be written as

$$P_3[K_{\phi(pq)+1}, K_{p-1}, K_{q-1}].$$

Here, $\mathcal{H} = P_3, n_1 = \phi(pq) + 1, n_2 = p - 1$ and $n_3 = q - 1$. Let v_1, v_2, v_3 be the vertices of P_3 with degree of vertex v_1 equal to two, then $c_2 = \phi(pq) + 1 = c_3$;

$$\begin{aligned}\tau_{12} &= \frac{\phi(d_1)}{2} \left(\frac{n-1}{\phi(d_1)+c_2-1} + \frac{\phi(d_1)+c_2-1}{n-1} \right) = \frac{p-1}{2} \left(\frac{n-1}{p+\phi(pq)-1} + \frac{p+\phi(pq)-1}{n-1} \right), \\ \tau_{13} &= \frac{\phi(d_2)}{2} \left(\frac{n-1}{\phi(d_2)+c_3-1} + \frac{\phi(d_2)+c_3-1}{n-1} \right) = \frac{q-1}{2} \left(\frac{n-1}{q+\phi(pq)-1} + \frac{q+\phi(pq)-1}{n-1} \right), \\ \tau_{21} &= \frac{\phi(n)+1}{2} \left(\frac{n-1}{\phi(d_1)+c_2-1} + \frac{\phi(d_1)+c_2-1}{n-1} \right) = \frac{\phi(pq)+1}{2} \left(\frac{n-1}{p+\phi(pq)-1} + \frac{p+\phi(pq)-1}{n-1} \right), \\ \tau_{31} &= \frac{\phi(n)+1}{2} \left(\frac{n-1}{\phi(d_2)+c_3-1} + \frac{\phi(d_2)+c_3-1}{n-1} \right) = \frac{\phi(pq)+1}{2} \left(\frac{n-1}{q+\phi(pq)-1} + \frac{q+\phi(pq)-1}{n-1} \right),\end{aligned}$$

and $\tau_{23} = 0 = \tau_{32}$. Now, using Theorem 6, it follows that the A_{ex} -spectrum of $\mathcal{P}(\mathbb{Z}_n)$ consists of the eigenvalue -1 with multiplicity $n-3$. The other three A_{ex} -eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ are the eigenvalues of the following matrix:

$$\begin{pmatrix} \phi(pq) & \frac{p-1}{2} \left(\frac{n-1}{p+\phi(pq)-1} + \frac{p+\phi(pq)-1}{n-1} \right) & \frac{q-1}{2} \left(\frac{n-1}{q+\phi(pq)-1} + \frac{q+\phi(pq)-1}{n-1} \right) \\ \frac{\phi(pq)+1}{2} \left(\frac{n-1}{p+\phi(pq)-1} + \frac{p+\phi(pq)-1}{n-1} \right) & p-2 & 0 \\ \frac{\phi(pq)+1}{2} \left(\frac{n-1}{q+\phi(pq)-1} + \frac{q+\phi(pq)-1}{n-1} \right) & 0 & q-2 \end{pmatrix}.$$

From the above discussion, it is clear that if $n = pq$, where $p < q$ are primes, then we can completely determine all except three A_{ex} -eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$.

Example 9. Consider the finite abelian group $\mathbb{Z}_{15}$ shown in Figure 6. Here, $n = 15, p = 3, q = 5$, and $\phi(pq) = \phi(15) = \phi(3 \times 5) = \phi(3)\phi(5) = 8$. Also, $\frac{p-1}{2} \left(\frac{n-1}{p+\phi(pq)-1} + \frac{p+\phi(pq)-1}{n-1} \right) = \frac{74}{35}$, $\frac{q-1}{2} \left(\frac{n-1}{q+\phi(pq)-1} + \frac{q+\phi(pq)-1}{n-1} \right) = \frac{85}{21}$, $\frac{\phi(pq)+1}{2} \left(\frac{n-1}{p+\phi(pq)-1} + \frac{p+\phi(pq)-1}{n-1} \right) = \frac{333}{35}$ and $\frac{\phi(pq)+1}{2} \left(\frac{n-1}{q+\phi(pq)-1} + \frac{q+\phi(pq)-1}{n-1} \right) = \frac{765}{84}$. Therefore, the A_{ex} -spectrum of $\mathcal{P}(\mathbb{Z}_{15})$ consists of the eigenvalue -1 with multiplicity 12 and the 3 eigenvalues of the following matrix.

$$\begin{pmatrix} 8 & \frac{74}{35} & \frac{85}{21} \\ \frac{333}{35} & 1 & 0 \\ \frac{765}{84} & 0 & 3 \end{pmatrix}.$$

It is easy to see that the characteristic polynomial of this matrix is $-x^3 + 12x^2 + \frac{107693}{4900}x - \frac{7321}{100}$, so the remaining three eigenvalues are approximately 13.2422117204, -3.0530403064 , and 1.8108285859.

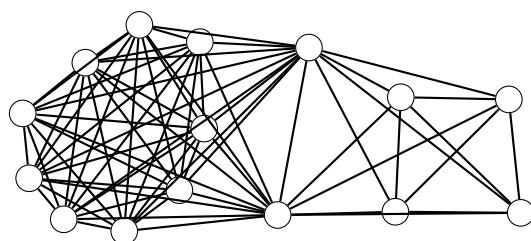


Figure 6. The graph $P_3[K_9, K_2, K_4]$, which is the power graph of $\mathbb{Z}_{15}$.

Example 10. Consider the finite abelian group $\mathbb{Z}_{35}$. Here, $n = 35, p = 5, q = 7$ and $\phi(pq) = \phi(35) = \phi(5 \times 7) = \phi(5)\phi(7) = 24$. Also, $\frac{p-1}{2} \left(\frac{n-1}{p+\phi(pq)-1} + \frac{p+\phi(pq)-1}{n-1} \right) = \frac{485}{119}$, $\frac{q-1}{2} \left(\frac{n-1}{q+\phi(pq)-1} + \frac{q+\phi(pq)-1}{n-1} \right) = \frac{514}{85}$, $\frac{\phi(pq)+1}{2} \left(\frac{n-1}{p+\phi(pq)-1} + \frac{p+\phi(pq)-1}{n-1} \right) = \frac{12125}{476}$ and

$\frac{\phi(pq)+1}{2} \left(\frac{n-1}{q+\phi(pq)-1} + \frac{q+\phi(pq)-1}{n-1} \right) = \frac{1285}{51}$. Therefore, the A_{ex} -spectrum of $\mathcal{P}(\mathbb{Z}_{35})$ consists of the eigenvalue -1 with multiplicity 32 and the 3 eigenvalues of the following matrix:

$$\begin{pmatrix} 24 & \frac{485}{119} & \frac{514}{85} \\ \frac{12125}{476} & 3 & 0 \\ \frac{1285}{51} & 0 & 5 \end{pmatrix}.$$

It is easy to see that the characteristic polynomial of this matrix is $-x^3 + 32x^2 + \frac{8357159}{169932}x - \frac{34902493}{56644}$, so the remaining three eigenvalues are approximately 32.9252815124, -4.813308841 , and 3.88802732.

If $n = pqr$, where p, q, r are prime numbers with $p < q < r$, based on the definition of $\mathcal{D}_n$, the vertex set and edge set of $\mathcal{D}_n$ are $\{p, q, r, pq, pr, qr\}$ and $\{(p, pq), (p, pr), (q, pq), (q, qr), (r, pr), (r, qr)\}$, respectively. Let $\mathcal{H} = K_1 \vee \mathcal{D}_n$. Then, $\mathcal{P}(\mathbb{Z}_n) = \mathcal{H}[K_{\phi(n)+1}, K_{\phi(d_1)}, \dots, K_{\phi(d_6)}]$, where $d_1 = p, d_2 = q, d_3 = r, d_4 = pq, d_5 = pr$, and $d_6 = qr$. Now, proceeding to the case $n = pq$, we can also determine the A_{ex} -eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$.

5. Conclusions

In this article, we have determined the A_{ex} -spectrum of the joined union of regular graphs in terms of their adjacency spectrum and the eigenvalues of an auxiliary matrix. As a consequence, we have determined the A_{ex} -eigenvalues of the join of two regular graphs, the lexicographic product of regular graphs, and the A_{ex} -eigenvalues of some well-known families of graphs. We have considered the problem of building graphs with different A_{ex} -eigenvalues but the same A_{ex} -energy. As applications of our results, we have constructed different classes of infinite families of extended adjacency equienergetic graphs. We also compared the energy and the A_{ex} -energy of the join of two regular graphs, and we showed that the A_{ex} -energy of the join of two regular graphs exceeds their energy. In addition, we discussed the A_{ex} -eigenvalues of the power graph of finite abelian groups. We showed that all except the $t + 1$ A_{ex} -eigenvalues of the power graph of finite abelian groups are completely determined. We considered the cases when $n = p^z, pq$ and pqr , where p, q, r are distinct primes, and we discussed the A_{ex} -eigenvalues of the power graph of $\mathbb{Z}_n$. The results presented in Section 4 concerning the power graph of $\mathbb{Z}_n$ have been rigorously verified using Python 3 and the NumPy library, as detailed in Appendix A. The following interesting problem can be considered for future research.

Problem 3. Discuss the A_{ex} -eigenvalues of the power graph of $\mathbb{Z}_n$ when $n = p^t q, p^t q^s, p^t q^s r^k$, where p, q, r are distinct primes and t, s, k are positive integers.

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Abbreviations

The following abbreviations are used in this manuscript:

A_{ex} -eigenvalues	extended adjacency eigenvalues
A_{ex} -spectrum	extended adjacency spectrum
A_{ex} -energy	extended adjacency energy
A_{ex} -spectral radius	extended adjacency spectral radius

Appendix A

The following algorithm summarizes the computational procedure used to generate the power graph $\mathcal{P}(\mathbb{Z}_n)$ and to compute the A_{ex} -eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$. This implementation was carried out in Python 3 using the NumPy library to reproduce the spectral results presented in Section 4.

Algorithm A1 Power Graph Generation and Extended Adjacency Spectrum Analysis

Input:

- n : an integer representing the modulus for the additive group $\mathbb{Z}_n$.

Output:

- The eigenvalues of the extended adjacency matrix $A_{ex}(\mathcal{P}(\mathbb{Z}_n))$ with their multiplicities.

Steps:

1. Initialization:

- Define $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$.
- Initialize an empty simple graph G .

2. Graph Construction:

- For each pair of distinct elements (a, b) in $\mathbb{Z}_n$, add an edge between a and b in G if $a \equiv kb \pmod{n}$ or $b \equiv ka \pmod{n}$ for some integer $k \geq 2$.
- The resulting graph G represents the power graph $\mathcal{P}(\mathbb{Z}_n)$.

3. Extended Adjacency Matrix:

- Construct the matrix $A_{ex}(G) = [b_{ij}]_{n \times n}$, where

$$b_{ij} = \begin{cases} \frac{1}{2} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right), & \text{if } v_i, v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

4. Eigenvalue Computation:

- Compute all eigenvalues of $A_{ex}(G)$ using a numerical routine (e.g., `numpy.linalg.eigvals()`).
- Report the eigenvalues and their multiplicities.

End Algorithm

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