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Analytical Solutions and Controllability of Delay Differential Matrix Equations via Kronecker Product and Delayed Matrix Functions

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Abstract

This work introduces a unified framework for analyzing linear delay differential Sylvester matrix equations with noncommuting coefficients. The methodology employs a Kronecker product-based vectorization to transform the system, yielding explicit closed-form solutions via a novel delayed perturbation matrix function. Additionally, a delay-adapted Gramian matrix is formulated to derive necessary and sufficient controllability criteria. The approach's efficacy is confirmed through a numerical example, demonstrating its capability in complex, noncommutative scenarios where classical methods are inapplicable.

Keywords: exact solutions; delay differential matrix equation; delayed matrix function; controllability; Kronecker product; vector operator

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1. Introduction

The study of delay differential Sylvester matrix equations (DSEs) constitutes a cornerstone in modern systems theory and mathematical modeling of complex dynamical systems with memory effects, filter design theory, model reduction problems, and robust control problems; see ref. [1–3] and the references therein.

In the theory of ordinary differential equations with delays, one of the key tools used to solve initial problems is the method of steps; see, for instance, ref. [4–7]. This method is especially efficient if linear equations are considered. This concept is implicitly used below to formulate the solution of the matrix equation:

$$\begin{aligned} X'(\vartheta) &= \mathbb{B}X(\vartheta - \sigma) + X(\vartheta - \sigma)\mathbb{C} + F(\vartheta), \quad \text{for } \vartheta \in [0, \infty), \\ X(\vartheta) &= \Phi(\vartheta), \quad \text{for } \vartheta \in [-\sigma, 0], \end{aligned} \quad (1)$$

where ϑ is the independent variable, σ is a positive constant delay, X is an $n \times n$ unknown real dependent variable matrix, $\mathbb{B}, \mathbb{C}$ are given $n \times n$ constant real matrices, F is a continuous $n \times n$ real matrix on $[0, \infty)$, and Φ is a continuously differentiable $n \times n$ real matrix, defined on $[-\sigma, 0]$. Equation (1) has a unique solution, $X = X(\vartheta)$, in the sense that $X(\vartheta)$ is continuous on $[-\sigma, \infty)$ and continuously differentiable on $[0, \infty)$. Recently, much attention has been attracted to derive representations of solutions of linear differential equations with delays by using a concept of so-called delayed matrix functions, with their definitions

suggested by the step method. We can refer to papers [8,9], where these investigations are applied to first-order [9] and second-order [8] linear delayed differential equations with a single delay. Let us refer, for instance, to papers [10–14] substantially contributing to this field, and extending these results to much more complicated systems. These results are widely applied, for example, to stability investigations, iterative learning control, or relative controllability; let us mention at least [15–18].

On the other hand, control theory plays an important role in various applications, including vibration control in mechanical systems [19,20], stability analysis of power networks [21], fractional order epidemic models [22], and robust control design for time-delay systems [23,24]. Several authors have studied the controllability of the differential Sylvester matrix equations

$$\begin{aligned} X'(\vartheta) &= \mathbb{A} X(\vartheta) + X(\vartheta) \mathbb{B} + \mathbb{E}U(\vartheta)\mathbb{F}, \quad \text{for } \vartheta \in [0, \infty), \\ X(\vartheta) &= \Phi(\vartheta), \quad \text{for } \vartheta \in [-\sigma, 0], \end{aligned}$$

where $X = (X_1, X_2, \dots, X_n)^T : [-\sigma, \infty) \rightarrow \mathbb{R}^{n \times p}$ is a solution satisfying (2) for every $\vartheta \in [0, \infty)$, $U(\vartheta) \in \mathbb{R}^{m_1 \times m_2}$ shows the control matrix, $\mathbb{A}$, $\mathbb{B}$, $\mathbb{E}$, and $\mathbb{F}$ are $n \times n$, $p \times p$, $n \times m_1$, and $m_2 \times p$ constant real nonzero matrices, respectively, and $\Phi : [-\sigma, 0) \rightarrow \mathbb{R}^{n \times p}$ determines initial conditions. Murty et al. [25] established fundamental controllability criteria for linear matrix Lyapunov systems using rank conditions and matrix function techniques. Murty and SureshKumar [26] addressed the controllability of linear impulsive matrix Lyapunov differential systems with delays present in the control function, highlighting the impact of delayed control inputs on the steerability of the system. Dubey and George [27] investigated the controllability of semilinear matrix Lyapunov systems, establishing sufficient conditions under which such systems can be steered to a desired state. In a related study, Murty et al. [28] analyzed the controllability of Lyapunov fuzzy dynamical matrix systems, extending classical control concepts to systems governed by fuzzy logic. Appa Rao and Prasad [29] examined the controllability of dynamical systems of the Sylvester matrix on time scales, unifying continuous and discrete formulations within a single framework, while the controllability of Sylvester-type systems has been explored in the absence of delays; see [25,27,29]. Moreover, Sadek et al. [30] presented global extended Krylov subspace methods for the numerical solution of large-scale differential Sylvester matrix equations, offering efficient computational tools for high-dimensional systems. Finally, Diblik [31] studied the representation of solutions to linear matrix first-order differential equations with delay, establishing explicit solution formulas under commutativity conditions. Our work extends these results to the more challenging noncommutative case while incorporating control inputs and obtaining complete controllability criteria.

However, to the best of our knowledge, the closed-form solutions and controllability analysis of delay differential Sylvester matrix equations with distributed delays and control inputs under non-permutable matrix coefficients remain largely unaddressed, leaving a critical gap in the literature. Therefore, in this paper, we consider solutions and controllability of the nonhomogeneous linear delay differential Sylvester matrix equation (DSE) of the form

$$\begin{aligned} X'(\vartheta) &= \mathbb{A} X(\vartheta) + X(\vartheta) \mathbb{B} + \mathbb{S} X(\vartheta - \sigma) + X(\vartheta - \sigma) \mathbb{J} \\ &\quad + \mathbb{E}U(\vartheta)\mathbb{F}, \quad \text{for } \vartheta \in [0, \infty), \\ X(\vartheta) &= \Phi(\vartheta), \quad \text{for } \vartheta \in [-\sigma, 0], \end{aligned} \quad (2)$$

where $X = (X_1, X_2, \dots, X_n)^T : [-\sigma, \infty) \rightarrow \mathbb{R}^{n \times p}$ is a solution that satisfies (2) for every $\vartheta \in [0, \infty)$, $U(\vartheta) \in \mathbb{R}^{m_1 \times m_2}$ represents the control input, $\mathbb{A}$, $\mathbb{B}$, $\mathbb{S}$, $\mathbb{J}$, $\mathbb{E}$, and $\mathbb{F}$ are $n \times n$, $p \times p$, $n \times n$, $p \times p$, $n \times m_1$, and $m_2 \times p$ constant real nonzero matrices, respectively, and $\Phi : [-\sigma, 0) \rightarrow \mathbb{R}^{n \times p}$ determines the initial conditions. The presence of delay terms $\mathbb{S} X(\vartheta - \sigma)$

and $X(\vartheta - \sigma) \mathbb{J}$ introduces infinite-dimensional dynamics, rendering classical Kalman-type rank conditions insufficient and necessitating novel functional-analytic or operator-theoretic approaches [2,6,7].

This paper aims to bridge this theoretical gap by establishing the following:

- (i) Explicit representation formulas for the solution of (2) under both permutable and non-permutable assumptions on the coefficient matrices, utilizing a novel delayed matrix exponential framework adapted to the Sylvester structure;
- (ii) Necessary and sufficient conditions for the controllability of (2), formulated in terms of the system matrices $\mathbb{A}$, $\mathbb{B}$, $\mathbb{S}$, $\mathbb{J}$, $\mathbb{E}$, and $\mathbb{F}$, and the delay σ , without requiring commutativity;
- (iii) A constructive methodology based on Kronecker product transformations and delayed matrix function expansions, enabling both theoretical analysis and numerical implementation;
- (iv) The derived results not only generalize existing literature on scalar and vector delay systems, but also provide foundational tools for the control and stabilization of large-scale matrix dynamical systems arising in multi-agent networks, tensor-based models, and coupled PDE-ODE systems with memory.

To provide a clear and structured overview of the methodological framework developed in this paper, we present a systematic block diagram in Figure 1.

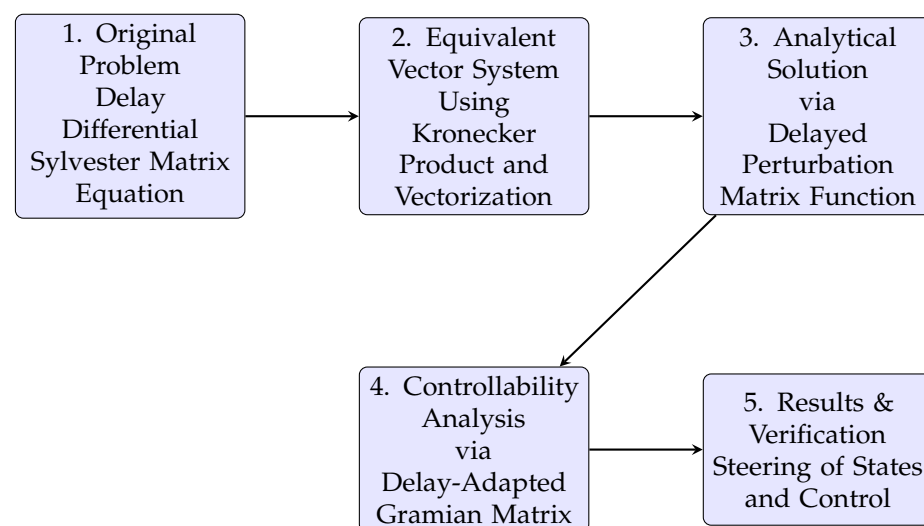


Figure 1. A schematic overview of the proposed methodology for solving and analyzing the controllability of delay differential Sylvester matrix equations.

The remainder of this paper is organized as follows: In Section 2, we introduce necessary notation and preliminary results on a delayed matrix function. In Section 3, we derive the explicit solution representation for (2). In Section 4, we establish controllability criteria. Finally, we provide illustrative examples.

2. Preliminaries

In this section, we present some properties and rules for Kronecker products, a definition of the Vec operator, and basic results related to the delayed perturbation matrix function.

Definition 1 ([32]). Let $\mathbb{O} = [o_{ij}] \in \mathbb{R}^{m \times n}$ and $\mathbb{H} \in \mathbb{R}^{p \times q}$; then, the Kronecker product of $\mathbb{O}$ and $\mathbb{H}$ written $\mathbb{O} \otimes \mathbb{H}$ is defined to be the matrix

$$\mathbb{O} \otimes \mathbb{H} = \begin{bmatrix} o_{11}\mathbb{H} & o_{12}\mathbb{H} & \cdots & o_{1n}\mathbb{H} \\ o_{21}\mathbb{H} & o_{22}\mathbb{H} & \cdots & o_{2n}\mathbb{H} \\ \vdots & \vdots & \ddots & \vdots \\ o_{m1}\mathbb{H} & o_{m2}\mathbb{H} & \cdots & o_{mn}\mathbb{H} \end{bmatrix},$$

which is an $mp \times nq$ matrix.

Definition 2 ([32]). Let $\mathbb{Z} = [Z_{ij}] \in \mathbb{R}^{n \times s}$. The vector $\text{Vec}(\mathbb{Z})$ is defined as

$$\text{Vec}(\mathbb{Z}) = [Z_{11}, Z_{21}, \dots, Z_{n1}, \dots, Z_{1s}, Z_{2s}, \dots, Z_{ns}]^T \in \mathbb{R}^{ns}.$$

Moreover, we have the following properties of the Kronecker product:

1. $(\mathbb{A} \otimes \mathbb{B})^T = \mathbb{A}^T \otimes \mathbb{B}^T$.
2. $(\mathbb{A} \otimes \mathbb{B})(\mathbb{C} \otimes \mathbb{D}) = (\mathbb{A}\mathbb{C} \otimes \mathbb{B}\mathbb{D})$.
3. $\text{Vec}(\mathbb{A}\mathbb{B}) = (\mathbb{B}^T \otimes \mathbb{A})\text{Vec}(\mathbb{Y})$.

Definition 3. Consider constant real matrices $\mathbb{C}, \mathbb{W} \in \mathbb{R}^{n \times n}$ (both nonzero). The matrix sequence $Y^{\mathbb{C}, \mathbb{W}}(k; s)$ in $\mathbb{R}^{n \times n}$ satisfies the delayed recurrence relation

$$Y^{\mathbb{C}, \mathbb{W}}(k+1; s) = \mathbb{C}Y^{\mathbb{C}, \mathbb{W}}(k; s) + \mathbb{W}Y^{\mathbb{C}, \mathbb{W}}(k; s - \sigma),$$

with initial conditions of $Y^{\mathbb{C}, \mathbb{W}}(0; s) = \Theta$, $Y^{\mathbb{C}, \mathbb{W}}(k; -\sigma) = \Theta$, and $Y^{\mathbb{C}, \mathbb{W}}(1; 0) = \mathbb{I}_n$, where $k = 1, 2, \dots$ and $s = 0, \sigma, 2\sigma, \dots$, Θ represents the $n \times n$ zero matrix, and $\mathbb{I}_n$ denotes the $n \times n$ identity matrix.

Definition 4 ([33]). The delayed perturbation matrix function $\mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}$ is defined as follows:

$$\mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta) := \begin{cases} \Theta, & -\sigma \leq \vartheta < 0, \\ \mathbb{I}_n, & \vartheta = 0, \\ \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} \frac{(\vartheta - \sigma j)^i}{i!} Y^{\mathbb{C}, \mathbb{W}}(i+1; j\sigma), & (p-1)\sigma < \vartheta \leq p\sigma. \end{cases}$$

Lemma 1 ([33]). The delayed perturbation matrix function $\mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta)$, defined in Definition 4, is a solution of the homogeneous linear delay differential equation

$$\frac{d}{d\vartheta} \mathbb{Z}(\vartheta) = \mathbb{C}\mathbb{Z}(\vartheta) + \mathbb{W}\mathbb{Z}(\vartheta - \sigma), \quad \text{for } \vartheta > 0,$$

with the initial condition $\mathbb{Z}(\vartheta) = \mathbb{I}_n$ for $\vartheta = 0$ and $\mathbb{Z}(\vartheta) = \Theta$ for $\vartheta \in [-\sigma, 0)$.

The explicit solution provided in Theorem 2 relies on the delayed perturbation matrix function $\mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta)$, defined in Definition 4 as an infinite series. For the solution to be well-defined, the convergence of this series must be established. The following proposition provides a convergence guarantee.

Proposition 1 (Convergence Criteria). Let $\mathbb{C}, \mathbb{W} \in \mathbb{R}^{n \times n}$ be arbitrary constant matrices (not necessarily permutable), and let $\sigma > 0$ be the delay. The matrix series defining $\mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta)$ in Definition 4 converges absolutely and uniformly for all ϑ on any compact interval.

Proof. Consider the general term of the series for a given $\vartheta \in [(p-1)\sigma, p\sigma]$:

$$a_i = \sum_{j=0}^{p-1} \frac{|\vartheta - j\sigma|^i}{i!} \|Y^{\mathbb{C}, \mathbb{W}}(i+1; j\sigma)\|.$$

The matrix sequence $Y^{\mathbb{C}, \mathbb{W}}(i+1; j\sigma)$ is defined by a linear recurrence. Therefore, there exists a constant $M > 0$ (dependent on $\mathbb{C}, \mathbb{W}, p$) and a matrix norm $\|\cdot\|$ such that $\|Y^{\mathbb{C}, \mathbb{W}}(i+1; j\sigma)\| \leq M^i$ for sufficiently large i . Let $R = \max_{0 \leq j \leq p-1} |\vartheta - j\sigma|$. Then,

$$a_i \leq p \frac{R^i}{i!} M^i = p \frac{(MR)^i}{i!}.$$

The series $\sum_{i=0}^{\infty} \frac{(MR)^i}{i!}$ is the Taylor series for e^{MR} , which converges for any finite M, R . By the Weierstrass M-test, the original matrix series converges absolutely and uniformly. $\square$

Remark 1. 1. *Implications for Large Delay (σ):* From Proposition 1, the convergence radius is infinite for sufficiently large delay σ . The function $\mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta)$ is an entire function of ϑ for any given σ . However, for a fixed computational budget (truncating the series at $i = N$), the number of terms required for a desired accuracy in the interval $[(p-1)\sigma, p\sigma]$ grows with $R = \max_j |\vartheta - j\sigma|$, which itself grows with σ . Therefore, the series defining $\mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta)$ converges for all finite values of ϑ , meaning it has an infinite radius of convergence. This guarantees that the solution remains valid over any finite time interval, regardless of the delay magnitude σ .

2. *Non-Permutable Matrices:* The proof of Proposition 1 makes **no assumption** on the commutativity of $\mathbb{C}$ and $\mathbb{W}$. The bound relies only on the growth of the matrix sequence $Y^{\mathbb{C}, \mathbb{W}}$, which is guaranteed to be at most exponential due to its linear recurrence structure. This confirms that the framework and the resulting solutions are valid for the general non-permutable case, which is the central focus of this work.

Lemma 2 (Properties of the Delayed Perturbation Matrix Function). *Let $\mathbb{C}, \mathbb{W} \in \mathbb{R}^{n \times n}$ be arbitrary constant matrices (not necessarily permutable), and let $\sigma > 0$. The delayed perturbation matrix function $\mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta)$ defined in Definition 4 satisfies the following properties on any finite interval $[-\sigma, T]$:*

1. *Existence and Continuity:* $\mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta)$ exists and is continuous for all $\vartheta \in [-\sigma, T]$.
2. *Boundedness:* There exists a constant $K_T > 0$ such that $\|\mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta)\| \leq K_T$ for all $\vartheta \in [-\sigma, T]$.
3. *Differentiability:* $\mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta)$ is continuously differentiable on the intervals $((k-1)\sigma, k\sigma)$ for $k \in \mathbb{Z}^+$, and its derivative satisfies the delayed differential equation:

$$\frac{d}{d\vartheta} \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta) = \mathbb{C} \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta) + \mathbb{W} \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - \sigma), \quad \text{for } \vartheta > 0.$$

Proof. We prove each property in sequence.

1. *Existence and Continuity:* From Proposition 1, the series defining $\mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta)$ converges uniformly on any compact interval. Since each partial sum is a continuous function of ϑ (being a finite sum of continuous functions), the uniform limit $\mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta)$ is itself continuous on $[-\sigma, T]$.

2. *Boundedness:* The uniform convergence on the compact interval $[-\sigma, T]$ implies that $\mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta)$ is bounded on this interval. We can take $K_T = \max_{\vartheta \in [-\sigma, T]} \|\mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta)\| < \infty$.

3. *Differentiability:* For $\vartheta \in ((k-1)\sigma, k\sigma)$, the series can be differentiated term-by-term due to the uniform convergence of the derivative series on any closed subinterval. Differentiating the series representation,

$$\begin{aligned}
\frac{d}{d\vartheta} \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta) &= \sum_{i=1}^{\infty} \sum_{j=0}^{p-1} \frac{(\vartheta - j\sigma)^{i-1}}{(i-1)!} Y^{\mathbb{C}, \mathbb{W}}(i+1; j\sigma) \\
&= \mathbb{C} \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} \frac{(\vartheta - j\sigma)^i}{i!} Y^{\mathbb{C}, \mathbb{W}}(i+1; j\sigma) + \mathbb{W} \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} \frac{(\vartheta - (j+1)\sigma)^i}{i!} Y^{\mathbb{C}, \mathbb{W}}(i+1; j\sigma) \\
&= \mathbb{C} \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta) + \mathbb{W} \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - \sigma),
\end{aligned}$$

where we have used the recurrence relation from Definition 3: $Y^{\mathbb{C}, \mathbb{W}}(i+1; j\sigma) = \mathbb{C} Y^{\mathbb{C}, \mathbb{W}}(i; j\sigma) + \mathbb{W} Y^{\mathbb{C}, \mathbb{W}}(i; (j-1)\sigma)$. The derivative is continuous on each open interval $((k-1)\sigma, k\sigma)$. $\square$

3. Exact Solutions of a Delay Differential Matrix Equation

This section is devoted to deriving the explicit form of the solution to problem (2). The following theorem shows an equivalence between the delay differential Sylvester matrix Equations (2) for size $n \times p$ and the delay differential system (3) for size $np \times 1$.

Theorem 1. Assume that $\mathbb{A}, \mathbb{B}, \mathbb{S}, \mathbb{J}$ are non-permutable $n \times n, p \times p, n \times n, p \times p$ constant real nonzero matrices, respectively. Let $\mathbb{X}(\vartheta) = \text{Vec}(X(\vartheta))$, $\mathbb{U}(\vartheta) = \text{Vec}(U(\vartheta))$, and $\Psi(\vartheta) = \text{Vec}(\Phi(\vartheta))$. Then the delay differential Sylvester matrix Equation (2) is equivalent to system

$$\begin{aligned}
\mathbb{X}'(\vartheta) &= \mathbb{C} \mathbb{X}(\vartheta) + \mathbb{W} \mathbb{X}(\vartheta - \sigma) + (\mathbb{F}^T \otimes \mathbb{E}) \mathbb{U}(\vartheta), \quad \text{for } \vartheta \in [0, \infty), \\
\mathbb{X}(\vartheta) &= \Psi(\vartheta), \quad \text{for } \vartheta \in [-\sigma, 0],
\end{aligned} \tag{3}$$

where $\mathbb{C} = (\mathbb{B}^T \otimes \mathbb{I}_n) + (\mathbb{I}_p \otimes \mathbb{A})$, $\mathbb{W} = (\mathbb{J}^T \otimes \mathbb{I}_n) + (\mathbb{I}_p \otimes \mathbb{S})$, and $\mathbb{I}_n$ is the identity matrix.

Proof. We apply the Vec operator to Equation (2), and using the above properties of the Kronecker product, we have

$$\begin{aligned}
\mathbb{X}'(\vartheta) &= \left[(\mathbb{B}^T \otimes \mathbb{I}_n) + (\mathbb{I}_p \otimes \mathbb{A}) \right] \text{Vec}(X(\vartheta)) \\
&\quad + \left[(\mathbb{J}^T \otimes \mathbb{I}_n) + (\mathbb{I}_p \otimes \mathbb{S}) \right] \text{Vec}(X(\vartheta - \sigma)) + \text{Vec}(\mathbb{E}U(\vartheta)\mathbb{F}) \\
&= \mathbb{C} \mathbb{X}(\vartheta) + \mathbb{W} \mathbb{X}(\vartheta - \sigma) + (\mathbb{F}^T \otimes \mathbb{E}) \mathbb{U}(\vartheta),
\end{aligned}$$

and

$$\text{Vec}(X(\vartheta)) = \text{Vec}(\Phi(\vartheta)) = \Psi(\vartheta).$$

This completes the proof. $\square$

Theorem 2. The unique solution of (3) with the initial condition $\mathbb{X}(\vartheta) = \Psi(\vartheta)$ is

$$\begin{aligned}
\mathbb{X}(\vartheta) &= \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta + \sigma) \Psi(-\sigma) + \int_{-\sigma}^0 \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - s) (\Psi'(s) - \mathbb{C} \Psi(s)) ds \\
&\quad + \int_0^{\vartheta} \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - s) (\mathbb{F}^T \otimes \mathbb{E}) \mathbb{U}(s) ds,
\end{aligned} \tag{4}$$

or

$$\begin{aligned}
\mathbb{X}(\vartheta) &= \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta) \Psi(0) + \mathbb{W} \int_{-\sigma}^0 \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - \sigma - s) \Psi(s) ds \\
&\quad + \int_0^{\vartheta} \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - s) (\mathbb{F}^T \otimes \mathbb{E}) \mathbb{U}(s) ds.
\end{aligned} \tag{5}$$

Proof. Using Lemma 1, the proof of Formula (4) is similar to the proof of Corollary 1 in [33]. Using summation by parts, we rewrite the term in (5) as follows:

$$\begin{aligned} & \int_{-\sigma}^0 \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - s) \Psi'(s) ds \\ &= \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta) \Psi(0) - \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta + \sigma) \Psi(-\sigma) \\ & \quad + \int_{-\sigma}^0 \left[\mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - s) \right]' \Psi(s) ds \\ &= \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta) \Psi(0) - \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta + \sigma) \Psi(-\sigma) \\ & \quad + \int_{-\sigma}^0 \left[\mathbb{C} \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - s) + \mathbb{W} \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - \sigma - s) \right] \Psi(s) ds. \end{aligned}$$

Hence

$$\begin{aligned} & \int_{-\sigma}^0 \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - s) [\Psi'(s) - \mathbb{C} \Psi(s)] ds \\ &= \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta) \Psi(0) - \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta + \sigma) \Psi(-\sigma) \\ & \quad + \mathbb{W} \int_{-\sigma}^0 \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - \sigma - s) \Psi(s) ds. \end{aligned} \quad (6)$$

Substituting (6) into (4), we obtain (5). This completes the proof. $\square$

Corollary 1. Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{S}$, $\mathbb{J}$ be permutable $n \times n$, $p \times p$, $n \times n$, $p \times p$ constant real nonzero matrices, respectively. Then the unique solution of (3) with the initial condition $\mathbb{X}(\vartheta) = \Psi(\vartheta)$ is

$$\begin{aligned} \mathbb{X}(\vartheta) &= \mathcal{R}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta + \sigma) \Psi(-\sigma) + \int_{-\sigma}^0 \mathcal{R}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - s) (\Psi'(s) - \mathbb{C} \Psi(s)) ds \\ & \quad + \int_0^{\vartheta} \mathcal{R}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - s) (\mathbb{F}^T \otimes \mathbb{E}) \mathbb{U}(s) ds, \end{aligned}$$

or

$$\begin{aligned} \mathbb{X}(\vartheta) &= \mathcal{R}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta) \Psi(0) + \mathbb{W} \int_{-\sigma}^0 \mathcal{R}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - \sigma - s) \Psi(s) ds \\ & \quad + \int_0^{\vartheta} \mathcal{R}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta - s) (\mathbb{F}^T \otimes \mathbb{E}) \mathbb{U}(s) ds, \end{aligned}$$

where

$$\mathcal{R}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta) := \begin{cases} \Theta, & -\sigma \leq \vartheta \leq 0, \\ \mathbb{I}_n, & 0 \leq \vartheta \leq \sigma, \\ \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} \frac{(\vartheta - \sigma j)^i}{i!} \binom{i}{j} \mathbb{C}^{i-j} \mathbb{W}^j, & (p-1)\sigma \leq \vartheta \leq p\sigma. \end{cases}$$

Corollary 2. Let $\mathbb{A} = \mathbb{B} = \Theta$, which leads to $\mathbb{C} = \Theta$ in (3). Suppose that $\mathbb{S}$ and $\mathbb{J}$ are non-permutable $n \times n$ and $p \times p$ constant real non-zero matrices, respectively. Then the solution $\mathbb{Y}(\vartheta)$ of (3) can be expressed as

$$\begin{aligned} \mathbb{X}(\vartheta) &= \mathcal{F}_{\sigma}^{\Theta, \mathbb{W}}(\vartheta + \sigma) \Psi(-\sigma) + \int_{-\sigma}^0 \mathcal{F}_{\sigma}^{\Theta, \mathbb{W}}(\vartheta - s) \Psi'(s) ds \\ & \quad + \int_0^{\vartheta} \mathcal{F}_{\sigma}^{\Theta, \mathbb{W}}(\vartheta - s) (\mathbb{F}^T \otimes \mathbb{E}) \mathbb{U}(s) ds, \end{aligned}$$

or

$$\begin{aligned}\mathbb{X}(\vartheta) &= \mathcal{F}_{\sigma}^{\Theta, \mathbb{W}}(\vartheta) \Psi(0) + \mathbb{W} \int_{-\sigma}^0 \mathcal{F}_{\sigma}^{\Theta, \mathbb{W}}(\vartheta - \sigma - s) \Psi(s) ds \\ &\quad + \int_0^{\vartheta} \mathcal{F}_{\sigma}^{\Theta, \mathbb{W}}(\vartheta - s) (\mathbb{F}^T \otimes \mathbb{E}) \mathbb{U}(s) ds,\end{aligned}$$

where

$$\begin{aligned}\mathcal{F}_{\sigma}^{\Theta, \mathbb{W}}(\vartheta) &= \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} \frac{(\vartheta - \sigma j)^i}{i!} Y^{\Theta, \mathbb{W}}(i+1; j\sigma) \\ &= \sum_{i=0}^{\lfloor \frac{\vartheta}{\sigma+1} \rfloor} \frac{(\vartheta - \sigma i)^i}{i!} \mathbb{W}^i = e^{(\vartheta - \sigma i)}.\end{aligned}$$

Remark 2. Corollary 2 improves the corresponding results in [31] by removing the commutativity assumptions.

4. Controllability

In this section, we obtain the necessary and sufficient conditions for the controllability of the system (3).

Definition 5 ([34]). The systems (3) are controllable over the interval $[0, \vartheta_f]$ if they possess the ability to transit any initial state $\mathbb{X}(\vartheta) = \mathbb{X}_0$, $-\sigma \leq \vartheta \leq 0$ to a designated final state $\mathbb{X}_f$ through the application of a control law $\mathbb{U}(\vartheta) \in \mathbb{R}^{m_1 \times m_2}$ valid within the interval $[0, \vartheta_f]$, such that (3) has a solution $\mathbb{X} : [-\sigma, \vartheta_f] \rightarrow \mathbb{R}^n$ that satisfies $\mathbb{X}(\vartheta) = \mathbb{X}_0$, $-\sigma \leq \vartheta \leq 0$, and $\mathbb{X}(\vartheta_f) = \mathbb{X}_f$ for all $\mathbb{X}_0, \mathbb{X}_f \in \mathbb{R}^{n \times p}$.

Definition 6. The delay Gramian matrix of system (3) is defined as follows:

$$\Pi_{\sigma}[0, \vartheta_f] = \int_0^{\vartheta_f} \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta_f - s) (\mathbb{F}^T \otimes \mathbb{E}) (\mathbb{F} \otimes \mathbb{E}^T) \mathcal{F}_{\sigma}^{\mathbb{C}^T, \mathbb{W}^T}(\vartheta_f - s) ds, \quad (7)$$

where $\cdot^T$ denotes the transpose of the matrix.

Theorem 3. The system (3) is controllable in $[0, \vartheta_f]$ if and only if the $np \times np$ symmetric controllability delay matrix Π_{σ} is positive definite. In this case, the control

$$\mathbb{U}(\vartheta) = (\mathbb{F} \otimes \mathbb{E}^T) \mathcal{F}_{\sigma}^{\mathbb{C}^T, \mathbb{W}^T}(\vartheta_f - \vartheta) \Pi_{\sigma}^{-1}[0, \vartheta_f] \mu, \quad (8)$$

where

$$\mu = \mathbb{X}_f - \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta_f) \Psi(0) - \mathbb{W} \int_{-\sigma}^0 \mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta_f - \sigma - s) \Psi(s) ds,$$

defined in $0 \leq \vartheta \leq \vartheta_f$, transfers $\mathbb{X}(\vartheta) = \mathbb{X}_0$, $-\sigma \leq \vartheta \leq 0$ to $\mathbb{X}(\vartheta_f) = \mathbb{X}_f$.

Proof. Assume that $\Pi_{\sigma}[0, \vartheta_f]$ is positive definite. Then it is a nonsingular matrix, which guarantees that $\Pi_{\sigma}^{-1}[0, \vartheta_f]$ exists. By Lemma 2, $\mathcal{F}_{\sigma}^{\mathbb{C}, \mathbb{W}}(\vartheta)$ is continuous and bounded on $[0, \vartheta_f]$, which ensures that the control function defined by (8) is well-defined and the

integrals in the solution representation (5) exist. Now, substituting (8) into (5) with $\vartheta = \vartheta_f$, and using the properties of $\mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta)$ yields

$$\begin{aligned}\mathbb{X}(\vartheta_f) &= \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f)\Psi(0) + \mathbb{W} \int_{-\sigma}^0 \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f - \sigma - s)\Psi(s)ds \\ &\quad + \int_0^{\vartheta_f} \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f - s) \left(\mathbb{F}^T \otimes \mathbb{E} \right) (\mathbb{F} \otimes \mathbb{E}^T) \mathcal{F}_\sigma^{\mathbb{C}^T, \mathbb{W}^T}(\vartheta_f - s) ds \Pi_\sigma^{-1}[0, \vartheta_f] \mu \\ &= \mathbb{X}_f.\end{aligned}$$

Hence, (3) is controllable on $[0, \vartheta_f]$. Conversely, assume that (3) is controllable on $[0, \vartheta_f]$; then, we have to show that $\Pi_\sigma[0, \vartheta_f]$ is nonsingular. The continuity and boundedness of $\mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta)$ (Lemma 2) guarantee that the Gramian matrix $\Pi_\sigma[0, \vartheta_f]$ is well-defined. Suppose that $\Pi_\sigma[0, \vartheta_f]$ is symmetric and positive semidefinite. Thus, there exists at least one non-zero arbitrary constant state $z \in \mathbb{R}^{np}$ such that

$$\begin{aligned}0 &= z^T \Pi_\sigma[0, \vartheta_f] z = \int_0^{\vartheta_f} z^T \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f - s) \left(\mathbb{F}^T \otimes \mathbb{E} \right) (\mathbb{F} \otimes \mathbb{E}^T) \mathcal{F}_\sigma^{\mathbb{C}^T, \mathbb{W}^T}(\vartheta_f - s) z ds \\ &= \int_0^{\vartheta_f} \theta(s, \vartheta_f) \theta^T(s, \vartheta_f) ds = \int_0^{\vartheta_f} \|\theta\|^2 ds,\end{aligned}\quad (9)$$

where

$$\theta(s, \vartheta_f) = z^T \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f - s) \left(\mathbb{F}^T \otimes \mathbb{E} \right),$$

which implies that

$$\int_0^{\vartheta_f} \|\theta\|^2 ds = 0,$$

then

$$\theta(s, \vartheta_f) = z^T \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f - s) \left(\mathbb{F}^T \otimes \mathbb{E} \right) := \mathbf{0}^T, \quad 0 \leq \vartheta \leq \vartheta_f, \quad (10)$$

where $\mathbf{0}$ signifies the np -dimensional null vector. Since (3) is controllable on $[0, \vartheta_f]$, there exists a control $\mathbb{U}_1(\vartheta)$ that drives the initial state to zero state at ϑ_f . That is,

$$\begin{aligned}\mathbb{X}(\vartheta_f) &= \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f)\Psi(0) + \mathbb{W} \int_{-\sigma}^0 \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f - \sigma - s)\Psi(s)ds \\ &\quad + \int_0^{\vartheta_f} \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f - s) \left(\mathbb{F}^T \otimes \mathbb{E} \right) \mathbb{U}_1(s) ds \\ &= \mathbf{0},\end{aligned}\quad (11)$$

similarly, there exists a control $\mathbb{U}_2(\vartheta)$ that drives the initial state to $z \neq 0$ state at ϑ_f . That is,

$$\begin{aligned}\mathbb{X}(\vartheta_f) &= \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f)\Psi(0) + \mathbb{W} \int_{-\sigma}^0 \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f - \sigma - s)\Psi(s)ds \\ &\quad + \int_0^{\vartheta_f} \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f - s) \left(\mathbb{F}^T \otimes \mathbb{E} \right) \mathbb{U}_2(s) ds \\ &= z.\end{aligned}\quad (12)$$

From (11) and (12), we have

$$z = \int_0^{\vartheta_f} \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f - s) \left(\mathbb{F}^T \otimes \mathbb{E} \right) [\mathbb{U}_2(s) - \mathbb{U}_1(s)] ds. \quad (13)$$

Multiplying (13) by z^T , and using (10), we have

$$z^T z = \int_0^{\vartheta_f} z^T \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(\vartheta_f - s) \left(\mathbb{F}^T \otimes \mathbb{E} \right) [\mathbb{U}_2(s) - \mathbb{U}_1(s)] ds = 0.$$

We conclude that $z^T z = 0$, which contradicts the assumption $z \neq 0$. Therefore, $\Pi_{\sigma}[0, \vartheta_f]$ is positive definite and consequently nonsingular. The proof is complete. $\square$

5. An Example

The following example illustrates the theoretical results.

Example 1. Consider the nonhomogeneous delay differential Sylvester matrix equation:

$$\begin{aligned} X'(\vartheta) &= \mathbb{A} X(\vartheta) + X(\vartheta) \mathbb{B} + \mathbb{S} X(\vartheta - 1) + X(\vartheta - 1) \mathbb{J} + \mathbb{E} U(\vartheta) \mathbb{F}, \quad \text{for } \vartheta \in [0, 2], \\ X(\vartheta) &= \Phi(\vartheta) = \begin{pmatrix} \vartheta & 0 \\ 0 & \vartheta \end{pmatrix}, \quad \text{for } \vartheta \in [-1, 0], \end{aligned} \quad (14)$$

where

$$\begin{aligned} \mathbb{A} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathbb{E} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ \mathbb{S} &= \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, \quad \mathbb{J} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}, \quad \mathbb{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \Psi(\vartheta) &= \begin{pmatrix} \vartheta \\ 0 \\ 0 \\ \vartheta \end{pmatrix}, \\ (\mathbb{F}^T \otimes \mathbb{E}) &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (\mathbb{F} \otimes \mathbb{E}^T) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \mathbb{C} &= \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \mathbb{W} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ -1 & 2 & 0 & 2 \\ 3 & 0 & 1 & 0 \\ 0 & 3 & -1 & 2 \end{pmatrix}. \end{aligned}$$

Using Definitions 3 and 4, we compute

$$\mathcal{F}_1^{\mathbb{C}, \mathbb{W}}(\vartheta) := \begin{cases} \Theta, & -1 \leq \vartheta < 0, \\ \mathbb{I}, & \vartheta = 0, \\ \sum_{i=0}^{\infty} \frac{\vartheta^i}{i!} \Upsilon^{\mathbb{C}, \mathbb{W}}(i+1; 0), & 0 < \vartheta \leq 1, \\ \sum_{i=0}^{\infty} \sum_{j=0}^1 \frac{(\vartheta-j)^i}{i!} \Upsilon^{\mathbb{C}, \mathbb{W}}(i+1; j), & 1 < \vartheta \leq 2, \end{cases}$$

and, for $i \geq 0$, we deduce that

$$\Upsilon^{\mathbb{C}, \mathbb{W}}(i+1; 0) = \mathbb{C}^i, \quad \Upsilon^{\mathbb{C}, \mathbb{W}}(i+1; 1) = i\mathbb{W}\mathbb{C}^{i-1},$$

such that

$$\Upsilon^{\mathbb{C}, \mathbb{W}}(0; s) = \Upsilon^{\mathbb{C}, \mathbb{W}}(k; -1) = \Theta, \quad \Upsilon^{\mathbb{C}, \mathbb{W}}(1; 0) = \mathbb{I}.$$

Thus

$$\mathcal{F}_1^{\mathbb{C}, \mathbb{W}}(\vartheta) = \sum_{i=0}^{\infty} \sum_{j=0}^1 \frac{(\vartheta-j)^i}{i!} \Upsilon^{\mathbb{C}, \mathbb{W}}(i+1; j) = e^{\mathbb{C}\vartheta} + (\vartheta-1)\mathbb{W}e^{\mathbb{C}(\vartheta-1)}.$$

By Theorem 2, the explicit solutions $\mathbb{X}(\vartheta)$ of (14) for $\vartheta \in [0, 2]$ are given by

$$\begin{aligned}\mathbb{X}(\vartheta) &= \mathcal{F}_1^{\mathbb{C}, \mathbb{W}}(\vartheta) \Psi(0) + \mathbb{W} \int_{-1}^0 \mathcal{F}_1^{\mathbb{C}, \mathbb{W}}(\vartheta - s - 1) \Psi(s) ds \\ &\quad + \int_0^\vartheta \mathcal{F}_1^{\mathbb{C}, \mathbb{W}}(\vartheta - s) (\mathbb{F}^T \otimes \mathbb{E}) \mathbb{U}(s) ds \\ &= \mathbb{W} \int_{-1}^0 \left[e^{\mathbb{C}(\vartheta-s-1)} + (\vartheta - s - 2) e^{\mathbb{C}(\vartheta-s-2)} \mathbb{W} \right] \Psi(s) ds \\ &\quad + \int_0^\vartheta \left[e^{\mathbb{C}(\vartheta-s)} + (\vartheta - s - 1) e^{\mathbb{C}(\vartheta-s-1)} \mathbb{W} \right] (\mathbb{F}^T \otimes \mathbb{E}) \mathbb{U}(s) ds.\end{aligned}$$

Moreover, the delay Gramian matrix corresponding to (14) is constructed using (7), giving

$$\begin{aligned}\Pi_1[0, 2] &= \int_0^2 \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(2-s) (\mathbb{F}^T \otimes \mathbb{E}) (\mathbb{F} \otimes \mathbb{E}^T) \mathcal{F}_\sigma^{\mathbb{C}^T, \mathbb{W}^T}(2-s) ds \\ &= \begin{pmatrix} 74.76 & 114.96 & 43.01 & 86.78 \\ 114.96 & 249.85 & 64.14 & 163.56 \\ 43.01 & 64.14 & 35.07 & 70.53 \\ 86.78 & 163.56 & 70.53 & 168.73 \end{pmatrix} > 0,\end{aligned}$$

and

$$\Pi_1^{-1}[0, 2] = \begin{pmatrix} 0.14 & -0.06 & -0.21 & 0.07 \\ -0.06 & 0.03 & 0.09 & -0.04 \\ -0.21 & 0.09 & 0.51 & -0.19 \\ 0.07 & -0.04 & -0.19 & 0.09 \end{pmatrix}.$$

Since the numerical properties of the delay Gramian matrix provide concrete, quantitative metrics to validate the theoretical controllability results, the computed Gramian $\Pi_1[0, 2]$ is positive definite, as verified by its eigenvalues:

$$\lambda(\Pi_1[0, 2]) \approx \{1.463, 19.118, 47.427, 460.402\}.$$

All eigenvalues being positive provides a direct numerical confirmation of the nonsingularity required by Theorem 3. As a result, for any finite terminal conditions $\mathbb{X}_f \in \mathbb{R}^{2 \times 2}$ such that $\mathbb{X}(2) = \mathbb{X}_f = (X_1, X_2, X_3, X_4)^T$, the control function $\mathbb{U}(\vartheta) = (U_1, U_2, U_3, U_4)^T$ is given by

$$\mathbb{U}(\vartheta) = (\mathbb{F} \otimes \mathbb{E}^T) \mathcal{F}_\sigma^{\mathbb{C}^T, \mathbb{W}^T}(2 - \vartheta) \Pi_\sigma^{-1}[0, 2] \mu,$$

where

$$\mu = \mathbb{X}_f - \mathbb{W} \int_{-1}^0 \mathcal{F}_\sigma^{\mathbb{C}, \mathbb{W}}(1-s) \Psi(s) ds,$$

Hence, the system (14) is controllable on $[0, 2]$ by Theorem 3. The successful computation of the control input $\mathbb{U}(\vartheta)$ that drives the system from the initial state to the specified final state $\mathbb{X}_f$ for various initial conditions, as visualized in the subsequent figures, is the ultimate quantitative demonstration of the system's controllability over the interval $[0, 2]$. To visualize the system behavior, we plot the state trajectories and control inputs for two different initial conditions. Figure 2 shows the state evolution $X(\vartheta)$ for the initial vector $X = (2, 1, 3, 4)^T$, while Figure 3 displays the corresponding control functions $U_1(\vartheta), U_2(\vartheta), U_3(\vartheta), U_4(\vartheta)$ over the interval $[0, 2]$. Similarly, Figure 4 illustrates the state $X(\vartheta)$ for $X = (1, 7, 6, 0)^T$, and Figure 5 shows the associated control inputs.

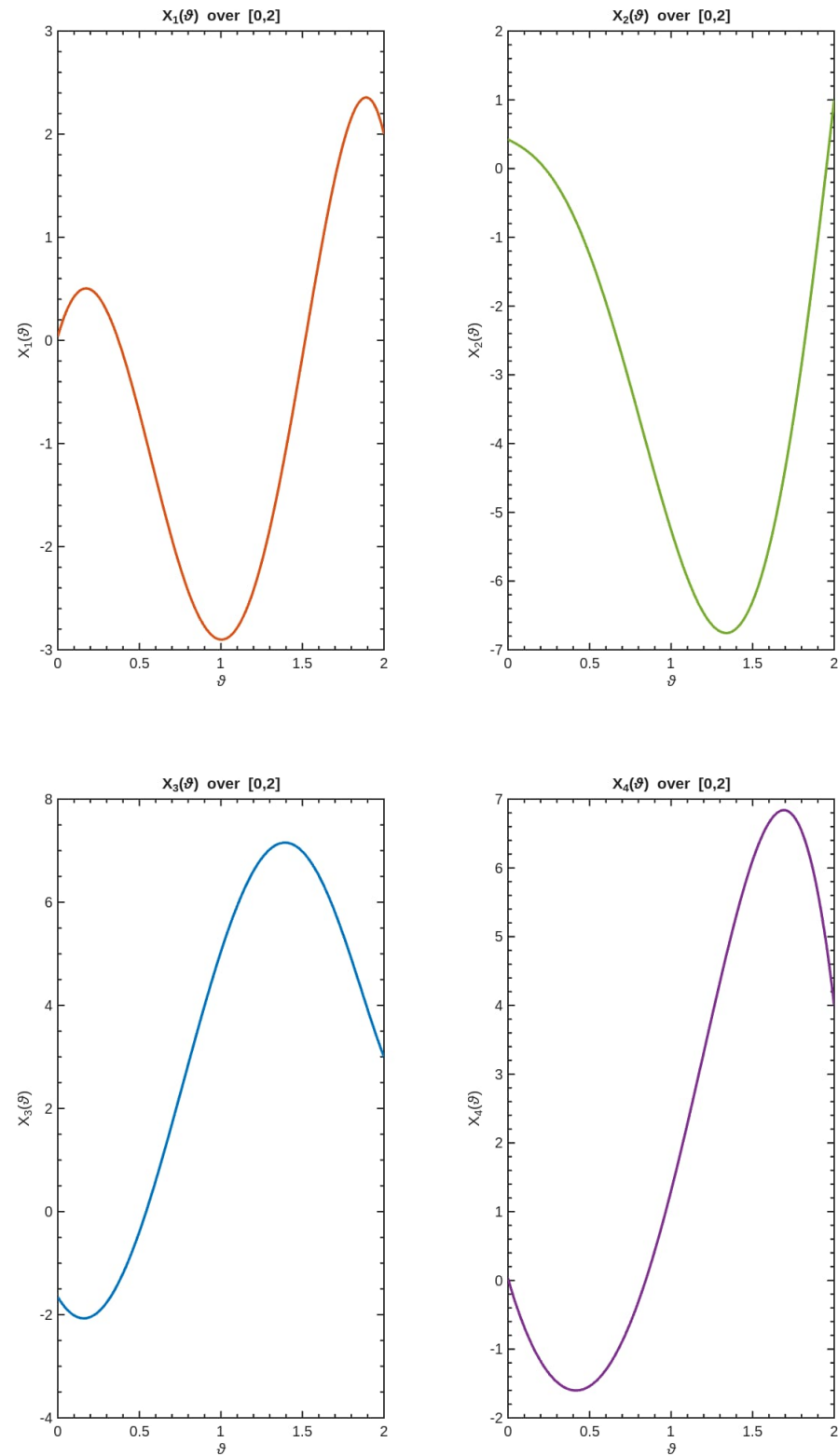


Figure 2. The trajectory of the state $X(\theta)$ of system (14) in the interval $[0, 2]$ when $X = (X_1, X_2, X_3, X_4)^\top = (2, 1, 3, 4)^\top$.

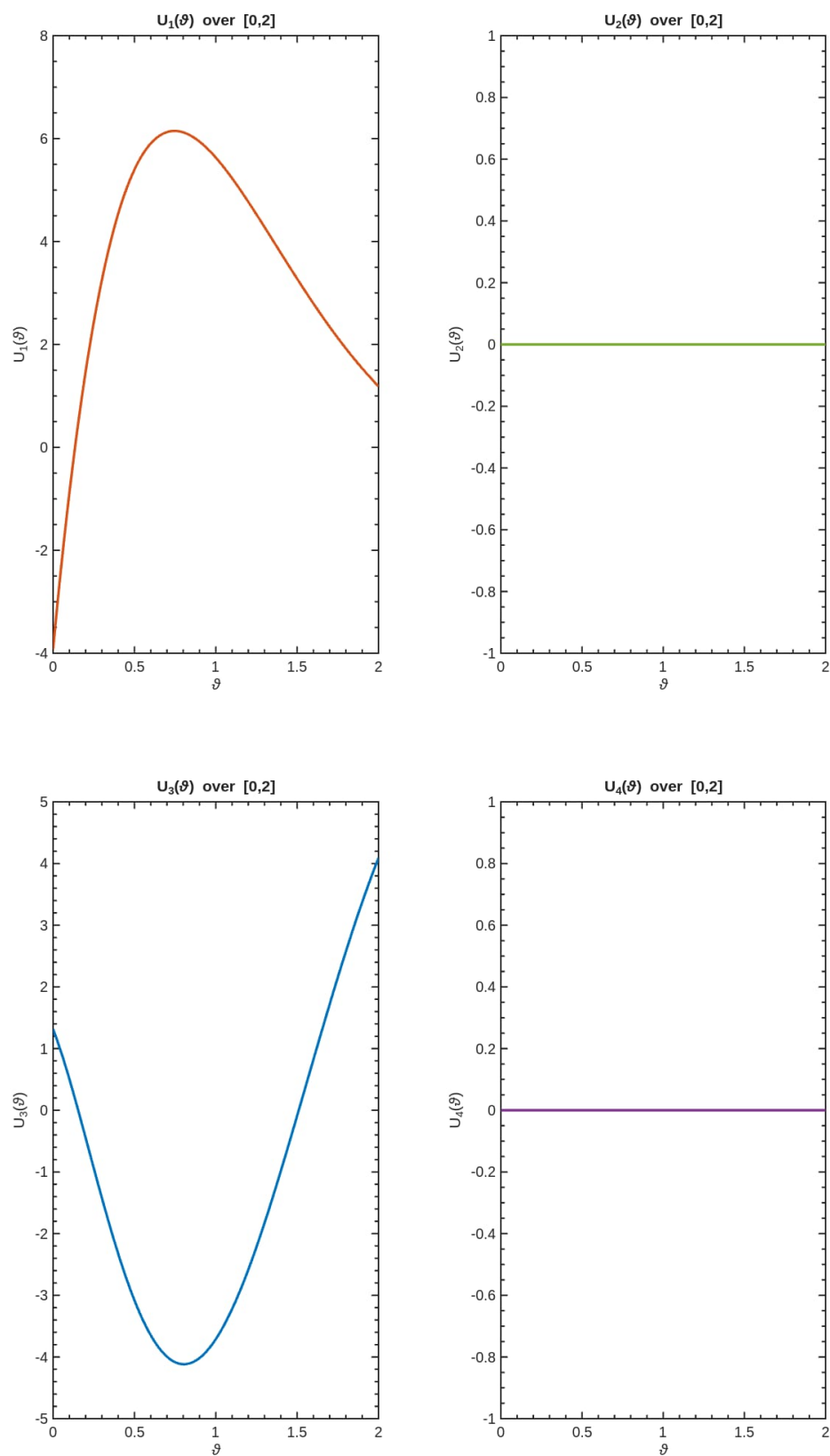


Figure 3. The trajectory of the control functions $U_1(\vartheta)$, $U_2(\vartheta)$, $U_3(\vartheta)$, $U_4(\vartheta)$ for $\vartheta \in [0, 2]$ when $X = (X_1, X_2, X_3, X_4)^\top = (2, 1, 3, 4)^\top$.

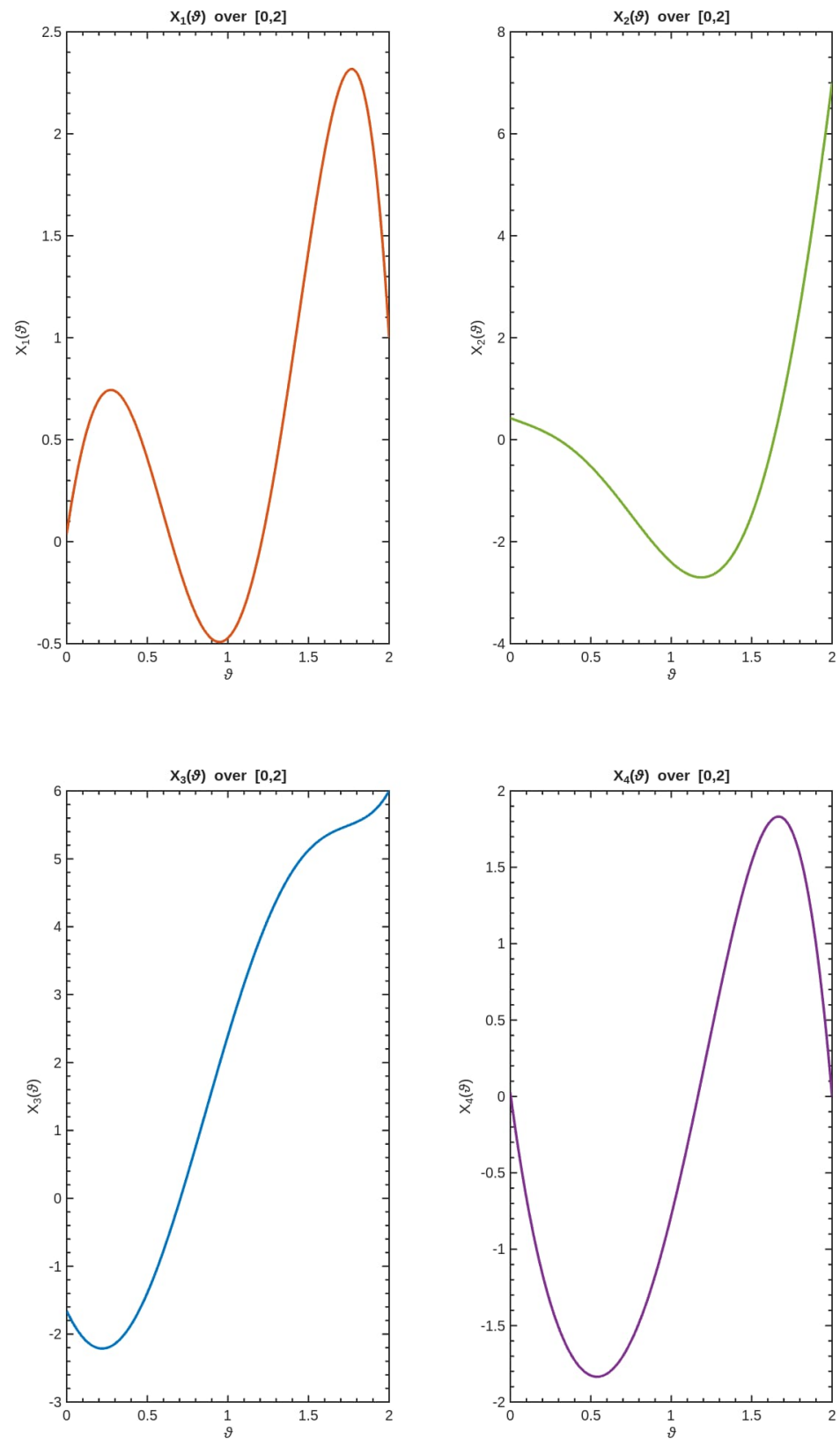


Figure 4. The trajectory of the state $X(\theta)$ of system (14) in the interval $[0, 2]$ when $X = (X_1, X_2, X_3, X_4)^\top = (1, 7, 6, 0)^\top$.

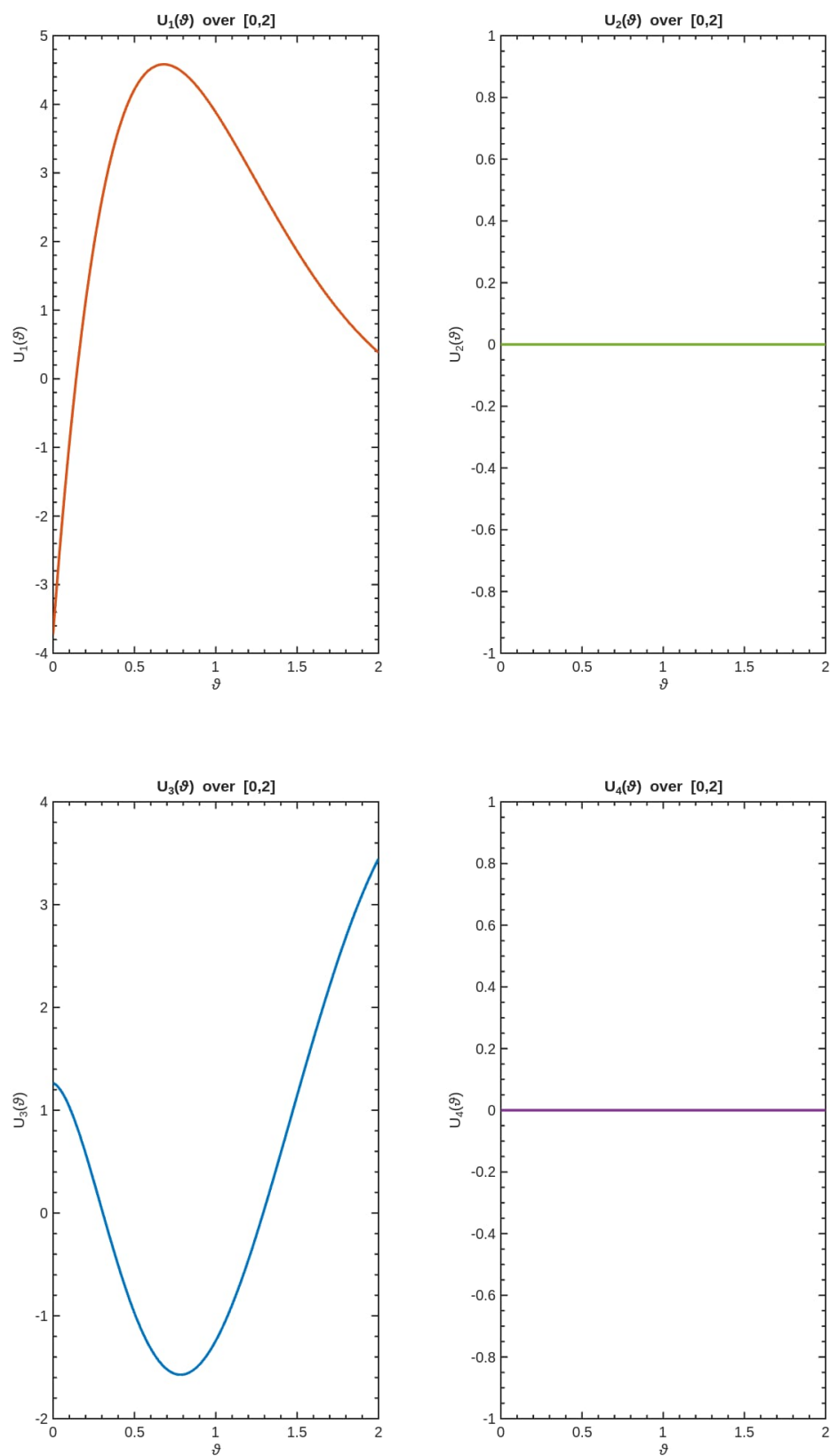


Figure 5. The trajectory of the control functions $U_1(\vartheta)$, $U_2(\vartheta)$, $U_3(\vartheta)$, $U_4(\vartheta)$ for $\vartheta \in [0, 2]$ when $X = (X_1, X_2, X_3, X_4)^\top = (1, 7, 6, 0)^\top$.

Remark 3. *It is seen that the conclusion of Theorem 5 in [31] is not applicable to the system (14) because our results impose minimal restrictions on the coefficient matrices: they are not required to commute. Therefore, our results improve some existing ones.*

6. Conclusions

This study has successfully established a unified and computationally tractable framework for analyzing the solutions and controllability of linear delay differential Sylvester matrix systems even in the absence of commutativity among coefficient matrices. By leveraging the vectorization operator and Kronecker product algebra, we transformed the original matrix system into an equivalent vector differential equation, thereby circumventing the analytical complexities inherent in noncommuting matrix structures. This reduction enabled us to derive explicit closed-form expressions for the general solution using a delayed perturbation matrix function induced by the time delay.

A key strength of our approach lies in its generality: the assumptions on the system matrices are deliberately minimal, allowing the inclusion of numerous practically relevant cases from single matrix delay equations to coupled systems as special instances of our formulation. Moreover, we introduced a delay-adapted Gramian matrix and used it to derive necessary and sufficient conditions for controllability.

The effectiveness of the proposed methodology was demonstrated through carefully constructed examples, which confirm that our solution representation and controllability test remain valid and efficient even when traditional approaches break down due to noncommutativity or delay coupling. These results not only resolve a significant theoretical gap but also provide engineers and applied scientists with practical tools for designing controllers in matrix-structured systems such as those arising in networked control, multi-agent coordination, vibration suppression, and tensor-based dynamical modeling.

Future research directions include extending the framework to nonlinear perturbations, stochastic disturbances, fractional-order derivatives, and systems with multiple or state-dependent delays all under nonpermutable matrix structures. Additionally, developing efficient numerical algorithms based on the derived closed-form solutions, along with a rigorous analysis of their computational complexity and a comparative study against standard DDE solvers, and integrating these results with Krylov subspace methods [3,30] for high-dimensional systems present a promising avenue for scalable numerical solutions.

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References

1. Abou-Kandil, H.; Frelling, G.; Ionescu, V.; Jank, G. *Matrix Riccati Equations in Control and Systems Theory*; Systems and Control: Foundations and Applications; Birkhäuser: Basel, Switzerland, 2003.
2. Corless, M.J.; Frazho, A.E. *Linear Systems and Control—An Operator Perspective*; Pure and Applied Mathematics; Marcel Dekker: New York, NY, USA, 2003.
3. Hached, M.; Jbilou, K. Computational Krylov-based methods for large-scale differential Sylvester matrix problems. *Numer. Linear Algebra Appl.* **2018**, *25*, e2187. [\[CrossRef\]](#)
4. Driver, R.D. *Ordinary and Delay Differential Equations*; Springer: New York, NY, USA, 1977.
5. Élsgolts, L.E.; Norkin, S.B. *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*; Mathematics in Science and Engineering; Academic Press: New York, NY, USA, 1973; Volume 105.
6. Hale, J.K.; Verduyn Lunel, S.M. *Introduction to Functional-Differential Equations*; Applied Mathematical Sciences; Springer: New York, NY, USA, 1993; Volume 99.
7. Kolmanovskii, V.; Myshkis, A. *Introduction to the Theory and Applications of Functional Differential Equations*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1999.
8. Khusainov, D.Y.; Diblík, J.; Růžicková, M.; Lukáčová, J. A representation of the solution of the Cauchy problem for an oscillatory system with pure delay. *Nonlinear Oscil.* **2008**, *11*, 276–285. [\[CrossRef\]](#)
9. Khusainov, D.Y.; Shuklin, G.V. Linear autonomous time-delay system with permutation matrices solving. *Math. Ser.* **2003**, *17*, 101–108.
10. Pospíšil, M.; Jaros, F. On the representation of solutions of delayed differential equations via Laplace transform. *Electron. J. Qual. Theory Differ. Equ.* **2016**, *2016*, 117. [\[CrossRef\]](#)
11. Mahmudov, N.I. A novel fractional delayed matrix cosine and sine. *Appl. Math. Lett.* **2019**, *92*, 41–48. [\[CrossRef\]](#)
12. Pan, R.; Fan, Z. Analyses of solutions of Riemann-Liouville fractional oscillatory differential equations with pure delay. *Math. Methods Appl. Sci.* **2023**, *46*, 10450–10464. [\[CrossRef\]](#)
13. Elshenhab, A.M.; Wang, X.T. Representation of solutions of linear differential systems with pure delay and multiple delays with linear parts given by non-permutable matrices. *Appl. Math. Comput.* **2021**, *410*, 126443. [\[CrossRef\]](#)
14. Mahmudov, N.I. Multi-delayed perturbation of Mittag-Leffler type matrix functions. *J. Math. Anal. Appl.* **2022**, *505*, 125589. [\[CrossRef\]](#)
15. Lakshmi Priya, P.K.; Kaliraj, K. A study on the finite time stability and controllability of time delay fractional model. *Arab. J. Math.* **2025**, *14*, 155–170. [\[CrossRef\]](#)
16. Elshenhab, A.; Wang, X.T. Representation of solutions of delayed linear discrete systems with permutable or nonpermutable matrices and second-order differences. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2022**, *116*, 58. [\[CrossRef\]](#)
17. Pervaiz, B.; Zada, A.; Popa, I.-L.; Ben Moussa, S.; Kallekh, A. Exponential stability and relative controllability of first-order delayed integro-differential systems with impulses. *Math. Methods Appl. Sci.* **2024**, *47*, 7590–7615. [\[CrossRef\]](#)
18. Liang, C.; Wang, J. Analysis of iterative learning control for an oscillating control system with two delays. *Trans. Inst. Meas. Control* **2018**, *40*, 1757–1765. [\[CrossRef\]](#)
19. Amer, Y.; El-Sayed, A.; Ahmed, E.E. Vibration reduction of a non-linear ship model using positive position feedback controllers. *Int. J. Dyn. Control* **2022**, *10*, 409–426. [\[CrossRef\]](#)
20. Zhang, Y.; Zhang, X.; Xiong, F.-R.; Liu, L.; Zhang, S. Active vibration control of typical piping system of a nuclear power plant based on fractional PI controller. *Int. J. Dyn. Control* **2022**, *10*, 2111–2123. [\[CrossRef\]](#)
21. Biswal, S.S.; Swain, D.R.; Rout, P.K. Inter-area and intra-area oscillation damping for UPFC in a multi-machine power system based on tuned fractional PI controllers. *Int. J. Dyn. Control* **2022**, *10*, 1594–1612. [\[CrossRef\]](#)
22. Das, M.; Samanta, G. Optimal control of a fractional order epidemic model with carriers. *Int. J. Dyn. Control* **2022**, *10*, 598–619. [\[CrossRef\]](#)
23. Bouazza, L.; Mourillon, B.; Makhlouf, A.; Birouche, A. Controllability and observability of formal perturbed linear time invariant systems. *Int. J. Dyn. Control* **2021**, *9*, 1444–1455. [\[CrossRef\]](#)
24. Boudjerida, A.; Seba, D. Controllability of nonlocal Hilfer fractional delay dynamic inclusions with non-instantaneous impulses and non-dense domain. *Int. J. Dyn. Control* **2022**, *10*, 1613–1625. [\[CrossRef\]](#)
25. Murty, M.; Appa, R.B.; Suresh, K.G. Controllability, observability, and realizability of matrix Lyapunov systems. *Bull. Korean Math. Soc.* **2006**, *43*, 149–159. [\[CrossRef\]](#)
26. Murty, M.; SureshKumar, G. On ψ -bounded solutions for non-homogeneous matrix Lyapunov systems on $\mathbb{R}$. *Electron. J. Qual. Theory Differ. Equ.* **2009**, *2009*, 1–12. [\[CrossRef\]](#)
27. Dubey, B.; George, R.K. Controllability of semilinear matrix Lyapunov systems. *Electron. J. Differ. Equ.* **2013**, *2013*, 1–12.
28. Murty, M.; Kumar, G.S.; Rao, B.A.; Prasad, K. On controllability of fuzzy dynamical matrix Lyapunov systems. *Ann. West Univ. Timișoara-Math. Comput. Sci.* **2013**, *51*, 73–87. [\[CrossRef\]](#)

29. Appa Rao, B.V.; Prasad, K.A.S.N.Y. Controllability and observability of Sylvester matrix dynamical systems on time scales. *Kyungpook Math. J.* **2016**, *56*, 529–539. [[CrossRef](#)]
30. Sadek, E.M.; Bentbib, A.H.; Sadek, L.; Talibi Alaoui, H. Global extended Krylov subspace methods for large-scale differential Sylvester matrix equations. *J. Appl. Math. Comput.* **2020**, *62*, 157–177. [[CrossRef](#)]
31. Diblík, J. Representation of solutions to a linear matrix first-order differential equation with delay. *Bull. Math. Sci.* **2025**, 1–13. . [[CrossRef](#)]
32. Graham, A. *Kronecker Products and Matrix Calculus with Applications*; Dover Publications: Mineola, NY, USA, 2018.
33. Mahmudov, N.I. Delayed perturbation of Mittag-Leffler functions and their applications to fractional linear delay differential equations. *Math. Methods Appl. Sci.* **2019**, *42*, 5489–5497. [[CrossRef](#)]
34. Wang, J.; Luo, Z.; Fečkan, M. Relative controllability of semilinear delay differential systems with linear parts defined by permutable matrices. *Eur. J. Control* **2017**, *38*, 39–46. [[CrossRef](#)]

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