

Article

Decision Making Under Uncertainty: A Z-Number-Based Regret Principle

Ramiz Alekperov ^{1,*} , Vugar Salahli ¹  and Rahib Imamguluyev ² ¹ Department of Mathematics and Digital Technologies, Odlar Yurdu University, Baku AZ1072, Azerbaijan; vuqar.salahli@oyu.edu.az² Department of Information Technologies, Baku Business University, 88a H. Zardabi St., Baku AZ1122, Azerbaijan; rahib.imamguluyev@bbu.edu.az

* Correspondence: elekberov.ramiz@oyu.edu.az

Abstract

Decision-making theory has developed over many decades at the intersection of economics, mathematics, psychology, and engineering. Its classical foundations include Bernoulli's expected utility theory, von Neumann and Morgenstern's rational choice theory, and the criteria proposed by Savage, Wald, Hurwicz, and others. However, in real-world contexts, decisions are made under uncertainty, incompleteness, and unreliability of information, which classical approaches do not adequately address. To overcome these limitations, modern multi-criteria decision-making methods such as Analytic Hierarchy Process (AHP), Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), VlseKriterijumska Optimizacija I Kompromisno Resenje (Compromise solution approach) (VIKOR), and EElimination Et Choix Traduisant la REalité (Elimination and Choice Expressing Reality) (ELECTRE), as well as their fuzzy and Z-number extensions, are widely applied to the modeling and evaluation of complex systems. These Z-number extensions are based on the concept of Z-numbers introduced by Lotfi Zadeh in 2011 to formalize higher-order uncertainty. This study introduces the Z-Regret principle, which extends Savage's regret criterion through the use of Z-numbers. Supported by Rafik Aliev's mathematical justifications concerning arithmetic operations on Z-numbers, the model evaluates regret not only as a loss relative to the best alternative but also by incorporating the degree of confidence and reliability of this evaluation. Calculations for the selection of digital advertising platforms in terms of performance assessment under various scenarios demonstrate that the Z-Regret principle enables more stable and well-founded decision-making under uncertainty.

Keywords: Z-Regret principle; decision-making under uncertainty; regret theory; Z-numbers; fuzzy sets; multi-criteria decision-making; digital advertising performance

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1. Introduction

Decision-making theory has developed over many decades at the intersection of economics, mathematics, psychology, and engineering. Its classical foundations include Bernoulli's expected utility theory, von Neumann and Morgenstern's rational choice theory, and the criteria of Savage, Wald, Hurwicz, and others. However, in real-world conditions, decisions are made under uncertainty, incompleteness, and unreliability of information, which classical approaches do not adequately address.

To overcome these limitations, modern multi-criteria decision-making methods such as AHP, TOPSIS, VIKOR, and ELECTRE, as well as their fuzzy and Z-number extensions, are widely applied in modeling complex systems. The Z-number concept, introduced by Lotfi Zadeh in 2011, enables the representation of both uncertainty and the reliability of information within a unified framework.

Motivated by the need to account for the reliability of information in regret-based models, this study proposes the Z-Regret principle—an extension of Savage’s regret criterion based on Z-numbers. By combining the fuzzy evaluation component (A) with the reliability component (B), the Z-Regret approach evaluates regret not only as a deviation from the best alternative but also by incorporating the decision-maker’s confidence in the available data.

Drawing on Rafik Aliev’s methods for arithmetic operations on Z-numbers, the proposed model is applied to the selection of digital advertising platforms, demonstrating that the Z-Regret principle ensures more consistent and well-grounded decisions under uncertainty.

2. Literature Review

2.1. Classical Decision-Making Theories

The main distinguishing criterion in the theory of decision-making is the degree of awareness of the probabilities of cases of nature corresponding to possible consequences. For cases where the probabilities are objectively known (risk conditions), Bernoulli [1] proposed the idea of measuring the benefit and calculating the expected value, while von Neumann and Morgenstern [2] summarized this approach with the axiomatic basis of rationality. For cases where probabilities are not known exactly but can be estimated by subjective assessment, Savage [3] developed the concept of subjective probability, substantiating the application of benefit theory under conditions of uncertainty. Finally, for cases where probabilities are not known at all, decision-making was carried out according to Wald’s Maximin approach [4] and Hurwicz’s criterion [5], as well as the minimax regret principle introduced by Savage in the same work [3].

Classical decision-making principles are usually based on the degree of knowledge of probabilities. However, the growing complexity of decision-making processes has necessitated the development of new methods to address problems under multi-criteria and multi-alternative conditions. In this context, the multi-attribute utility theory introduced by Keeney and Raiffa [6] enabled the evaluation of alternatives across multiple objectives. The AHP, developed by Saaty [7], provided a hierarchical structure and a mechanism for pairwise comparison of criteria. The TOPSIS method proposed by Hwang and Yoon [8] facilitated decision-making in multi-alternative settings by selecting solutions based on their relative closeness to the ideal and distance from the anti-ideal.

Within social choice theory, Arrow’s impossibility theorem [9] and the Gibbard–Satterthwaite results [10,11] revealed inherent limitations in collective choice. In game theory, Nash [12,13], Harsanyi [14], and Selten [15] established a theoretical foundation for modeling interdependent decisions. Finally, Simon’s concept of “bounded rationality” [16] and Kahneman and Tversky’s Prospect Theory [17] highlighted the impact of psychological constraints and behavioral factors on decision-making processes.

2.2. Fuzzy Sets and Their Extensions

A major turning point in the modeling of uncertainty was the introduction of fuzzy sets by Zadeh [18]. Based on this theoretical foundation, a number of extensions were subsequently developed: the possibility theory of Dubois and Prade [19], Shafer’s evi-

dence theory [20], and Atanassov's intuitionistic fuzzy sets [21] are regarded as important contributions to the modeling of uncertain knowledge.

In addition, new extensions of fuzzy sets have been proposed in recent years, such as $Q[\varepsilon]$ -fuzzy sets based on the Hardy field and complex fuzzy sets, which allow the modeling of both algebraic and complex-valued uncertainty [22,23].

One of the key milestones in the evolution of fuzzy approaches has been their integration with classical multi-criteria decision-making methods. For example, Saaty's Analytic Hierarchy Process has been extended into the fuzzy-AHP form [24], while the TOPSIS method of Hwang and Yoon has been generalized into fuzzy-TOPSIS [25]. These approaches have enabled the incorporation of uncertain and incomplete information into the decision-making process, allowing for more realistic and reliable results in the fields of economics [26–29] and management [30–33].

Applications include the development of multi-criteria approaches for retail-outlet location selection [26], the selection of market structures [27], the segmentation of economic regions [28], the improvement of enterprise-management efficiency [29], and the processing of flexible queries in commercial-enterprise databases [33].

2.3. Theoretical Foundations of Z-Numbers

Z-numbers allow not only the representation of uncertainty in terms of its content but also take into account the degree of reliability of the information. However, in most existing studies, the computational procedures have been carried out by converting Z-numbers either into fuzzy numbers or into crisp values. For example, Kang, Wei, Li, and Deng [34] proposed a method for transforming Z-numbers into classical fuzzy numbers. Although such approaches provide computational simplicity, the additional information carried by Z-numbers—particularly the reliability component—is lost.

Recent research has sought to overcome this limitation. Jiang et al. [35] provided a comprehensive mathematical exploration of the expected value, variance, and semi-variance of Z-numbers, demonstrating their statistical interpretability and computational feasibility. Their work contributed to strengthening the mathematical basis of Z-number theory by formalizing its core properties and clarifying its relationship with credibility theory.

In a fundamentally new direction, Rafik Aliev developed an original mathematical framework for performing direct operations on Z-numbers without converting them into classical fuzzy numbers [36–38]. His approach introduced unique mechanisms for comparison, normalization, distance measurement, and aggregation directly in the Z-number space, thus forming a new branch in fuzzy decision theory. His monograph *Fundamentals of the Fuzzy Logic-Based Generalized Theory of Decisions* [38] systematized these developments and expanded the scope of Z-number applications across economics, management, and engineering.

2.4. Application Examples of Z-Numbers

In the subsequent stage, international studies have demonstrated the real application potential of Z-numbers across various fields. Liao [39] presented a comprehensive review of theoretical and practical applications of Z-number-based decision-making methods. Wang [40] et al. developed a new approach to multi-attribute group decision-making based on Z-numbers and the Choquet integral, while Su et al. [41] introduced the Z-DEMATEL (Decision Making Trial and Evaluation Laboratory (DEMATEL)) method in the energy sector. Alam and Badrul [42] applied the integration of Z-numbers into fuzzy decision-making processes under uncertainty, and Wu et al. [43] proposed a spherical Z-number-based model for multi-attribute group decision-making. Alam et al. [44] proposed

a synergic ranking approach for fuzzy Z-numbers based on vectorial distance and the spread criterion. The method integrates the content and reliability components of Z-numbers, enabling more precise comparisons and broadening their applicability in multi-criteria decision-making processes. The work *The Arithmetic of Z-numbers* by Aliev and colleagues [36] further deepened the theoretical foundations of these approaches and expanded their applicability in economics and engineering.

Applications of Z-numbers have also become widespread in the management of economic and business processes. Various studies have employed this approach in assessing employee motivation, processing survey data, detecting anomalies, and managing risks in energy consumption, often in combination with soft-computing methods [45–49]. Furthermore, applications have been reported in evaluating enterprise development levels [49], applying Z-information-based scenarios in energy policy development [50]. In addition, the integration of Z-numbers with fuzzy multi-criteria decision-making (MCDM) approaches has yielded new results in modeling the renewable energy transition [51].

3. Necessity of the Z-Regret Model for Robust Decisions Under Uncertainty

Savage's regret principle [3] is based on minimizing the maximum possible regret under uncertainty and serves as an important alternative criterion in classical decision theory. However, regret relies solely on the comparison of outcomes and does not take into account the accuracy or reliability of the information. This limitation is highly significant in real-life contexts, since uncertainty arises not only from the imprecision of outcomes but also from the degree of reliability of the available information.

Z-numbers were introduced to overcome this limitation. A Z-number, expressed as $Z = (A, B)$, consists of two components: A , representing the fuzzy assessment, and B , denoting the degree of reliability of that assessment [52]. Thus, when regret is formulated through Z-numbers, it reflects not only the loss relative to the best outcome but also the extent to which this loss is observed with a certain degree of reliability. This constitutes the essence of the Z-Regret model.

The Z-Regret approach enables more robust decision-making under uncertainty, as it simultaneously evaluates both the fuzzy analysis of outcomes and the reliability of the information. This approach represents a logical continuation of Rafik Aliev's generalized decision theory based on Z-numbers [38] and lays the foundation for a new stage in the development of decision theory.

4. Preliminaries

This section systematically outlines the foundational concepts, beginning with classical decision-making theories and progressing through fuzzy approaches to the formulation of Z-numbers. The objective is to construct a coherent theoretical framework that logically culminates in the introduction of the Z-Regret principle.

Definition 1 (Alternatives and States–Scenarios [1–4]). *A decision problem is defined by two sets:*

$$A = \{a_1, a_2, \dots, a_m\}, S = \{s_1, s_2, \dots, s_n\} \quad (1)$$

Here, A denotes the set of alternatives, and S denotes the set of states of nature (scenarios). For each pair (a_i, s_j) , the outcome function is defined as:

$$L : A \times S \rightarrow R, (a, s) \mapsto L(a, s) \quad (2)$$

Here, $L(a, s)$ denotes the outcome (utility or cost) of alternative a under scenario s .

Definition 2 (Optimal Outcome by Scenario [3,4]). *For each scenario, the optimal outcome is defined depending on the objective as follows:*

Case of Minimization:

$$L^*(s) = \min_{a \in A} L(a, s) \quad (3)$$

Case of Maximization:

$$L^*(s) = \max_{a \in A} L(a, s) \quad (4)$$

Here, the minimization case is typically applied to cost or loss functions, whereas the maximization case applies to utility or benefit functions.

Definition 3 (Classical Regret and the Minimax Criterion [3]). *The regret for alternative a_i is defined as:*

$$R(a_i, s_j) = |L(a_i, s_j) - L^*(s_j)| \quad (5)$$

The worst-case outcome for each alternative:

$$R_{\max}(a_i) = \max_{s \in S} R(a_i, s) \quad (6)$$

According to the minimax principle, the optimal alternative is selected as:

$$a^* = \operatorname{argmin}_{a \in A} R_{\max}(a) \quad (7)$$

Here, the operator “arg” denotes the argument of the minimum function, that is, the value of the decision variable (alternative a_i) for which the function $R_{\max}(a_i)$ attains its minimum value. In other words, a^* represents the alternative that minimizes the maximum regret among all possible alternatives.

The classical regret approach embodies the principle of “minimizing the loss under the worst-case scenario.” However, its fundamental limitation is the inability to incorporate the fuzziness of outcomes and the reliability of information, which are critical factors in real-world decision-making under uncertainty.

Definition 4 (Fuzzy Numbers [18,38,53]). *A fuzzy number A on a universe X is defined as follows:*

$$A = \{(x, \mu_A(x)) \mid x \in X\}, \quad \mu_A : X \rightarrow [0, 1] \quad (8a)$$

Here, $\mu_A(x)$ is the membership function.

Figure 1a illustrates a triangular fuzzy number. Here, uncertainty is represented by a single maximum point. A triangular fuzzy number is characterized by three parameters: $A = (a, c, b)$.

Here:

- a —the starting point (left support), where the membership value is $\mu(x) = 0$.
- c —the peak of the triangle, where the membership function reaches its maximum value— $\mu(x) = 1$.
- b —the end point (right support), where again $\mu(x) = 0$.

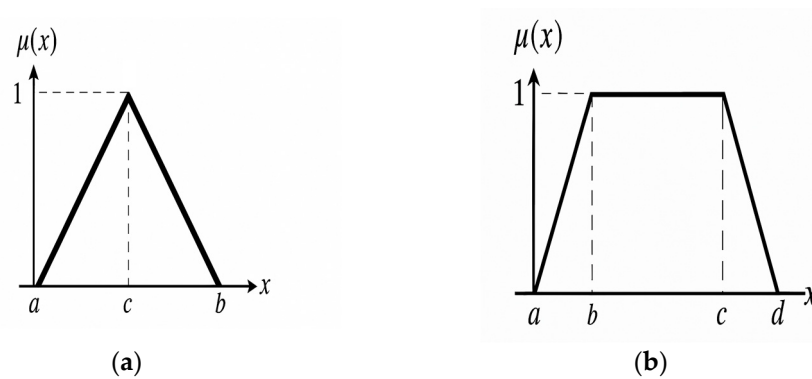


Figure 1. Membership functions of triangular (a) and trapezoidal (b) fuzzy numbers.

The membership function is mathematically defined as follows:

$$\mu_A(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{c-a}, & a < x \leq c \\ \frac{b-x}{b-c}, & c < x < b \\ 0, & x \geq b \end{cases} \quad (8b)$$

Since this function is triangular in shape, it represents one of the simplest fuzzy numbers and is often used in modeling uncertainties.

Figure 1b illustrates a trapezoidal fuzzy number, in which uncertainty is represented by a constant maximum interval. A trapezoidal fuzzy number is defined by four parameters: $B = (a, b, c, d)$.

Here:

- a —the starting point (left support), where $\mu(x) = 0$.
- b —the left inflection point, where the membership function increases from 0 to 1.
- c —the right inflection point, where the membership function begins to decrease from 1 to 0.
- d —the end point (right support), where $\mu(x) = 0$.

The membership function is mathematically expressed as follows:

$$\mu_B(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a < x \leq b \\ 1, & b < x \leq c \\ \frac{d-x}{d-c}, & c < x < d \\ 0, & x \geq d \end{cases} \quad (8c)$$

Trapezoidal fuzzy numbers are more flexible compared to triangular fuzzy numbers, as they incorporate a region of constancy ($b \leq x \leq c$). In this interval, the membership value is always equal to 1.

Both membership functions are considered among the most commonly used functions for defining fuzzy sets.

Definition 5 (Z-numbers [36–38,52]). A Z-number is defined as a three-component construct.

$$Z = (X, A, B) \quad (9)$$

where

- X denotes the variable or linguistic descriptor under consideration;
- A is a fuzzy restriction on the values of X , representing the outcome or evaluation;

- B is a fuzzy number, interval, or probability distribution that expresses the reliability of the statement $X=A$.

Note: Here, A denotes the fuzzy restriction component of the Z -number, not to be confused with the set of alternatives.

The main differences between classical numbers, fuzzy numbers, and Z -numbers are presented in Table 1.

Table 1. Comparison of classical numbers, fuzzy numbers, and Z -numbers in terms of outcomes, reliability, and uncertainty.

Model	Evaluation	Reliability	Level of Uncertainty
Classical numbers	Crisp, deterministic value	Assumed to be fully reliable (implicit = 1)	No uncertainty (only exact outcomes)
Fuzzy numbers	Fuzzy evaluation with membership values	Not explicitly represented	Models vagueness and imprecision
Z -numbers	Fuzzy evaluation ($\tilde{A}$)	Reliability component (B), fuzzy or probabilistic	Models both uncertainty and reliability

Definition 6 (The distance function between Z -numbers [37,38]). *The distance function between two Z -numbers is defined as follows:*

$$d((A_1, B_1), (A_2, B_2)) = \alpha d_A(A_1, A_2) + (1 - \alpha) d_B(B_1, B_2), \alpha \in [0, 1] \quad (10)$$

Here, d_A and d_B are the distances between the A and B components of the Z -numbers, respectively. α represents the degree of preference assigned to either the A or B component of the Z -number.

Definition 7 (Arithmetic Operations on Z -numbers and the Closure Property [36,38]). *Arithmetic operations on Z -numbers are defined so as to preserve the closure property, i.e., the result of any operation between two Z -numbers is itself a Z -number. The theoretical foundations of this property were developed in [37,38]. In the calculations presented here, the concept of a discrete Z^+ number is used. Given a discrete $Z = (A, B)$ the corresponding discrete Z^+ number is defined as $Z^+ = (A, R)$, where A is a discrete fuzzy number (as in the case of simple Z -numbers), and R is a probability distribution p that satisfies the condition $(A) = \sum_{i=1}^n \mu_A(x_i) p(x_i)$, $P(A) \in \text{supp}(B)$. Table 2 below presents a step-by-step mathematical derivation of the subtraction operation (see Equation (11)) applied to two Z -numbers $Z_{ij} = (A_{ij}, B_{ij})$ and $Z_j^* = (A_j^*, B_j^*)$ used in the computation of the regret function:*

$$R_Z(a_i, s_j) = (A_{ij}, B_{ij}) \ominus (A_j^*, B_j^*) = (A_{ij}^R, B_{ij}^R) \quad (11)$$

The calculations corresponding to Equation (11) are performed in the sequential order shown in Table 2, which illustrates the execution of the subtraction operation on Z -numbers.

Table 2. Execution of the subtraction operation on Z-numbers.

Step	Mathematical Expression	Explanation
1	$\mu_{A_{ij}^R}(v) = \sup_u (\mu_{A_{ij}}(u) \wedge \mu_{A_j^*}(v - u))$	The fuzzy component A_{ij}^R is computed in accordance with the subtraction rule for fuzzy numbers.
2	$p_{R_X - R_Y}(v) = \int_R p_{R_X}(u) p_{R_Y}(v - u) du$	Intermediate step: subtraction based on probability-density convolution, used for computing the reliability component (B).
3	$\mu_{p_Z}(p_Z) = \sup_{p_X, p_Y} (\mu_{B_X}(\int_R \mu_{A_X}(u) p_X(u) du) \wedge \mu_{B_Y}(\int_R \mu_{A_Y}(u) p_Y(u) du))$	Intermediate stage of computing B_{ij}^R through the aggregated probability distribution p_Z .
4	$\mu_{B_Z}(w) = \sup_{p_Z} (\mu_{p_Z}(p_Z))$	The membership function of the reliability component (B_{ij}^R) is obtained.
5	$b_{jl} = \sum_{k=1}^{n_j} \mu_{A_j}(x_{jk}) p_{jl}(x_{jk})$ subject to: $\sum_{k=1}^{n_j} p_{jl}(x_{jk}) = 1,$ $p_{jl}(x_{jk}) \geq 0, k = 1, \dots, n_j$ This is solved as a linear programming problem: $v_1 + c_2 v_2 + \dots + c_n v_n \rightarrow b_{jl},$ subject to $v v_1 + v_2 + \dots + v_n = 1, \quad v_k \geq 0,$ where $c_k = \mu_{A_j}(x_{jk}), \quad v_k = p_{jl}(x_{jk})$	Determination of the elements of B_{ij}^R by solving a linear-programming problem that ensures probabilistic consistency between fuzzy and reliability components.
6	$w = \int_R \mu_{A_z}(u) p_{Z(u)} du$	Ensures the consistency condition A_{ij}^R and B_{ij}^R . The final result $R_Z(a_i, s_j) = (A_{ij}^R, B_{ij}^R)$ is a Z-number, confirming the closure property.

Notation and Interpretation:

- A_{ij}^R : fuzzy number obtained as the difference between A_{ij} and A_j^*
- B_{ij}^R : reliability component derived through the convolution between the fuzzy and probabilistic components, following the computational procedure outlined in Table 2.
- $\mu_A(u), \mu_B(w)$: membership functions of fuzzy and reliability components.
- $\wedge$: fuzzy intersection minimum operator.
- $\sup$: supremum operator (equivalent to “max” in fuzzy computations).
- $p_{R_X}(u), p_{R_Y}(v)$: probability-density functions of Z_{ij} and Z_j^* .
- $p_{R_X - R_Y}(v)$: convolution-based difference in probability distributions.
- $p_Z(u)$: resulting probability distribution after subtraction.
- b_{jl} : discrete support values of B_j ; $\mu_{B_j}(b_{jl})$: their membership degrees.
- $p_{jl}(x_{jk})$: probability value at x_{jk} .
- $c_k = \mu_{A_j}(x_{jk})$: membership degree of the fuzzy component A_j^* at the same point.
- $v_k = p_{jl}(x_{jk})$: unknown probability variables in the optimization problem.
- w : mean value verifying the compatibility of A_z and B_z components.

The subtraction of two Z-numbers is performed through the sequence of operations described in Table 2. Each intermediate step produces either a fuzzy component or a probabilistic reliability component. Their integration yields a new Z-number (A_{ij}^R, B_{ij}^R) , confirming that the result of a subtraction between Z-numbers is itself a Z-number.

Hence, the closure property of Z-numbers under arithmetic operations is preserved.

In the addition, subtraction, multiplication, and division of Z-numbers, the computation of the B component (representing the degree of reliability) is carried out according to the same principle in all operations. The computation of the A component (representing the fuzzy outcome) follows the standard arithmetic rules for the addition, subtraction, multiplication, and division of fuzzy numbers.

Example 1. For illustration, an example is presented below: Let $Z_1 = (A_1, B_1) = (\{1.20, 1.70, 2.00, 2.00\}, \{0.38, 0.50, 0.50, 0.62\})$ and $Z_2 = (A_2, B_2) = (\{9.20, 13.575, 16.425, 16.85\}, \{0.12, 0.20, 0.20, 0.30\})$.

The arithmetic operations performed on these two Z-numbers yield the following results:

- Addition: $Z_3 = Z_1 + Z_2 = (\{10.40, 15.275, 18.425, 18.85\}, \{0.10, 0.17, 0.17, 0.27\})$.
- Subtraction: $Z_3 = Z_1 - Z_2 = (\{-15.65, -14.725, -11.575, -7.20\}, \{0.10, 0.17, 0.17, 0.27\})$.
- Multiplication: $Z_3 = Z_1 \times Z_2 = (\{11.04, 23.077, 32.85, 33.70\}, \{0.10, 0.17, 0.17, 0.27\})$.
- Division: $Z_3 = Z_1 \div Z_2 = (\{0.0712, 0.1035, 0.1473, 0.2174\}, \{0.10, 0.17, 0.17, 0.27\})$.

The subtraction (and other arithmetic operations) of Z-numbers follow multi-stage fuzzy–probabilistic procedures. Due to their complexity, only the general formulation is presented here, while the detailed computational algorithm can be found in [36].

Having established this property, we now proceed to define the minimum and maximum of discrete Z-numbers (Definition 8).

Definition 8 (Finding the minimum and maximum of discrete Z-numbers [37,38]).

Let us assume that,

$$Z_1 = (A_1, B_1), Z_2 = (A_2, B_2) \quad (12)$$

are discrete Z-numbers. The minimum is defined as follows:

$$Z_{12} = \min(Z_1, Z_2) = (A_{12}, B_{12}) \quad (13)$$

Here,

$$A_{12} = \min(A_1, A_2), p_{12} = p_1 \circ_{\min} p_2 \quad (14)$$

The B reliability component is constructed based on the membership degrees of fuzzy probability distributions.

$$\mu_{B_{12}}(b_{12s}) = \sup \mu_{p_{12s}}(p_{12s}) \text{ subject to } b_{12s} = \sum_k p_{12s}(x_k) \mu_{A_{12}}(x_k) \quad (15)$$

The B component is calculated in the same way as described in Definition 7. In other words, as in the computational example provided in Definition 7, the B component yields the same result under the corresponding arithmetic operations.

In the case of the maximum,

$$Z_{12} = \max(Z_1, Z_2) = (\max(A_1, A_2), B_{12}) \quad (16)$$

and the convolution operation is performed in the “max” form.

These rules demonstrate that Z-numbers possess a closed structure under operations [36,38]: both the minimum and the maximum also yield a Z-number.

Definition 9 (Conversion of Z-numbers into classical fuzzy numbers: Kang et al. [34]). *proposed a method for transforming Z-numbers into classical fuzzy numbers. Based on this method, the conversion of Z_{ij} into a classical fuzzy number is explained step by step as follows:*

1. Scalarization of the reliability component.

$$\alpha = \frac{\int x \mu_{B_{ij}}(x) dx}{\int \mu_{B_{ij}}(x) dx} \quad (17)$$

Here, $\mu_{B_{ij}}(x)$ denotes the membership function of the fuzzy number B_{ij} , while α represents the scalar value that determines the degree of reliability.

2. The fuzzy number A_{ij} is transformed into a weighted fuzzy number based on the reliability degree α .

$$A_{ij}^{\alpha} = \left\{ \left\langle x, \alpha \mu_{A_{ij}}(x) \right\rangle \mid x \in X \right\},$$

Here, $\mu_{A_{ij}}(x)$ represents the membership function of the fuzzy number A_{ij} .

The weighted fuzzy number A_{ij}^{α} is transformed into a classical fuzzy number as follows:

$$A'_{ij} = \left\{ \left\langle x, \mu_{A_{ij}}\left(\frac{x}{\sqrt{\alpha}}\right) \right\rangle \mid x \in \sqrt{\alpha}X \right\} \quad (18)$$

As a result, the classical fuzzy number obtained from Z_{ij} is expressed as follows:

$$Z_{ij} \parallel \longrightarrow \parallel A'_{ij}, \quad (19)$$

Example 2. For illustration, an example is presented below: Let $Z_1 = (A_1, B_1) = (\{1.20, 1.70, 2.00, 2.00\}, \{0.38, 0.50, 0.50, 0.62\})$, where each component of the number Z is interpreted as a trapezoidal fuzzy number (a, b, c, d) .

We first transform the trapezoidal fuzzy number B_1 into a crisp number according to Equation (17): $\alpha = \frac{a+2b+2c+d}{6} = \frac{0.38+2(0.50)+2(0.50)+0.62}{6} = \frac{3.00}{6} = 0.50$ and therefore, $\sqrt{\alpha} = \sqrt{0.50} \approx 0.7071$.

Next, we transform the fuzzy component of A_1 according to Equation (18), $A_1 = (a\sqrt{\alpha_1}, b\sqrt{\alpha_1}, c\sqrt{\alpha_1}, d\sqrt{\alpha_1}) = (1.20 \times 0.7071, 1.70 \times 0.7071, 2.00 \times 0.7071, 2.00 \times 0.7071) = (0.8485, 1.2021, 1.4142, 1.4142)$.

Finally, from Equations (17)–(19), we obtain: $Z_1 \longrightarrow A_1 = (0.8485, 1.2021, 1.4142, 1.4142)$, $\alpha = 0.50$.

Definition 10 (Ranking of Classical Fuzzy Numbers: Wang Method [54]). Wang et al. [54] introduced the centroid formula for generalized fuzzy numbers. Specifically, for a generalized trapezoidal fuzzy number:

$$\tilde{A} = (a, b, c, d; w) \quad (20)$$

the centroid coordinates are given as follows:

$$(\bar{x}_0, \bar{y}_0) = \left(\frac{a+b+c+d}{4} - \frac{dc-ab}{(d+c)-(a+b)} \cdot \frac{w}{3} \left(1 + \frac{c-b}{(d+c)-(a+b)} \right) \right) \quad (21)$$

and the ranking function corresponding to the fuzzy number $\tilde{A}$ is defined as follows:

$$R(\tilde{A}) = \sqrt{\bar{x}_0^2 + \bar{y}_0^2} \quad (22)$$

The ranking rules for two fuzzy numbers $\tilde{A}_i$ and $\tilde{A}_j$ should be carried out as follows:

- If $R(\tilde{A}_i) > R(\tilde{A}_j)$, then $\tilde{A}_i > \tilde{A}_j$;
- If $R(\tilde{A}_i) < R(\tilde{A}_j)$, then $\tilde{A}_i < \tilde{A}_j$;
- If $R(\tilde{A}_i) = R(\tilde{A}_j)$, then $\tilde{A}_i = \tilde{A}_j$.

This method has been proposed to ensure a more stable and accurate comparison of fuzzy numbers.

5. Z-Regret Principle

The classical regret principle (Definitions 1 and 2) expresses utility functions $L(a_i, s_j)$ (see Equation (2)) using precise scalar values and is based on the criterion of “minimizing the worst-case loss.” While this approach is mathematically straightforward, it does not take into account the uncertainty and reliability factors commonly observed in human decision-making processes. In real-world scenarios, however, decision-makers often assess outcomes not through exact numerical values, but rather through fuzzy evaluations accompanied by degrees of reliability. In this regard, Z-numbers, defined as $Z = (A, B)$, provide a more adequate framework by jointly modeling both the fuzzy assessment of the outcome (A) and the associated degree of reliability (B).

This paper proposes two approaches to extend the classical regret principle based on Z-numbers:

1. **Z-Regret principle.** The first approach is based on the closedness principle of Z-numbers and incorporates the computational operations on Z-numbers developed by Rafik Aliev [36–38]. In this approach, for each scenario, the ideal Z-number $Z_j^* = (A_j^*, B_j^*)$ is selected, and the calculation of the regret matrix is carried out on the basis of the subtraction operation and distance measures on Z-numbers (Definition 6 and Definition 7).
2. **Z-Regret Principle.** The second approach is based on the method of transforming Z-numbers into classical fuzzy numbers [34]. In this approach, the reliability component B_{ij} is first scalarized, and then the outcome component A_{ij} is transformed into a weighted fuzzy number. The resulting classical fuzzy numbers are then ranked using the method proposed by Wang [54], after which the regret-based calculations are carried out in the traditional manner.
3. Although both approaches are supported by algorithmic procedures and mathematical formalism, the primary aim of this study is to demonstrate that the approach based on the closure property of Z-numbers represents a more fundamental and theoretically sound formulation of the Z-regret principle. Accordingly, the second approach serves an auxiliary and comparative role, while the first approach constitutes the mathematical foundation of the Z-regret model.
4. Sections 5.1 and 5.2 provide a step-by-step explanation of both approaches.

5.1. Z-Regret Principle, Approach 1

In this approach, all algorithms and operations carried out in accordance with the classical regret principle are executed on Z-numbers, based on the principle of closure.

Step 1. Construction of the decision matrix. In the decision matrix, the utility (or cost) for each alternative and scenario is given in the form of a Z-number (Definition 5):

1. A_{ij} —the fuzzy evaluation of utility, represented in the form of fuzzy numbers (Definition 4);
2. B_{ij} —the degree of reliability of the fuzzy evaluation of utility, represented in the form of fuzzy numbers;
3. $Z_{ij} = (A_{ij}, B_{ij})$ denotes the Z-utility of alternative i under scenario j . Table 1, constructed on the basis of Z-numbers, is referred to as the Z-number-based utility matrix (Table 3).

Table 3. A Z-number-based utility matrix.

Alternatives/Scenarios	s_1	s_2	...	s_m
a_1	$Z_{11} = (A_{11}, B_{11})$	$Z_{12} = (A_{12}, B_{12})$	...	$Z_{1m} = (A_{1m}, B_{1m})$
a_2	$Z_{21} = (A_{21}, B_{21})$	$Z_{22} = (A_{22}, B_{22})$	...	$Z_{2m} = (A_{2m}, B_{2m})$
...	...	...	...	...
a_n	$Z_{n1} = (A_{n1}, B_{n1})$	$Z_{n2} = (A_{n2}, B_{n2})$	...	$Z_{nm} = (A_{nm}, B_{nm})$

Step 2. Identification of the scenarios with maximum utility. For each scenario j , the best Z-number is selected among all alternatives, and this optimal value is denoted as $Z_j^* = (A_j^*, B_j^*)$. Here, the term “best” (i.e., closest to the ideal) is determined based on the minimum distance between the Z-numbers Z_j^* and Z_{ij} . The calculation of distances between Z-numbers is carried out using Definition 11 proposed below.

Definition 11 (Proposed: Calculating Regrets Based on Z-numbers). Suppose that two Z-numbers are given:

$$Z_{ij} = (A_{ij}, B_{ij}), Z_{s_j}^* = (A_{s_j}^*, B_{s_j}^*) \quad (23)$$

Here, $Z_{s_j}^*$ denote the ideal Z-number for a given scenario s_j .

The computation of regrets based on Z-numbers is carried out through the calculation of distances between Z-numbers, according to the following equation (assuming that the components of Z-numbers are trapezoidal fuzzy numbers, as defined in Definition 4):

$$D_Z(a_i, s_j) = d((A_{ij}, B_{ij}), (A_{s_j}^*, B_{s_j}^*)) = 0.5 \cdot \left(\sum_{m=1}^4 |a_{ms_j}^* - a_{ij}| + \sum_{k=1}^4 |b_{ks_j}^* - b_{ij}| \right) \quad (24)$$

For each scenario s_j , the scenarios with maximum utility $R_Z(a_i, s_j)$ are determined based on the ranking of distances $D_Z^*(a_i, s_j)$ computed according to the following equation:

$$D_Z^*(a_i, s_j) = \min_{s_j \in S} D_Z(a_i, s_j), i \in \overline{1, \dots, M}, j \in \overline{1, \dots, N} \quad (25)$$

Here, M denotes the number of scenarios, and N denotes the number of alternatives. The meaning of Equation (25) is that, since the ideal Z-number is chosen as the best—that is, the maximum Z-number—the Z-number closest to it in terms of distance is regarded as optimal. In other words, since the distances $D_Z^*(a_i, s_j)$ to the ideal Z-numbers are the smallest, they are regarded as the Z-numbers possessing the maximum utility across the scenarios. For the purposes of subsequent calculations, let us denote these Z-numbers as $Z_{s_j}^{max}(a_i, s_j)$.

Step 3. Calculation of the regret matrix. Traditionally, the regret matrix is obtained by calculating the differences between the possible maximum outcome for each state and

the current outcome. In our case, since each scenario (state) is represented in the form of Z-numbers, and taking into account the closure principle of Z-numbers, the determination of the regret matrix is carried out by subtracting ($\ominus$) the Z-numbers of each scenario, $Z_{ij} = (A_{ij}, B_{ij})$, from the Z-numbers with maximum utility across the scenarios, $Z_{s_j}^{\max}(a_i, s_j)$. The subtraction operation is performed according to Equation (11) in Definition 7.

Table 4 presents the procedure for calculating the regret values for each alternative and scenario based on the Z-regret principle.

Table 4. Z-Regret Matrix (Approach 1).

Alternatives/Scenarios	s_1	s_2	...	s_m
a_1	$R_z^{11} = Z_{s_1}^{\max} \ominus Z^{11}$	$R_z^{12} = Z_{s_2}^{\max} \ominus Z^{12}$	...	$R_z^{1m} = Z_{s_m}^{\max} \ominus Z^{1m}$
a_2	$R_z^{21} = Z_{s_1}^{\max} \ominus Z^{21}$	$R_z^{22} = Z_{s_2}^{\max} \ominus Z^{22}$	...	$R_z^{2m} = Z_{s_m}^{\max} \ominus Z^{2m}$
...	...	...	...	...
a_n	$R_z^{n1} = Z_{s_1}^{\max} \ominus Z^{n1}$	$R_z^{n2} = Z_{s_2}^{\max} \ominus Z^{n2}$	...	$R_z^{nm} = Z_{s_m}^{\max} \ominus Z^{nm}$

Step 4. Determination of the maximum regrets across scenarios for each alternative.

Since each element of the regret matrix is a Z-number, in accordance with Step 2 and Equation (26), for each alternative the maximum Z-number across scenarios, i.e., $R_{minz}^{ij} = (R_{Amin}^{ij}, R_{Bmin}^{ij})$ is determined.

$$R_{minZ}^i = \min_{j=1, \dots, m} R_Z^{ij} = (R_{Amin}^i, R_{Bmin}^i) \quad (26)$$

Since being closest to the ideal Z-numbers implies having the minimum distance, unlike in the classical regret principle, instead of using the term “maximum,” we denote the maximum regrets as minimum by R_{minZ}^i .

Step 5. Selection of the minimum among the maximum regrets. In accordance with Equation (24), by calculating the distances from the ideal Z-number and based on the definition of 12, the minimum of the maximum regret values is determined.

Definition 12 (Proposed: Formal Expression of the Z-Regret Principle). *The selection of the situation with the minimum value among the maximum regret values, in the classical regret principle, is carried out using Equation (7). In the Z-Regret principle, however, the value corresponding to the farthest point (i.e., the maximum distance) from the ideal Z-number is considered as the minimum regret value (see Equation (27)).*

$$a^* = \operatorname{argmax}_{a_i \in A} \{R_{minZ}^i\} = (A_{ij}^{a^*}, B_{ij}^{a^*}) \quad (27)$$

Since a^* itself is a Z-number, the obtained result simultaneously reflects both the minimum utility and its associated reliability.

Thus, in this approach, the limitation of the classical regret principle is overcome, as both uncertainty and the corresponding reliability are taken into account.

5.2. Z-Regret Principle, Approach 2

This principle is based on the transformation of Z-numbers into ordinary fuzzy numbers and is carried out in the following sequence:

Step 1. Construction of the decision matrix. This step is identical to Step 1 in Approach 1.

Step 2. Based on Definition 9, by applying Equations (17) and (18), each Z_{ij} number is transformed into ordinary Z_{ij}^* -fuzzy numbers, and the results are presented as shown in Table 5.

Table 5. A Z-number-based utility matrix (Approach 2).

Alternatives/Scenarios	s_1	s_2	...	s_m
a_1	$r_{11'} = Z_{11}(A_{11}, B_{11})$	$A'_{12} = Z_{12}(A_{12}, B_{12})$	...	$A'_{1m} = Z_{1m}(A_{1m}, B_{1m})$
a_2	$Z_{21'} = Z_{21}(A_{21}, B_{21})$	$Z_{22'} = Z_{22}(A_{22}, B_{22})$	...	$Z_{2m'} = Z_{2m}(A_{2m}, B_{2m})$
...	...	...	...	...
a_n	$Z_{n1'} = Z_{n1}(A_{n1}, B_{n1})$	$A'_{n2} = Z_{n2}(A_{n2}, B_{n2})$	...	$A'_{nm} = Z_{nm}(A_{nm}, B_{nm})$

Step 3. Determination of the ranks of the ordinary fuzzy numbers $Z_{ij'}$. For each ordinary fuzzy number $Z_{ij'}$, the Wang ranks are calculated based on Definition 10 as $r_{ij} = R(Z_{ij'})$ (Table 6).

Table 6. Wang ranking matrix.

Alternatives/Scenarios	s_1	s_2	...	s_m
a_1	r_{11}	r_{12}	...	r_{1m}
a_2	r_{21}	r_{22}	...	r_{2m}
...	...	...	...	...
a_n	r_{n1}	r_{n2}	...	r_{nm}

Step 4. By applying the classical regret algorithm, the maximum ranks across scenarios are determined for each alternative based on Table 6.

$$M_j = \max_{i \in \{1, \dots, n\}} r_{ij}, \quad j \in 1, \dots, M \quad (28)$$

Step 4. Construction of the regret matrix.

- Sub-step 4.1. Based on Equation (29), the regret matrix is constructed in accordance with Table 6.

$$Reg_{ij} = M_j - r_{ij} \geq 0 \quad (29)$$

- Sub-step 4.2. Calculation of the “maximum regret” values for the alternatives. For each alternative, the “maximum regret” values are calculated based on Equation (30).

$$Reg_i^{\max} = \max_{j \in \{1, \dots, m\}} Reg_{ij} \quad (30)$$

Step 5. Selection of the minimum among the maximum regrets. Based on Equation (31), the alternative with the minimum “maximum regret” value is identified according to the classical regret principle.

$$a^* \in \operatorname{argmin}_i Reg_i^{\max} \quad (31)$$

6. Practical Application of the Z-Regret Algorithm

Problem Statement. The assessment of advertising effectiveness is one of the most important issues in the strategic management of companies in the modern digital market. Advertising expenses usually constitute the largest part of the marketing budget, and any misjudgment may lead to significant financial losses directed toward inefficient channels [55,56]. Effective performance measurement enables companies to choose channels that provide the highest return on investment (ROI). Therefore, the issue of evaluating advertising effectiveness is of critical importance not only from an economic perspective but also from the standpoint of risk management [57].

A specific feature of the advertising environment is that it is characterized by a high degree of uncertainty. For example:

- In cases of Bot Attacks, a large proportion of clicks do not belong to real users, and as a result, high Click-Through Rate (CTR), Cost Per Click (CPC), and Conversion Rate indicators lead to incorrect decisions [58,59].
- Seasonal fluctuations in consumer demand and retail sales lead to significant deviations in performance indicators, reflecting the long-term persistence of seasonal effects in economic activity [60].
- Platform differences also create additional risks. For example, in TikTok advertising, the CTR may be very high, but if data reliability is low (i.e., the share of bots is significant), managing the budget based on this indicator can be highly risky [61].

Suppose that, in the evaluation of performance in a digital advertising strategy, three alternative platforms and three scenarios are given:

- Alternatives (platforms): $A = \{a_1 = \text{Google}, a_2 = \text{Facebook}, a_3 = \text{TikTok}\}$.
- Scenarios (states): $S = \{s_1: \text{Normal}, s_2: \text{Bot Attack}, s_3: \text{Seasonal Variation}\}$.
- For each pair (a_i, s_j) , the advertising performance is represented as a Z-number (the decision matrix): $Z_{ij} = (A_{ij}, B_{ij})$. Here, A_{ij} denotes the evaluation of performance under different scenarios (e.g., a composite score of CTR/ROI) as a trapezoidal fuzzy number within the interval $[0, 100]$, B_{ij} denotes the evaluation of the reliability of A_{ij} as a trapezoidal fuzzy number within the interval $(0, 1)$. Trapezoidal fuzzy numbers provide a numerical representation of linguistic terms such as low, medium, high, very high, or expressions of certainty like medium, high, full certainty (Definition 4).

The initial dataset (the decision matrix) was assessed by experts using actual advertising performance indicators (CTR, CPC, Conversion Rate). Subsequently, through linguistic terms, the data were transformed into trapezoidal fuzzy numbers and represented in the form of Z-numbers, as illustrated in Table 3. In other words, Step 1 of Algorithms 5.1 and 5.2 has been implemented, and the corresponding results are provided in Table 7.

Table 7. The decision matrix. Initial performance evaluations of advertising platforms (Google, Facebook, TikTok) expressed as Z-numbers under three scenarios. The underlying data were provided by domain experts as described in the paper; therefore, no additional external sources are cited.

Alternatives/Scenarios	S ₁ : Normal	S ₂ : Bot Attack	S ₃ : Seasonal Variation
A ₁ (Google)	$Z_1^1 = A$: Very high $\rightarrow [70, 75, 85, 90]$ B : Full certainty $\rightarrow [0.95, 1.00, 1.00, 1.00]$	$Z_1^2 = A$: Low $\rightarrow [40, 45, 55, 60]$ B : Moderate certainty $\rightarrow [0.70, 0.80, 0.85, 0.90]$	$Z_1^3 = A$: High $\rightarrow [60, 65, 75, 80]$ B : High certainty $\rightarrow [0.85, 0.90, 0.95, 1.00]$
A ₂ (Facebook)	$Z_2^1 = A$: High $\rightarrow [65, 70, 80, 85]$ B : High certainty $\rightarrow [0.85, 0.90, 0.95, 1.00]$	$Z_2^2 = A$: Medium $\rightarrow [50, 55, 65, 70]$ B : Moderate certainty $\rightarrow [0.70, 0.80, 0.85, 0.90]$	$Z_2^3 = A$: Medium $\rightarrow [55, 60, 70, 75]$ B : Moderate certainty $\rightarrow [0.70, 0.80, 0.85, 0.90]$
A ₃ (TikTok)	$Z_3^1 = A$: High $\rightarrow [60, 65, 75, 80]$ B : Moderate certainty $\rightarrow [0.70, 0.80, 0.85, 0.90]$	$Z_3^2 = A$: Low $\rightarrow [45, 50, 60, 65]$ B : Moderate certainty $\rightarrow [0.70, 0.80, 0.85, 0.90]$	$Z_3^3 = A$: Medium $\rightarrow [50, 55, 65, 70]$ B : Moderate certainty $\rightarrow [0.70, 0.80, 0.85, 0.90]$

Assumptions and conditions:

- Principle of Closedness: The set of Z-numbers is closed under arithmetic and comparison operations; that is, the results of such operations are again represented in the form of a Z-number;
- The distance function in Definition 6 (see Equation (10)) takes into account both the performance component (A) and the reliability component (B), and the weights α can be selected according to the requirements;
- Table 7 presents the alternatives and scenarios as Z-numbers, expressed in terms of pairs of performance (A) and reliability (B). Each of these components is adequately modeled using trapezoidal fuzzy numbers.

A solution to this problem is required. The purpose of this study is to evaluate digital advertising performance under conditions of uncertainty. To this end, the limitations of classical indicators (CTR, CPC, Conversion Rate) are addressed by employing a Z-number

approach, which incorporates both the performance component (A) and the degree of reliability (B) of information A .

On the basis of Z-numbers, the proposed Z-Regret principle is used, enabling the comparison of alternatives (Google, Facebook, TikTok) across different scenarios (Normal, Bot Attack, Seasonal Variation). Consequently, the selection of the optimal advertising channel is carried out in accordance with the requirements of the classical regret principle.

Solution to the Problem. To demonstrate the practical efficacy of the proposed Z-Regret principle, we now present a step-by-step procedure for solving this problem based on this principle.

The problem was solved by considering both approaches proposed within the framework of the Z-principle.

6.1. Solution of the Formulated Problem Under Approach 1

Step 1: In the problem formulation section, the decision matrix is presented in Table 7.

Step 2: At this stage, for each scenario $j \in \overline{1 \dots 3}$, the maximum (optimal) Z_{sj}^* - values are to be identified. For simplicity, for each scenario we assume the overall ideal as $Z_S^* = [100, 100, 100, 100], [0.95, 1.00, 1.00, 1.00]$. In the comparison, the Z-number that is at the smallest distance from the ideal Z-number is considered the maximum Z-number. The distance comparisons between Z-numbers and the ideal Z-number are carried out on the basis of Definition 11 and Equation (10).

Now, for each scenario, let us calculate the distance from this ideal number according to Equation (10) (where $\alpha = 0.5$ is assumed).

The calculations yield the following results:

1. Scenario S_1 —the scenario of ‘Normal Conditions’:
 - Distances: $d(Z_S^*, Z_1^1) = 40.00$, $d(Z_S^*, Z_2^1) = 50.125$, $d(Z_S^*, Z_3^1) = 60.35$.
 - Result: the smallest distance from the ideal Z_S^* is $d(Z_S^*, Z_1) = 40.00$. For the ‘Normal Condition’ scenario, the maximum is $Z_{S1}^* = Z_1$ (Google) = $\{A = [70, 75, 85, 90], B = [0.95, 1.00, 1.00, 1.00]\}$.
2. Scenario S_2 —the scenario of ‘Bot Attack’:
 - Distances: $d(Z_S^*, Z_1^2) = 100.35$, $d(Z_S^*, Z_2^2) = 80.35$, $d(Z_S^*, Z_3^2) = 90.35$.
 - Result: For the ‘Bot Attack’ scenario, the maximum is $Z_{S2}^* = Z_2$ (Facebook) = $\{A = [50, 55, 65, 70], B = [0.70, 0.80, 0.85, 0.90]\}$.
3. Scenario S_3 —the scenario of ‘Seasonal Variation’:
 - Distances: $d(Z_S^*, Z_1^3) = 60.125$, $d(Z_S^*, Z_2^3) = 70.35$, $d(Z_S^*, Z_3^3) = 80.35$.
 - Result: or the ‘Seasonal Variation’ scenario, the maximum is $Z_{S3}^* = Z_1$ (Google) = $\{A = [60, 65, 75, 80], B = [0.85, 0.90, 0.95, 1.00]\}$.

The results obtained in Step 2 are presented in Table 8.

Table 8. Maximum Z-numbers for each scenario obtained by applying Approach 1.

Scenarios	$d(Z_S^*, Z_1^j)$	$d(Z_S^*, Z_2^j)$	$d(Z_S^*, Z_3^j)$	The Minimum Distance	Maximum Z_{sj}^*	Z_{sj}^{max}
S_1 — Normal Conditions	40.00	50.125	60.35	40.00	Z_1^1 (Google)	$Z_{S1}^{max} = [70, 75, 85, 90], [0.95, 1.00, 1.00, 1.00]$
S_2 — BotAttack	100.35	80.35	90.35	80.35	Z_2^2 (Facebook)	$Z_{S2}^{max} = [50, 55, 65, 70], [0.70, 0.80, 0.85, 0.90]$
S_3 — Seasonal Variation	60.125	70.35	80.35	60.125	Z_1^3 (Google)	$Z_{S2}^{max} = [60, 65, 75, 80], [0.85, 0.90, 0.95, 1.00]$

Step 3: At this step, the regret matrix is computed:

- Sub-step 3.1: For each i -th alternative and j -th scenario, the corresponding Z_{ij} is subtracted from the maximum Z_{sj}^{max} -for that scenario according to Equation (14), based

on the computational operations over Z-numbers proposed by Rafik Aliev [36–38] (Definition 7). As a result, the regret matrix presented in Table 9 is obtained.

Table 9. Regret matrix of alternatives under different scenarios (Approach 1).

Alternatives/Scenarios	S ₁ : Normal	S ₂ : Bot Attack	S ₃ : Seasonal Variation
A ₁ (Google)	$Z_1^1 = \{[-20, -10, 10, 20], [0.92, 1.00, 1.00, 1.00]\}$	$Z_1^2 = \{[-10, 0, 20, 30], [0.60, 0.71, 0.77, 0.84]\}$	$Z_1^3 = \{[-20, -10, 10, 20], [0.77, 0.84, 0.91, 1.00]\}$
A ₂ (Facebook)	$Z_2^1 = \{[-15, -5, 15, 25], [0.83, 0.90, 0.95, 1.00]\}$	$Z_2^2 = \{[-20, -10, 10, 20], [0.60, 0.71, 0.77, 0.84]\}$	$Z_2^3 = \{[-15, -5, 15, 25], [0.66, 0.76, 0.82, 0.90]\}$
A ₃ (TikTok)	$Z_3^1 = \{[-10, 0, 20, 30], [0.57, 0.70, 0.75, 0.81]\}$	$Z_3^2 = \{[-15, 5, 15, 25], [0.60, 0.71, 0.77, 0.84]\}$	$Z_3^3 = \{[-10, 0, 20, 30], [0.66, 0.76, 0.82, 0.90]\}$

- Sub-step 3.2: Now, using Definition 11 and Equation (24), we calculate the distances for the regret matrix for each scenario (with the overall ideal value $Z_S^* = [100, 100, 100, 100], [0.95, 1.00, 1.00, 1.00]$ and $\alpha = 0.5$), and determine the maximum values for each scenario (Table 10).

Table 10. Maximum regret distances for alternatives under Normal, Bot Attack, and Seasonal scenarios (Approach 1).

Scenarios	$d(Z_S^*, Z_1^i)$	$d(Z_S^*, Z_2^i)$	$d(Z_S^*, Z_3^i)$	The Minimum Distance	Maximum $Z_{S_j}^*$	$Z_{S_j}^{max}$
A ₁ (Google)	75.00	70.10	115.46	70.10	Z_1^2 (Bot Attack)	$Z_{S_1}^{max} = [-15, -5, 15, 25], [0.83, 0.90, 0.95, 1.00]$
A ₂ (Facebook)	100.35	105.35	110.35	100.35	Z_2^2 (Normal)	$Z_{S_2}^{max} = [-20, -10, 10, 20], [0.60, 0.71, 0.77, 0.84]$
A ₃ (TikTok)	95.04	105.29	115.29	95.04	Z_1^3 (Normal)	$Z_{S_3}^{max} = [-20, -10, 10, 20], [0.77, 0.84, 0.91, 1.00]$

Under Approach 1, the evaluations based on the proximity distances from the ideal state are graphically illustrated in Figure 2.

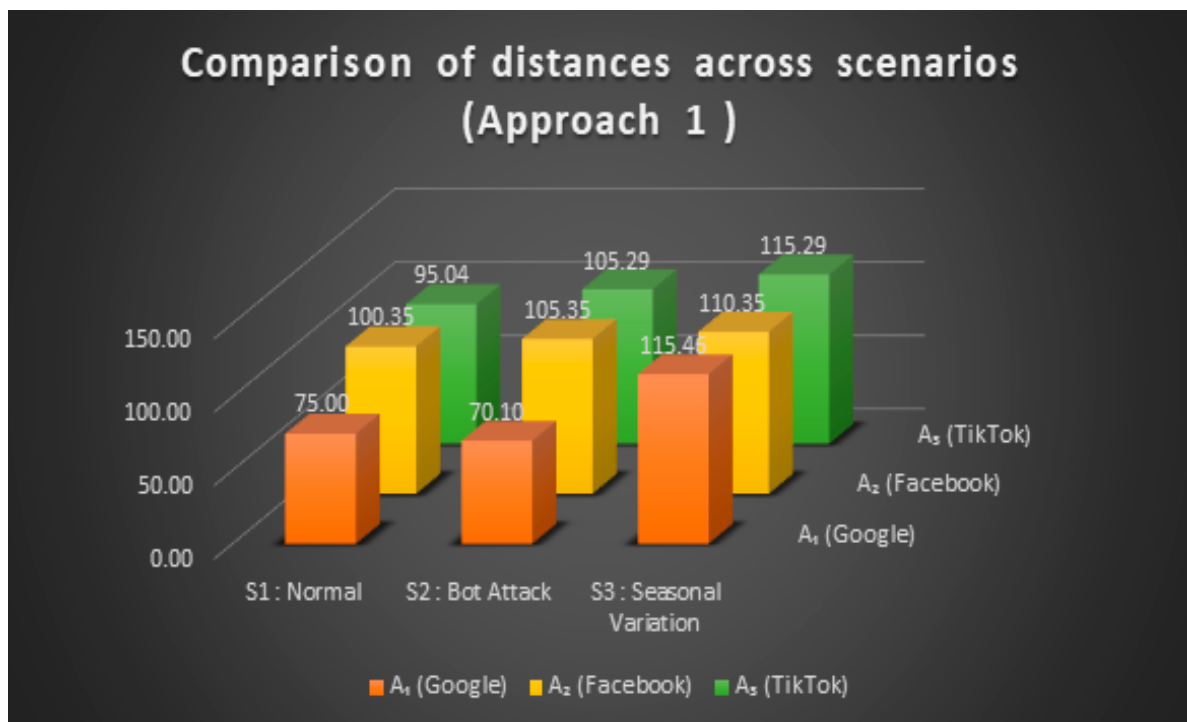


Figure 2. Maximum regret distances (Approach 1).

Step 4. In this step, the $Z_{S_j}^*$ values are ranked with respect to their proximity to the ideal Z_S^* number, in accordance with Definition 11 and Equation (25). The obtained results are presented in Table 11.

Table 11. Ranking of alternatives based on Z-Regret principle (Approach 1).

The Maximum Z-Number	$D(Z_S^*, Z_{S_j}^{max})$	The Ranks
$Z_{S_1}^{max}$ (Google)	70.10	S_2 – BotAttack
$Z_{S_3}^{max}$ (TikTok)	95.04	S_1 – Normal
$Z_{S_2}^{max}$ (Facebook)	105.35	S_1 – Normal

Step 5. On the basis of Equation (27) provided in Definition 12, the alternative that exhibits the farthest distance among $D(Z_S^*, Z_{S_j}^{max})$ is regarded as the alternative with the least maximum regret. Thus, by examining this table, we can state that, $Z_{S_1}^{max}$ (Facebook) $\prec$ $Z_{S_1}^{max}$ (TikTok) $\prec$ $Z_{S_2}^{max}$ (Google). Thus, the least maximum regret corresponds to $Z_{S_1}^{max}$ (Facebook) = Z_2^1 = A: High $\rightarrow$ [65, 70, 80, 85], B: High certainty $\rightarrow$ [0.85, 0.90, 0.95, 1.00], which indicates the Facebook alternative. This means that, with high confidence (B: High certainty), it can be stated that among the available advertising platforms, the selection of Facebook for advertising ensures that, even under the worst-case scenario, the potential losses will remain minimal.

6.2. Solution of the Formulated Problem Under Approach 2

Now let us calculate the regrets on the basis of the second approach. The main objective here is to transform the Z-numbers into ordinary fuzzy numbers using the method given in [34], and then convert them into crisp numbers by applying the Wang Ranking method [54] for ranking. Subsequently, the classical regret principle is applied to compute the least maximum regret value.

Step 1. Step 1 is carried out in the same way as in Approach 1.

Step 2. Transformation of Z-numbers into ordinary fuzzy numbers. Using Equations (17) and (18) in Definition 9, all Z-numbers listed in Table 7 are converted into ordinary fuzzy numbers, with the outcomes of the transformation shown in Table 12.

Table 12. Transformation of the Z-numbers specified for each alternative and scenario (Table 7) into classical fuzzy numbers (Approach 2).

Alternatives/Scenarios	S_1 : Normal	S_2 : Bot Attack	S_3 : Seasonal Variation
A_1 (Google)	$\alpha = 0.9833$ $Z_{11'} = [69.41, 74.37, 84.29, 89.25]$	$\alpha = 0.8125$ $Z_{12'} = [36.08, 40.59, 49.87, 54.38]$	$\alpha = 0.9250$ $Z_{13'} = [57.05, 61.76, 71.37, 76.18]$
A_2 (Facebook)	$\alpha = 0.9250$ $Z_{21'} = [62.52, 67.33, 76.94, 81.76]$	$\alpha = 0.8125$ $Z_{22'} = [45.15, 49.66, 58.94, 63.45]$	$\alpha = 0.8125$ $Z_{23'} = [49.75, 54.27, 63.32, 67.84]$
A_3 (TikTok)	$\alpha = 0.8100$ $Z_{31'} = [54.00, 58.50, 67.50, 72.00]$	$\alpha = 0.8125$ $Z_{32'} = [41.41, 46.01, 55.29, 59.89]$	$\alpha = 0.8125$ $Z_{33'} = [45.23, 49.75, 58.81, 63.33]$

Step 3. Calculation of the ranks of ordinary fuzzy numbers. By applying Equations (20)–(22) from Definition 10, and assuming $w = 1$, the ranks of each fuzzy number in Table 12 are calculated. The obtained results are presented in Table 13.

Step 4. The application of the classical regret algorithm involves calculating the regret values for each scenario, identifying the maximum values, and constructing the regret matrix.

This procedure is implemented through the following two sub-steps:

- Sub-step 4.1. The maximum values calculated across the scenarios are presented in Table 14.

Table 13. The results obtained from the ranking assessment of fuzzy numbers by applying the Wang method (Approach 2).

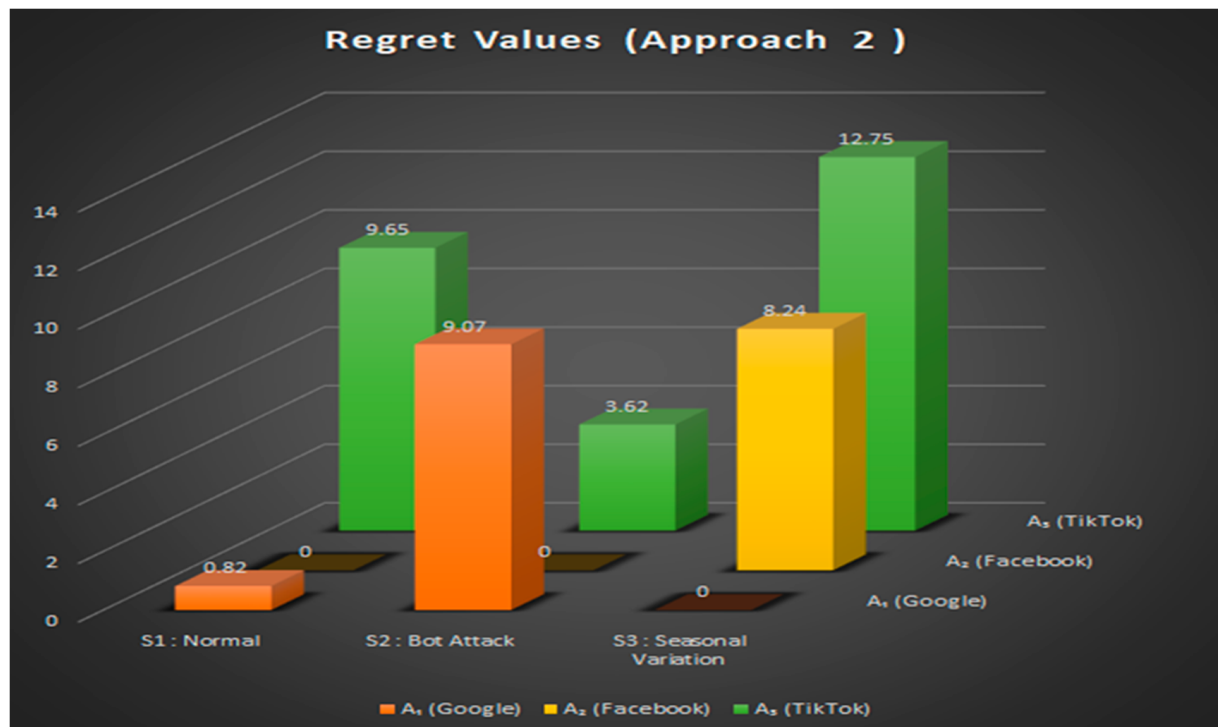
Alternatives/Scenarios	S ₁ : Normal	S ₂ : Bot Attack	S ₃ : Seasonal Variation
A ₁ (Google)	Centroid: $(x_0, y_0) = (79.33, 0.5)$	Centroid: $(x_0, y_0) = (52.92, 0.5)$	Centroid: $(x_0, y_0) = (74.58, 0.5)$
	Wang Rank : $R(Z_{11'}) = R(\tilde{A}) = 79.58$	Wang Rank : $R(Z_{12'}) = R(\tilde{A}) = 53.17$	Wang Rank : $R(Z_{13'}) = R(\tilde{A}) = 74.83$
A ₂ (Facebook)	Centroid: $(x_0, y_0) = (80.15, 0.5)$	Centroid: $(x_0, y_0) = (61.99, 0.5)$	Centroid: $(x_0, y_0) = (66.34, 0.5)$
	Wang Rank : $R(Z_{21'}) = R(\tilde{A}) = 80.40$	Wang Rank : $R(Z_{22'}) = R(\tilde{A}) = 62.24$	Wang Rank : $R(Z_{23'}) = R(\tilde{A}) = 66.59$
A ₃ (TikTok)	Centroid: $(x_0, y_0) = (70.50, 0.5)$	Centroid: $(x_0, y_0) = (58.37, 0.5)$	Centroid: $(x_0, y_0) = (61.83, 0.5)$
	Wang Rank : $R(Z_{31'}) = R(\tilde{A}) = 70.75$	Wang Rank : $R(Z_{32'}) = R(\tilde{A}) = 58.62$	Wang Rank : $R(Z_{33'}) = R(\tilde{A}) = 62.08$

Table 14. Maximum values by scenarios (Approach 2).

Scenarios	Maximum Values for the Scenarios
S ₁ : Normal	80.40
S ₂ : Bot Attack	62.24
S ₃ : Seasonal Variation	74.83

- Sub-step 4.2. Based on the maximum values obtained for the scenarios, the regret matrix elements are derived using Equation (5). The results of these computations are shown in Table 15.

Figure 3 provides a graphical illustration of the regret matrix, visually comparing the regret values of the three alternatives across all scenarios. This visualization offers a clearer perception of the comparative differences among alternatives and complements the numerical results in Table 15, thereby improving the overall interpretability of the analysis.

**Figure 3.** A graphical representation of the regret matrix (Approach 2).

Step 5. At this stage, considering the maximum regret values listed in the Maximum Regret column of Table 15, the least maximum regret value is obtained through the application of Equation (7) in Definition 3. The corresponding result is provided in Table 16.

Table 15. The regret matrix (Approach 2).

Alternatives/Scenarios	S ₁ : Normal	S ₂ : Bot Attack	S ₃ : Seasonal Variation	Maximum Regrets
A ₁ (Google)	0.82	9.07	0.00	9.07
A ₂ (Facebook)	0.00	0.00	8.24	8.24
A ₃ (TikTok)	9.65	3.62	12.75	12.75

Table 16. Final minimax ranking of alternatives under classical regret (Approach 2).

Alternatives	Maximum Regrets	Scenarios	The Rankings
A ₂ (Facebook)	8.24	S ₃ : Seasonal Variation	1
A ₁ (Google)	9.07	S ₂ : Bot Attack	2
A ₃ (TikTok)	12.75	S ₃ : Seasonal Variation	3

Table 16 shows that A₂—Facebook, under Approach 2, represents the alternative with the least maximum regret, as its maximum regret value is lower than that of the other alternatives.

6.3. Comparative Evaluation of Similar Decision-Making Approaches

For comparative validation, the proposed Z-Regret principle was evaluated against the classical Savage Regret, TOPSIS, and Fuzzy-TOPSIS methods using the same decision matrix across three scenarios (S₁—Normal condition; S₂—Bot attack; S₃—Seasonal variation), as shown in Table 7.

The performance component of each Z-number, $A = (a, b, c, d)$, was defuzzified into crisp values according to the weighted average formula $\alpha = \frac{a+2b+2c+d}{6}$, which assigns higher weight to the core interval (b, c) to reduce uncertainty.

Following this transformation, the mean performance values for each alternative–scenario pair were derived and utilized in the comparative evaluation of classical and fuzzy decision-making methods. The outcomes of this comparison are summarized in Table 17.

Table 17. Comparison results of similar decision-making methods.

Nº	Method	Considered Components	Key Indicators and Results
1	Z-Regret (Approach 1)	(A, B) (full Z-environment)	Distance between the ideal Z-number and each alternative: Facebook (105.35) > TikTok (95.04) > Google (70.10)
2	Z-Regret (Approach 2)	(A) (values transformed and defuzzified based on B)	Minimum possible loss under the worst-case scenario: Facebook (8.24) < Google (9.07) < TikTok (12.75)
3	Savage Regret	(A) (defuzzified values)	Maximum loss principle: Facebook (5.00) < Google (10.00) = TikTok (10.00); after tie-break analysis, the final ranking is Facebook > Google > TikTok
4	TOPSIS	(A) (defuzzified values)	Closeness coefficient to the ideal solution: Facebook (0.672) > Google (0.528) > TikTok (0.289)
5	Fuzzy-TOPSIS	(A) (defuzzified values)	Fuzzy closeness coefficient: Facebook (0.637) > Google (0.578) > TikTok (0.229)

As indicated in Table 17, the consistent selection of Facebook (A₂) as the optimal alternative across all comparative methods confirms the convergence of results under different decision-making frameworks, thereby validating the computational stability of the proposed approach.

However, in the case of Z-Regret (Approach 1), the ranking of Facebook > TikTok > Google demonstrates that the model responds more sensitively to the reliability component (B).

- Model stability: Despite differences in normalization and distance functions, Facebook outperformed in both the performance (A) and reliability (B) components.
- Reliability effect: A high confidence level of B = [0.85, 0.90, 0.95, 1.00] reduced the distance function for Facebook, ensuring the most stable outcome. Although TikTok exhibited a similar A value, its lower reliability degree placed it second.

- Cross-method consistency: In all other methods, the ranking remained the same—Facebook > Google > TikTok—confirming the result stability and computational reliability of the Z-Regret model.

The Z-Regret principle, particularly Approach 1, forms stable, confident, and risk-resistant decisions by integrating both the performance (A) and reliability (B) components into the decision-making process.

This principle models the information quality and degree of confidence jointly in decisions made under uncertainty, thereby demonstrating the practical strength and epistemic reliability of Z-numbers.

7. Results

The experimental results revealed that both approaches produced the same final outcome; that is, in accordance with the Z-Regret principle, the superior alternative was identified as A_2 —(Facebook) (Figure 4).

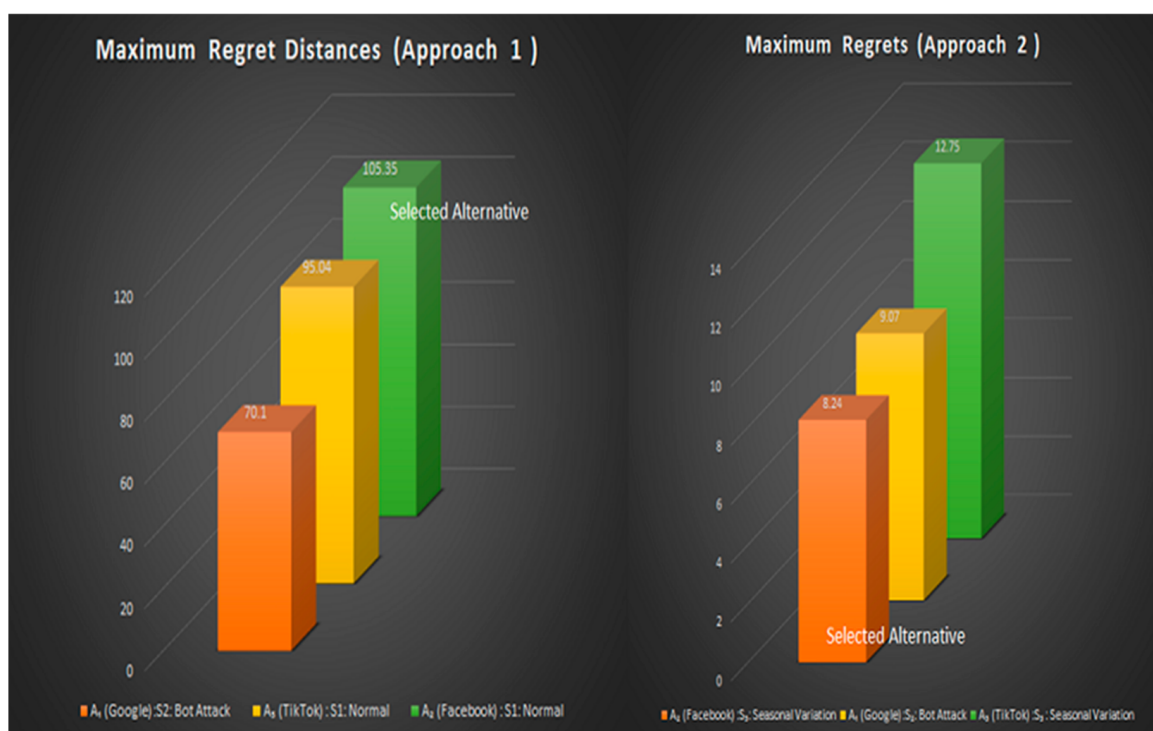


Figure 4. Comparative analysis of the results obtained from Approach 1 and Approach 2.

However, the reasons for the selection differ in accordance with the theoretical essence of the computational algorithms proposed for the Z-Regret principle.

Approach 1, by jointly considering both the performance values (A) and their reliability components (B), provides a more reliable and well-grounded result in the Normal Condition scenario where Facebook is selected. Precisely under normal conditions, both the performance indicators are stable and their reliability degree is highly evaluated. As a result, they are expressed in the form of Z-numbers. For this reason, the obtained result is balanced and allows us to make an optimal decision with maximum reliability. This approach shows that in real conditions it is not sufficient to rely only on the nominal values of performance indicators; in parallel, it is necessary to take into account the quality, stability, and risk factors of the data. Thus, Approach 1 ensures the most robust and risk-resistant decision-making, especially in environments where the information is incomplete, subject to manipulation, and characterized by uncertainty.

In Approach 2, the Z-numbers were first converted into fuzzy numbers and then into crisp values. As a result of this transformation, the influence of the reliability component was weakened, leading to a certain loss of information. In this case, decision-making relied primarily on performance indicators. For this reason, the superiority of Facebook was identified in the *Seasonal Variation* scenario, as it demonstrated relatively more stable performance values.

Thus, although both approaches coincide in terms of the overall outcome, the decision provided by Approach 1 should be regarded as more well-founded and sustainable. This is because, without incurring any loss of information, the alternatives are evaluated both by their performance indicators and by their level of reliability. This result confirms the superiority of the Z-Regret principle: while the classical regret principle compares only the outcomes, the Z-Regret principle enables more balanced decision-making by taking into account both the outcome and the quality of information.

Managerial Implications. The findings of this study provide significant contributions not only from a theoretical perspective but also for management practice:

1. Optimization of advertising budgets. Companies should not rely solely on high CTR and CPC indicators but also take into account the degree of reliability of these indicators. The Z-Regret approach ensures this balance.
2. Risk management. False indicators arising from bot attacks and click fraud are not considered in classical models. In contrast, Z-Regret enables more resilient decision-making in such cases.
3. Scenario planning. The fact that the justifications of the results differ across scenarios demonstrates that companies should evaluate not only the “best alternative” but also how that alternative behaves under different scenarios.
4. Strategy formulation. By applying the Z-Regret approach, managers can select advertising platforms that ensure long-term stability and resilience.

Approach 1 computations were carried out directly in the Z-environment, integrating both performance (A) and reliability (B) components. Hence, Facebook’s selection under the Normal Condition scenario is considered more justified, as this context ensures both stability of indicators and high reliability. Under normal conditions, performance measures such as CTR, CPC, and Conversion Rate are not subject to disruptive fluctuations, and their associated reliability levels remain consistently high. This integration allows Z-Regret to provide a balanced and maximally reliable decision outcome, demonstrating that the most trustworthy results are achieved precisely when both performance and reliability are jointly taken into account.

In Approach 2, the Z-numbers were first converted into fuzzy numbers and then into crisp values. As a result of this transformation, the influence of the reliability component (B) was weakened, leading to a certain loss of information. Since the decision was based on adjusted performance indicators, the superiority of Facebook was justified in the Seasonal Variation scenario due to its relatively stable performance values. However, because seasonal variation is subject to sharp fluctuations caused by campaigns and holidays, this weakens the confidence in the decision reached.

Thus, although both approaches coincide in terms of the overall outcome, the decision provided by Approach 1 should be considered more well-founded. This is because the alternatives are evaluated not only according to their performance indicators but also with respect to their reliability levels. This feature is particularly critical in risk-prone environments—such as advertising bot attacks or manipulation cases—where robustness and credibility of decisions are of paramount importance.

The analysis of the results further shows that the Z-Regret model not only validates the final decision but also explains how alternatives behave under different scenarios. In

Approach 1, the selection of Facebook in the *Normal Condition* scenario is based on its higher reliability indicators. In Approach 2, however, the same alternative is chosen primarily due to its stability under Seasonal Variation conditions. This difference demonstrates that the Z-Regret principle ensures a more objective evaluation in real decision-making processes by jointly considering the combination of “outcome + reliability” in a balanced manner.

These findings constitute the main contribution of the study: by extending the classical regret theory with a reliability component, the Z-Regret principle enables more robust and well-founded decision-making under conditions of uncertainty.

8. Discussion

The results of the conducted experiments showed that, in both approaches, the final choice was the Facebook (A_2) alternative. However, the justification of the selections across scenarios differed in line with the theoretical essence of each approach. This fact clearly demonstrates the main advantage of the Z-Regret principle in both approaches—namely, the consideration of not only outcomes but also the degree of reliability.

Since Approach 1 takes into account both the evaluation (A) and its reliability (B), the selection of Facebook is explained by its ability to provide more reliable results precisely in the *Normal Condition* scenario. In practical terms, this means that the algorithm filters out manipulated or unreliable indicators, thereby enabling more resilient decision-making.

Approach 2, however, weakened the B component due to the transformation of Z-numbers first into fuzzy and then into crisp values. For this reason, the superiority of Facebook was justified in the Seasonal Variation scenario based on its more stable performance indicators.

This comparison shows that the Z-Regret principle not only identifies the optimal alternative but also reveals its resilience profile under different scenarios. This feature is critically important in practical management, especially in fields such as digital advertising that are highly vulnerable to manipulation.

9. Conclusions

In this study, for the first time, the regret principle has been extended with Z-numbers, and the Z-Regret principle has been formally proposed. The proposed model combines Savage’s classical minimax regret criterion with Zadeh’s Z-numbers, thereby allowing consideration of both the outcome (A) and the degree of reliability of information (B). This approach provides a basis for making decisions that are more well-founded and resilient under conditions of uncertainty.

The practical application conducted—namely, the evaluation of the performance of digital advertising platforms—has demonstrated that in both approaches the final choice was the Facebook (A_2) alternative. However, the underlying reasons for this selection differed:

- Approach 1 (Z-Regret algorithm based on the closedness principle of Z-numbers) revealed that Facebook provided more reliable results in the Normal Condition scenario.
- Approach 2 (Z-Regret algorithm based on the transformation of Z-numbers), on the other hand, showed that Facebook was superior under Seasonal Variation conditions due to its stable performance.

These results confirm that, while the classical regret principle relies solely on comparing evaluations, the Z-Regret principle enables more balanced and risk-resilient decision-making by considering both the evaluations themselves and their levels of reliability.

Theoretical Contributions:

- The Z-Regret principle has been scientifically formalized for the first time.

- In conducting regret calculations on Z-numbers, the importance of preserving the principle of closedness (i.e., the result of operations on Z-numbers must itself be a Z-number) has been demonstrated.
- The comparison of two different approaches has substantiated the theoretical and practical advantages of incorporating the reliability factor.

Practical Contributions:

- Companies can optimize their advertising strategies not only on the basis of nominal performance but also by considering the degree of reliability of the data.
- In cases such as bot attacks and click fraud, the model based on the Z-Regret principle enables more resilient decision-making.
- The analysis of results across scenarios allows managers to see not only the best alternative but also how that alternative behaves under different conditions.

Managerial Implications:

- Managers can manage risks more effectively by paying attention not only to the magnitude of outcomes but also to their degree of reliability.
- Considering different scenarios makes the selection of the optimal alternative more reliable and enhances the strategic sustainability of decisions.
- Limitations: The applicability of the proposed Z-Regret principle depends on the accurate definition of fuzzy membership and reliability functions. In large-scale decision problems, the computational complexity of arithmetic operations on Z-numbers may increase significantly, which can limit its practical use.
- Validation of Efficiency and Reliability: The efficiency and reliability of the proposed Z-Regret principle were demonstrated through a comparative case study involving digital advertising platforms. The method consistently identified the optimal alternative (Facebook) under different scenarios, while maintaining stability across fuzzy and probabilistic components. This confirms that the model provides both computational efficiency and decision reliability under uncertainty.

Future Research:

- The application of the Z-Regret principle can be explored in other uncertain environments such as financial markets, energy consumption, and healthcare.
- For big data processing, computational optimization of algorithms developed under the Z-Regret principle should be carried out.
- Comparing the Z-Regret principle with other multi-criteria methods such as Fuzzy-AHP, TOPSIS, and VIKOR can more clearly highlight its distinctions.

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Abbreviations

The following abbreviations are used in this manuscript:

MCDM	Multi-Criteria Decision-Making
AHP	Analytic Hierarchy Process
TOPSIS	Technique for Order Preference by Similarity to Ideal Solution
VIKOR	VlseKriterijumska Optimizacija I Kompromisno Resenje (Compromise solution approach)
DEMATEL	Decision Making Trial and Evaluation Laboratory
CTR	Click-Through Rate (the ratio of clicks to impressions)
CPC	Cost Per Click (the cost incurred for each click)
ROI	Return on Investment (a measure of the profitability of an investment)

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