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K -g-Fusion Frames on Cartesian Products of Two Hilbert C^* -Modules

Sanae Touaiher ¹, Maryam G. Alshehri ^{2,*} and Mohamed Rossafi ³

¹ Laboratory Analysis, Geometry and Applications, Faculty of Science, University of Ibn Tofail, P.O. Box 133, Kenitra 14000, Morocco; sanae.touaiher@uit.ac.ma

² Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia

³ Laboratory Analysis, Geometry and Applications, Higher School of Education and Training, University of Ibn Tofail, P.O. Box 242, Kenitra 14000, Morocco; mohamed.rossafi1@uit.ac.ma

* Correspondence: mgalshehri@ut.edu.sa

Abstract

In this paper, we introduce and investigate the concept of K -g-fusion frames in the Cartesian product of two Hilbert C^* -modules over the same unital C^* -algebra. Our main result establishes that the Cartesian product of two K -g-fusion frames remains a K -g-fusion frame for the direct-sum module. We give explicit formulae for the associated synthesis, analysis, and frame operators and prove natural relations (direct-sum decomposition of the frame operator). Furthermore, we prove a perturbation theorem showing that small perturbations of the component families, measured in the operator or norm sense, still yield a K -g-fusion frame for the product module, with explicit new frame bounds obtained.

Keywords: Hilbert C^* -module; fusion frame; generalized fusion frame; Cartesian product; perturbation

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1. Introduction

The concept of frames, first introduced by Duffin and Schaeffer [1], provides stable yet redundant representations of vectors in Hilbert spaces. Since its inception, frame theory has become a fundamental tool with wide-ranging applications in harmonic analysis, wavelet theory, signal processing, sampling theory, and operator theory (see [2]).

Several extensions of frame theory have been proposed to address increasingly sophisticated settings, including g -frames [3], and fusion frames [4], among others [5]. Each of these generalizations enhances the flexibility of frame representations while preserving their fundamental stability properties. In this context, the notion of K -g-fusion frames, which unifies the features of K -frames, g -frames, and fusion frames, offers a powerful framework for studying operator-related decompositions in Hilbert spaces and beyond.

A natural direction of research has been the extension of frame theory to Hilbert C^* -modules, initiated by Frank and Larson [6]. In contrast to Hilbert spaces, Hilbert C^* -modules present significant challenges, arising from the absence of projections onto arbitrary closed submodules and the presence of a C^* -algebra-valued inner product. Despite these difficulties, frame concepts have been successfully adapted, leading to a variety of results in this setting.

The aim of this paper is to advance the theory of K -g-fusion frames on Cartesian products of Hilbert C^* -modules. Such products naturally emerge in operator algebras,

module decompositions, and block-matrix methods, and hence provide a rich framework for our study.

The paper is organized as follows. Section 2 reviews the fundamental notions of Hilbert C^* -modules and adjointable operators, and introduces the concept of K -g-fusion frames together with their operator-theoretic features. Section 3 contains the main result concerning Cartesian products of K -g-fusion frames. In Section 4, we establish perturbation results, while the final section is devoted to concluding remarks and illustrative examples.

2. Preliminaries

We briefly recall the basic definitions and facts about Hilbert C^* -modules needed in the sequel.

Definition 1 ([7]). Let $\mathcal{A}$ be a unital C^* -algebra. A left Hilbert C^* -module over $\mathcal{A}$ is a left $\mathcal{A}$ -module $\mathcal{H}$ equipped with a map

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{A}$$

called the $\mathcal{A}$ -valued inner product, satisfying

- (1) $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ for all $a \in \mathcal{A}$, $x, y, z \in \mathcal{H}$;
- (2) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in \mathcal{H}$;
- (3) $\langle x, x \rangle \geq 0$ in $\mathcal{A}$, and $\langle x, x \rangle = 0 \iff x = 0$.

The associated norm is defined by $\|x\| := \|\langle x, x \rangle\|^{1/2}$, and completeness with respect to this norm is assumed.

For Hilbert $\mathcal{A}$ -modules $\mathcal{H}, \mathcal{K}$, we denote by $\text{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ the set of adjointable operators from $\mathcal{H}$ into $\mathcal{K}$, i.e., those operators $T : \mathcal{H} \rightarrow \mathcal{K}$ for which there exists an adjoint $T^* : \mathcal{K} \rightarrow \mathcal{H}$ satisfying

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \text{for all } x \in \mathcal{H}, y \in \mathcal{K}.$$

If $\mathcal{K} = \mathcal{H}$, then we simply write $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ instead of $\text{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H})$. An operator $T \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$ is called *positive*—written $T \geq 0$ —if

$$\langle Tx, x \rangle \geq 0 \quad \text{for all } x \in \mathcal{H}.$$

The partial order on self-adjoint operators is determined by this cone.

For a closed submodule $W \subset \mathcal{H}$, an *orthogonal projection* $P_W \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$ satisfies

$$P_W^2 = P_W = P_W^* \quad \text{and} \quad \text{Ran}(P_W) = W.$$

Unlike the Hilbert space case, not every closed submodule is complemented in $\mathcal{H}$.

In this work, we restrict attention to orthogonally complemented submodules.

In what follows, all sums indexed by a countable set I are assumed to converge in norm in $\mathcal{A}$ whenever convergence is asserted.

These preliminaries allow us to introduce K -g-fusion frames in Hilbert C^* -modules in the next.

Given two Hilbert $\mathcal{A}$ -modules $\mathcal{H}_1$ and $\mathcal{H}_2$, their external direct sum (or product) is defined as

$$\mathcal{H}_1 \oplus \mathcal{H}_2 := \{(x_1, x_2) : x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2\},$$

with the natural left $\mathcal{A}$ -module action

$$a \cdot (x_1, x_2) := (ax_1, ax_2), \quad a \in \mathcal{A},$$

and $\mathcal{A}$ -valued inner product

$$\langle (x_1, x_2), (y_1, y_2) \rangle := \langle x_1, y_1 \rangle_{\mathcal{H}_1} + \langle x_2, y_2 \rangle_{\mathcal{H}_2}.$$

With this structure, $\mathcal{H}_1 \oplus \mathcal{H}_2$ is a Hilbert $\mathcal{A}$ -module.

Moreover, if $V \subset \mathcal{H}_1$ and $W \subset \mathcal{H}_2$ are orthogonally complemented submodules, then their direct sum $V \oplus W$ is an orthogonally complemented submodule of $\mathcal{H}_1 \oplus \mathcal{H}_2$, with the projection operator

$$P_{V \oplus W} = P_V \oplus P_W.$$

This observation will be essential in constructing product families of frames and proving stability under perturbations.

Definition 2 ([8]). Let $\mathcal{H}$ and $\mathcal{K}$ be countably generated Hilbert $\mathcal{A}$ -modules. Suppose that

- (1) $\{v_i\}_{i \in I}$ is a family of positive invertible elements from the center of $\mathcal{A}$;
- (2) $\{W_i\}_{i \in I}$ is a family of orthogonally complemented closed submodules of $\mathcal{H}$;
- (3) $\{\mathcal{H}_i\}_{i \in I}$ is a family of closed submodules of $\mathcal{K}$;
- (4) for each $i \in I$, $\Lambda_i \in \text{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i)$;
- (5) $K \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$.

We say that $\Lambda = (W_i, \Lambda_i, v_i)_{i \in I}$ is a K -g-fusion frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$ if there exist scalars $0 < A \leq B < \infty$, such that

$$A \langle K^* f, K^* f \rangle \leq \sum_{i \in I} v_i^2 \langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f \rangle \leq B \langle f, f \rangle, \quad (1)$$

for all $f \in \mathcal{H}$. The constants A and B are called the lower and upper bounds of the K -g-fusion frame. In addition,

- If the inequalities hold with $K = I_{\mathcal{H}}$, then Λ is a g-fusion frame, i.e.,

$$A \langle f, f \rangle \leq \sum_{i \in I} v_i^2 \langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f \rangle \leq B \langle f, f \rangle, \quad \forall f \in \mathcal{H}.$$

- If, in addition, $K = I_{\mathcal{H}}$ and $\Lambda_i = P_{W_i}$ for all $i \in I$, then Λ reduces to a fusion frame for $\mathcal{H}$.

Now, for a K -g-fusion frame $\Lambda = (W_i, \Lambda_i, v_i)_{i \in I}$ of $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$,

- The analysis operator

$$T_{\Lambda}^* : \mathcal{H} \longrightarrow \ell^2(\{\mathcal{H}_i\}_{i \in I})$$

is defined by

$$T_{\Lambda}^* f = (v_i \Lambda_i P_{W_i} f)_{i \in I}, \quad f \in \mathcal{H}. \quad (2)$$

- The synthesis operator

$$T_{\Lambda} : \ell^2(\{\mathcal{H}_i\}_{i \in I}) \longrightarrow \mathcal{H}$$

is the adjoint of T_{Λ}^* and is given by

$$T_{\Lambda}((f_i)_{i \in I}) = \sum_{i \in I} v_i P_{W_i} \Lambda_i^* f_i, \quad (f_i)_{i \in I} \in \ell^2(\{\mathcal{H}_i\}_{i \in I}). \quad (3)$$

- The frame operator

$$S_{\Lambda} : \mathcal{H} \longrightarrow \mathcal{H}$$

is defined by

$$S_{\Lambda} f = T_{\Lambda} T_{\Lambda}^* f = \sum_{i \in I} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} f, \quad f \in \mathcal{H}. \quad (4)$$

3. Product K -g-Fusion Frames and Main Theorem

Let $\mathcal{A}$ be a unital C^* -algebra and let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert $\mathcal{A}$ -modules. For each $i \in I$, let $W_i \subset \mathcal{H}_1$ and $V_i \subset \mathcal{H}_2$ be orthogonally complemented closed submodules with projections $P_{W_i} \in \text{End}_{\mathcal{A}}^*(\mathcal{H}_1)$ and $P_{V_i} \in \text{End}_{\mathcal{A}}^*(\mathcal{H}_2)$. Let $\mathcal{H}_{1,i}, \mathcal{H}_{2,i}$ be Hilbert $\mathcal{A}$ -modules and let $\Lambda_i \in \text{End}_{\mathcal{A}}^*(W_i, \mathcal{H}_{1,i})$, $\Gamma_i \in \text{End}_{\mathcal{A}}^*(V_i, \mathcal{H}_{2,i})$ be adjointable maps. Assume that $(W_i, \Lambda_i, v_i)_{i \in I}$ is a K_1 -g-fusion frame for $\mathcal{H}_1$ with bounds $A_1, B_1 > 0$ and that $(V_i, \Gamma_i, v_i)_{i \in I}$ is a K_2 -g-fusion frame for $\mathcal{H}_2$ with bounds $A_2, B_2 > 0$, where $K_i \in \text{End}_{\mathcal{A}}^*(\mathcal{H}_i)$ ($i = 1, 2$). Define, for each $i \in I$,

$$\Theta_i : \mathcal{H}_1 \oplus \mathcal{H}_2 \longrightarrow \mathcal{H}_{1,i} \oplus \mathcal{H}_{2,i}, \quad \Theta_i(x, y) = (\Lambda_i P_{W_i} x, \Gamma_i P_{V_i} y). \quad (5)$$

Then, we have the following theorem:

Theorem 1. Assume that $(W_i, \Lambda_i, v_i)_{i \in I}$ is a K_1 -g-fusion frame for $\mathcal{H}_1$ with bounds $A_1, B_1 > 0$ and that $(V_i, \Gamma_i, v_i)_{i \in I}$ is a K_2 -g-fusion frame for $\mathcal{H}_2$ with bounds $A_2, B_2 > 0$, then the family $\{(W_i \oplus V_i, \Theta_i, v_i)\}_{i \in I}$ is a $(K_1 \oplus K_2)$ -g-fusion frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$ with bounds $A = \min\{A_1, A_2\}$ and $B = \max\{B_1, B_2\}$. Furthermore, if $T^{(1)}$ and $T^{(2)}$ are respectively the synthesis operators of $(W_i, \Lambda_i, v_i)_{i \in I}$ and $(V_i, \Gamma_i, v_i)_{i \in I}$ and $S^{(1)}$ and $S^{(2)}$ their frame operators, then the synthesis operator T of the product satisfies $T = T^{(1)} \oplus T^{(2)}$, and the frame operator satisfies $S = S^{(1)} \oplus S^{(2)}$.

Proof. Since $(W_i, \Lambda_i, v_i)_{i \in I}$ is a K_1 -g-fusion frame for $\mathcal{H}_1$, it is in particular a Bessel family. Thus, there exists a scalar $B_1 > 0$ such that for all $x \in \mathcal{H}_1$,

$$\sum_{i \in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x \rangle \leq B_1 \langle x, x \rangle,$$

where the series converges in norm in $\mathcal{A}$. Similarly, for (V_i, Γ_i, v_i) , there is $B_2 > 0$ with analogous norm-convergent series

$$\sum_{i \in I} v_i^2 \langle \Gamma_i P_{V_i} y, \Gamma_i P_{V_i} y \rangle \leq B_2 \langle y, y \rangle,$$

for all $y \in \mathcal{H}_2$. Therefore for any $(x, y) \in \mathcal{H}_1 \oplus \mathcal{H}_2$,

$$\begin{aligned} \sum_{i \in I} v_i^2 \langle \Theta_i(x, y), \Theta_i(x, y) \rangle &= \sum_{i \in I} v_i^2 \left(\langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x \rangle + \langle \Gamma_i P_{V_i} y, \Gamma_i P_{V_i} y \rangle \right) \\ &= \left(\sum_{i \in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x \rangle \right) + \left(\sum_{i \in I} v_i^2 \langle \Gamma_i P_{V_i} y, \Gamma_i P_{V_i} y \rangle \right), \end{aligned}$$

with each summand being norm-convergent in $\mathcal{A}$; hence the whole sum converges in norm. Moreover,

$$\sum_{i \in I} v_i^2 \langle \Theta_i(x, y), \Theta_i(x, y) \rangle \leq B_1 \langle x, x \rangle + B_2 \langle y, y \rangle \leq \max\{B_1, B_2\} \langle (x, y), (x, y) \rangle$$

shows the desired uniform Bessel bound on the product.

By the K_1 -g-fusion inequality on $\mathcal{H}_1$, we have the $\mathcal{A}$ -valued inequality

$$A_1 \langle K_1^* x, K_1^* x \rangle \leq \sum_{i \in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x \rangle \leq B_1 \langle x, x \rangle,$$

and similarly for $\mathcal{H}_2$:

$$A_2 \langle K_2^* y, K_2^* y \rangle \leq \sum_{i \in I} v_i^2 \langle \Gamma_i P_{V_i} y, \Gamma_i P_{V_i} y \rangle \leq B_2 \langle y, y \rangle.$$

We can see that these two $\mathcal{A}$ -valued inequalities yield

$$A_1 \langle K_1^* x, K_1^* x \rangle + A_2 \langle K_2^* y, K_2^* y \rangle \leq \sum_{i \in I} v_i^2 \langle \Theta_i(x, y), \Theta_i(x, y) \rangle \leq B_1 \langle x, x \rangle + B_2 \langle y, y \rangle.$$

Since A_1, A_2 are positive real scalars,

$$A_1 \langle K_1^* x, K_1^* x \rangle + A_2 \langle K_2^* y, K_2^* y \rangle \geq \min\{A_1, A_2\} (\langle K_1^* x, K_1^* x \rangle + \langle K_2^* y, K_2^* y \rangle),$$

and likewise

$$B_1 \langle x, x \rangle + B_2 \langle y, y \rangle \leq \max\{B_1, B_2\} (\langle x, x \rangle + \langle y, y \rangle).$$

Observing that $(K_1 \oplus K_2)^*(x, y) = (K_1^* x, K_2^* y)$ and that $\langle (x, y), (x, y) \rangle = \langle x, x \rangle + \langle y, y \rangle$, we obtain the claimed inequalities with $A = \min\{A_1, A_2\}$ and $B = \max\{B_1, B_2\}$, that is

$$\begin{aligned} A \langle (K_1 \oplus K_2)^*(x, y), (K_1 \oplus K_2)^*(x, y) \rangle &\leq \sum_{i \in I} v_i^2 \langle \Theta_i(x, y), \Theta_i(x, y) \rangle \\ &\leq B \langle (x, y), (x, y) \rangle. \end{aligned} \quad (6)$$

Now, compute the adjoint $\Theta_i^* : \mathcal{H}_{1,i} \oplus \mathcal{H}_{2,i} \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$. For $(u, v) \in \mathcal{H}_{1,i} \oplus \mathcal{H}_{2,i}$ and $(x, y) \in \mathcal{H}_1 \oplus \mathcal{H}_2$, one has

$$\begin{aligned} \langle \Theta_i(x, y), (u, v) \rangle &= \langle \Lambda_i P_{W_i} x, u \rangle_{\mathcal{H}_{1,i}} + \langle \Gamma_i P_{V_i} y, v \rangle_{\mathcal{H}_{2,i}} \\ &= \langle (x, y), (P_{W_i} \Lambda_i^* u, P_{V_i} \Gamma_i^* v) \rangle; \end{aligned} \quad (7)$$

hence,

$$\Theta_i^*(u, v) = (P_{W_i} \Lambda_i^* u, P_{V_i} \Gamma_i^* v). \quad (8)$$

Therefore, the operator $\Theta_i^* \Theta_i$ acts on (x, y) by

$$\Theta_i^* \Theta_i(x, y) = (P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} x, P_{V_i} \Gamma_i^* \Gamma_i P_{V_i} y).$$

Multiplying by the scalar weight v_i^2 and summing over i gives the frame operator on the product:

$$S(x, y) = \sum_{i \in I} v_i^2 \Theta_i^* \Theta_i(x, y) = \left(\sum_{i \in I} v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} x, \sum_{i \in I} v_i^2 P_{V_i} \Gamma_i^* \Gamma_i P_{V_i} y \right).$$

The right-hand side is precisely $(S^{(1)}x, S^{(2)}y)$ where

$$S^{(1)} = \sum_i v_i^2 P_{W_i} \Lambda_i^* \Lambda_i P_{W_i} \text{ and } S^{(2)} = \sum_i v_i^2 P_{V_i} \Gamma_i^* \Gamma_i P_{V_i} \quad (9)$$

are the frame operators of the component families. Thus, $S = S^{(1)} \oplus S^{(2)}$. In particular, S is positive and the operator inequalities

$$A(K_1 K_1^* \oplus K_2 K_2^*) \leq S \leq B I_{\mathcal{H}_1 \oplus \mathcal{H}_2} \quad (10)$$

hold in $\text{End}_{\mathcal{A}}^*(\mathcal{H}_1 \oplus \mathcal{H}_2)$. This completes the proof. $\square$

Example 1. Let $\mathcal{A} = \mathbb{C}^2$ with coordinate-wise operations and the usual involution. Consider the left $\mathcal{A}$ -modules $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{A}^6$. Any element of $\mathcal{H}_1$ or $\mathcal{H}_2$ can be written as $x = (x_1, \dots, x_6)$, where $x_m = (a_m, b_m) \in \mathcal{A} = \mathbb{C}^2$, for $m = 1, \dots, 6$. The $\mathcal{A}$ -valued inner product is given by

$$\langle x, y \rangle = \left(\sum_{m=1}^6 a_m \overline{c_m}, \sum_{m=1}^6 b_m \overline{d_m} \right),$$

for $y = (y_1, \dots, y_6)$ with $y_m = (c_m, d_m)$. In particular,

$$\langle x, x \rangle = \left(\sum_{m=1}^6 |a_m|^2, \sum_{m=1}^6 |b_m|^2 \right).$$

Define two diagonal adjointable operators K_1 and K_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively by

$$K_1(x_1, \dots, x_6) = (x_1, 2x_2, x_3, 2x_4, x_5, 2x_6),$$

$$K_2(y_1, \dots, y_6) = (y_1, 3y_2, y_3, 3y_4, y_5, 3y_6).$$

Both are self-adjoint, so $K_i^* = K_i \in \text{End}_{\mathcal{A}}(\mathcal{H}_i)$ for $i = 1, 2$.

Denote $e_1 = (1, 0, 0, 0, 0, 0), \dots, e_6 = (0, 0, 0, 0, 0, 1)$ as the canonical elements of $\mathcal{H}_j$, $j = 1, 2$. For the Hilbert $\mathcal{C}^*$ -module $\mathcal{H}_1$, define

$$W_1 = \text{span}_{\mathcal{A}}\{e_1, e_2\}, \quad W_2 = \text{span}_{\mathcal{A}}\{e_3, e_4\}, \quad W_3 = \text{span}_{\mathcal{A}}\{e_5, e_6\},$$

and let

$$\Lambda_i : W_i \rightarrow \mathcal{A}^2, \quad \Lambda_i(x_{2i-1}, x_{2i}) = (x_{2i-1}, x_{2i}).$$

For $\mathcal{H}_2$ choose the submodules

$$V_1 = \text{span}_{\mathcal{A}}\{e_1, e_3\}, \quad V_2 = \text{span}_{\mathcal{A}}\{e_2, e_5\}, \quad V_3 = \text{span}_{\mathcal{A}}\{e_4, e_6\},$$

and define

$$\Gamma_i : V_i \rightarrow \mathcal{A}^2, \quad \Gamma_i(x_j, x_k) = (x_j, x_k).$$

All weights v_i , $i = 1, 2, 3$ are chosen to be equal to 1.

Now observe that for all $x \in \mathcal{H}_1$ and all $y \in \mathcal{H}_2$, we have

$$\begin{aligned} \frac{1}{4} \langle K_1^* x, K_1^* x \rangle &\leq \sum_{i=1}^3 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x \rangle = \langle x, x \rangle, \\ \frac{1}{9} \langle K_2^* y, K_2^* y \rangle &\leq \sum_{i=1}^3 \langle \Gamma_i P_{V_i} y, \Gamma_i P_{V_i} y \rangle = \langle y, y \rangle. \end{aligned}$$

So, (V_i, Γ_i) is a K_2 -g-fusion frame with bounds $A_2 = \frac{1}{9}$, $B_2 = 1$.

The product family

$$\{(W_i \oplus V_i, \Theta_i, 1)\}_{i=1}^3, \quad \Theta_i(x, y) = (\Lambda_i P_{W_i} x, \Gamma_i P_{V_i} y),$$

is a $(K_1 \oplus K_2)$ -g-fusion frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$ with bounds

$$A = \min\{A_1, A_2\} = \frac{1}{9}, \quad B = \max\{B_1, B_2\} = 1.$$

That is, for all $(x, y) \in \mathcal{H}_1 \oplus \mathcal{H}_2$,

$$\frac{1}{9} \langle (K_1 \oplus K_2)^*(x, y), (K_1 \oplus K_2)^*(x, y) \rangle \leq \sum_{i=1}^3 \langle \Theta_i(x, y), \Theta_i(x, y) \rangle \leq \langle (x, y), (x, y) \rangle.$$

4. Perturbation Theorem

Let $K_i \in \text{End}_{\mathcal{A}}^*(\mathcal{H}_i)$ ($i = 1, 2$), and assume that $\{(W_i, \Lambda_i, v_i)\}_{i \in I}$ is a K_1 -g-fusion frame for $\mathcal{H}_1$ with frame bounds $0 < A_1 \leq B_1 < \infty$, and that $\{(V_i, \Gamma_i, v_i)\}_{i \in I}$ is a K_2 -g-fusion frame for $\mathcal{H}_2$ with frame bounds $0 < A_2 \leq B_2 < \infty$. Denote their product frame by $\mathcal{F} = \{(W_i \oplus V_i, \Theta_i, v_i)\}_{i \in I}$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, by taking the common weights v_i ($i \in I$). The following theorem gives a perturbation result saying that if each component of the family is a K -g-fusion frame and each component perturbation is small, then the perturbed product family is again a K -g-fusion frame on the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$.

Theorem 2. Let $\mathcal{F}' = \{(W'_i, V'_i, \Lambda'_i, \Gamma'_i, v_i)\}_{i \in I}$ be a perturbed family with the same weights v_i , where $W'_i \subset \mathcal{H}_1$ and $V'_i \subset \mathcal{H}_2$ are orthogonally complemented submodules, and Λ'_i, Γ'_i are adjointable operators. Assume there exist scalars $r_1, r_2 > 0$ such that, for all $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$,

$$\begin{cases} \sum_{i \in I} v_i^2 \langle (\Lambda_i P_{W_i} - \Lambda'_i P_{W'_i})x_1, (\Lambda_i P_{W_i} - \Lambda'_i P_{W'_i})x_1 \rangle \leq r_1 \langle K_1^* x_1, K_1^* x_1 \rangle, \\ \sum_{i \in I} v_i^2 \langle (\Gamma_i P_{V_i} - \Gamma'_i P_{V'_i})x_2, (\Gamma_i P_{V_i} - \Gamma'_i P_{V'_i})x_2 \rangle \leq r_2 \langle K_2^* x_2, K_2^* x_2 \rangle. \end{cases} \quad (11)$$

If $A_1 > r_1$ and $A_2 > r_2$, then the perturbed product frame $\mathcal{F}'$ is a $(K_1 \oplus K_2)$ -g-fusion frame for $\mathcal{H}$ with frame bounds

$$\begin{aligned} A' &:= \min\{(\sqrt{A_1} - \sqrt{r_1})^2, (\sqrt{A_2} - \sqrt{r_2})^2\} \\ \text{and} \\ B' &:= \max\{2B_1 + 2r_1 \|K_1^*\|^2, 2B_2 + 2r_2 \|K_2^*\|^2\}. \end{aligned} \quad (12)$$

Proof. For $x_1 \in \mathcal{H}_1$, note that $\Lambda'_i P_{W'_i} x_1 = \Lambda_i P_{W_i} x_1 + (\Lambda'_i P_{W'_i} - \Lambda_i P_{W_i}) x_1$. For $x_2 \in \mathcal{H}_2$, set similarly $\Gamma'_i P_{V'_i} x_2 = \Gamma_i P_{V_i} x_2 + (\Gamma'_i P_{V'_i} - \Gamma_i P_{V_i}) x_2$.

First, from the two g-fusion frame inequalities (11) and Theorem 1, we obtain immediately, with $A = \min\{A_1, A_2\}$ and $B = \max\{B_1, B_2\}$, that the product family $\mathcal{F}$ is a $(K_1 \oplus K_2)$ -g-fusion frame with bounds A, B . Thus, this satisfies

$$\begin{aligned} A \langle (K_1 \oplus K_2)^*(x_1, x_2), (K_1 \oplus K_2)^*(x_1, x_2) \rangle &\leq \sum_{i \in I} v_i^2 \langle \Theta_i(x_1, x_2), \Theta_i(x_1, x_2) \rangle \\ &= \sum_{i \in I} v_i^2 (\langle \Lambda_i P_{W_i} x_1, \Lambda_i P_{W_i} x_1 \rangle + \langle \Gamma_i P_{V_i} x_2, \Gamma_i P_{V_i} x_2 \rangle) \\ &\leq B \langle (x_1, x_2), (x_1, x_2) \rangle. \end{aligned}$$

Now, fix $x_1 \in \mathcal{H}_1$. Consider the perturbed left-component sum

$$\begin{aligned} S'_1(x_1) &:= \sum_{i \in I} v_i^2 \langle \Lambda'_i P_{W'_i} x_1, \Lambda'_i P_{W'_i} x_1 \rangle \\ &= \sum_{i \in I} v_i^2 \langle \Lambda_i P_{W_i} x_1 + (\Lambda'_i P_{W'_i} - \Lambda_i P_{W_i}) x_1, \Lambda_i P_{W_i} x_1 + (\Lambda'_i P_{W'_i} - \Lambda_i P_{W_i}) x_1 \rangle. \end{aligned}$$

For the sake of readability, we denote X and Y as the elements of the Hilbert C^* -module $\ell^2(I, \mathcal{A})$ defined by

$$X := (v_i \Lambda_i P_{W_i} x_1)_{i \in I}, \quad Y := (v_i (\Lambda'_i P_{W'_i} - \Lambda_i P_{W_i}) x_1)_{i \in I}. \quad (13)$$

Thus

$$S'_1(x_1) = \langle X, X \rangle_{\ell^2(I, \mathcal{A})} + \langle Y, Y \rangle_{\ell^2(I, \mathcal{A})} + \langle X, Y \rangle_{\ell^2(I, \mathcal{A})} + \langle X, Y \rangle_{\ell^2(I, \mathcal{A})}^*. \quad (14)$$

It follows from the computations that

$$\begin{aligned} \langle X, Y \rangle_{\ell^2(I, \mathcal{A})} + \langle X, Y \rangle_{\ell^2(I, \mathcal{A})}^* &= \sum_{i \in I} v_i^2 (\langle \Lambda_i P_{W_i} x_1, (\Lambda'_i P_{W'_i} - \Lambda_i P_{W_i}) x_1 \rangle + \langle (\Lambda'_i P_{W'_i} - \Lambda_i P_{W_i}) x_1, \Lambda_i P_{W_i} x_1 \rangle) \\ \langle X, X \rangle_{\ell^2(I, \mathcal{A})} &= \sum_{i \in I} v_i^2 \langle \Lambda_i P_{W_i} x_1, \Lambda_i P_{W_i} x_1 \rangle \\ \langle Y, Y \rangle_{\ell^2(I, \mathcal{A})} &= \sum_{i \in I} v_i^2 \langle (\Lambda'_i P_{W'_i} - \Lambda_i P_{W_i}) x_1, (\Lambda'_i P_{W'_i} - \Lambda_i P_{W_i}) x_1 \rangle, \end{aligned}$$

Now, since $\langle X + Y, X + Y \rangle_{\ell^2(I, \mathcal{A})}$ and $\langle X - Y, X - Y \rangle_{\ell^2(I, \mathcal{A})}$ are non-negative in $\mathcal{A}$, we have

$$\begin{aligned} -\sqrt{\frac{r_1}{A_1}} \langle X, X \rangle_{\ell^2(I, \mathcal{A})} - \sqrt{\frac{A_1}{r_1}} \langle Y, Y \rangle_{\ell^2(I, \mathcal{A})} &\leq \langle X, Y \rangle_{\ell^2(I, \mathcal{A})} + \langle X, Y \rangle_{\ell^2(I, \mathcal{A})}^* \\ &\leq \langle X, X \rangle_{\ell^2(I, \mathcal{A})} + \langle Y, Y \rangle_{\ell^2(I, \mathcal{A})} \end{aligned} \quad (15)$$

Since $S'_1(x_1) = \langle X + Y, X + Y \rangle_{\ell^2(I, \mathcal{A})}$, we deduce from (11), (15) and (14) that

$$\begin{aligned} S'_1(x_1) &\leq 2\langle X, X \rangle_{\ell^2(I, \mathcal{A})} + 2\langle Y, Y \rangle_{\ell^2(I, \mathcal{A})} \\ &\leq 2B_1 \langle x_1, x_1 \rangle + 2r_1 \|K_1^*\|^2 \langle x_1, x_1 \rangle \\ &= (2B_1 + 2r_1 \|K_1^*\|^2) \langle x_1, x_1 \rangle \end{aligned}$$

For the other inequality, using (11), (14), (15) and the hypothesis $\sqrt{\frac{r_1}{A_1}} < 1$, we obtain that

$$\begin{aligned} S'_1(x_1) &\geq \langle X, X \rangle_{\ell^2(I, \mathcal{A})} + \langle Y, Y \rangle_{\ell^2(I, \mathcal{A})} - \sqrt{\frac{r_1}{A_1}} \langle X, X \rangle_{\ell^2(I, \mathcal{A})} - \sqrt{\frac{A_1}{r_1}} \langle Y, Y \rangle_{\ell^2(I, \mathcal{A})} \\ &= (1 - \sqrt{\frac{r_1}{A_1}}) \langle X, X \rangle_{\ell^2(I, \mathcal{A})} - (\sqrt{\frac{A_1}{r_1}} - 1) \langle Y, Y \rangle_{\ell^2(I, \mathcal{A})} \\ &\geq (A_1(1 - \sqrt{\frac{r_1}{A_1}}) \langle K_1^* x_1, K_1^* x_1 \rangle - r_1(\sqrt{\frac{A_1}{r_1}} - 1) \langle K_1^* x_1, K_1^* x_1 \rangle) \\ &\geq (A_1(1 - \sqrt{\frac{r_1}{A_1}}) + r_1(1 - \sqrt{\frac{A_1}{r_1}})) \langle K_1^* x_1, K_1^* x_1 \rangle \\ &= (\sqrt{A_1} - \sqrt{r_1})^2 \langle K_1^* x_1, K_1^* x_1 \rangle \end{aligned}$$

The same argument applied to the second component yields

$$(\sqrt{A_2} - \sqrt{r_2})^2 \langle K_2^* x_2, K_2^* x_2 \rangle \leq S'_2(x_2) \leq (2B_2 + 2r_2 \|K_2^*\|^2) \langle x_2, x_2 \rangle.$$

Finally, for $(x_1, x_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2$, we have

$$S'(x_1, x_2) := \sum_i v_i^2 \langle \Theta'_i(x_1, x_2), \Theta'_i(x_1, x_2) \rangle = S'_1(x_1) + S'_2(x_2).$$

Thus, combining the component-wise upper bounds yields

$$\begin{aligned} S'(x_1, x_2) &\leq (2B_1 + 2r_1 \|K_1^*\|^2) \langle x_1, x_1 \rangle + (2B_2 + 2r_2 \|K_2^*\|^2) \langle x_2, x_2 \rangle \\ &\leq B' \langle (x_1, x_2), (x_1, x_2) \rangle, \end{aligned}$$

with $B' = \max\{(2B_1 + 2r_1 \|K_1^*\|^2), (2B_2 + 2r_2 \|K_2^*\|^2)\}$.

Similarly, combining the lower bounds, we get

$$\begin{aligned} S'(x_1, x_2) &\geq (\sqrt{A_1} - \sqrt{r_1})^2 \langle K_1^* x_1, K_1^* x_1 \rangle + (\sqrt{A_2} - \sqrt{r_2})^2 \langle K_2^* x_2, K_2^* x_2 \rangle \\ &\geq A' \langle (K_1 \oplus K_2)^*(x_1, x_2), (K_1 \oplus K_2)^*(x_1, x_2) \rangle, \end{aligned}$$

with $A' = \min\{(\sqrt{A_1} - \sqrt{r_1})^2, (\sqrt{A_2} - \sqrt{r_2})^2\}$. This yields the claimed two-sided $\mathcal{A}$ -order inequalities and completes the proof that $\mathcal{F}'$ is a $(K_1 \oplus K_2)$ -g-fusion frame with bounds A', B' . $\square$

5. Concluding Remark and Examples Illustrating Theorems 1 and 2

In this section, we show that the conditions of our results are not restrictive. We also provide illustrative examples of our two *main theorems*, along with some comparative observations.

Let us begin by noting that, to the best of our knowledge, our perturbation theorem is new, even in the case of the perturbation of a frame in a single classical Hilbert space or in a Hilbert C^* -module $\mathcal{H}$. We can state it as follows:

Theorem 3. *Let $K \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$, and assume that $\{(W_i, \Lambda_i, v_i)\}_{i \in I}$ is a K -g-fusion frame for $\mathcal{H}$ and $\mathcal{F}' = \{(W'_i, \Lambda'_i, v_i)\}_{i \in I}$ is a perturbed family with the same weights v_i , where $W'_i \subset \mathcal{H}$ are orthogonally complemented submodules, $i \in \mathcal{N}$, and $\Lambda'_i, i \in \mathcal{N}$, are adjointable operators. Assume there exist scalars $r > 0$ such that, for all $x \in \mathcal{H}$,*

$$\sum_{i \in I} v_i^2 \langle (\Lambda_i P_{W_i} - \Lambda'_i P_{W'_i})x, (\Lambda_i P_{W_i} - \Lambda'_i P_{W'_i})x \rangle \leq r \langle K^*x, K^*x \rangle, \quad (16)$$

If $A > r$, then the perturbed product frame $\mathcal{F}'$ is a K -g-fusion frame for $\mathcal{H}$ with frame bounds $A' := (\sqrt{A} - \sqrt{r})^2$ and $B' := 2B + 2r\|K^*\|^2$.

5.1. Complemented Submodules in Hilbert C^* -Modules: Structural Justification

The study of orthogonally complemented submodules in Hilbert C^* -modules relies on deep structural properties of the underlying C^* -algebra. As established by Magajna [9], every closed submodule of a Hilbert module is orthogonally complemented *if and only if* the C^* -algebra B is $*$ -isomorphic to a (possibly non-unital) C^* -algebra of compact operators. Hence, the existence of orthogonal complements is not an additional assumption, but an intrinsic property of such algebras.

Typical unital examples include finite direct sums of matrix algebras

$$B = \bigoplus_{i=1}^p M_{n_i}(\mathbb{C}),$$

which play an essential role in the theory of operator algebras. They represent a natural and important generalization of the field $\mathbb{C}$, corresponding to the case $M_1(\mathbb{C})$. These algebras preserve the orthogonal complementation property and form the structural backbone of many constructions in Hilbert C^* -module theory.

Moreover, classical operator-theoretic results, such as Lance's Theorem 3.9 [10], provide a constructive framework for generating complemented submodules. For any adjointable operator T with closed range, one has

$$\text{Ran}(T) = \text{Ran}(TT^*) \quad \text{and} \quad \text{Ran}(T)^\perp = \ker T^*, \quad (17)$$

so that $\text{Ran}(T)$ is a closed orthogonally complemented submodule of X . Thus, the ranges of adjointable operators yield a systematic method for constructing complemented submodules within this setting.

Additionally, given a C^* -algebra $\mathcal{A}$ and a family of Hilbert C^* -modules $\{(\mathcal{H}_i, \langle \cdot, \cdot \rangle)\}_{i \in I}$, the ℓ^2 -direct sum

$$\ell^2((\mathcal{H}_i)_{i \in I}) = \left\{ (x_i)_{i \in I} \in \prod_{i \in I} \mathcal{H}_i : \sum_{i \in I} \left\| \langle x_i, x_i \rangle_i \right\|_{\mathcal{A}} < \infty \right\},$$

equipped with $\langle (x_i)_{i \in I}, (y_i)_{i \in I} \rangle := \sum_{i \in I} \langle x_i, y_i \rangle_i$, is a Hilbert C^* -module with a rich set of closed complemented submodules. For any nonempty subset $J \subseteq I$, the submodule $\ell^2((\mathcal{H}_i)_{i \in J})$ is itself closed and orthogonally complemented. This construction offers a canonical and flexible source of complemented submodules in the general Hilbert C^* -module framework.

Consequently, restricting attention to Hilbert modules over C^* -algebras with orthogonally complemented submodules is both natural and well-founded; it precisely corresponds to the class of Hilbert modules over C^* -algebras of compact operators (including finite direct sums of matrix algebras and ℓ^2 -direct sums of Hilbert modules), and it is further supported by the structural theory of adjointable operators developed by Lance. Therefore, the theoretical scope of our study is both coherent and rigorously grounded in the existing literature.

5.2. Examples Illustrating Theorems 1 and 2

Let X be a compact Hausdorff space and let $\mathcal{A} = C(X)$. Consider the Hilbert $\mathcal{A}$ -module

$$\mathcal{H} = \ell^2(\mathcal{A}) = \left\{ f := (f_n)_{n \geq 0}, f_n \in \mathcal{A} : \sum_{n=0}^{\infty} f_n^* f_n \text{ converges in } \mathcal{A} \right\}, \quad (18)$$

equipped with the $\mathcal{A}$ -valued inner product

$$\langle f, g \rangle(x) = \sum_{n=0}^{\infty} f_n(x) \overline{g_n(x)}, \quad x \in X.$$

In particular,

$$\langle f, f \rangle(x) = \sum_{n=0}^{\infty} |f_n(x)|^2.$$

For each $i \in \mathbb{N}$, we define

$$e_i := (\delta_{in})_{n \in \mathbb{N}}, \quad (19)$$

where δ_{in} denotes the Kronecker symbol and 1 is the unit element of $\mathcal{A}$.

Example 2 (Product frame—Theorem 1). For each $i \in \mathbb{N}$, define two families of closed submodules:

$$W_i = \text{span}_{\mathcal{A}}\{e_{2i}, e_{2i+1}\}, \quad V_i = \text{span}_{\mathcal{A}}\{e_{3i}, e_{3i+1}, e_{3i+2}\},$$

with orthogonal projections P_{W_i}, P_{V_i} , $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, define operators

$$\Lambda_i : W_i \rightarrow \mathcal{A}^2, \quad \Lambda_i(ae_{2i}, be_{2i+1}) = (a + b, a),$$

$$\Gamma_i : V_i \rightarrow \mathcal{A}^2, \quad \Gamma_i(ae_{3i}, be_{3i+1}, ce_{3i+2}) = (a + b + c, a).$$

for all $a, b, c \in \mathcal{A}$. Set all weights $v_i = 1$.

For $f = (f_n) \in \ell^2(\mathcal{A})$, and for each $x \in X$, we have

$$|\Lambda_i P_{W_i} f(x)|^2 = |f_{2i}(x) + f_{2i+1}(x)|^2 + |f_{2i}(x)|^2 \leq 3(|f_{2i}(x)|^2 + |f_{2i+1}(x)|^2).$$

for all $i \in \mathbb{N}$. Thus we have

$$\begin{aligned} \sum_{i \in \mathbb{N}} |f_{2i}(x)|^2 &\leq \langle \Lambda_i P_{W_i} f, \Lambda_i P_{W_i} f \rangle(x) \\ &= \sum_{i \in \mathbb{N}} |\Lambda_i P_{W_i} f(x)|^2 \\ &\leq 3 \sum_{i \in \mathbb{N}} |f_i(x)|^2 \\ &= 3 \langle f, f \rangle_{\mathcal{A}}(x). \end{aligned}$$

for all $x \in X$. Similarly, by the inequality $|a + b + c|^2 \leq 3(|a|^2 + |b|^2 + |c|^2)$,

$$\begin{aligned} \sum_{i \in \mathbb{N}} |f_{3i}(x)|^2 &\leq \langle \Gamma_i P_{V_i} f, \Gamma_i P_{V_i} f \rangle(x) \\ &= \sum_{i \in \mathbb{N}} |\Gamma_i P_{V_i} f(x)|^2 \\ &\leq 4 \sum_{i \in \mathbb{N}} |f_i(x)|^2 \\ &= 4 \langle f, f \rangle_{\mathcal{A}}(x). \end{aligned}$$

for all $x \in X$. Now, let K_1 and K_2 be the elements of $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ defined by

$$\begin{aligned} K_1(e_{2i}) &= e_{2i} \quad \text{and} \quad K_1(e_{2i+1}) = 0 \\ K_2(e_{3i}) &= e_{3i} \quad \text{and} \quad K_2(e_{3i+1}) = K_2(e_{3i+2}) = 0 \end{aligned} \tag{20}$$

for all $i \in \mathbb{N}$. Thus $K_1^* = K_1$ and $K_2^* = K_2$, and we see that $\Lambda := (W_i, \Lambda_i, 1)_{i \in \mathbb{N}}$ is a K_1 -g-fusion frame with bounds $A_1 = 1$, $B_1 = 3$, and $\Gamma := (V_i, \Gamma_i, 1)_{i \in \mathbb{N}}$ is a K_2 -g-fusion frame with bounds $A_2 = 1$, $B_2 = 4$.

Application of Theorem 1: The product family

$$(W_i \oplus V_i, \Lambda_i \oplus \Gamma_i, 1)_{i \in \mathbb{N}}$$

is a $(K_1 \oplus K_2)$ -g-fusion frame for $\mathcal{H} \oplus \mathcal{H}$ with bounds

$$A = \min(A_1, A_2) = 1, \quad B = \max(B_1, B_2) = 4.$$

This concretely illustrates Theorem 1.

Example 3 (Perturbation—Theorem 2). Keeping the same bounds, for all $i \in \mathbb{N}$, W_i, V_i as above. Define a perturbation on W_i as

$$\Theta_i : W_i \rightarrow \mathcal{A}^2, \quad \Theta_i(ae_{2i}, be_{2i+1}) = (0, a).$$

for all $a, b \in \mathcal{A}$. For $\varepsilon > 0$, set

$$\Lambda'_i = \Lambda_i + \varepsilon \Theta_i.$$

Thus, for $f \in \mathcal{H}$ and for each $x \in X$,

$$|(\Lambda_i - \Lambda'_i)P_{W_i} f(x)|^2 = \varepsilon^2 |f_{2i}(x)|^2 \leq \varepsilon^2 (|f_{2i}(x)|^2)$$

for all $i \in \mathbb{N}$. This implies that

$$\sum_{i \in \mathbb{N}} |(\Lambda_i - \Lambda'_i)P_{W_i} f(x)|^2 \leq \varepsilon^2 \langle K_1(f), K_1(f) \rangle_{\mathcal{A}}(x). \tag{21}$$

Similarly, for any $i \in \mathbb{N}$, define $\Gamma'_i = \Gamma_i + \Phi_i$, where

$$\Phi_i : V_i \rightarrow \mathcal{A}^2, \quad \Phi_i(ae_{3i}, be_{3i+1}, ce_{3i+2}) = (0, a)$$

for all $a, b, c \in \mathcal{A}$. Thus

$$\sum_{i \in \mathbb{N}} |(\Gamma_i - \Gamma'_i)P_{W_i}f(x)|^2 \leq \varepsilon^2 \langle K_2(f), K_2(f) \rangle_{\mathcal{A}}(x). \quad (22)$$

Thus the perturbation constant is $r = \varepsilon^2$, and if $r < 1$, i.e., $\varepsilon < 1$, the hypotheses of Theorem 2 are satisfied.

Application of Theorem 2: The perturbed family

$$(W_i \oplus V_i, \Lambda'_i \oplus \Gamma'_i, 1)_{i \in \mathbb{N}} \quad (23)$$

remains a $(K_1 \oplus K_2)$ -g-fusion frame for $\mathcal{H} \oplus \mathcal{H}$ with frame bounds

$$\begin{aligned} A' &:= \min\{(\sqrt{A_1} - \sqrt{r})^2, (\sqrt{A_2} - \sqrt{r})^2\} \\ &= (1 - \varepsilon)^2 \\ B' &:= \max\{2B_1 + 2r\|K_1^*\|^2, 2B_2 + 2r\|K_2^*\|^2\} \\ &= \max\{2 \cdot 3 + 2r, 2 \cdot 4 + 2r\} \\ &= 8 + 2\varepsilon^2. \end{aligned} \quad (24)$$

Below is a direct computation using Theorem 2:

$$\begin{aligned} (1 + \varepsilon^2) \sum_{i \in \mathbb{N}} |f_{2i}(x)|^2 &\leq \langle \Lambda'_i P_{W_i} f, \Lambda'_i P_{W_i} f \rangle(x) \\ &= \sum_{i \in \mathbb{N}} |\Lambda'_i P_{W_i} f(x)|^2 \\ &\leq (3 + \varepsilon^2) \sum_{i \in \mathbb{N}} |f_i(x)|^2 \\ &= (3 + \varepsilon^2) \langle f, f \rangle_{\mathcal{A}}(x) \end{aligned}$$

for all $f \in \mathcal{H}$ and $x \in X$. Similarly,

$$\begin{aligned} (1 + \varepsilon^2) \sum_{i \in \mathbb{N}} |f_{3i}(x)|^2 &\leq \langle \Gamma'_i P_{V_i} f, \Gamma'_i P_{V_i} f \rangle(x) \\ &= \sum_{i \in \mathbb{N}} |\Gamma'_i P_{V_i} f(x)|^2 \\ &\leq (4 + \varepsilon^2) \sum_{i \in \mathbb{N}} |f_i(x)|^2 \\ &= (4 + \varepsilon^2) \langle f, f \rangle_{\mathcal{A}}(x) \end{aligned}$$

for all $f \in \mathcal{H}$ and $x \in X$. By Theorem 1, we see that $(W_i \oplus V_i, \Lambda'_i \oplus \Gamma'_i, 1)_{i \in \mathbb{N}}$ is a $(K_1 \oplus K_2)$ -g-fusion frame for $\mathcal{H} \oplus \mathcal{H}$ with frame bounds $(1 + \varepsilon^2)$ and $(4 + \varepsilon^2)$. Observe that in this situation, we can choose the positive scalar ε without the condition $\varepsilon < 1$.

Remark on the Comparison of Frame Bounds

We observe that the frame bounds obtained by direct computation are better than those obtained by applying Theorem 2 (see (24)). Now, if we replace, in the current example, Θ_i by $-\Theta_i$ and Φ_i by $-\Phi_i$ for some, but not all, indices $i \in \mathbb{N}$, then the frame bounds obtained by applying Theorem 2 remain the same, whereas those obtained by direct computation become $(1 - \varepsilon^2)$ and $(4 + \varepsilon^2)$. Note that in this situation the condition $\varepsilon < 1$ is necessary.

This is natural, since Theorem 2 provides only an estimate of the frame bounds in a general context, where the computation may be delicate and dependent on the nature of the perturbed frames.

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