

Article

A New Hardy–Hilbert-Type Integral Inequality Involving General Homogeneous Kernel and Two Derivative Functions of Higher Order

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Abstract

In this paper, by introducing a general homogeneous kernel function and several parameters, we establish a new Hardy–Hilbert-type integral inequality involving two derivative functions of higher-order. For the resulting inequality, we determine the equivalent conditions of the best possible constant factor related to the parameters. As applications, we demonstrate that a lot of new Hardy–Hilbert-type integral inequalities can be derived by choosing specific homogeneous kernel functions.

Keywords: Hardy–Hilbert-type integral inequality; general homogeneous kernel function; higher-order derivative function; best possible constant factor

MSC: 26D15; 26D10; 26A42

1. Introduction

The classical Hardy–Hilbert inequality reads as follows:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}, \quad (1)$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible ([1], Theorem 315).

Corresponding to inequality (1), the following inequality is known in the literature as the integral Hardy–Hilbert inequality.

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{\infty} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y) dy \right)^{\frac{1}{q}} \quad (2)$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $0 < \int_0^{\infty} f^p(x) dx < \infty$ and $0 < \int_0^{\infty} g^q(y) dy < \infty$, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best value ([1], Theorem 316).

The Hardy–Hilbert inequalities (1) and (2) have significant applications in mathematical analysis and approximation theory (see [1,2]).



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In 2006, Krnić and J. Pečarić [3] provided a meaningful generalization of the Hardy–Hilbert’s inequality by introducing two parameters, as follows:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1, \lambda_2) \left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q \right]^{\frac{1}{q}}, \quad (3)$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_i \in (0, 2]$ ($i = 1, 2$), $\lambda_1 + \lambda_2 = \lambda \in (0, 4]$, the constant factor $B(\lambda_1, \lambda_2)$ given by the Beta function, $B(u, v) := \int_0^{\infty} \frac{t^{u-1}}{(1+t)^{u+v}} dt$ ($u, v > 0$), is the best value.

In 2019, Adiyasuren et al. [4] proposed a new extension of the Hardy–Hilbert’s inequality by introducing two partial sums.

During 2016–2025, Hong et al. [5–9], He et al. [10], and Liu [11] dealt with various extensions of the Hardy–Hilbert integral inequality and discussed the equivalent characterizations of the best possible constant factor related to the parameters. For a more detailed discussion of integral inequalities of Hardy–Hilbert type, we refer the reader to the monographs [12,13].

In 2024, Zeng and Yang [14] established a Hardy–Hilbert-type integral inequality involving two derivative functions of n -order by introducing the homogeneous kernel function $\frac{1}{(x+y)^{\lambda+2n}}$, as follows:

$$\begin{aligned} \int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{(x+y)^{\lambda+2n}} dx dy &< \frac{1}{\prod_{k=0}^{2n-1} (\lambda+k)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\ &\times \left[\int_0^{\infty} x^{p(1-\hat{\lambda}_1)-1} (f^{(n)}(x))^p dx \right]^{\frac{1}{p}} \left[\int_0^{\infty} y^{q(1-\hat{\lambda}_2)-1} (g^{(n)}(y))^q dy \right]^{\frac{1}{q}}, \end{aligned} \quad (4)$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda_1, \lambda_2 < \lambda$, $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, $f^{(i)}(x), g^{(j)}(y) \geq 0$, $f^{(i)}(0+) = g^{(j)}(0+) = 0$ ($i = 0, 1, \dots, n$), $0 < \int_0^{\infty} x^{p(1-\hat{\lambda}_1)-1} (f^{(n)}(x))^p dx < \infty$ and $0 < \int_0^{\infty} y^{q(1-\hat{\lambda}_2)-1} (g^{(n)}(y))^q dy < \infty$.

In the same year, Yang [15] gave a further generalization of the above inequality by constructing the homogeneous kernel function $\frac{1}{(x+y)^{\lambda+m+n}}$, i.e.,

$$\begin{aligned} \int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{(x+y)^{\lambda+m+n}} dx dy &< \frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)} B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \\ &\times \left[\int_0^{\infty} x^{p(1-\hat{\lambda}_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}} \left[\int_0^{\infty} y^{q(1-\hat{\lambda}_2)-1} (g^{(n)}(y))^q dy \right]^{\frac{1}{q}}, \end{aligned} \quad (5)$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda_1, \lambda_2 < \lambda$, $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, $f^{(i)}(x), g^{(j)}(y) \geq 0$, $m, n \in \{0, 1, 2, \dots\}$, $f^{(i-1)}(u) = g^{(j-1)}(u) = o(e^{tu})$ ($t > 0; u \rightarrow \infty$), $f^{(i-1)}(0+) = g^{(j-1)}(0+) = 0$ ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$), $0 < \int_0^{\infty} x^{p(1-\hat{\lambda}_1)-1} (f^{(m)}(x))^p dx < \infty$ and $0 < \int_0^{\infty} y^{q(1-\hat{\lambda}_2)-1} (g^{(n)}(y))^q dy < \infty$.

The results presented in [14,15] motivate us to consider a unified generalization of inequalities (4) and (5). In this paper, by using the technique of constructing weight functions and the idea of introducing parameters, we establish a new Hardy–Hilbert-type integral inequality containing the general homogeneous kernel $k_{\lambda+m+n}(x, y)$ and two derivative functions of higher order, the result obtained is a generalization of the above-mentioned inequalities (4) and (5). Moreover, we characterize the equivalent conditions of

the best possible constant factor related to several parameters and illustrate the applications of the newly established Hardy–Hilbert type integral inequality.

2. Preliminaries and Lemmas

We first state the following assumption (H1), which will be used frequently in the subsequent discussion.

H1. $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $N := \{1, 2, \dots\}$, $0 < \lambda_1, \lambda_2 < \lambda$, $m, n \in N_0 := N \cup \{0\}$, $f^{(i)}(x) \geq 0$, $g^{(j)}(y) \geq 0$ ($i = 0, 1, \dots, m; j = 0, 1, \dots, n$), $f^{(m)}(x)$ and $g^{(n)}(y)$ are continuous function unless at finite points in $R_+ := (0, \infty)$ satisfying $0 < \int_0^\infty x^{p(1-\hat{\lambda}_1)-1} f^{(m)}(x) dx < \infty$ and $0 < \int_0^\infty y^{q(1-\hat{\lambda}_2)-1} g^{(n)}(y) dy < \infty$. Moreover, in order to make the form of the inequality in the main results more concise, we introduce the symbols $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$ and $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$ for the simplification of the parameter expressions.

Lemma 1. Under the assumption (H1), let $s \in (0, \infty)$, $s_1, s_2 \in (0, s)$, and let $k_s(x, y) (\geq 0)$ be a homogeneous function of s -degree, which satisfies $k_s(ux, uy) = u^{-s} k_s(x, y)$ ($u, x, y > 0$), and

$$k_s(\sigma) := \int_0^\infty k_s(t, 1) t^{\sigma-1} dt \in R_+ (\sigma = s_1, s - s_2).$$

The homogeneous function $k_s(x, y)$ is called the kernel function. To characterize the integral properties of the kernel function $k_s(x, y)$, we define the following weight functions:

$$\omega_s(s_1, y) := y^{s-s_1} \int_0^\infty k_s(x, y) x^{s_1-1} dx (y \in R_+), \quad (6)$$

$$\omega_s(s_2, x) := x^{s-s_2} \int_0^\infty k_s(x, y) y^{s_2-1} dy (x \in R_+). \quad (7)$$

Then, the following expressions hold true:

$$\omega_s(s_1, y) = k_s(s_1) (y \in R_+), \quad (8)$$

$$\omega_s(s_2, x) = k_s(s - s_2) (x \in R_+). \quad (9)$$

Proof. In (6), for fixed $y \in R_+$, by setting $t = \frac{x}{y}$, we find

$$\omega_s(s_1, y) = y^{s-s_1} \int_0^\infty k_s(ty, y) (ty)^{s_1-1} y dt = \int_0^\infty k_s(t, 1) t^{s_1-1} dt,$$

this yields $\omega_s(s_1, y) = k_s(s_1)$.

In (7), for fixed $x \in R_+$, by the same way as above, setting $t = \frac{x}{y}$, we obtain

$$\begin{aligned} \omega_s(s_2, x) &= -x^{s-s_2} \int_\infty^0 k_s(x, \frac{x}{t}) (\frac{x}{t})^{s_2-1} \frac{x}{t^2} dt \\ &= x^{s-s_2} \int_0^\infty (\frac{x}{t})^{-s} k_s(t, 1) (\frac{x}{t})^{s_2-1} \frac{x}{t^2} dt = \int_0^\infty k_s(t, 1) t^{(s-s_2)-1} dt, \end{aligned}$$

which implies that $\omega_s(s_2, x) = k_s(s - s_2)$. The Lemma 1 is proved. $\square$

Remark 1. Consider some special cases of the homogeneous kernel function $k_s(x, y)$, one has the following particular homogeneous functions satisfying the conditions of Lemma 1.

For $s \in (0, \infty)$, $\sigma = s_1, s - s_2 \in (0, s)$,

(i) Choosing $k_s(x, y) = \frac{1}{(x+y)^s} (x, y > 0)$, we obtain

$$k_s(\sigma) = \int_0^\infty \frac{1}{(t+1)^s} t^{\sigma-1} dt = B(\sigma, s-\sigma) \in \mathbb{R}_+;$$

(ii) Choosing $k_s(x, y) = \frac{\ln(x/y)}{x^s - y^s} (x, y > 0)$, we obtain

$$k_s(\sigma) = \int_0^\infty \frac{\ln t}{t^s - 1} t^{\sigma-1} dt = \frac{1}{s^2} \int_0^\infty \frac{\ln u}{u-1} u^{\frac{\sigma}{s}-1} du = \left[\frac{\pi}{\sin(\pi\sigma/s)} \right]^2 \in \mathbb{R}_+;$$

(iii) Choosing $k_s(x, y) = \frac{1}{(\max\{x, y\})^s} (x, y > 0)$, we acquire

$$k_s(\sigma) = \int_0^\infty \frac{t^{\sigma-1} dt}{(\max\{t, 1\})^s} = \int_0^1 t^{\sigma-1} dt + \int_1^\infty \frac{t^{\sigma-1}}{t^s} dt = \frac{1}{\sigma(s-\sigma)} \in \mathbb{R}_+.$$

Lemma 2. Under the assumption (H1), we have the following inequalities:

$$\int_0^\infty x^{p(1-m-\hat{\lambda}_1)-1} f^p(x) dx \leq \prod_{i=0}^{m-1} (i + \hat{\lambda}_1)^{-p} \int_0^\infty x^{p(1-\hat{\lambda}_1)-1} (f^{(m)}(x))^p dx, \quad (10)$$

$$\int_0^\infty y^{q(1-n-\hat{\lambda}_2)-1} g^q(y) dy \leq \prod_{j=0}^{n-1} (j + \hat{\lambda}_2)^{-q} \int_0^\infty y^{q(1-\hat{\lambda}_2)-1} (g^{(n)}(y))^q dy, \quad (11)$$

where for $m = 0$ and $n = 0$, we denote $\prod_{i=0}^{m-1} (i + \hat{\lambda}_1) := 1$, $\prod_{j=0}^{n-1} (j + \hat{\lambda}_2) := 1$, respectively.

Proof. For $m = 0$, $f(x) = f^{(0)}(x)$, the inequality (10) keeps the form of an equality, and the conclusion is evident.

For $m \in \mathbb{N}$, $i = 0, 1, \dots, m-1$, $\hat{\lambda}_1 > 0$, using the classical Hardy's integral inequality ([1], Theorem 330)

$$\int_0^\infty x^{-r} \left(\int_0^x h(t) dt \right)^p dx \leq \left(\frac{p}{r-1} \right)^p \int_0^\infty x^{p-r} h^p(x) dx,$$

where $h(x) (\geq 0) \in L(\mathbb{R}_+)$, $r > 1$, and setting

$$r = p(i + \hat{\lambda}_1) + 1 (> 1), \quad h(x) = f^{(m-i)}(x), \quad \int_0^x h(t) dt = f^{(m-i-1)}(x),$$

it follows that

$$\int_0^\infty x^{p(-i-\hat{\lambda}_1)-1} (f^{(m-1-i)}(x))^p dx \leq (i + \hat{\lambda}_1)^{-p} \int_0^\infty x^{p(1-i-\hat{\lambda}_1)-1} (f^{(m-i)}(x))^p dx.$$

Substituting $i = 0, 1, \dots, m-1$, in turn, in the above inequality, and then by simplifications, we obtain the inequality (10). For the case of $\hat{\lambda}_2 > 0$, in the same way as described above, we can derive inequality (11). The proof of Lemma 2 is complete. $\square$

Lemma 3. Under assumption (H1), for $k_{\lambda+m+n}(\lambda_1 + m), k_{\lambda+m+n}(\lambda + n - \lambda_2) \in \mathbb{R}_+$, the following inequality holds

$$\begin{aligned} I := \int_0^\infty \int_0^\infty k_{\lambda+m+n}(x, y) f(x) g(y) dx dy &< k_{\lambda+m+n}^{\frac{1}{p}}(\lambda + m - \lambda_2) k_{\lambda+m+n}^{\frac{1}{q}}(\lambda_1 + m) \\ &\times \left[\int_0^\infty x^{p(1-m-\hat{\lambda}_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-n-\hat{\lambda}_2)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (12)$$

Proof. By applying equality (10) and the assumption (H1), we have

$$L := \int_0^\infty x^{p(1-m-\hat{\lambda}_1)-1} f^p(x) dx < \infty.$$

If $L = 0$, then $f(x) = 0$ a.e. in $\mathbb{R}_+$, thereby $f^{(m)}(x) = 0$ a.e. in $\mathbb{R}_+$, which contradicts the fact that $\int_0^\infty x^{p(1-\hat{\lambda}_1)-1} (f^{(m)}(x))^p dx > 0$. Hence, we conclude $0 < L < \infty$. In the same way as above, we can deduce that

$$0 < \int_0^\infty y^{q(1-n-\hat{\lambda}_2)-1} g^q(y) dy < \infty.$$

Note that $\frac{1}{p} + \frac{1}{q} = 1$, $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, using the classical Hölder's inequality and Fubini theorem ([16,17]), it follows that

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty k_{\lambda+m+n}(x, y) \left[\frac{x^{(1-m-\lambda_1)/q}}{y^{(1-n-\lambda_2)/p}} f(x) \right] \left[\frac{y^{(1-n-\lambda_2)/p}}{x^{(1-m-\lambda_1)/q}} g(y) \right] dx dy \\ &\leq \left\{ \int_0^\infty \int_0^\infty k_{\lambda+m+n}(x, y) \frac{x^{(1-m-\lambda_1)(p-1)}}{y^{1-n-\lambda_2}} f^p(x) dx dy \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_0^\infty \int_0^\infty k_{\lambda+m+n}(x, y) \frac{y^{(1-n-\lambda_2)(q-1)}}{x^{1-m-\lambda_1}} g^q(y) dx dy \right\}^{\frac{1}{q}} \\ &= \left\{ \int_0^\infty [x^{\lambda+m-\lambda_2} \int_0^\infty k_{\lambda+m+n}(x, y) \frac{dy}{y^{1-n-\lambda_2}}] x^{(1-m-\lambda_1)(p-1)-\lambda-m+\lambda_2} f^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_0^\infty [y^{\lambda+n-\lambda_1} \int_0^\infty k_{\lambda+m+n}(x, y) \frac{dx}{x^{1-m-\lambda_1}}] y^{(1-n-\lambda_2)(q-1)-\lambda-n+\lambda_1} g^q(y) dy \right\}^{\frac{1}{q}} \\ &= \left\{ \int_0^\infty [x^{\lambda+m-\lambda_2} \int_0^\infty k_{\lambda+m+n}(x, y) \frac{dy}{y^{1-n-\lambda_2}}] x^{p(1-m-\hat{\lambda}_1)-1} f^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_0^\infty [y^{\lambda+n-\lambda_1} \int_0^\infty k_{\lambda+m+n}(x, y) \frac{dx}{x^{1-m-\lambda_1}}] y^{q(1-n-\hat{\lambda}_2)-1} g^q(y) dy \right\}^{\frac{1}{q}}. \end{aligned}$$

By the aid of the definitions of $\omega_s(s_1, y)$ and $\omega_s(s_2, x)$, together with $\omega_s(s_1, y) = k_s(s_1)$, $\omega_s(s_2, x) = k_s(s - s_2)$ asserted by Lemma 1, for $s = \lambda + m + n$, $s_1 = \lambda_1 + m$, $s_2 = \lambda_2 + n$, $k_{\lambda+m+n}(\lambda_1 + m), k_{\lambda+m+n}(\lambda + n - \lambda_2) \in \mathbb{R}_+$, we have

$$\begin{aligned} &\left\{ \int_0^\infty [x^{\lambda+m-\lambda_2} \int_0^\infty k_{\lambda+m+n}(x, y) \frac{dy}{y^{1-n-\lambda_2}}] x^{p(1-m-\hat{\lambda}_1)-1} f^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_0^\infty [y^{\lambda+n-\lambda_1} \int_0^\infty k_{\lambda+m+n}(x, y) \frac{dx}{x^{1-m-\lambda_1}}] y^{q(1-n-\hat{\lambda}_2)-1} g^q(y) dy \right\}^{\frac{1}{q}} \\ &= [\int_0^\infty \omega_{\lambda+m+n}(\lambda_2 + n, x) x^{p(1-m-\hat{\lambda}_1)-1} f^p(x) dx]^{\frac{1}{p}} \\ &\quad \times [\int_0^\infty \omega_{\lambda+m+n}(\lambda_1 + m, y) y^{q(1-n-\hat{\lambda}_2)-1} g^q(y) dy]^{\frac{1}{q}} \\ &= k_{\lambda+m+n}^{\frac{1}{p}}(\lambda + m - \lambda_2) k_{\lambda+m+n}^{\frac{1}{q}}(\lambda_1 + m) \\ &\quad \times [\int_0^\infty x^{p(1-m-\hat{\lambda}_1)-1} f^p(x) dx]^{\frac{1}{p}} [\int_0^\infty y^{q(1-n-\hat{\lambda}_2)-1} g^q(y) dy]^{\frac{1}{q}}. \end{aligned}$$

Hence

$$I \leq k_{\lambda+m+n}^{\frac{1}{p}}(\lambda+m-\lambda_2) k_{\lambda+m+n}^{\frac{1}{q}}(\lambda_1+m) \times \left[\int_0^\infty x^{p(1-m-\hat{\lambda}_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-n-\hat{\lambda}_2)-1} g^q(y) dy \right]^{\frac{1}{q}}. \quad (13)$$

Below, we verify that inequality (13) holds strictly (no equality case). Otherwise, if inequality (13) takes the form of an equality, then there exist constants A and B such that they are not both zero, and (see [16])

$$A \frac{x^{(1-m-\lambda_1)(p-1)}}{y^{1-n-\lambda_2}} f^p(x) = B \frac{y^{(1-n-\lambda_2)(q-1)}}{x^{1-m-\lambda_1}} g^q(y) \text{ a.e. in } \mathbb{R}_+ \times \mathbb{R}_+.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, the above expression is equivalent to

$$A x^{p(1-m-\hat{\lambda}_1)-1} f^p(x) = [B y^{q(1-n-\hat{\lambda}_2)+\lambda-\lambda_1-\lambda_2} g^q(y)] x^{-1-(\lambda-\lambda_1-\lambda_2)} \text{ a.e. in } \mathbb{R}_+.$$

Assuming that $A \neq 0$, then there exists a $y \in \mathbb{R}_+$ such that

$$x^{p(1-m-\hat{\lambda}_1)-1} f^p(x) = \left[\frac{B}{A} y^{q(1-n-\hat{\lambda}_2)+\lambda-\lambda_1-\lambda_2} g^q(y) \right] x^{-1-(\lambda-\lambda_1-\lambda_2)} \text{ a.e. in } \mathbb{R}_+.$$

Note that for any $\lambda - \lambda_1 - \lambda_2 \in \mathbb{R}$, $\int_0^\infty x^{-1-(\lambda-\lambda_1-\lambda_2)} dx = \int_0^1 x^{-1-(\lambda-\lambda_1-\lambda_2)} dx + \int_1^\infty x^{-1-(\lambda-\lambda_1-\lambda_2)} dx = \infty$, which contradicts the fact that $0 < L < \infty$. So, inequality (13) holds strictly. This completes the proof of Lemma 3. $\square$

3. Main Results

By using inequalities (10)–(12), we obtain

Theorem 1. Under assumption (H1) and that of Lemma 3, we have the following inequality

$$I = \int_0^\infty \int_0^\infty k_{\lambda+m+n}(x, y) f(x) g(y) dx dy < \prod_{i=0}^{m-1} (i + \hat{\lambda}_1)^{-1} \prod_{j=0}^{n-1} (j + \hat{\lambda}_2)^{-1} k_{\lambda+m+n}^{\frac{1}{p}}(\lambda+m-\lambda_2) k_{\lambda+m+n}^{\frac{1}{q}}(\lambda_1+m) \times \left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\hat{\lambda}_2)-1} (g^{(n)}(y))^q dy \right]^{\frac{1}{q}}. \quad (14)$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, in view of (H1), it implies

$$0 < \int_0^\infty x^{p(1-\lambda_1)-1} (f^{(m)}(x))^p dx < \infty, \quad 0 < \int_0^\infty y^{q(1-\lambda_2)-1} (g^{(n)}(y))^q dy < \infty.$$

Then, the following inequality holds true

$$\int_0^\infty \int_0^\infty k_{\lambda+m+n}(x, y) f(x) g(y) dx dy < \prod_{i=0}^{m-1} (i + \lambda_1)^{-1} \prod_{j=0}^{n-1} (j + \lambda_2)^{-1} k_{\lambda+m+n}(\lambda_1+m) \times \left[\int_0^\infty x^{p(1-\lambda_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} (g^{(n)}(y))^q dy \right]^{\frac{1}{q}}. \quad (15)$$

Proof. Using Lemmas 2 and 3 gives

$$\int_0^\infty x^{p(1-m-\hat{\lambda}_1)-1} f^p(x) dx \leq \prod_{i=0}^{m-1} (i + \hat{\lambda}_1)^{-p} \int_0^\infty x^{p(1-\hat{\lambda}_1)-1} (f^{(m)}(x))^p dx,$$

$$\int_0^\infty y^{q(1-n-\hat{\lambda}_2)-1} g^q(y) dy \leq \prod_{j=0}^{n-1} (j + \hat{\lambda}_2)^{-q} \int_0^\infty y^{q(1-\hat{\lambda}_2)-1} (g^{(n)}(y))^q dy$$

and

$$I = \int_0^\infty \int_0^\infty k_{\lambda+m+n}(x, y) f(x) g(y) dx dy < k_{\lambda+m+n}^{\frac{1}{p}}(\lambda + m - \lambda_2) k_{\lambda+m+n}^{\frac{1}{q}}(\lambda_1 + m) \\ \times \left[\int_0^\infty x^{p(1-m-\hat{\lambda}_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-n-\hat{\lambda}_2)-1} g^q(y) dy \right]^{\frac{1}{q}}.$$

Combining the above three inequalities leads to inequality (14). When $\lambda_1 + \lambda_2 = \lambda$, we have $\hat{\lambda}_1 = \lambda_1, \hat{\lambda}_2 = \lambda_2$, then we derive inequality (15) from inequality (14). The Theorem 1 is proved. $\square$

Theorem 2. Under assumption (H1) and that of Lemma 3, if $\lambda_1 + \lambda_2 = \lambda$, then the constant

$$\prod_{i=0}^{m-1} (i + \hat{\lambda}_1)^{-1} \prod_{j=0}^{n-1} (j + \hat{\lambda}_2)^{-1} k_{\lambda+m+n}^{\frac{1}{p}}(\lambda + m - \lambda_2) k_{\lambda+m+n}^{\frac{1}{q}}(\lambda_1 + m)$$

in (14) is the best possible.

Proof. In order to prove that the constant in (14) is the best possible, we need only to prove that the constant factor in (15) is the best value. Based on the characteristics of the above constants along with the structural form of the right-hand side of inequality (15), we will construct two higher-order derivative functions related to any $\varepsilon > 0$, and then use the limit process ($\varepsilon \rightarrow 0$) to prove that there does not exist a smaller constant factor than the one asserted by Theorem 2 that can make the inequality (15) hold true. $\square$

For any $0 < \varepsilon < \min\{p\lambda_1, q\lambda_2\}, m, n \in \mathbb{N}_0$, we construct the following functions:

$$\tilde{f}^{(m)}(x) := \begin{cases} 0, & 0 < x < 1, \\ \prod_{i=0}^{m-1} (i + \lambda_1 - \frac{\varepsilon}{p}) x^{\lambda_1 - \frac{\varepsilon}{p} - 1}, & x \geq 1 \end{cases},$$

$$\tilde{g}^{(n)}(y) := \begin{cases} 0, & 0 < y < 1, \\ \prod_{j=0}^{n-1} (j + \lambda_2 - \frac{\varepsilon}{q}) y^{\lambda_2 - \frac{\varepsilon}{q} - 1}, & y \geq 1 \end{cases}.$$

Then, for $m, n \in \mathbb{N}$, from the definitions of $\tilde{f}^{(m)}(x)$ and $\tilde{g}^{(n)}(y)$, we can derive $\tilde{f}(x)$ and $\tilde{g}(y)$, as follows:

$$\tilde{f}(x) = \prod_{i=0}^{m-1} (i + \lambda_1 - \frac{\varepsilon}{p}) \int_0^x (\int_0^{t_m} \dots \int_0^{t_2} \tilde{f}^{(m)}(t_1) dt_1 \dots dt_{m-1}) dt_m$$

$$= \begin{cases} 0, & 0 < x < 1, \\ \prod_{i=0}^{m-1} (i + \lambda_1 - \frac{\varepsilon}{p}) \int_1^x (\int_1^{t_m} \dots \int_1^{t_2} t_1^{\lambda_1 - \frac{\varepsilon}{p} - 1} dt_1 \dots dt_{m-1}) dt_m, & x \geq 1 \end{cases},$$

$$= \begin{cases} 0, & 0 < x < 1, \\ x^{\lambda_1 + m - \frac{\varepsilon}{p} - 1} - p_{m-1}(x), & x \geq 1 \end{cases},$$

where $p_{m-1}(x)$ is a polynomial of $(m-1)$ -degree with all the positive coefficients, which satisfies

$$p_{m-1}^{(i)}(1) = (x^{\lambda_1+m-\frac{\varepsilon}{p}-1})^{(i)}|_{x=1} (i = 0, \dots, m-1).$$

$$\tilde{g}(y) = \begin{cases} 0, & 0 < y < 1, \\ y^{\lambda_2+n-\frac{\varepsilon}{q}-1} - P_{n-1}(y), & y \geq 1 \end{cases}$$

where $P_{n-1}(y)$ is a polynomial of $(n-1)$ -degree with all the positive coefficients, which satisfies

$$P_{n-1}^{(j)}(1) = (y^{\lambda_2+n-\frac{\varepsilon}{q}-1})^{(j)}|_{y=1} (j = 0, \dots, n-1).$$

Moreover, we denote that $p_{-1}(x) = P_{-1}(y) := 0$, and the above expressions satisfy the case of $m = n = 0$.

Since $P_{n-1}(y)$ is a polynomial of $(n-1)$ -degree, we have $y^{1-n}P_{n-1}(y) \rightarrow C_0$ (constant) as $y \rightarrow \infty$. It is obvious that there exists a positive constant C , such that $y^{1-n}P_{n-1}(y) \leq C$ ($y \in [1, \infty)$). Similarly, one has $x^{1-m}p_{m-1}(x) \leq C_1$ (constant) for $x \in [1, \infty)$.

If there exists a positive constant M , satisfying

$$M \leq \prod_{i=0}^{m-1} (i + \lambda_1)^{-1} \prod_{j=0}^{n-1} (j + \lambda_2)^{-1} k_{\lambda+m+n}(\lambda_1 + m) \quad (16)$$

such that (15) is valid when we replace the constant factor by M , then, in particular, we have

$$\begin{aligned} \tilde{I} &:= \int_0^\infty \int_0^\infty k_{\lambda+m+n}(x, y) \tilde{f}(x) \tilde{g}(y) dx dy \\ &< M \left[\int_0^\infty x^{p(1-\lambda_1)-1} (\tilde{f}^{(m)}(x))^p dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} (\tilde{g}^{(n)}(y))^q dy \right]^{\frac{1}{q}} \\ &= M \prod_{i=0}^{m-1} \left(i + \lambda_1 - \frac{\varepsilon}{p} \right)^{\frac{n-1}{p}} \prod_{j=0}^{n-1} \left(j + \lambda_2 - \frac{\varepsilon}{q} \right)^{\frac{1}{p}} \left(\int_1^\infty x^{-\varepsilon-1} dx \right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\varepsilon-1} dy \right)^{\frac{1}{q}} \\ &= \frac{M^{m-1}}{\varepsilon} \prod_{i=0}^{m-1} \left(i + \lambda_1 - \frac{\varepsilon}{p} \right)^{\frac{n-1}{p}} \prod_{j=0}^{n-1} \left(j + \lambda_2 - \frac{\varepsilon}{q} \right)^{\frac{1}{p}}. \end{aligned}$$

On the other hand, by substituting $\tilde{f}(x)$ and $\tilde{g}(y)$ into the left-hand side of inequality (14) and utilizing the Fubini theorem [17], we obtain

$$\begin{aligned} \tilde{I} &= \int_1^\infty (y^{\lambda_2+n-\frac{\varepsilon}{q}-1} - P_{n-1}(y)) \left[\int_1^\infty k_{\lambda+m+n}(x, y) (x^{\lambda_1+m-\frac{\varepsilon}{p}-1} - p_{m-1}(x)) dx \right] dy \\ &\geq A - (A_1 + A_2), \end{aligned}$$

where A, A_1, A_2 are formulated by

$$\begin{aligned} A &:= \int_1^\infty y^{\lambda_2+n-\frac{\varepsilon}{q}-1} \left(\int_1^\infty k_{\lambda+m+n}(x, y) x^{\lambda_1+m-\frac{\varepsilon}{p}-1} dx \right) dy, \\ A_1 &:= \int_1^\infty P_{n-1}(y) \left(\int_1^\infty k_{\lambda+m+n}(x, y) x^{\lambda_1+m-\frac{\varepsilon}{p}-1} dx \right) dy, \\ A_2 &:= \int_1^\infty p_{m-1}(x) \left(\int_1^\infty k_{\lambda+m+n}(x, y) y^{\lambda_2+n-\frac{\varepsilon}{q}-1} dy \right) dx. \end{aligned}$$

Setting $u = \frac{x}{y}$, we deduce that

$$\begin{aligned}
 A &= \int_1^\infty y^{-\varepsilon-1} \left(\int_{\frac{1}{y}}^\infty k_{\lambda+m+n}(u, 1) u^{\lambda_1+m-\frac{\varepsilon}{p}-1} du \right) dy \\
 &= \int_1^\infty y^{-\varepsilon-1} \left(\int_{\frac{1}{y}}^1 k_{\lambda+m+n}(u, 1) u^{\lambda_1+m-\frac{\varepsilon}{p}-1} du \right) dy + \frac{1}{\varepsilon} \int_1^\infty k_{\lambda+m+n}(u, 1) u^{\lambda_1+m-\frac{\varepsilon}{p}-1} du \\
 &= \int_0^1 k_{\lambda+m+n}(u, 1) u^{\lambda_1+m-\frac{\varepsilon}{p}-1} \left(\int_{\frac{1}{u}}^\infty y^{-\varepsilon-1} dy \right) du + \frac{1}{\varepsilon} \int_1^\infty k_{\lambda+m+n}(u, 1) u^{\lambda_1+m-\frac{\varepsilon}{p}-1} du \\
 &= \frac{1}{\varepsilon} \left[\int_0^1 k_{\lambda+m+n}(u, 1) u^{\lambda_1+m+\frac{\varepsilon}{q}-1} du + \int_1^\infty k_{\lambda+m+n}(u, 1) u^{\lambda_1+m-\frac{\varepsilon}{p}-1} du \right], \\
 0 \leq A_1 &\leq \int_1^\infty y^{-\lambda_1-1} (y^{1-n} P_{n-1}(y)) \left(\int_{\frac{1}{y}}^\infty k_{\lambda+m+n}(u, 1) u^{\lambda_1+m-1} du \right) dy \\
 &\leq C \int_1^\infty y^{-\lambda_1-1} dy \int_0^\infty k_{\lambda+m+n}(u, 1) u^{\lambda_1+m-1} du \leq \frac{C}{\lambda_1} k_{\lambda+m+n}(\lambda_1 + m) \leq M_1 < \infty.
 \end{aligned}$$

In the same way as above, we can obtain $0 \leq A_2 \leq M_2 < \infty$.

Based on the above results, one has the following inequality:

$$\begin{aligned}
 &\int_0^1 k_{\lambda+m+n}(u, 1) u^{\lambda_1+m+\frac{\varepsilon}{q}-1} du + \int_1^\infty k_{\lambda+m+n}(u, 1) u^{\lambda_1+m-\frac{\varepsilon}{p}-1} du - \varepsilon(A_1 + A_2) \\
 &\leq \varepsilon \tilde{I} < M \prod_{i=0}^{m-1} \left(i + \lambda_1 - \frac{\varepsilon}{p} \right) \prod_{j=0}^{n-1} \left(j + \lambda_2 - \frac{\varepsilon}{q} \right).
 \end{aligned}$$

In the above inequality, letting $\varepsilon \rightarrow 0^+$, which implies $\varepsilon(A_1 + A_2) \rightarrow 0$, it follows from the Fatou lemma [16] that

$$\begin{aligned}
 k_{\lambda+m+n}(\lambda_1 + m) &= \int_0^1 k_{\lambda+m+n}(u, 1) u^{\lambda_1+m-1} du + \int_1^\infty k_{\lambda+m+n}(u, 1) u^{\lambda_1+m-1} du \\
 &\leq \lim_{\varepsilon \rightarrow 0^+} \left[\int_0^1 k_{\lambda+m+n}(u, 1) u^{\lambda_1+m+\frac{\varepsilon}{q}-1} du + \int_1^\infty k_{\lambda+m+n}(u, 1) u^{\lambda_1+m-\frac{\varepsilon}{p}-1} du - \varepsilon(A_1 + A_2) \right] \\
 &\leq \lim_{\varepsilon \rightarrow 0^+} M \prod_{i=0}^{m-1} \left(i + \lambda_1 - \frac{\varepsilon}{p} \right) \prod_{j=0}^{n-1} \left(j + \lambda_2 - \frac{\varepsilon}{q} \right) = M \prod_{i=0}^{m-1} (i + \lambda_1) \prod_{j=0}^{n-1} (j + \lambda_2),
 \end{aligned}$$

which yields that

$$\prod_{i=0}^{m-1} (i + \lambda_1)^{-1} \prod_{j=0}^{n-1} (j + \lambda_2)^{-1} k_{\lambda+m+n}(\lambda_1 + m) \leq M.$$

By the assumption in (16), we conclude that $M = \prod_{i=0}^{m-1} (i + \lambda_1)^{-1} \prod_{j=0}^{n-1} (j + \lambda_2)^{-1} k_{\lambda+m+n}(\lambda_1 + m)$ is the best possible constant factor for inequality (15). Also, it is the best constant factor for inequality (14) in the case when $\lambda_1 + \lambda_2 = \lambda$. The proof of Theorem 2 is complete.

Theorem 3. Under assumption (H1) and that of Lemma 3, if the constant factor

$$\prod_{i=0}^{m-1} (i + \hat{\lambda}_1)^{-1} \prod_{j=0}^{n-1} (j + \hat{\lambda}_2)^{-1} k_{\lambda+m+n}^{\frac{1}{p}}(\lambda + m - \lambda_2) k_{\lambda+m+n}^{\frac{1}{q}}(\lambda_1 + m)$$

in (14) is the best possible, then we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. For $\hat{\lambda}_1 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 = \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$, we have $\hat{\lambda}_1 + \hat{\lambda}_2 = \lambda$ with $0 < \hat{\lambda}_1, \hat{\lambda}_2 < \lambda$.
By employing Hölder's inequality [16], we find

$$\begin{aligned} k_{\lambda+m+n}(\hat{\lambda}_1 + m) &= \int_0^\infty k_{\lambda+m+n}(u, 1) u^{\hat{\lambda}_1+m-1} du \\ &= \int_0^\infty k_{\lambda+m+n}(u, 1) \left(u^{\frac{\lambda - \lambda_2 + m - 1}{p}} \right) \left(u^{\frac{\lambda_1 + m - 1}{q}} \right) du \\ &\leq \left(\int_0^\infty k_{\lambda+m+n}(u, 1) u^{\lambda - \lambda_2 + m - 1} du \right)^{\frac{1}{p}} \left(\int_0^\infty k_{\lambda+m+n}(u, 1) u^{\lambda_1 + m - 1} du \right)^{\frac{1}{q}} \\ &= k_{\lambda+m+n}^p(\lambda - \lambda_2 + m) k_{\lambda+m+n}^q(\lambda_1 + m) \in \mathbb{R}_+. \end{aligned} \quad (17)$$

Since the constant factor

$$\prod_{i=0}^{m-1} (i + \hat{\lambda}_1)^{-1} \prod_{j=0}^{n-1} (j + \hat{\lambda}_2)^{-1} k_{\lambda+m+n}^{\frac{1}{p}}(\lambda + m - \lambda_2) k_{\lambda+m+n}^{\frac{1}{q}}(\lambda_1 + m)$$

in (14) is the best value, comparing with the constant factors in (14) and (15) (for $\lambda_i = \hat{\lambda}_i$, $i = 1, 2$), we obtain

$$\begin{aligned} &\prod_{i=0}^{m-1} (i + \hat{\lambda}_1)^{-1} \prod_{j=0}^{n-1} (j + \hat{\lambda}_2)^{-1} k_{\lambda+m+n}^{\frac{1}{p}}(\lambda + m - \lambda_2) k_{\lambda+m+n}^{\frac{1}{q}}(\lambda_1 + m) \\ &\leq \prod_{i=0}^{m-1} (i + \hat{\lambda}_1)^{-1} \prod_{j=0}^{n-1} (j + \hat{\lambda}_2)^{-1} k_{\lambda+m+n}(\hat{\lambda}_1 + m) \end{aligned}$$

which yields

$$k_{\lambda+m+n}^{\frac{1}{p}}(\lambda + m - \lambda_2) k_{\lambda+m+n}^{\frac{1}{q}}(\lambda_1 + m) \leq k_{\lambda+m+n}(\hat{\lambda}_1 + m)$$

By comparing the above inequality with inequality (17), we conclude that inequality (17) keeps the form of an equality. From the derivation process of inequality (17), it follows that the necessary and sufficient condition for taking an equal sign is that there exist constants A and B such that they are not both zero and $Au^{\lambda - \lambda_2 + m - 1} = Bu^{\lambda_1 + m - 1}$ a.e. in $\mathbb{R}_+$ (see [16]). Assuming that $A \neq 0$, we have $u^{\lambda - \lambda_1 - \lambda_2} = \frac{B}{A}$ a.e. in $\mathbb{R}_+$, which implies that $\lambda - \lambda_1 - \lambda_2 = 0$, that is $\lambda_1 + \lambda_2 = \lambda$. This completes the proof of Theorem 3. $\square$

4. Application to Generating New Hardy–Hilbert-Type Integral Inequalities

Example 1. Using Remark 1 (ii), by taking $k_{\lambda+m+n}(x, y) = \frac{\ln(x/y)}{x^{\lambda+m+n} - y^{\lambda+m+n}}$ in (15) of Theorem 2, we obtain

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{\ln(x/y)}{x^{\lambda+m+n} - y^{\lambda+m+n}} f(x) g(y) dx dy \\ &< \prod_{i=0}^{m-1} (i + \lambda_1)^{-1} \prod_{j=0}^{n-1} (j + \lambda_2)^{-1} \left[\frac{\pi}{\sin \pi(\lambda_1 + m)/(\lambda + m + n)} \right]^2 \\ &\quad \times \left[\int_0^\infty x^{p(1-\lambda_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} (g^{(n)}(y))^q dy \right]^{\frac{1}{q}}, \end{aligned} \quad (18)$$

where the constant factor $\prod_{i=0}^{m-1} (i + \lambda_1)^{-1} \prod_{j=0}^{n-1} (j + \lambda_2)^{-1} \left[\frac{\pi}{\sin \pi(\lambda_1 + m)/(\lambda + m + n)} \right]^2$ ($0 < \lambda_1$, $\lambda_2 < \lambda$, $\lambda_1 + \lambda_2 = \lambda$) is the best possible.

Moreover, as a direct application of the main results, choosing $m = n = 0$ in Theorems 1–3, we obtain the following new Hardy–Hilbert-type integral inequalities.

Corollary 1. Under assumption (H1), the following inequality is valid

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dx dy < k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \times \left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\hat{\lambda}_2)-1} g^q(y) dy \right]^{\frac{1}{q}}. \quad (19)$$

Corollary 2. Under assumption (H1), if $\lambda_1 + \lambda_2 = \lambda$, then the following inequality is valid

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dx dy < k_\lambda(\lambda_1) \left[\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} g^q(y) dy \right]^{\frac{1}{q}}. \quad (20)$$

Corollary 3. If $\lambda_1 + \lambda_2 = \lambda$, then the constant factor $k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (19) is the best value. If the constant factor $k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (19) is the best value, then $\lambda_1 + \lambda_2 = \lambda$.

Remark 2. In view of the discussion in Remark 1, if we employ various homogeneous kernel functions in Theorem 1, we can derive a number of new Hardy–Hilbert-type integral inequalities. For example, in Theorem 1, if we choose the homogeneous kernel functions $k_{\lambda+2n}(x, y) = \frac{1}{(x+y)^{\lambda+2n}}$ and $k_{\lambda+m+n}(x, y) = \frac{1}{(x+y)^{\lambda+m+n}}$, respectively, we recover inequalities (4) and (5), which were originally presented in [14] and [15] respectively. Moreover, if we choose different homogeneous kernel functions such as $k_{\lambda+m+n}(x, y) = \frac{\ln(x/y)}{x^{\lambda+m+n} - y^{\lambda+m+n}}$ (see Example 1), $k_{\lambda+m+n}(x, y) = \frac{1}{(\max\{x, y\})^{\lambda+m+n}}$ and the others, then a wide variety of new Hardy–Hilbert-type integral inequalities can be obtained directly.

5. Conclusions

In this article, we investigate a very extensive generalization form of the Hardy–Hilbert integral inequality. By introducing a general homogeneous kernel function $k_{\lambda+m+n}(x, y)$ and using the technique of constructing weight functions, we establish a new Hardy–Hilbert-type integral inequality in Theorem 1. Subsequently, in Theorems 2 and 3, we determine the equivalent conditions of the best possible constant factor related to several parameters. Finally, in Corollaries 1, 2 and Remark 2, we illustrate that a lot of new Hardy–Hilbert-type integral inequalities can be derived by choosing different homogeneous kernel functions. The main contribution of our work is the development of a novel method for extending the Hardy–Hilbert integral inequality through the construction of general homogeneous kernel functions. From the perspective of practical applications, the results of this work enrich the study of mathematical inequalities, providing valuable cases and materials for elementary mathematics education through the mathematical thinking and methods presented.

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