

Article

Diffusion in Heterogeneous Media with Stochastic Resetting and Pauses

Ervin K. Lenzi ^{1,2} , Luciano R. da Silva ^{2,3}  and Marcelo K. Lenzi ^{4,*} ¹ Departamento de Física, Universidade Estadual de Maringá, Av. Colombo 5790, Maringá 87020-900, PR, Brazil; eklenzi@uem.br² National Institute of Science and Technology for Complex Systems, Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro 22290-180, RJ, Brazil; luciano@fisica.ufrn.br³ Departamento de Física, Universidade Federal do Rio Grande do Norte, Natal 59078-900, RN, Brazil⁴ Departamento de Engenharia Química, Universidade Federal do Paraná, Rua Coronel Francisco Heráclito dos Santos 100, Curitiba 81531-980, PR, Brazil

* Correspondence: lenzi@ufpr.br

Abstract

Diffusion in heterogeneous environments is usually governed by unusual dynamics, exhibiting sub- or superdiffusive scaling depending on the structural complexity and memory effects. In many systems, diffusing particles may alternate between periods of motion and rest, or may undergo stochastic resetting to a preferred position. While each of these mechanisms has been studied independently, their combined effect in a heterogeneous medium has been insufficiently investigated. We formulate and solve a coupled set of one dimension diffusion equations for the probability densities of moving and resting particles, accounting for space-dependent diffusivity and stochastic resetting. We obtain expressions for the probability distribution and show the behavior of the survival probability, mean-square displacement, and first-passage time. The results reveal a diverse range of behaviors with distinct diffusion regimes. One of them is obtained for small times, which can be connected to the heterogeneity present in the system, and another for intermediate times related to the intermittent process produced by the moving and pauses before the system reaches the stationary state.

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1. Introduction

One of the most fundamental phenomena in nature is diffusion, which can occur in various scenarios characterized by either usual [1,2] or anomalous [3–5]. For usual diffusion, we have the Markovian processes essentially characterized by a linear time dependence for the mean square displacement $\langle (x - \langle x \rangle)^2 \rangle \sim t$ with the distribution of probabilities expressed in terms of a Gaussian distribution [1,2]. For anomalous diffusion, we have a nonlinear time dependence of the mean square displacement, e.g., $\langle (x - \langle x \rangle)^2 \rangle \sim t^\delta$, where $\delta < 1$ and $\delta > 1$ are connected with the sub- and superdiffusion, typical of non-Markovian processes [5,6]. Other dependencies for the mean square displacement are possible, such as $\langle (x - \langle x \rangle)^2 \rangle \sim \ln^\delta t$, which characterizes an ultraslow diffusion [7,8]. We also have the Lévy distributions, which are asymptotically characterized by power laws, and the mean square displacement is not defined. These behaviors have been

verified in many scenarios, such as electrical response [6], anomalous transport [9–11], ecology and movement patterns [12,13], and heterogeneous [14], bacterial motion in porous media [15], active particles in complex environments [16], and disordered [17,18] media. Another aspect of the diffusion processes concerns the feature that, in general, they can appear combined with others, such as reactions [19], adsorption–desorption [20–22], and stochastic resetting [23–26]. The interaction of particles with the substrate, where they are diffusing, can be represented by reactions or adsorption–desorption. This process promotes immobilization and, after a characteristic time, allows them to transition to motion, characterizing an intermittent process. These dynamics are observed in transport through porous media, molecular motors exhibiting pause phases, or random searchers alternating between relocation and inspection [27,28]. The stochastic resetting introduces processes in which particles in a diffusive system are periodically and randomly returned to a starting point, altering the system's long-term behavior and creating non-equilibrium stationary states. This enables applications in areas such as search theory [29,30], biological processes [31], and economic modeling [32,33]. The interplay among these processes in a heterogeneous medium can provide insights into several physical and biological phenomena that involve complex structures and processes.

Here, we analyze intricate dynamics that arise from the combination of stochastic resetting and diffusion occurring within spatially heterogeneous environments with pauses for the one-dimensional case. For this, we consider the distribution $\rho(x, t) = \rho_1(x, t) + \rho_2(x, t)$, with $\rho_1(x, t)$ and $\rho_2(x, t)$ defined by the following set of equations:

$$\begin{aligned} \frac{\partial}{\partial t} \rho_1(x, t) &= \frac{\partial}{\partial x} \left\{ D(x) \frac{\partial}{\partial x} \rho_1(x, t) \right\} + r(S_1(t) \delta(x - x') - \rho_1(x, t)) \\ &\quad - \zeta_1 \rho_1(x, t) + \zeta_2 \rho_2(x, t), \\ \frac{\partial}{\partial t} \rho_2(x, t) &= \zeta_1 \rho_1(x, t) - \zeta_2 \rho_2(x, t) \end{aligned} \quad (1)$$

where $\rho_1(x, t)$ and $\rho_2(x, t)$ represent the particles in the bulk, where $\rho_1(x, t)$ represents the particles diffusing and $\rho_2(x, t)$ represents the particles immobile (paused), and $S_1(t)$ is the survival probability connected to the particles that are diffusing. The diffusion considered here is $D(x) = D|x|^{-\eta}$, which has been applied in several contexts such as diffusion on fractals [34], disordered media [18,35], and turbulent flux [36,37]. A different form of heterogeneity in the context of stochastic resetting has been explored in Refs. [38–40]. It is worth mentioning that for a space-dependent diffusivity $D(x)$, the Fokker–Planck equation corresponding to the stochastic process $dX_t = a(X_t) dt + \sqrt{2D(X_t)} \circ_\alpha dW_t$ depends on the stochastic interpretation parameter α ($\alpha = 0$: Itô; $\alpha = \frac{1}{2}$: Stratonovich; $\alpha = 1$: Hänggi–Klimontovich). The resulting FP operator includes an additional term $(1 - \alpha)D'(x)\rho$, known as the spurious drift [41–43]. In our formulation, the diffusive term present in Equation (1) with spatial dependence corresponds to the Hänggi–Klimontovich convention. For Equation (1), we obtain the solution and the mean square displacement by considering the boundary conditions $\rho_1(\pm\infty, t) = 0$, $\rho_2(\pm\infty, t) = 0$, and the initial conditions $\rho_1(x, 0) = \varphi(x)$ and $\rho_2(x, 0) = 0$. To analyze the mean first passage time, we consider the system subjected to the boundary conditions $\rho_1(0, t) = \rho_1(\infty, t) = 0$ with $\rho_2(0, t) = \rho_2(\infty, t) = 0$, which characterizes an absorbing surface in $x = 0$. These developments are performed in Section 2. In Section 3, we present our discussion and conclusions.

2. Diffusion and Stochastic Resetting

Let us start our analysis of Equation (1) by considering the eigenfunctions of the Sturm–Liouville problem related to the following differential equation:

$$\frac{\partial}{\partial x} \left\{ |x|^{-\eta} \frac{\partial}{\partial x} \psi_{\pm}(x, k) \right\} = -|k|^{2+\eta} \psi_{\pm}(x, k), \quad (2)$$

with the condition $|\psi_{\pm}(\pm\infty, k)| < \infty$, which results in

$$\psi_{+}(x, k) = (|k||x|)^{\frac{1}{2}(1+\eta)} J_{-\nu} \left(2(|k||x|)^{\frac{1}{2}(2+\eta)} / (2+\eta) \right) \quad \text{and} \quad (3)$$

$$\psi_{-}(x, k) = xk(|k||x|)^{\frac{1}{2}(1+\eta)-1} J_{\nu} \left(2(|k||x|)^{\frac{1}{2}(2+\eta)} / (2+\eta) \right), \quad (4)$$

where the sub-indexes $+$ and $-$ refer to the even and odd solutions, $\nu = (1+\eta)/(2+\eta)$, and $J_{\nu}(x)$ is the Bessel function [44]. By using the eigenfunctions defined by Equations (3) and (4), it is possible to obtain the solution for Equation (1). To do this, we consider that the solution can be written as follows:

$$\rho(x, t) = \frac{1}{2} \int_{-\infty}^{\infty} \left[\psi_{+}(x, k) \tilde{\rho}_{+}(k, t) + \psi_{-}(x, k) \tilde{\rho}_{-}(k, t) \right] dk, \quad (5)$$

with

$$\tilde{\rho}_{\pm}(k, t) = \frac{1}{2} \int_{-\infty}^{\infty} \psi_{\pm}(x, k) \rho(x, t) dx, \quad (6)$$

where $\tilde{\rho}_{\pm}(k, t)$ is determined by Equation (1). By applying the previous equation in Equation (1), we find that

$$\frac{\partial}{\partial t} \tilde{\rho}_{1,\pm}(k, t) = -(D|k|^{2+\eta} + \zeta_1 + r) \tilde{\rho}_{1,\pm}(k, t) + \zeta_2 \tilde{\rho}_{2,\pm}(k, t) + \frac{1}{2} r S_1(t) \psi_{\pm}(x', k) \quad (7)$$

and

$$\frac{\partial}{\partial t} \tilde{\rho}_{2,\pm}(k, t) = \zeta_1 \tilde{\rho}_{1,\pm}(k, t) - \zeta_2 \tilde{\rho}_{2,\pm}(k, t). \quad (8)$$

By using the Laplace transform $(\mathcal{L}\{\rho_i(x, t); s\} = \int_0^{\infty} ds e^{-st} \rho_i(x, t) = \hat{\rho}_i(x, s)$ and $\mathcal{L}^{-1}\{\hat{\rho}_i(x, s); t\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds e^{st} \hat{\rho}_i(x, s) = \rho_i(x, t)$) and performing some additional calculations, it is possible to show that

$$\hat{\tilde{\rho}}_{1,\pm}(k, s) = \frac{\tilde{\varphi}_{1,\pm}(k) + (r/2) \hat{S}_1(s) \psi_{\pm}(x', k)}{s + D|k|^{2+\eta} + r + \zeta_1 - \hat{\Lambda}(s)}, \quad (9)$$

$$\hat{\tilde{\rho}}_{2,\pm}(k, s) = \frac{\zeta_1}{s + \zeta_2} \frac{\tilde{\varphi}_1(k) + (r/2) \hat{S}_1(s) \psi_{\pm}(x', k)}{s + D|k|^{2+\eta} + r + \zeta_1 - \hat{\Lambda}(s)}, \quad (10)$$

with $\hat{\Lambda}(s) = \zeta_1 \zeta_2 / (s + \zeta_2)$. By performing some calculations, it is possible to show that the

$$\hat{\rho}_1(x, s) = \frac{1}{2} \left[\int_{-\infty}^{\infty} \varphi(\bar{x}) \hat{\mathcal{G}}(x, \bar{x}, s) d\bar{x} + r \hat{S}_1(s) \hat{\mathcal{G}}(x, x', s) \right], \quad (11)$$

$$\hat{\rho}_2(x, s) = \frac{\zeta_1}{2(s + \zeta_2)} \left[\int_{-\infty}^{\infty} \varphi(\bar{x}) \hat{\mathcal{G}}(x, \bar{x}, s) d\bar{x} + r \hat{S}_1(s) \hat{\mathcal{G}}(x, x', s) \right], \quad (12)$$

with $\widehat{\mathcal{G}}(x, x', s)$ defined as follows:

$$\widehat{\mathcal{G}}(x, x', s) = \frac{2}{(2+\eta)D} |xx'|^{\frac{1}{2}(1+\eta)} \left[I_{-\nu} \left(\frac{2}{2+\eta} \widehat{\Phi}(s) |x|^{\frac{1}{2}(2+\eta)} \right) K_{-\nu} \left(\frac{2}{2+\eta} \widehat{\Phi}(s) |x'|^{\frac{1}{2}(2+\eta)} \right) + \frac{xx'}{|xx'|} I_{\nu} \left(\frac{2}{2+\eta} \widehat{\Phi}(s) |x|^{\frac{1}{2}(2+\eta)} \right) K_{\nu} \left(\frac{2}{2+\eta} \widehat{\Phi}(s) |x'|^{\frac{1}{2}(2+\eta)} \right) \right],$$

for $|x| < |x'|$, and

$$\widehat{\mathcal{G}}(x, x', s) = \frac{2}{(2+\eta)D} |xx'|^{\frac{1}{2}(1+\eta)} \left[I_{-\nu} \left(\frac{2}{2+\eta} \widehat{\Phi}(s) |x'|^{\frac{1}{2}(2+\eta)} \right) K_{-\nu} \left(\frac{2}{2+\eta} \widehat{\Phi}(s) |x|^{\frac{1}{2}(2+\eta)} \right) + \frac{xx'}{|xx'|} I_{\nu} \left(\frac{2}{2+\eta} \widehat{\Phi}(s) |x'|^{\frac{1}{2}(2+\eta)} \right) K_{\nu} \left(\frac{2}{2+\eta} \widehat{\Phi}(s) |x|^{\frac{1}{2}(2+\eta)} \right) \right],$$

for $|x| > |x'|$, where $\widehat{\Phi}(s) = \sqrt{(s + r + \zeta_1 - \widehat{\Lambda}(s))/D}$ and $K_{\nu}(x)$ is the Bessel function of modified argument of second kind [44].

The survival probability can be determined from the set of Equation (1). For this, we can perform an integration on the spatial variable and obtain the following set of equations:

$$\begin{aligned} \frac{\partial}{\partial t} S_1(t) &= -\zeta_1 S_1(t) + \zeta_2 S_2(t) \\ \frac{\partial}{\partial t} S_2(t) &= \zeta_1 S_1(t) - \zeta_2 S_2(t) \end{aligned} \quad (13)$$

where $S_1(t) = \int_{-\infty}^{\infty} dx \rho_1(x, t)$ and $S_2(t) = \int_{-\infty}^{\infty} dx \rho_2(x, t)$, which results in the following solutions for the survival probabilities:

$$S_1(t) = \frac{\zeta_2}{\zeta_2 + \zeta_1} + \frac{\zeta_1}{\zeta_2 + \zeta_1} e^{-t(\zeta_2 + \zeta_1)} \quad (14)$$

and

$$S_2(t) = \frac{\zeta_1}{\zeta_2 + \zeta_1} (1 - e^{-t(\zeta_2 + \zeta_1)}) . \quad (15)$$

Figure 1 shows the behavior of the distribution connected to the system for different values of η to illustrate how the heterogeneity combined with the stochastic resetting and pauses modifies the time evolution of the system. This feature is more clearly illustrated in Figure 2 with the time behavior of the mean square displacement. The heterogeneous diffusivity introduces a non-trivial scaling of the mean square displacement, $\sigma_x^2(t) \propto t^{2/(2+\eta)}$ for small times (see Figure 2a), which interpolates between sub- and superdiffusive regimes depending on η . The inclusion of pauses effectively tempers the transport exponent (see Figure 2b), while stochastic resetting establishes a nonequilibrium stationary state. Figure 3a,b show the behavior of the distribution for different times by considering $\eta = 1/2$ and $\eta = -1/2$. It is worth mentioning that the distribution has different behaviors depending on the choices performed for η , which is different from the standard case with stochastic resetting [24]. Particularly, it is possible to show that $\rho_{st}(x) = \lim_{t \rightarrow \infty} \rho(x, t)$ is given by

$$\rho_{st}(x) = \frac{|x|^{\frac{1}{2}(1+\eta)}}{\Gamma\left(\frac{1}{\eta+2}\right)} \sqrt{\frac{r}{D}} \left(\frac{1}{2+\eta} \sqrt{\frac{r}{D}} \right)^{\frac{1}{\eta+2}} K_{\nu} \left(\frac{2}{2+\eta} \sqrt{\frac{r}{D}} |x|^{\frac{\eta+2}{2}} \right) \quad (16)$$

for the initial condition used in Figure 3, which implies that the parameters ζ_1 and ζ_2 influence only the time evolution of the system until it reaches the stationary state.

This result also shows that the asymptotic behavior of mean square displacement, i.e., $\sigma_{x,st} = \lim_{t \rightarrow \infty} \sigma_x(t)$, in this scenario, for the system is independent of the ζ_1 and ζ_2 , yielding

$$\sigma_{x,st} = \frac{2\Gamma\left(\frac{2}{2+\eta}\right)}{(2+\eta)\Gamma\left(\frac{1}{\eta+2}\right)} \Gamma\left(\frac{3}{\eta+2}\right) \left[(\eta+2)\sqrt{\frac{D}{r}}\right]^{\frac{4}{2+\eta}}. \quad (17)$$

For intermediate times, the influence of ζ_1 and ζ_2 is evident as show in Figure 2a,b.

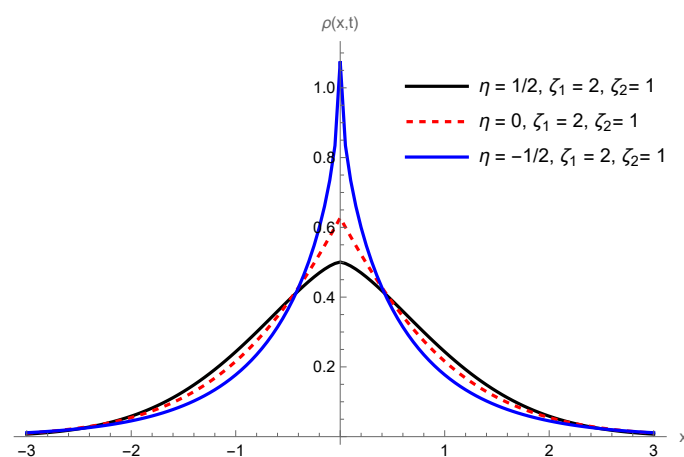


Figure 1. Behavior of the distribution versus x for different values of η by considering the diffusion process with pauses and stochastic resetting. We consider, for simplicity, $r = 1$, $t = 1$, $D = 1$, and the initial condition $\varphi(x) = \delta(x - \bar{x})$ with $\bar{x} = 0$.

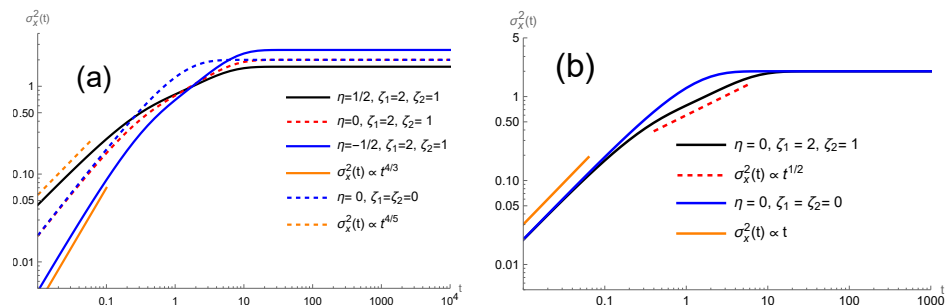


Figure 2. (a) shows the behavior of the mean square displacement versus t for different values of η by considering the diffusion process with pauses and stochastic resetting. (b) compares the behavior of the mean square displacement versus t for $\eta = 0$ with pauses and without pauses to show the intermediate behavior introduced by the intermittent motion promoted by the pauses. We consider, for simplicity, $r = 1$, $D = 1$, $x' = 0$, and the initial condition $\varphi(x) = \delta(x - \bar{x})$ with $\bar{x} = 0$.

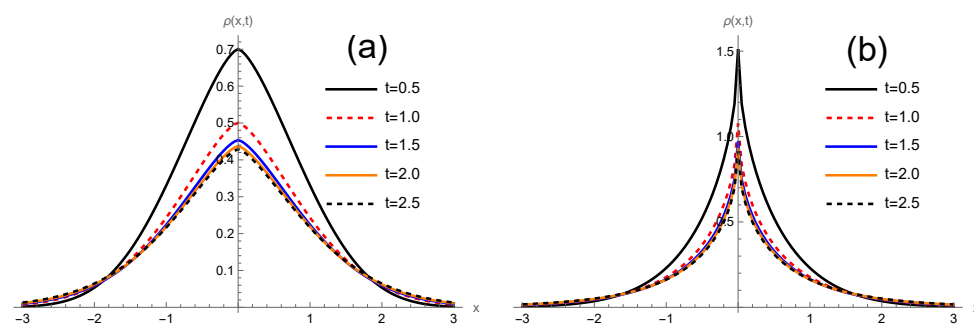


Figure 3. (a,b) show the behavior of the distribution for different times by considering $\eta = 1/2$ and $\eta = -1/2$. We consider, for simplicity, $r = 1$, $t = 1$, $D = 1$, $\zeta_1 = 2$, $\zeta_2 = 1$, $x' = 0$, and the initial condition $\varphi(x) = \delta(x - \bar{x})$ with $\bar{x} = 0$.

Now, we consider the system subjected to the boundary conditions $\rho_1(0, t) = \rho_1(\infty, t) = 0$ with $\rho_2(0, t) = \rho_2(\infty, t) = 0$, which characterizes an absorbing surface in $x = 0$. The system's behavior in this context is analyzed to determine the mean first-passage time, accounting for stochastic resetting and pauses in diffusion processes within a heterogeneous medium. By using the considerations for the previous case, it is possible to show that the solution is given by

$$\hat{\rho}_{1,-}(k, s) = \frac{\tilde{\varphi}_{1,-}(k) + (r/2)\hat{S}_1(s)\psi_{-}(x', k)}{s + D|k|^{2+\eta} + r + \zeta_1 - \hat{\Lambda}(s)}, \quad (18)$$

$$\hat{\rho}_{2,-}(k, s) = \frac{\zeta_1}{s + \zeta_2} \frac{\tilde{\varphi}_1(k) + (r/2)\hat{S}_1(s)\psi_{-}(x', k)}{s + D|k|^{2+\eta} + r + \zeta_1 - \hat{\Lambda}(s)}. \quad (19)$$

By performing some calculations, it is possible to show that the

$$\hat{\rho}_1(x, s) = \int_0^\infty \varphi(\bar{x})\hat{\mathcal{G}}_{-}(x, \bar{x}, s)d\bar{x} + r\hat{S}_1(s)\hat{\mathcal{G}}_{-}(x, x', s), \quad (20)$$

$$\hat{\rho}_2(x, s) = \frac{\zeta_1}{s + \zeta_2} \left[\int_0^\infty \varphi(\bar{x})\hat{\mathcal{G}}_{-}(x, \bar{x}, s)d\bar{x} + r\hat{S}_1(s)\hat{\mathcal{G}}_{-}(x, x', s) \right], \quad (21)$$

with $\hat{\mathcal{G}}_{-}(x, x', s)$ defined as follows:

$$\hat{\mathcal{G}}_{-}(x, x', s) = \frac{2}{(2+\eta)D} |xx'|^{\frac{1}{2}(1+\eta)} \text{I}_\nu \left(\frac{2}{2+\eta} \hat{\Phi}(s) |x|^{\frac{1}{2}(2+\eta)} \right) \text{K}_\nu \left(\frac{2}{2+\eta} \hat{\Phi}(s) |x'|^{\frac{1}{2}(2+\eta)} \right), \quad (22)$$

for $|x| < |x'|$, and

$$\hat{\mathcal{G}}_{-}(x, x', s) = \frac{2}{(2+\eta)D} |xx'|^{\frac{1}{2}(1+\eta)} \text{I}_\nu \left(\frac{2}{2+\eta} \hat{\Phi}(s) |x'|^{\frac{1}{2}(2+\eta)} \right) \text{K}_\nu \left(\frac{2}{2+\eta} \hat{\Phi}(s) |x|^{\frac{1}{2}(2+\eta)} \right), \quad (23)$$

for $|x| > |x'|$.

Figure 4 shows the behavior of the survival probability for the system when subjected to an absorbent surface for different values of η , when subjected to the stochastic resetting and pauses. By using the previous equations, it is possible to obtain the mean first passage time, i.e.,

$$\begin{aligned} T(\bar{x}, x') &= \frac{\zeta_1 + r\zeta_2}{\frac{2r\zeta_2}{D\Gamma(\nu)} \left(\frac{1}{2+\eta} \sqrt{\frac{r}{D}} \right)^\nu x'^{\frac{1}{2}(1+\eta)} \text{K}_\nu \left(\frac{2}{2+\eta} \sqrt{\frac{r}{D}} x'^{\frac{1}{2}(2+\eta)} \right)} \\ &\times \left(1 - \frac{2}{D\Gamma(\nu)} \left(\frac{1}{2+\eta} \sqrt{\frac{r}{D}} \right)^\nu \bar{x}^{\frac{1}{2}(1+\eta)} \text{K}_\nu \left(\frac{2}{2+\eta} \sqrt{\frac{r}{D}} \bar{x}^{\frac{1}{2}(2+\eta)} \right) \right). \end{aligned} \quad (24)$$

Figure 5 shows the behavior of the mean first passage time for different scenarios. Figure 5a shows the behavior of the mean first passage time with the stochastic resetting rate r for different values of η in the presence of pauses. This intermittent motion increases the first-passage time relative to the standard case. Figure 5b shows the behavior of the mean first passage time with the stochastic resetting rate r for different values of η in the presence of pauses for different values of the resetting position.

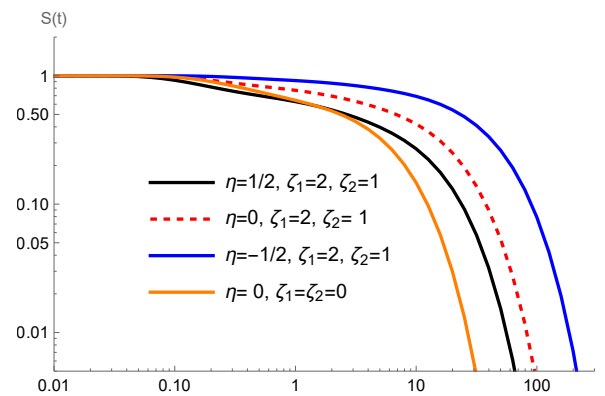


Figure 4. Behavior of the survival probability versus t for different values of η by considering the diffusion process with pauses and stochastic resetting. We consider, for simplicity, $r = 1$, $D = 1$, and the initial condition $\varphi(x) = \delta(x - \bar{x})$ with $\bar{x} = 1$.

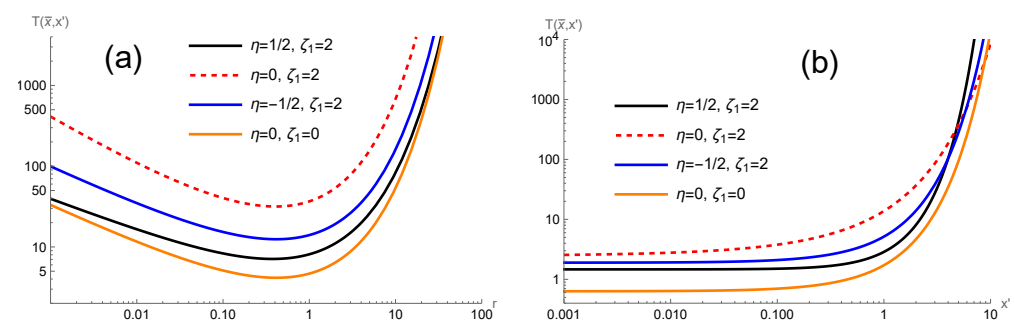


Figure 5. Behavior of the mean first passage time for different values of η by considering the diffusion process with pauses and stochastic resetting. (a) shows the behavior with the resetting rate r for $x' = 2$ and (b) shows the behavior with the resetting position x' for a given initial condition $\bar{x}$ with $r = 1$. We consider, for simplicity, $D = 1$ and the initial condition $\varphi(x) = \delta(x - \bar{x})$ with $\bar{x} = 1$.

3. Discussion and Conclusions

We have analyzed the combination of heterogeneous diffusion, pauses, and stochastic resetting. The results obtained here enable a rich dynamical behavior, governed by the interplay of spatial inhomogeneity and temporal intermittency. The heterogeneous diffusivity considered here modifies the local spreading rate. For $\eta > 0$, diffusion is hindered near the origin, generating an effective trapping potential, while $\eta < 0$ favors outward propagation and superdiffusive trends. These features are connected to the behaviors observed for the mean square displacement, particularly for small times, which allows us to obtain $\sigma_x^2(t) \propto t^{2/(2+\eta)}$. Introducing pauses during the process slows the growth of the mean-squared displacement, leading to a crossover between the active and immobilized regimes as the system responds to the intermittent process. The resetting process establishes a nonequilibrium stationary distribution whose shape depends sensitively on η . The stochastic resetting also alters the first-passage statistics, producing a finite mean first-passage time as verified for the case of an absorbing surface. The results obtained for the mean first-passage time $T(x, x')$ exhibit a familiar non-monotonic dependence on r , which for small resetting rates allows for long excursions, while very high r confines the dynamics excessively. This behavior yields an optimal resetting rate r_{opt} that minimizes search time, revealing how spatial heterogeneity with intermittent motion modifies the optimal search efficiency. This point can be verified by inspection in Figure 5. This interplay may explain, for example, adaptive strategies in active matter and living systems where movement intermittency and environmental inhomogeneity coexist. Finally, we hope that the results presented here will help to analyze diffusion in systems where the medium's heterogeneity is combined with stochastic resetting and intermittent motion.

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References

1. Gardiner, C.W. *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences (Springer Series in Synergetics)*; Springer: Berlin/Heidelberg, Germany, 1985.
2. Risken, H. *The Fokker-Planck Equation: Methods of Solution and Applications*; Springer: Berlin/Heidelberg, Germany, 1989.
3. Pekalski, A.; Sznajd-Weron, K. *Anomalous Diffusion from Basics to Applications*; Springer: Berlin/Heidelberg, Germany, 1999.
4. Méndez, V.; Campos, D.; Bartumeus, F. *Stochastic Foundations in Movement Ecology: Anomalous Diffusion, Front Propagation and Random Searches*; Springer: Berlin/Heidelberg, Germany, 2014.
5. Metzler, R.; Klafter, J. The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **2000**, *339*, 1–77. [[CrossRef](#)]
6. Evangelista, L.R.; Lenzi, E.K. *Fractional Diffusion Equations and Anomalous Diffusion*; Cambridge University Press: Cambridge, UK, 2018.
7. Havlin, S.; Weiss, G.H. A new class of long-tailed pausing time densities for the CTRW. *J. Stat. Phys.* **1990**, *58*, 1267–1273. [[CrossRef](#)]
8. Bodrova, A.S.; Chechkin, A.V.; Cherstvy, A.G.; Metzler, R. Ultraslow scaled Brownian motion. *New J. Phys.* **2015**, *17*, 063038. [[CrossRef](#)]
9. Höfling, F.; Franosch, T. Anomalous transport in the crowded world of biological cells. *Rep. Prog. Phys.* **2013**, *76*, 046602. [[CrossRef](#)] [[PubMed](#)]
10. Klages, R.; Radons, G.; Sokolov, I.M. *Anomalous Transport*; Wiley Online Library: Hoboken, NJ, USA, 2008.
11. Zaslavsky, G.M. Chaos, fractional kinetics, and anomalous transport. *Phys. Rep.* **2002**, *371*, 461–580. [[CrossRef](#)]
12. Vilks, O.; Charter, M.; Toledo, S.; Barkai, E.; Nathan, R. Strong Anomalous Diffusion for Free-Ranging Birds. *PRX Life* **2025**, *3*, 033020. [[CrossRef](#)]
13. Li, Y.; Suleiman, K.; Xu, Y. Anomalous diffusion, non-Gaussianity, nonergodicity, and confinement in stochastic-scaled Brownian motion with diffusing diffusivity dynamics. *Phys. Rev. E* **2024**, *109*, 014139. [[CrossRef](#)]
14. Barletta, A.; Rees, D.; Celli, M.; Brandão, P. Open Boundaries, Anomalous Diffusion and the Darcy-Bénard Instability. *Transp. Porous Media* **2025**, *152*, 27. [[CrossRef](#)]
15. Bhattacharjee, T.; Datta, S.S. Bacterial hopping and trapping in porous media. *Nat. Commun.* **2019**, *10*, 2075. [[CrossRef](#)]
16. Bechinger, C.; Di Leonardo, R.; Löwen, H.; Reichhardt, C.; Volpe, G.; Volpe, G. Active particles in complex and crowded environments. *Rev. Mod. Phys.* **2016**, *88*, 045006. [[CrossRef](#)]
17. Havlin, S.; Ben-Avraham, D. Diffusion in disordered media. *Adv. Phys.* **1987**, *36*, 695–798. [[CrossRef](#)]
18. Ben-Avraham, D.; Havlin, S. *Diffusion and Reactions in Fractals and Disordered Systems*; Cambridge University Press: Cambridge, UK, 2000.
19. Méndez, V.; Fedotov, S.; Horsthemke, W. *Reaction-Transport Systems: Mesoscopic Foundations, Fronts, and Spatial Instabilities*; Springer Science & Business Media: Cham, Switzerland, 2010.
20. Lenzi, E.K.; Yednak, C.A.R.; Evangelista, L.R. Non-Markovian diffusion and the adsorption-desorption process. *Phys. Rev. E* **2010**, *81*, 011116. [[CrossRef](#)] [[PubMed](#)]
21. Ledesma-Durán, A.; Hernández, S.; Santamaría-Holek, I. Effect of Surface Diffusion on Adsorption-Desorption and Catalytic Kinetics in Irregular Pores. I. Local Kinetics. *J. Phys. Chem. C* **2017**, *121*, 14544–14556. [[CrossRef](#)]
22. Ledesma-Durán, A.; Hernández, S.I.; Santamaría-Holek, I. Effect of Surface Diffusion on Adsorption-Desorption and Catalytic Kinetics in Irregular Pores. II. Macro-Kinetics. *J. Phys. Chem. C* **2017**, *121*, 14557–14565. [[CrossRef](#)]
23. Evans, M.R.; Majumdar, S.N. Diffusion with stochastic resetting. *Phys. Rev. Lett.* **2011**, *106*, 160601. [[CrossRef](#)]
24. Evans, M.R.; Majumdar, S.N.; Schehr, G. Stochastic resetting and applications. *J. Phys. A Math. Theor.* **2020**, *53*, 193001. [[CrossRef](#)]
25. Bressloff, P.C. Search processes with stochastic resetting and multiple targets. *Phys. Rev. E* **2020**, *102*, 022115. [[CrossRef](#)]
26. Shkilev, V.P. Continuous-time random walk under time-dependent resetting. *Phys. Rev. E* **2017**, *96*, 012126. [[CrossRef](#)]

27. Bénichou, O.; Loverdo, C.; Moreau, M.; Voituriez, R. Intermittent search strategies. *Rev. Mod. Phys.* **2011**, *83*, 81–130. [[CrossRef](#)]
28. Fedotov, S. Subdiffusion, chemotaxis, and anomalous aggregation of self-attracting particles. *Phys. Rev. E* **2011**, *83*, 021110. [[CrossRef](#)]
29. Belan, S. Restart could optimize the probability of success in a Bernoulli trial. *Phys. Rev. Lett.* **2018**, *120*, 080601. [[CrossRef](#)]
30. Stanislavsky, A.; Weron, A. Optimal non-Gaussian search with stochastic resetting. *Phys. Rev. E* **2021**, *104*, 014125. [[CrossRef](#)] [[PubMed](#)]
31. Roldán, É.; Lisica, A.; Sánchez-Taltavull, D.; Grill, S.W. Stochastic resetting in backtrack recovery by RNA polymerases. *Phys. Rev. E* **2016**, *93*, 062411. [[CrossRef](#)]
32. Santra, I. Effect of tax dynamics on linearly growing processes under stochastic resetting: A possible economic model. *Europhys. Lett.* **2022**, *137*, 52001. [[CrossRef](#)]
33. Jolakoski, P.; Pal, A.; Sandev, T.; Kocarev, L.; Metzler, R.; Stojkoski, V. A first passage under resetting approach to income dynamics. *Chaos Solitons Fractals* **2023**, *175*, 113921. [[CrossRef](#)]
34. O’Shaughnessy, B.; Procaccia, I. Analytical solutions for diffusion on fractal objects. *Phys. Rev. Lett.* **1985**, *54*, 455. [[CrossRef](#)]
35. O’Shaughnessy, B.; Procaccia, I. Diffusion on fractals. *Phys. Rev. A* **1985**, *32*, 3073. [[CrossRef](#)] [[PubMed](#)]
36. Richardson, L.F. Atmospheric diffusion shown on a distance-neighbour graph. *Proc. R. Soc. Lond. Ser. A Contain. Pap. A Math. Phys. Character* **1926**, *110*, 709–737.
37. Salazar, J.P.; Collins, L.R. Two-particle dispersion in isotropic turbulent flows. *Annu. Rev. Fluid Mech.* **2009**, *41*, 405–432. [[CrossRef](#)]
38. Sandev, T.; Kocarev, L.; Metzler, R.; Chechkin, A. Stochastic dynamics with multiplicative dichotomic noise: Heterogeneous telegrapher’s equation, anomalous crossovers and resetting. *Chaos Solitons Fractals* **2022**, *165*, 112878. [[CrossRef](#)]
39. Sandev, T.; Domazetoski, V.; Kocarev, L.; Metzler, R.; Chechkin, A. Heterogeneous diffusion with stochastic resetting. *J. Phys. A Math. Theor.* **2022**, *55*, 074003. [[CrossRef](#)]
40. Ray, S. Space-dependent diffusion with stochastic resetting: A first-passage study. *J. Chem. Phys.* **2020**, *153*. [[CrossRef](#)] [[PubMed](#)]
41. Hanggi, P. Nonlinear fluctuations: The problem of deterministic limit and reconstruction of stochastic dynamics. *Phys. Rev. A* **1982**, *25*, 1130. [[CrossRef](#)]
42. Klimontovich, Y.L. Ito, Stratonovich and kinetic forms of stochastic equations. *Physica A* **1990**, *163*, 515–532. [[CrossRef](#)]
43. Volpe, G.; Wehr, J. Effective drifts in dynamical systems with multiplicative noise: A review of recent progress. *Rep. Prog. Phys.* **2016**, *79*, 053901. [[CrossRef](#)]
44. Wyld, H.W.; Powell, G. *Mathematical Methods for Physics*; CRC Press: Boca Raton, FL, USA, 2020.

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