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Rapid Deterioration of Convergence in Taylor Expansions of Linearizing Maps of Hénon Maps at Hyperbolic Fixed Points

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Abstract

In this paper, we prove that the Taylor expansions of analytic functions appearing in the linearization of quadratic maps at hyperbolic fixed points do not successfully approximate invariant manifolds, such as stable and unstable manifolds, when higher-order terms are truncated. This fact was pointed out by Newhouse et al. in their numerical experiments, and implies that the Taylor expansions are inadequate for quantitatively studying dynamical systems such as quadratic maps. In fact, it is shown that the computational complexity for the approximations by the Taylor expansions grows exponentially.

Keywords: discrete dynamical systems; Hénon maps; linearization; Taylor expansions; hyperbolic fixed points; computational complexity

MSC: 37D45; 37M21; 68Q17; 39A13; 41A10



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1. Introduction

The Hénon map $f : \mathbb{C}^2 \to \mathbb{C}^2$ is a quadratic map defined by

$$f : (x, y) \mapsto \Big(1 + y - ax^2, bx\Big),$$

where $a, b \in \mathbb{C}$ are parameters, and $a \neq 0, b \neq 0$ ([1]). Let $P = (x_f, y_f)$ be one of the fixed points of f, and let α_1, α_2 be the eigenvalues of the derivative $D_P f$. Throughout this paper, we assume

$$|\alpha_1| \neq 1$$

and set $\alpha = \alpha_1$. Then

$$\alpha^2 - \lambda\alpha - b = 0$$

holds, where $\lambda = -2ax_f$. Two cases occur: $0 < |\alpha| < 1$ and $|\alpha| > 1$. We recall that P is *hyperbolic* when $|\alpha_2| \neq 1$ in addition. If $|\alpha| < 1 < |\alpha_2|$ or $|\alpha_2| < 1 < |\alpha|$, then P is a *saddle*. This paper treats the case satisfying the following condition:

$$\alpha \neq \alpha_2,$$

which is more general than the saddle case.

The following result on the linearization of dynamical systems is well-known (cf. [2]).

Proposition 1. (Poincaré [3]) *Under the above assumptions, for any eigenvector $v \neq 0$ for α, there uniquely exists a holomorphic map $\mathcal{P}_\alpha : \mathbb{C} \to \mathbb{C}^2$ satisfying $\mathcal{P}_\alpha(0) = P$ and $\mathcal{P}'_\alpha(0) = v$ such that*

$$f \circ \mathcal{P}_\alpha(t) = \mathcal{P}_\alpha(\alpha t) \tag{1}$$

for all $t \in \mathbb{C}$. Furthermore, $\mathcal{P}_\alpha : \mathbb{C} \to \mathbb{C}^2$ is injective and for all $t \in \mathbb{C}$, $\mathcal{P}'_\alpha(t) \neq 0$.

The map $\mathcal{P}_\alpha : \mathbb{C} \to \mathbb{C}^2$ in Proposition 1 is called *Poincaré's map*. The image

$$W_\alpha(P) = \mathcal{P}_\alpha(\mathbb{C})$$

is the *invariant manifold associated with* α, which coincides with the stable or unstable manifold when P is a saddle. By Proposition 1, $W_\alpha(P)$ is actually f-invariant (i.e., $f(W_\alpha(P)) = W_\alpha(P)$) and an injective immersed submanifold of $\mathbb{C}^2$ through P. We equip the intrinsic metric on $W_\alpha(P)$ induced by $\mathcal{P}_\alpha$.

Franceschini and Russo [4] used Poincaré's map for numerical calculations to detect homoclinic points in the case where a and b are the original Hénon's parameter values. Also, Fornaess and Gavosto [5,6] investigated the existence of homoclinic tangencies using Poincaré's map. The existence of higher-dimensional Poincaré's maps was investigated by Cabré, Fontich, and de la Llave [7–9]. Anastassiou, Bountis, and Bäcker [10] used their result to detect homoclinic points for the 4-dimensional maps. Also, Newhouse, Berz, Grote and Makino [11] estimated the topological entropy of the Hénon map using symbolic dynamics via the Taylor models in addition to Poincaré's map. We emphasize that Poincaré's maps were expressed in Taylor expansions in those studies.

It is important to clarify how the Taylor expansions work for quantitative studies of dynamical systems, such as numerically calculating the topological entropy or detecting the homoclinic set (cf. [11–13]). In this paper, we prove that the computational complexity for the approximations by the Taylor expansions grows exponentially, which is precisely stated as follows.

Theorem 1. *When $0 < |\alpha| < 1$ (resp. $|\alpha| > 1$), if the disk centered at P with radius $|\alpha|^{-L}$ (resp. $|\alpha|^{L}$) in the invariant manifold $W_\alpha(P)$ with respect to the intrinsic metric is computationally realized with high computational accuracy in $\mathbb{C}^2$ by the map $\mathcal{P}_\alpha$, then the order of the truncation terms in the Taylor expansions of $\mathcal{P}_\alpha$ must be greater than N satisfying the condition*

$$N \sim 2^L, \tag{2}$$

where L is a natural number given arbitrarily.

This result implies that the Taylor expansions are inadequate for quantitative studies such as numerically calculating the topological entropy or detecting the homoclinic set, which was pointed out by Newhouse, Berz, Grote, and Makino [11] in their numerical experiments.

2. Main Results

To state the main results of the current paper, we mention the construction of Poincaré's map $\mathcal{P}_\alpha$. For simplicity, let us consider a conjugate map $f : \mathbb{C}^2 \to \mathbb{C}^2$ of the above map by the translation $(x + x_0, y + y_0) \mapsto (x, y)$ on $\mathbb{C}^2$ sending P to the origin $O = (0, 0)$, i.e.,

$$f : (x, y) \ \mapsto \Big(\lambda x + y - ax^2, bx\Big).$$

Set

$$\mathcal{P}_\alpha(s) = \big(x_p(s), y_p(s)\big) \qquad (s \in \mathbb{C}).$$

Then, from the relation (1), we have

$$x_p(\alpha t) - \lambda x_p(s) - b x_p(\alpha^{-1} s) = -a\{x_p(s)\}^2 \tag{3}$$

and $y_p(s) = b x_p(\alpha^{-1} s)$. Setting a series expansion

$$x_p(s) = \sum_{n=1}^{\infty} a_n s^n, \tag{4}$$

and substituting this into the above relation (3), we obtain by the coefficient comparison

$$a_n = -\frac{a}{D_n} \sum_{k=1}^{n-1} a_k a_{n-k} \quad (\forall n \geq 2), \tag{5}$$

where $D_n = \alpha^n - \lambda - b\alpha^{-n}$. Note that the non-resonance condition that $D_n \neq 0$ for all $n \geq 2$ is assumed here. This condition is automatically satisfied when P is a saddle. The first-order coefficient a_1 is determined by the condition $\mathcal{P}'(0) = v$, and $a_1 \neq 0$.

Choosing ζ_1 as a branch of $-\log \alpha$, we introduce a new variable $t \in \mathbb{C}$ by the transformation

$$s = \mathrm{e}^{-\zeta_1 t},$$

and define $x(t) = x_p(s)$. Let $y(t) = bx(t-1)$. We define a holomorphic map $\mathcal{S} : \mathbb{C} \to \mathbb{C}^2$ by $\mathcal{S}(t) = (x(t), y(t))$. Then, from the relation (1), it follows that by the l-th iteration ($l \in \mathbb{Z}$) of f, any point in the invariant manifold $W_\alpha(P)$ is mapped as follows:

$$f^l \circ S(t) = S(t + l). \tag{6}$$

For each $n \geq 1$, let

$$x_n(t) = a_n \mathrm{e}^{-\zeta_1 n t}.$$

Then from (4), we have

$$x(t) = x_p(s) = \sum_{n=1}^{\infty} x_n(t). \tag{7}$$

We discuss the convergence of the series (7), assuming that $t \in \mathbb{C}$ are given arbitrarily.

2.1. Rapid Deterioration of Convergence in Taylor Expansions

Write $-\zeta_1 = \zeta_{1r} + i\zeta_{1i}$ and $t = t_r + it_i$ in the form of real and imaginary parts. For simplicity, we set

$$E_1 = \frac{\log(|2aa_1|e^{-\zeta_{1i}t_i})}{\log|\alpha|},$$

and let

$$r_{\alpha,t} = \begin{cases} e^{-E_1\log 2 - 1 - t_r\log 2} & \text{if } 0 < |\alpha| < 1, \\ e^{E_1\log 2 - 1 + t_r\log 2} & \text{if } |\alpha| > 1. \end{cases} \tag{8}$$

Define a natural number n_t as

$$n_t = [r_{\alpha,t}],$$

where $[\,\cdot\,]$ denotes the Gauss's symbol. We notice that n_t grows exponentially with respect to t_r, if t_r is chosen to be negative when $0 < |\alpha| < 1$, and to be positive when $|\alpha| > 1$, respectively.

The purpose of this paper is to give a lower bound to the magnitude of each higher-order term of the series (7) as follows.

Theorem 2. *There exists a constant N_1 such that if $n_t > N_1$ and $N_1 < n_t + k < 2n_t$ ($k \in \mathbb{Z}$) are satisfied, and if $\delta = k/n_t$ is close to zero, then there is ϵ with $0 \le \epsilon \ll n_t$ such that*

$$|x_{n_t+k}(t)| \ge \begin{cases} |\alpha|^{-\frac{n_t}{\log 2}(1-\frac{\delta^2}{2})+\epsilon} & \text{if } 0 < |\alpha| < 1, \\ |\alpha|^{\frac{n_t}{\log 2}(1-\frac{\delta^2}{2})-\epsilon} & \text{if } |\alpha| > 1. \end{cases} \tag{9}$$

The above result gives a mathematical description of the fact that the approximation of $x_p(s)$ described by the Taylor expansion (4) or (7) rapidly deteriorates as the value $|s|$ becomes large when higher-order terms are truncated.

To state how the series (4) converges, let us denote by $E_N(s)$ the remainder term:

$$E_N(s) = \sum_{n=N}^{\infty} a_n s^n = \sum_{n=N}^{\infty} x_n(t), \qquad s = e^{-\zeta_1 t}, \tag{10}$$

where N is a natural number. The following estimate holds.

Theorem 3. *Let $0 < |\alpha| < 1$. Then there exists a constant $K > 1$ such that for every $R > 0$, if a natural number N with $N > N_1$ is large enough, the remainder term $E_N(s)$ is estimated as*

$$\frac{1}{2}|2aa_1|^N|s|^N|\alpha|^{\frac{N\log N}{\log 2}} \le |E_N(s)| \le 2K^N|s|^N|\alpha|^{\frac{N(\log N - \log\log N)}{\log 2}} \tag{11}$$

for any s with $|s| \le R$, where N_1 is the constant stated in Theorem 2. When $|\alpha| > 1$, a similar statement holds if α is replaced with α^{-1}.

Theorem 1 is obtained from Theorem 3 as follows.

Proof of Theorem 1. Let $|s| \sim 1$ for $0 < |\alpha| < 1$. If the higher-order terms in the series (4) are truncated after N-th, the error E for the point $\mathcal{P}(\alpha^{-L}s)$ is estimated as

$$E \sim \left(|\alpha|^{\frac{\log N}{\log 2}}|\alpha^{-L}|\right)^N$$

by (11). If the right-hand side of this estimate is small enough, we need the condition that

$$\frac{\log N}{\log 2} - L > 0,$$

which is equivalent to $N > 2^L$. Thus, we obtain the necessary condition (2). In the same way as above, we obtain the conclusion for the case of $|\alpha| > 1$. The proof of Theorem 1 is complete. □

2.2. Discussion: Beyond Taylor Series

The following examples 1 and 2 specifically illustrate how pessimistic the implications of Theorems 2 and 3 are.

Example 1. *Let $0 \le t_0 \le 1$ be an initial time, and let $0 < |\alpha| < 1$. When $l = -10$ and $t = t_0 + l$, we have from (8) that $n_t = [r_{\alpha.t}] \sim 400$ (we set $E_1 = 0$ and $t_i = 0$). Then (9) gives*

$$|x_{n_t}(t_0 - 10)| \ge |\alpha|^{\frac{-400}{\log 2}} \sim |\alpha|^{-570}.$$

In this case, the computation is yet computationally feasible. On the other hand, when $l = -20$ and $t = t_0 + l$, (8) gives $n_t \sim 4.4 \times 10^5$ and by (9) we have

$$|x_{n_t}(t_0 - 20)| \ge |\alpha|^{\frac{-4.4\times 10^5}{\log 2}} \sim |\alpha|^{-6\times 10^5},$$

which implies that the computation is infeasible.

Example 2. *Here, we present an example of stable and unstable manifolds depicted by using the Taylor expansion (7). Figure 1 shows a chaotic system encircling KAM tori in a symplectic case. We select the eigenvalues α as $\alpha = \alpha_1 = \mathrm{e}^{-1.2}$ for the stable manifold and $\alpha = \alpha_2 = \mathrm{e}^{1.2}$ for the unstable manifold, respectively. Then, the dynamical parameter $b = -\alpha_1\alpha_2$ satisfies $b = -1$. Also λ and a in (3) are given as $\lambda = 2\cosh(1.2)$ and $a = -1 + (\lambda/2 - 1)^2 \sim -0.34284$, respectively. For these parameters, one fixed point is elliptic ($|\alpha| = 1$) and the other is a saddle point. The eigenvalues α_1 and α_2 at the saddle point [$(x, y) = (0, 0)$ here] are $\alpha_1 = 0.30119$ (stable) and $\alpha_2 = 3.32012$ (unstable). For $\alpha = \alpha_1$, the branch ζ_1 is chosen as $\zeta_1 = 1.2$ and the variable t is selected to be real numbers with ranges $t \ge -9.65$ and $t \ge -1.65$ by setting $a_1 = -\frac{\sinh(-1.2)}{2a}$ and $a_1 = \frac{\sinh(-1.2)}{2a}$, respectively. When $t = -9.65$, (8) gives $n_t \sim 375$. We similarly choose the branch ζ_1 and the variable t for $\alpha = \alpha_2$. In this numerical computation, we take the number of truncation N in the series (7) as $N = 4000$. Concentric KAM tori (and periodic orbits) exist inside the chaotic system. The number N_1 in Theorem 2 in this example is $N_1 \sim 560$ for $\alpha = \alpha_1$.*

To improve the pessimistic situation as stated above, Newhouse, Berz, Grote, and Makino [11] introduced the notion of the Taylor models by restricting the parameter s such that the set $\{\mathcal{P}_\alpha(\alpha^\ell s)\}_{\ell=0}^{\infty}$ ($0 < |\alpha| < 1$) or $\{\mathcal{P}_\alpha(\alpha^{-\ell} s)\}_{\ell=0}^{\infty}$ ($|\alpha| > 1$) is contained in a given bounded set, in which they rigorously estimated the topological entropy of Hénon maps using symbolic dynamics. See also Ishii [12].

In correspondence to such a development, it has been obtained very recently by the authors that the holomorphic maps $f^{-\ell} \circ \mathcal{P}_{\alpha,N}(\alpha^\ell s)$ defined by taking conjugates by iteration maps of the Hénon map and of the derivative give much better approximations of $\mathcal{P}_\alpha$ theoretically and numerically when the parameter s is restricted as above, where

$\mathcal{P}_{\alpha,N} : \mathbb{C} \to \mathbb{C}^2$ is the polynomial map with degree N when higher-order terms are truncated from $\mathcal{P}_\alpha$.

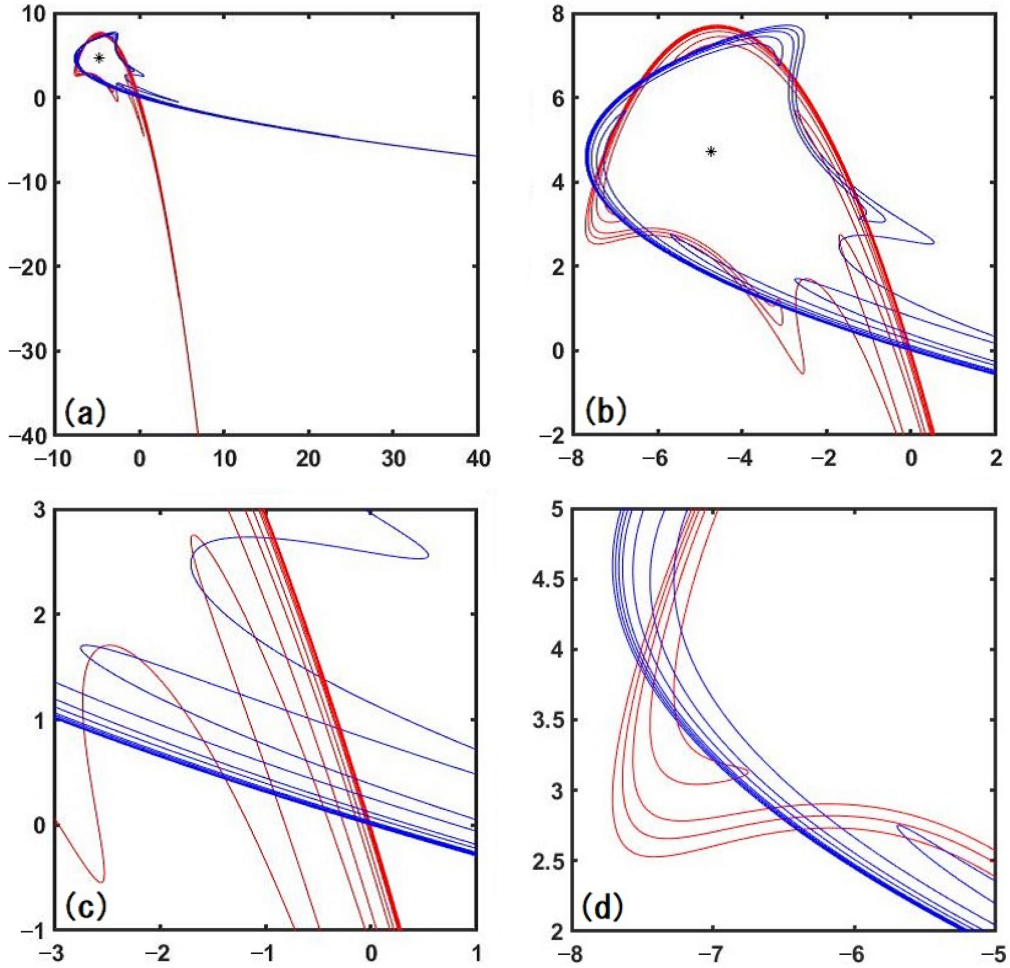


Figure 1. A chaotic system encircling KAM tori, where $a = -0.34284$ and $b = -1$. The asterisk denotes an elliptic fixed point. Panel (**b**) is the enlarged figure of (**a**), and panels (**c**,**d**) are further enlarged views of (**b**). The red and blue curves denote the stable and unstable manifolds, respectively.

It is well-known that the method of contraction mappings is useful in studying invariant manifolds and dynamical stability (cf. [14]). Such a method might also work for the approximation of $\mathcal{P}_\alpha$ theoretically and numerically.

The invariant manifold $W_\alpha(P)$ can be described by a divergent series called asymptotic expansion in Poincaré's sense, and a phenomenon similar to the Stokes phenomenon and to differential equation theory occurs. For details, we can refer to Tovbis [15], Lazutkin , Schachmannski and Tabanov [16], Gelfreich and Sauzin [17], Lazutkin, Segur and Hakim [16,18,19], Écalle [20–22], Sternin and Shatalov [23], Voros [24], and Matsuoka and Hiraide [13,25–27]. The results of those studies have made it possible to quantitatively investigate the dynamical system of the Hénon map f with extremely high precision using computers, which suggests that the asymptotic expansion gives us much better performance than the ones stated in Theorems 2 and 3. The authors hope that the research area will be developed in discrete dynamical systems.

3. Proofs of Theorems 2 and 3

To prove Theorems 2 and 3, we prepare the following Lemmas 1 and 2, which gives sharp estimates on upper and lower bounds to absolute values of the coefficients a_n ($n \geq 1$) (cf. [5,6]). In the following, suppose that $0 < |\alpha| < 1$.

Lemma 1. *There exists a constant $K > 1$ such that*

$$|a_n| < K^n |\alpha|^{\frac{n(\log n - \log\log_+ n)}{\log 2}} \tag{12}$$

holds for all $n \geq 1$, where $\log_+ n = \max\{\log n, 1\}$.

Lemma 2. *There exists a natural number $N_1 \geq 1$ such that*

$$|a_n| \geq |2aa_1|^n |\alpha|^{\frac{n\log n}{\log 2}} \tag{13}$$

holds for all $n \geq N_1$.

The above results are stronger than the estimate obtained by Fornaess and Gavosto [5,6] via complex analysis.

Lemmas 1 and 2 also hold for $|\alpha| > 1$, if α is replaced with α^{-1}. The proofs of Lemmas 1 and 2 are provided in Sections 4 and 5, respectively.

Define a real function

$$F(r,t) = E_1 r + \frac{r\log r}{\log 2} + t_r r \quad (r \geq 1).$$

Then, we have

$$\frac{\partial F}{\partial r}(r_{\alpha,t}, t) = 0,$$

which is equivalent to the relation (8) because the case of $0 < |\alpha| < 1$ is considered. It follows that $F(r,t)$ takes its minimum value at $r = r_{\alpha,t}$ for a fixed t.

Let

$$E_2 = \frac{\log(Ke^{-\zeta_{1i}t_i})}{\log|\alpha|},$$

where $K > 1$ is the constant stated in Lemma 1, and define a real function

$$G(r,t) = E_2 r + \frac{r(\log r - \log\log_+ r)}{\log 2} + t_r r \quad (r \geq 1).$$

By using Lemmas 1 and 2, the following Lemma 3 is easily checked.

Lemma 3. *If $N_1 \geq 1$ is the number as in Lemma 2,*

$$|\alpha|^{F(n,t)} \leq |x_n(t)| \leq |\alpha|^{G(n,t)} \tag{14}$$

holds for any $n \geq N_1$.

Proof of Theorem 2. Let $\epsilon' = \log r_{\alpha,t} - \log n_t$. Obviously, $0 \leq \epsilon' \ll 1$. From (8), we have

$$\log n_t = -E_1 \log 2 - 1 - t_r \log 2 - \epsilon',$$

which implies that

$$\frac{n_t \log n_t}{\log 2} = -E_1 n_t - \frac{n_t}{\log 2} - t_r n_t - \frac{\epsilon' n_t}{\log 2}.$$

Let $N_1 \geq 1$ be the number as in Lemma 2. Since

$$\frac{(n_t+k)\log(n_t+k)}{\log 2} = \frac{n_t \log n_t}{\log 2} + \frac{n_t \log\left(1+\frac{k}{n_t}\right)}{\log 2} + \frac{k \log n_t}{\log 2} + \frac{k \log\left(1+\frac{k}{n_t}\right)}{\log 2},$$

by Lemma 3 we obtain

$$\begin{aligned} |x_{n_t+k}(t)| &\geq |\alpha|^{F(n_t+k,t)} \\ &= |\alpha|^{E_1 k - \frac{n_t}{\log 2} + \frac{n_t \log\left(1+\frac{k}{n_t}\right)}{\log 2} + \frac{k \log n_t}{\log 2} + \frac{k \log\left(1+\frac{k}{n_t}\right)}{\log 2} - t_r k - \frac{\epsilon' n_t}{\log 2}} \\ &= |\alpha|^{\frac{n_t \log\left(1+\frac{k}{n_t}\right)}{\log 2} - \frac{(n_t+k)}{\log 2} + \frac{k \log\left(1+\frac{k}{n_t}\right)}{\log 2} - \frac{\epsilon'(n_t+k)}{\log 2}}. \end{aligned} \tag{15}$$

Taking $|k| \ll n_t$ into account, we have

$$\begin{aligned} n_t \log\left(1+\frac{k}{n_t}\right) - k &= n_t\left(\frac{k}{n_t} - \frac{1}{2}\left(\frac{k}{n_t}\right)^2 + \cdots\right) - k = n_t\left(-\frac{1}{2}\left(\frac{k}{n_t}\right)^2 + O\left[\left(\frac{k}{n_t}\right)^3\right]\right), \\ k \log\left(1+\frac{k}{n_t}\right) &= k\left(\frac{k}{n_t} - \frac{1}{2}\left(\frac{k}{n_t}\right)^2 + \cdots\right) = n_t\left(\left(\frac{k}{n_t}\right)^2 + O\left[\left(\frac{k}{n_t}\right)^3\right]\right). \end{aligned}$$

Then (15) yields the inequality (9). In the same way as above, we obtain the conclusion for the case of $|\alpha| > 1$. The proof of Theorem 2 is complete. □

Proof of Theorem 3. Let us express $|E_p(s)|$ as

$$|E_p(s)| = \left|a_N s^N\right| |1 + h(s)|,$$

where

$$\begin{aligned} h(s) &= \frac{a_{N+1}}{a_N} s + \frac{a_{N+2}}{a_N} s^2 + \cdots \\ &= \frac{x_{N+1}(t)}{x_N(t)} + \frac{x_{N+2}(t)}{x_N(t)} + \cdots. \end{aligned}$$

Let $N \geq N_1$. Then, from Lemma 3 we can estimate $|h(s)|$ as

$$|h(s)| \leq \sum_{m=1}^{\infty} |\alpha|^{G(N+m,t)-F(N,t)}. \tag{16}$$

Let $R > 0$ be given arbitrarily. Choosing $N > N_1$ to be sufficiently large, we have that if

$$|s| = e^{-\zeta_{1i} t_i + \zeta_{1r} t_r} \leq R$$

is satisfied, then

$$G(N+m,t) - F(N,t) \geq (N+m)\log(N+m) - N \log N$$

holds for any $m \geq 1$. Then, from (16) it follows that

$$|h(s)| \leq \sum_{m=1}^{\infty} \left(\frac{1}{(N+1)^{\log|\alpha^{-1}|}} \right)^m.$$

Since

$$\left|a_N s^N\right|(1-|h(s)|) \leq |E_p(s)| \leq \left|a_N s^N\right|(1+|h(s)|), \tag{17}$$

the conclusion is obtained from Lemmas 1 and 2. In the same way as above, we obtain the conclusion for the case of $|\alpha| > 1$. The proof of Theorem 3 is complete. □

4. Proof of Lemma 1

In this section, we give the proof of Lemma 1

Since $0 < |\alpha| < 1$, by the definition, there exists a constant $C > 1$ such that

$$\left|\frac{a}{D_n}\right||\alpha|^{-n} < C \tag{18}$$

holds for all $n \geq 1$, and then we can choose $n_0 \geq 5$ such that

$$C|\alpha|^{\frac{n+1+\frac{\log n}{\log|\alpha|}\log(n+1)}{\log(n+1)}} \leq 1 \tag{19}$$

is satisfied for any $n \geq n_0$. Take a constant $K > 1$ such that the inequality (12) holds for $1 \leq n \leq n_0$. Let $n \geq n_0$, and assume that $a_1, a_2, \cdots, a_{n-1}$ and a_n satisfy the inequality (12).

Then we have

$$\begin{aligned} |a_{n+1}| &= \left|\frac{a}{D_{n+1}}\right||a_1a_n + a_2a_{n-1} + \cdots a_na_1| \leq \left|\frac{a}{D_{n+1}}\right|\sum_{k=1}^{n}|a_k a_{n+1-k}| \\ &\leq \left|\frac{a}{D_{n+1}}\right|\sum_{k=1}^{n} K^{n+1}|\alpha|^{\frac{k}{\log 2}(\log k - \log\log_+ k) + \frac{n+1-k}{\log 2}\left[\log(n+1-k) - \log\log_+(n+1-k)\right]}. \end{aligned}$$

Noticing $n \geq n_0 \geq 5$, we have the result that the exponent part of $|\alpha|$ in the above inequality takes the minimum value at $k = (n+1)/2$. Therefore,

$$\begin{aligned} |a_{n+1}| &\leq n\left|\frac{a}{D_{n+1}}\right|K^{n+1}|\alpha|^{\frac{n+1}{\log 2}\log\frac{n+1}{2} - \frac{n+1}{\log 2}\log\log_+\frac{n+1}{2}} \\ &= n\left|\frac{a}{D_{n+1}}\right|K^{n+1}|\alpha|^{-(n+1)+\frac{n+1}{\log 2}\log(n+1) - \frac{n+1}{\log 2}\log\log_+(n+1) - \frac{n+1}{\log 2}\log\left(1-\frac{\log 2}{\log(n+1)}\right)}. \end{aligned}$$

Using the expansion

$$\log\left(1 - \frac{\log 2}{\log(n+1)}\right) = -\frac{\log 2}{\log(n+1)} - \frac{1}{2}\left(\frac{\log 2}{\log(n+1)}\right)^2 - \cdots,$$

we obtain by (18) and (19) that

$$\begin{aligned} |a_{n+1}| &\leq n\left|\frac{a}{D_{n+1}}\right| K^{n+1}|\alpha|^{-(n+1)+\frac{n+1}{\log(n+1)}+\frac{n+1}{\log 2}\left[\log(n+1)-\log\log_+(n+1)\right]} \\ &\leq CK^{n+1}|\alpha|^{\frac{n+1+\frac{\log n}{\log|\alpha|}\log(n+1)}{\log(n+1)}}|\alpha|^{\frac{n+1}{\log 2}\left[\log(n+1)-\log\log_+(n+1)\right]} \\ &\leq K^{n+1}|\alpha|^{\frac{n+1}{\log 2}\left[\log(n+1)-\log\log_+(n+1)\right]}, \end{aligned}$$

which implies that the inequality (12) holds for $n+1$. Therefore, the conclusion is obtained. The proof of Lemma 1 is complete.

In the same way as above, we obtain Lemma 1 for $|\alpha| > 1$.

5. Lower Bounds to Absolute Values of Coefficients a_n's

We give in this section the proof of Lemma 2. We first introduce a function for measuring the growth of an entire function at infinity (cf. [28]). Let $g : \mathbb{C} \to \mathbb{C}$ be an entire function and define a function M_g on $\{r \in \mathbb{R} \mid r > 0\}$ by

$$M_g(r) = \max_{|s|=r} |g(s)|.$$

The *growth* ρ of g is defined by

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M_g(r)}{\log r}.$$

If g is expressed in the form of the power series $g(t) = \sum_{n=0}^{\infty} c_n t^n$, the growth ρ is characterized by

$$\rho = \overline{\lim_{n\to\infty}} \frac{n\log n}{\log\left(\frac{1}{|c_n|}\right)}.$$

For more details on the growth ρ, refer to [28], pp. 4, Theorem 2; [29], pp. 257, Theorem 9.4.

5.1. Growth of $x_p(s)$ at Infinity

To prove Lemma 2, we prepare the following Lemma 4 that is already known ([30]). For completeness, a simple proof is given.

Lemma 4. *Let* $0 < |\alpha| < 1$. *Then*

$$\rho_p = \frac{\log 2}{\log|\alpha^{-1}|} \tag{20}$$

holds. When $|\alpha| > 1$, *a similar statement holds if* α *is replaced with* α^{-1}.

Proof. For $r > 0$, we denote by D_r the open disk with radius r centered at the origin in $\mathbb{C}$. Then there exists a maximum value

$$M_r = \sup\{|x_p(s)| \mid s \in D_r\}.$$

Note that x_p is not a constant. Fix $r > 0$ to be sufficiently large. Then M_r is large enough. By the maximum modulus theorem, M_r is attained at some point s_0 in the boundary of D_r, i.e., $|x_p(s_0)| = M_r$. Since $|\alpha| < 1$, we have $\alpha s_0 \in D_r$, and so $|x_p(\alpha s_0)| < M_r$. Since M_r is large enough, it follows that

$$\begin{aligned} |x_p(\alpha^{-1}s_0)| &= \left|\lambda x_p(s_0) + b x_p(\alpha s_0) - a\left[x_p(s_0)\right]^2\right| \\ &\geq M_r^2\left(|a| - \frac{\lambda}{M_r} - \frac{|b|}{M_r}\right) \geq \left|\frac{a}{2}\right| M_r^2. \end{aligned}$$

Letting $N_r = |a/2|^{1/2} M_r$, we have $N_r > 1$. Then

$$|x_p(\alpha^{-1}s_0)| \geq N_r^2$$

holds. Let $n \geq 2$, and assume that

$$|x_p(\alpha^{-n}s_0)| \geq N_r^{2^n}. \tag{21}$$

Then we have

$$\begin{aligned} |x_p(\alpha^{-(n+1)}s_0)| &= \left|\lambda x_p(\alpha^{-n}s_0) + b x_p(\alpha^{-(n-1)}s_0) - a\left[x_p(\alpha^{-n}s_0)\right]^2\right| \\ &\geq \left(\left|\frac{a}{2}\right|^{2^{n-1}} M_r^{2^n}\right)^2 \left(|a| - \frac{\lambda}{M_r}\left|\frac{2}{a}\right|^n - \frac{|b|}{M_r}\left|\frac{2}{a}\right|^n\right) \\ &\geq \left|\frac{a}{2}\right|^{2^n} M_r^{2^{n+1}} = N_r^{2^{n+1}}. \end{aligned}$$

Therefore, the inequality (21) holds for all $n \geq 1$.

The estimate (21) yields

$$\begin{aligned} \lim_{|s|\to\infty} \frac{\log\log M_{x_p}(|s|)}{\log|s|} &\geq \lim_{n\to\infty} \frac{\log\log|x_p(\alpha^{-n}s_0)|}{\log|\alpha^{-n}s_0|} \\ = \lim_{n\to\infty} \frac{n\log 2 + c}{n\log|\alpha^{-1}| + \log|s_0|} &= \frac{\log 2}{\log|\alpha^{-1}|}, \end{aligned} \tag{22}$$

where c is a constant.

For any point s in the closure of D_r we have

$$|x_p(\alpha s)| \leq M_r,$$

and hence,

$$|x_p(\alpha^{-1}s)| \leq M_r^2\left(|a| + \frac{\lambda}{M_r} + \frac{|b|}{M_r}\right) \leq 2|a|M_r^2.$$

Repeating this procedure, we obtain the estimate

$$|x_p(\alpha^{-n}s)| \leq (2|a|)^{2^{n-1}} M_r^{2^n} = R_r^{2^n} \tag{23}$$

for any $n \geq 1$ and s in the closure of D_r, where $R_r = (2|a|)^{1/2} M_r$. From the inequality (23), we have

$$\log\log|x_p(\alpha^{-n}s)| \leq n\log 2 + c'$$

where c' is a constant. This yields

$$\lim_{|s|\to\infty} \frac{\log\log M_{x_p}(|s|)}{\log|s|} \le \varlimsup_{\substack{n\to\infty \\ \frac{r}{|\alpha^{-1}|}\le|s|\le r}} \frac{\log\log|x_p(\alpha^{-n}s)|}{\log|\alpha^{-n}s|} = \frac{\log 2}{\log|\alpha^{-1}|}. \tag{24}$$

From the estimates (22) and (24), the conclusion (20) is obtained. In the same way, we obtain the result for $|\alpha| > 1$. The proof is complete. □

5.2. Proof of Lemma 2

In the following, suppose that $0 < |\alpha| < 1$. Let $\{a_n\}_{n=1}^{\infty}$ be the sequence provided in (5). By Lemma 4, for each sufficiently small positive number $\epsilon \ll 1$, there exists a natural number n_ϵ large enough such that

$$|\alpha|^{\frac{n_\epsilon \log n_\epsilon}{\log 2-\epsilon\log|\alpha|}} < |a_{n_\epsilon}| < |\alpha|^{\frac{n_\epsilon \log n_\epsilon}{\log 2+\epsilon\log|\alpha|}}. \tag{25}$$

We note that the estimate (25) holds for a sequence $\{n_\epsilon\}$ of natural numbers, and n_ϵ goes to infinity as $\epsilon \to 0$. To show Lemma 2, the lower bounds in (25) are used.

Proof of Lemma 2. Let $N \ge 1$. For $\tilde{a}_1$ given, we define a sequence $\{\tilde{a}_n\}_{n=1}^{N}$ by the recurrence relation

$$\begin{aligned} \tilde{a}_{n+1} &= -\frac{a\alpha^{(n+1)\frac{(N-n-1)(N+n+2)}{2}-\frac{2(n+1)-1}{2}(N^2+N)}}{D_{n+1}} \\ &\quad \sum_{k=1}^{n} \alpha^{-k\frac{(N-k)(N+k+1)}{2}+\frac{2k-1}{2}(N^2+N)} \\ &\quad \times\ \alpha^{-(n+1-k)\frac{(N-n+k-1)(N+n-k+2)}{2}+\frac{2(n+1-k)-1}{2}(N^2+N)}\tilde{a}_k\tilde{a}_{n+1-k}. \end{aligned} \tag{26}$$

Set

$$b_1^{(N)} = \alpha^{-N-(N-1)-\cdots-4-3-2+\frac{N^2+N}{2}}\ \tilde{a}_1 = \alpha^{-\frac{(N-1)(N+2)}{2}+\frac{N^2+N}{2}}\ \tilde{a}_1 = \alpha\tilde{a}_1, \tag{27}$$

$$\begin{aligned} b_2^{(N)} &= \alpha^{-2N-2(N-1)-\cdots-2\cdot4-2\cdot3+\frac{3}{2}(N^2+N)}\ \tilde{a}_2 = \alpha^{-2\frac{(N-2)(N+3)}{2}+\frac{3}{2}(N^2+N)}\ \tilde{a}_2, \\ &\cdots \\ b_n^{(N)} &= \alpha^{-nN-n(N-1)-\cdots-n(n+1)+\frac{2n-1}{2}(N^2+N)}\ \tilde{a}_n \\ &= \alpha^{-n\frac{(N-n)(N+n+1)}{2}+\frac{2n-1}{2}(N^2+N)}\ \tilde{a}_n, \end{aligned} \tag{28}$$

$$\begin{aligned} &\cdots \\ b_{N-2}^{(N)} &= \alpha^{-(N-2)N-(N-2)(N-1)+\frac{2N-5}{2}(N^2+N)}\ \tilde{a}_{N-2} = \alpha^{-(N-2)(2N-1)+\frac{2N-5}{2}(N^2+N)}\ \tilde{a}_{N-2}, \\ b_{N-1}^{(N)} &= \alpha^{-(N-1)N+\frac{2N-3}{2}(N^2+N)}\ \tilde{a}_{N-1}, \\ b_N^{(N)} &= \alpha^{\frac{2N-1}{2}(N^2+N)}\ \tilde{a}_N. \end{aligned} \tag{29}$$

Here, we remark that the exponents of α are artificial factors to accelerate the divergence.

Using the relation (27), we have

$$b_{n+1}^{(N)} = -\frac{a}{D_{n+1}}\sum_{k=1}^{n} b_k^{(N)} b_{n+1-k}^{(N)}$$

for any n with $1 \leq n \leq N-1$. Therefore, if we set

$$b_1^{(N)} = a_1, \tag{30}$$

then

$$b_n^{(N)} = a_n \tag{31}$$

holds for every n with $1 \leq n \leq N$, and we especially have

$$b_N^{(N)} = a_N = \alpha^{\frac{2N-1}{2}(N^2+N)}\tilde{a}_N. \tag{32}$$

To prove Lemma 2, we discuss the divergence properties of the sequence $\{\tilde{a}_n\}_{n=1}^N$ from both upper and lower in the following.

Choose $N_1 > 1$ such that

$$\left|\frac{an}{D_{n+1}}\right| |\alpha|^{\frac{n}{\log 2}\log(\frac{n}{(n+1)\log_+ n}\log_+(n+1)) - \frac{1}{\log 2}\log\frac{n+1}{\log_+(n+1)}} \leq 1$$

for all $n \geq N_1$. It should be noted that $N \geq 1$ is given arbitrarily and the sequence $\{\tilde{a}_n\}_{n=1}^N$ is defined as above. We choose N as $N > N_1$. Then there exists a constant $C > 1$ such that

$$|\tilde{a}_n| \leq C^n |\tilde{a}_1|^n |\alpha|^{\frac{n}{\log 2}(\log n - \log\log_+ n) - n^2(n+1) - n\frac{N^2+N}{2}} \tag{33}$$

is satisfied for any n with $1 \leq n \leq N_1$.

Now, we show that the right-hand side of (33) gives an upper bound of $|\tilde{a}_n|$ for every n with $1 \leq n \leq N$. Let $n \geq N_1$ and assume that $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n$ satisfy the inequality (33). Since the minimum value of the function

$$\begin{aligned} g(k) &= \frac{k}{\log 2}(\log k - \log\log_+ k) + \frac{n+1-k}{\log 2}\left[\log(n+1-k) - \log\log_+(n+1-k)\right] \\ &- k\frac{(N-k)(N+k+1)}{2} - (n+1-k)\frac{(N-n+k-1)(N+n-k+2)}{2} \\ &- k^2(k+1) - (n+1-k)^2(n+2-k) \\ &= \frac{k}{\log 2}(\log k - \log\log_+ k) + \frac{n+1-k}{\log 2}\left[\log(n+1-k) - \log\log_+(n+1-k)\right] \\ &- \frac{k^2(k+1)}{2} - \frac{(n+1-k)^2(n+2-k)}{2} - (n+1)\frac{N^2+N}{2} \end{aligned}$$

occurs at $k = 1$ and $g(k)$ is monotonically increasing in the interval $1 \leq k \leq \frac{n+1}{2}$, using the relation (27), we have

$$\begin{aligned} |\tilde{a}_{n+1}| &\leq C^{n+1}\frac{|\tilde{a}_1|^{n+1}n|a|}{|D_{n+1}|}|\alpha|^{(n+1)\frac{(N-n-1)(N+n+2)}{2}} \\ &\times |\alpha|^{\frac{n}{\log 2}(\log n - \log\log_+ n) - \frac{(N-1)(N+2)}{2} - n\frac{(N-n)(N+n+1)}{2} - 2 - n^2(n+1) - (n+1)\frac{N^2+N}{2}} \\ &= C^{n+1}\left|\frac{an}{D_{n+1}}\right| |\alpha|^{\frac{n}{\log 2}\log(\frac{n}{(n+1)\log_+ n}\log_+(n+1)) - \frac{1}{\log 2}\log\frac{n+1}{\log_+(n+1)}} \\ &\times |\tilde{a}_1|^{n+1}|\alpha|^{\frac{n+1}{\log 2}(\log(n+1) - \log\log_+(n+1)) - (n+1)^2(n+2) + \frac{3n^2+5n}{2} - (n+1)\frac{N^2+N}{2}}, \\ &\leq C^{n+1}|\tilde{a}_1|^{n+1}|\alpha|^{\frac{n+1}{\log 2}(\log(n+1) - \log\log_+(n+1)) - (n+1)^2(n+2) - (n+1)\frac{N^2+N}{2}}, \end{aligned}$$

by which we have the inequality (33) for $\tilde{a}_{n+1}$. Therefore, the inequality (33) holds for any n with $1 \le n \le N$.

Let $\epsilon > 0$ be sufficiently small. From the lower bound of the estimate (25), by choosing n_ϵ to be large enough if necessary, it follows that

$$|a_{n\epsilon}| \ge |2aa_1|^{n_\epsilon} |\alpha|^{(1+\epsilon)\frac{n_\epsilon \log n_\epsilon}{\log 2}}. \tag{34}$$

To show the estimate (13), we set $N = n_\epsilon$ and let $b_1^{(N)} = a_1$ as in (30). Let $n+1 = N$. Since

$$a_N = \alpha^{\frac{2N-1}{2}(N^2+N)} \tilde{a}_N, \quad a_1 = \alpha \tilde{a}_1$$

by the relations (32) and (27), we have

$$|\tilde{a}_{n+1}| \ge |2a\tilde{a}_1|^{n+1} |\alpha|^{(1+\epsilon)\frac{(n+1)\log(n+1)}{\log 2} - \frac{(n+1)^2(n+2)}{2} + (n+1) - n\frac{N^2+N}{2}}. \tag{35}$$

From the relation (27) and the estimate (33), it follows that

$$\begin{aligned} & \left| \tilde{a}_{n+1} + \frac{2a\alpha^{(n+1)\frac{(N-n-1)(N+n+2)}{2}}}{D_{n+1}} \alpha^{-\frac{(N-1)(N+2)}{2} - n\frac{(N-n)(N+n+1)}{2}} \tilde{a}_1 \tilde{a}_n \right| \\ = & \left| \frac{a\alpha^{(n+1)\frac{(N-n-1)(N+n+2)}{2}}}{D_{n+1}} \right| \sum_{k=2}^{n-1} |\alpha|^{-k\frac{(N-k)(N+k+1)}{2}} |\alpha|^{-(n+1-k)\frac{(N-n+k-1)(N+n-k+2)}{2}} |\tilde{a}_k \tilde{a}_{n+1-k}| \\ \le & \frac{(n-2)|a|C^{n+1}|\tilde{a}_1|^{n+1}}{|D_{n+1}|} |\alpha|^{-4 - \frac{(n-1)^2 n}{2} - (n+1)\frac{N^2+N}{2} + \frac{n-1}{\log 2}(\log(n-1) - \log\log_+(n-1))}. \end{aligned} \tag{36}$$

Here, it should be noted that $0 < |\alpha| < 1$. Dividing (37) by $\tilde{a}_{n+1}$ and taking into account of (35), we have

$$\begin{aligned} & \left| 1 + \frac{2a\tilde{a}_1\alpha^{(n+1)\frac{(N-n-1)(N+n+2)}{2}}}{D_{n+1}} \alpha^{\frac{-(N-1)(N+2)}{2} - n\frac{(N-n)(N+n+1)}{2}} \frac{\tilde{a}_n}{\tilde{a}_{n+1}} \right| \\ \le & \left(\frac{1}{|2a|} \right)^{n+1} \frac{(n-2)|a|C^{n+1}}{|D_{n+1}|} \times |\alpha|^{\frac{5n^2-n}{2} - 5} \\ \times & |\alpha|^{\frac{n-1}{\log 2}(\log(n-1) - \log\log_+(n-1)) - (1+\epsilon)\frac{(n+1)\log(n+1)}{\log 2}}. \end{aligned}$$

Noticing that $|D_{n+1}|^{-1} \sim |\alpha|^{n+1}$, the right-hand side of the above inequality is less than

$$\left(\frac{C}{|2a|} \right)^{n+1} |\alpha|^{2n^2} \times |\alpha|^{\frac{n-1}{\log 2}(\log(n-1) - \log\log_+(n-1)) - (1+\epsilon)\frac{(n+1)\log(n+1)}{\log 2}}.$$

Since $|\alpha| < 1$, taking N_1 sufficiently large if necessary, we can choose a constant θ with $0 < \theta < 1$ such that

$$\left| 1 + \frac{2a\tilde{a}_1\alpha^{(n+1)\frac{(N-n-1)(N+n+2)}{2}}}{D_{n+1}} \alpha^{-\frac{(N-1)(N+2)}{2} - n\frac{(N-n)(N+n+1)}{2}} \frac{\tilde{a}_n}{\tilde{a}_{n+1}} \right| \le \theta^n. \tag{37}$$

From (35) and (37), we have

$$|\tilde{a}_n| > \frac{1-\theta^n}{|2a\tilde{a}_1|} |2a\tilde{a}_1|^{n+1} |\alpha|^{\frac{3n^2+5n}{2} + (1+\epsilon)\frac{(n+1)\log(n+1)}{\log 2} - \frac{(n+1)^2(n+2)}{2} + (n+1) - n\frac{N^2+N}{2}},$$

from which it follows that

$$|\tilde{a}_n| \geq |2a\tilde{a}_1|^n |\alpha|^{(1+\epsilon)\frac{n\log n}{\log 2} - \frac{n^2(n+1)}{2} + n - (n-1)\frac{N^2+N}{2}}.$$

Therefore, the inequality (35) holds for $n = N-1 = n_\epsilon - 1$.

Since

$$\tilde{a}_{N-1} = \alpha^{(N-1)N - \frac{2N-3}{2}(N^2+N)} a_{N-1}, \quad \tilde{a}_1 = \alpha^{-1} a_1$$

by the relations (29), (31), and (27), from the above result, we obtain the inequality (34) for $n = n_\epsilon - 1$. Next, set $N = n_\epsilon - 1$ and let $n+1 = N$. Then, in the same way as above, we obtain the inequality (34) for $n = n_\epsilon - 2$. Repeat this procedure. We obtain

$$|a_n| \geq |2aa_1|^n |\alpha|^{(1+\epsilon)\frac{n\log n}{\log 2}}$$

for any n satisfying $N_1 \leq n \leq n_\epsilon$. Since n_ϵ goes to infinity as $\epsilon \to 0$, the estimate (13) is obtained for any $n \geq N_1$. The proof of Lemma 2 is complete. □

In the same way as above, we obtain Lemma 2 for $|\alpha| > 1$.

6. Concluding Remarks

In this paper, we have clarified that the Taylor expansions of analytic functions appearing in the linearization of Hénon maps at hyperbolic fixed points do not successfully approximate invariant manifolds, such as stable and unstable manifolds, when higher-order terms are truncated, by proving that the computational complexity for the approximations by the Taylor expansions grows exponentially. The key findings are to obtain the upper and lower bounds to the absolute values of the coefficients in the Taylor expansions.

One of the future works is to clarify that the holomorphic maps defined by taking conjugates by iteration maps of the Hénon map and of the derivative give much better approximations of invariant manifolds than the Taylor expansions under some assumptions. The other is to show that the asymptotic expansion gives us much better performance as well.

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