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Some New Boole-Type Inequalities via Modified Convex Functions with Their Applications and Computational Analysis

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Abstract

In numerical analysis, the Boole's formula serves as a pivotal tool for approximating definite integrals. The approximation of the definite integrals has a big role in numerical methods for differential equations; in particular, in the finite volume method, we need to use the best approximation of the integrals to obtain better results. This paper presents a rigorous proof of integral inequalities for first-time differentiable *s*-convex functions in the second sense. This paper has two main goals. The first is that the use of *s*-convex function extends the results for convex functions which cover a large class of functions and the second is the best approximation. To prove the main inequalities, we drive integral identity for differentiable functions. Then, with the help of this identity, we prove the error bounds of Boole's formula for differentiable *s*-convex functions in the second sense. Some new midpoint-type inequalities for generalized convex functions are also given which can help us in finding better error bounds for midpoint integration formulas compared to the existing ones. Moreover, we provide some applications to quadrature formulas and special means for the real numbers of these newly established inequalities. Furthermore, we present numerical examples and computational analysis that show that these newly established inequalities are numerically valid.

Keywords: Boole's formula-type inequality; modified convex function; quadrature formulae; midpoint formula; error bounds

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1. Introduction

The concept of numerical integration was formally introduced in 1915 in a publication titled *A Course in Interpolation and Numerical Integration for the Mathematical Laboratory* by David Gibb [1]. In recent decades, numerical integration has become an essential component of scientific computing, engineering, and data analysis. The term *quadrature* is a traditional mathematical expression for finding areas. By constructing various interpolating polynomials, a broad class of quadrature rules can be derived. One of the simplest methods assumes the interpolating function to be constant (a zero-degree polynomial), known as

the midpoint or rectangle rule. When the interpolating function is linear, the corresponding method is called the trapezoidal rule. If the interpolating polynomial is of the second degree, the resulting method is Simpson's rule, named after mathematician Thomas Simpson (1710–1761), whose simplest form is known as Simpson's 1/3 rule.

To achieve higher accuracy with smaller error bounds, a more advanced form known as the Boole rule is used, also known as Simpson's 2/45 or third Simpson rule. Named after George Boole, this method extends Simpson's approach by using a higher-degree polynomial, thereby providing improved precision in the numerical approximation of integrals.

$$\int_g^h \mathcal{Y}(\omega)d\omega = \frac{2h}{45}\left[7\mathcal{Y}(g) + 32\mathcal{Y}\left(\frac{3g+h}{4}\right) + 12\mathcal{Y}\left(\frac{g+h}{2}\right) + 32\mathcal{Y}\left(\frac{g+3h}{4}\right) + 7\mathcal{Y}(h)\right] + E(\mathcal{Y}).$$

where $E(\mathcal{Y})$ is an error term. To know more about numerical integration and its applications one can refer to [2]. The error bound for Boole's rule approximation is described as

$$E(\mathcal{Y}) = -\frac{8(h-g)^8}{945}M, \left|\mathcal{Y}^{(vi)}(\omega)\right| \leq M.$$

Convexity is a property of a function where any line segment joining two points on its graph will always lie above or on the graph. Convexity plays a vital role in calculus, optimization, and mathematical analysis. Additional results and related inequalities for convex functions are discussed in [3–6]. The formal definition is as follows:

Definition 1 ([7]). *A function $\mathcal{Y} : [\mathrm{g}, \mathrm{h}] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$\mathcal{Y}(\mu\mathrm{g} + (1-\mu)\mathrm{h}) \leq \mu\mathcal{Y}(\mathrm{g}) + (1-\mu)\mathcal{Y}(\mathrm{h}), \tag{1}$$

for each value of $\mu \in [0, 1]$.

Convex functions are involved in important mathematical inequalities known as Hermite–Hadamard-type inequalities, which are expressed as

Theorem 1. *For a convex function $\mathcal{Y} : [\mathrm{g}, \mathrm{h}] \subseteq \mathbb{R} \to \mathbb{R}^+$ the interval $[\mathrm{g}, \mathrm{h}]$. Then, we have*

$$\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right) \leq \frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega)d\omega \leq \frac{\mathcal{Y}(\mathrm{g}) + \mathcal{Y}(\mathrm{h})}{2}. \tag{2}$$

The aforementioned inequalities likewise apply to concave functions, although they work in the other way. Kirmaci [8] introduced the following midpoint-type inequality along with its error bounds for differentiable convex mappings, one of which is as follows:

Theorem 2. *Let $\mathcal{Y} : [\mathrm{g}, \mathrm{h}] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on (g, h); then, we have*

$$\left|\frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega)d\omega - \mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right)\right| \leq \frac{(\mathrm{h}-\mathrm{g})}{8}\left[\left|\mathcal{Y}'(\mathrm{g})\right| + \left|\mathcal{Y}'(\mathrm{h})\right|\right]. \tag{3}$$

In the literature, a Simpson-type inequality is outlined as follows:

Theorem 3. *Let $\mathcal{Y} : [\mathrm{g}, \mathrm{h}] \subset \mathbb{R} \to \mathbb{R}$ be a four-times differentiable mapping on (g, h); then, we have*

$$\left|\frac{1}{6}\left(\mathcal{Y}(\mathrm{g}) + 4\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right) + \mathcal{Y}(\mathrm{h})\right) - \frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega)d\omega\right| \leq \frac{(\mathrm{h}-\mathrm{g})^{(4)}}{2880}\left|\left|\mathcal{Y}^{(4)}\right|\right|_\infty,$$

where $\left|\left|\mathcal{Y}^{(4)}\right|\right|_\infty = \sup_{\omega\in[\mathrm{g},\mathrm{h}]}\left|\mathcal{Y}^{(4)}(\omega)\right| < \infty.$

The study of error bounds for numerical quadrature formulas has attracted considerable attention in the literature. In their pioneering work, Dragomir and Agarwal [9] extended the classical inequality and obtained sharp estimates for the error term of the trapezoidal rule when the integrand is assumed to be a differentiable convex function. Building upon this framework, Kirmaci [8] focused on the midpoint rule and established corresponding error bounds under the same convexity setting. These initial contributions opened up a new direction for research, encouraging many scholars to investigate error estimates for various quadrature rules under broader assumptions and by means of different analytical tools. Subsequently, Alomari and Dragomir [10,11] considered the Simpson formula and derived error bounds not only for convex functions but also for the wider class of generalized convex functions. Their work also highlighted some practical applications, which emphasized the utility of such results. Parallel to these studies, other authors extended Simpson-type inequalities to multivariate settings. For example, in [12,13], the error bounds for Simpson's rule were analyzed for functions of two variables, thereby enriching the scope of quadrature error analysis in higher dimensions. Furthermore, Du [14] utilized a general form of convexity in order to establish new estimates for Simpson's formula, broadening the applicability of such inequalities beyond the standard convex case.

In a more comprehensive sense, a variety of fundamental inequalities, including those of a Hermite–Hadamard, Ostrowski, midpoint, Simpson, and trapezoidal type, provide the theoretical basis for deriving error bounds of integration formulas. Numerous results of this nature can be found in the works of [15–20], where these classical inequalities are employed as essential tools to analyze and refine the accuracy of numerical integration methods.

In 1979, Breckner introduced the notion of generalized convex functions [21], marking a substantial advancement in the field of mathematical analysis. This work was later expanded upon by Hudzik and collaborators [22], who investigated the concept of s-convexity in the first sense. Pycia [23] subsequently provided rigorous demonstrations of Breckner's propositions in 2001. Analytical investigations have established that for a parameter $s \in (0,1)$, the condition of s-convexity in the second sense imposes a more restrictive and potent constraint than classical convexity, frequently producing sharper and more refined results. Furthermore, this framework provides a natural generalization of conventional convex functions, as the standard case is recovered by setting $s = 1$. The class is formally characterized in the following manner:

Definition 2 ([22]). *Let $\mathcal{Y} : [0,\infty) \subset \mathbb{R}^+ \to \mathbb{R}$ be s-convex function in the second sense if*

$$\mathcal{Y}(\mu \mathrm{g} + (1-\mu)\mathrm{h}) \leq \mu^s \mathcal{Y}(\mathrm{g}) + (1-\mu)^s \mathcal{Y}(\mathrm{h}),\ \forall\, \mathrm{g},\mathrm{h} \in [0,\infty),\ \mu \in [0,1], \tag{4}$$

for $s \in (0,1]$. When $s = 1$ in the above-outlined inequality (4), then we obtained inequality (1).

In [24], the Hermite–Hadamard inequality for s-convex functions is stated as

Theorem 4. *If $\mathcal{Y} : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $0 < s < 1$, let $\mathrm{g},\mathrm{h} \in [0,\infty)$, $\mathrm{g} < \mathrm{h}$. If $\mathcal{Y} \in L^1[\mathrm{g},\mathrm{h}]$, then the following inequality holds:*

$$2^{s-1}\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right) \leq \frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega)d\omega \leq \frac{\mathcal{Y}(\mathrm{g})+\mathcal{Y}(\mathrm{h})}{s+1}. \tag{5}$$

Refer to [25–32] for more recent results and insights regarding Hadamard's inequality.

The main motivation for using s-convexity is that it broadens the class of admissible functions for which integral inequalities can be established. Many functions that fail to satisfy the classical convexity condition may still satisfy the s-convexity condition for some $s \in (0,1)$. This makes it possible to derive more general and sharper inequalities that

depend on the parameter s. The key advantages of employing s-convexity are summarized as follows:

- It provides a unified framework that includes the classical convex case as a special instance when $s = 1$;
- The parameter s offers flexibility to obtain sharper and more refined inequality bounds;
- It allows the extension of classical results such as Hermite–Hadamard-, Simpson-, and Boole-type inequalities and other Newton–Cotes formulae-type inequalities to a broader class of functions;
- The derived inequalities often yield improved error bounds in numerical integration and approximation theory.

Motivated by recent studies, a new identity for differentiable mappings based on generalized convexity is established. This identity is then used to derive Boole-type inequalities for differentiable convex functions, along with their applications. Numerical examples are presented to illustrate the validity of the results. This generalization unifies previous results and deepens the understanding of Boole-type inequalities for numerical integration and error estimation.

The structure of the paper is as follows. Section 2 presents the main results for differentiable s-convex functions in the second sense. Applications to numerical integration and special means are given in Section 3. Section 4 provides computational examples illustrating the new inequalities. Concluding remarks with future directions are given in Section 5.

2. Boole-Type Inequality for Modified Convex Function

We rely on the following Lemma to establish our main theorems for s-convex functions.

Lemma 1. *Consider $\mathcal{Y} : [\mathrm{g}, \mathrm{h}] \to \mathbb{R}$ to be a differentiable function whose derivative is continuous (g, h) and $\mathcal{Y}' \in L^1[\mathrm{g}, \mathrm{h}]$. Then, the subsequent equality is satisfied:*

$$\begin{aligned}
&\frac{1}{90}\left[7\mathcal{Y}(\mathrm{g}) + 32\mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{4}\right) + 12\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right) + 32\mathcal{Y}\left(\frac{\mathrm{g}+3\mathrm{h}}{4}\right) + 7\mathcal{Y}(\mathrm{h})\right] - \frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega)d\omega \\
&= \frac{\mathrm{h}-\mathrm{g}}{16}\left[\int_0^1 \left(\mu - \frac{28}{90}\right)\left[\mathcal{Y}'\left(\frac{4-\mu}{4}\mathrm{g} + \frac{\mu}{4}\mathrm{h}\right)\right]d\mu + \int_0^1 \left(\mu - \frac{44}{60}\right)\left[\mathcal{Y}'\left(\frac{3-\mu}{4}\mathrm{g} + \frac{1+\mu}{4}\mathrm{h}\right)\right]d\mu\right. \\
&\left. + \int_0^1 \left(\mu - \frac{24}{90}\right)\left[\mathcal{Y}'\left(\frac{2-\mu}{4}\mathrm{g} + \frac{2+\mu}{4}\mathrm{h}\right)\right]d\mu + \int_0^1 \left(\mu - \frac{62}{90}\right)\left[\mathcal{Y}'\left(\frac{1-\mu}{4}\mathrm{g} + \frac{3+\mu}{4}\mathrm{h}\right)\right]d\mu\right].
\end{aligned}$$

Proof. By considering the basic rules of integration, we have

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^1 \left(\mu - \tfrac{28}{90}\right)\left[\mathcal{Y}'\left(\tfrac{4-\mu}{4}\mathrm{g} + \tfrac{\mu}{4}\mathrm{h}\right)\right]d\mu \\
&= \left.\frac{4\left(\mu - \frac{28}{90}\right)\mathcal{Y}\left(\frac{4-\mu}{4}\mathrm{g} + \frac{\mu}{4}\mathrm{h}\right)}{\mathrm{h}-\mathrm{g}}\right|_0^1 - \frac{4}{\mathrm{h}-\mathrm{g}}\int_0^1 \mathcal{Y}\left(\tfrac{4-\mu}{4}\mathrm{g} + \tfrac{\mu}{4}\mathrm{h}\right)d\mu \\
&= \frac{4}{\mathrm{h}-\mathrm{g}}\left[\frac{62}{90}\mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{4}\right) + \frac{28}{90}\mathcal{Y}(\mathrm{g})\right] - \frac{4}{\mathrm{h}-\mathrm{g}}\int_0^1 \mathcal{Y}\left(\tfrac{4-\mu}{4}\mathrm{g} + \tfrac{\mu}{4}\mathrm{h}\right)d\mu.
\end{aligned} \tag{6}$$

Similarly,

$$\begin{aligned}
\mathcal{I}_2 &= \int_0^1 \left(\mu - \tfrac{44}{60}\right)\left[\mathcal{Y}'\left(\tfrac{3-\mu}{4}\mathrm{g} + \tfrac{1+\mu}{4}\mathrm{h}\right)\right]d\mu \\
&= \frac{4}{\mathrm{h}-\mathrm{g}}\left[\frac{44}{60}\mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{4}\right) + \frac{16}{90}\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right)\right] - \frac{4}{\mathrm{h}-\mathrm{g}}\int_0^1 \mathcal{Y}\left(\tfrac{3-\mu}{4}\mathrm{g} + \tfrac{1+\mu}{4}\mathrm{h}\right)d\mu,
\end{aligned} \tag{7}$$

$$\begin{aligned}\mathcal{I}_3 &= \int_0^1 \left(\mu - \tfrac{24}{90}\right)\left[\mathcal{Y}'\left(\tfrac{2-\mu}{4}\mathrm{g} + \tfrac{2+\mu}{4}\mathrm{h}\right)\right]d\mu \\ &= \tfrac{4}{\mathrm{h}-\mathrm{g}}\left[\tfrac{66}{90}\mathcal{Y}\left(\tfrac{\mathrm{g}+3\mathrm{h}}{4}\right) + \tfrac{24}{90}\mathcal{Y}\left(\tfrac{\mathrm{g}+\mathrm{h}}{2}\right)\right] - \tfrac{4}{\mathrm{h}-\mathrm{g}}\int_0^1 \mathcal{Y}\left(\tfrac{2-\mu}{4}\mathrm{g} + \tfrac{2+\mu}{4}\mathrm{h}\right)d\mu,\end{aligned} \tag{8}$$

and

$$\begin{aligned}\mathcal{I}_4 &= \int_0^1 \left(\mu - \tfrac{62}{90}\right)\left[\mathcal{Y}'\left(\tfrac{1-\mu}{4}\mathrm{g} + \tfrac{3+\mu}{4}\mathrm{h}\right)\right]d\mu \\ &= \tfrac{4}{\mathrm{h}-\mathrm{g}}\left[\tfrac{28}{90}\mathcal{Y}(\mathrm{h}) + \tfrac{62}{90}\mathcal{Y}\left(\tfrac{\mathrm{g}+3\mathrm{h}}{4}\right)\right] - \tfrac{4}{\mathrm{h}-\mathrm{g}}\int_0^1 \mathcal{Y}\left(\tfrac{1-\mu}{4}\mathrm{g} + \tfrac{3+\mu}{4}\mathrm{h}\right)d\mu.\end{aligned} \tag{9}$$

Thus, we obtain following equality by adding (6) and (9) and a change in variable $\omega = \frac{4-\mu}{4}\mathrm{g} + \frac{\mu}{4}\mathrm{h}$.

$$\begin{aligned}&\frac{\mathrm{h}-\mathrm{g}}{16}[\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4] \\ &= \frac{1}{90}\left[7\mathcal{Y}(\mathrm{g}) + 32\mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{4}\right) + 12\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right) + 32\mathcal{Y}\left(\frac{\mathrm{g}+3\mathrm{h}}{4}\right) + 7\mathcal{Y}(\mathrm{h})\right] - \frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega)d\omega.\end{aligned}$$

The desired equality is obtained. □

Theorem 5. *Assume that the premises established in Lemma 1 are fulfilled. If $|\mathcal{Y}'|$ is s-convex on $[\mathrm{g}, \mathrm{h}]$, for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned}&\left|\tfrac{1}{90}\left[7\mathcal{Y}(\mathrm{g}) + 32\mathcal{Y}\left(\tfrac{3\mathrm{g}+\mathrm{h}}{4}\right) + 12\mathcal{Y}\left(\tfrac{\mathrm{g}+\mathrm{h}}{2}\right) + 32\mathcal{Y}\left(\tfrac{\mathrm{g}+3\mathrm{h}}{4}\right) + 7\mathcal{Y}(\mathrm{h})\right] - \tfrac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega)d\omega\right| \\ &\leq \tfrac{(\mathrm{h}-\mathrm{g})}{16}\tfrac{2^{1-2s}}{(45)^{s+2}(s+1)(s+2)}\left(45^{s+1}\left(3^{s+1} + 7\,4^{s+1} - 1\right)s - 19\,4^{s+2}45^{s+1} - 47\,45^{s+1}\right. \\ &\left. -133\,135^{s+1} + 14^{s+2} + 78^{s+2} - 90^{s+2} + 102^{s+2} + 166^{s+2}\right)[|\mathcal{Y}'(\mathrm{g})| + |\mathcal{Y}'(\mathrm{h})|].\end{aligned} \tag{10}$$

Proof. From Lemma 1 and taking the modulus, we obtain

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathcal{Y}(\mathrm{g}) + 32\mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{4}\right) + 12\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right) + 32\mathcal{Y}\left(\frac{\mathrm{g}+3\mathrm{h}}{4}\right) + 7\mathcal{Y}(\mathrm{h})\right] - \frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega)d\omega\right| \\ &\leq \frac{\mathrm{h}-\mathrm{g}}{16}\left[\int_0^1 \left|\mu - \frac{28}{90}\right|\left|\mathcal{Y}'\left(\frac{4-\mu}{4}\mathrm{g} + \frac{\mu}{4}\mathrm{h}\right)\right|d\mu + \int_0^1 \left|\mu - \frac{44}{60}\right|\left|\mathcal{Y}'\left(\frac{3-\mu}{4}\mathrm{g} + \frac{1+\mu}{4}\mathrm{h}\right)\right|d\mu\right. \\ &\left. + \int_0^1 \left|\mu - \frac{24}{90}\right|\left|\mathcal{Y}'\left(\frac{2-\mu}{4}\mathrm{g} + \frac{2+\mu}{4}\mathrm{h}\right)\right|d\mu + \int_0^1 \left|\mu - \frac{62}{90}\right|\left|\mathcal{Y}'\left(\frac{1-\mu}{4}\mathrm{g} + \frac{3+\mu}{4}\mathrm{h}\right)\right|d\mu\right] \\ &= \frac{\mathrm{h}-\mathrm{g}}{16}\left[\int_0^1 \left|\mu - \frac{28}{90}\right|\left|\mathcal{Y}'\left(\frac{4-\mu}{4}\mathrm{g} + \left(1 - \frac{4-\mu}{4}\right)\mathrm{h}\right)\right|d\mu\right. \\ &+ \int_0^1 \left|\mu - \frac{44}{60}\right|\left|\mathcal{Y}'\left(\frac{3-\mu}{4}\mathrm{g} + \left(1 - \frac{3-\mu}{4}\right)\mathrm{h}\right)\right|d\mu \\ &+ \int_0^1 \left|\mu - \frac{24}{90}\right|\left|\mathcal{Y}'\left(\frac{2-\mu}{4}\mathrm{g} + \left(1 - \frac{2-\mu}{4}\right)\mathrm{h}\right)\right|d\mu \\ &\left. + \int_0^1 \left|\mu - \frac{62}{90}\right|\left|\mathcal{Y}'\left(\frac{1-\mu}{4}\mathrm{g} + \left(1 - \frac{1-\mu}{4}\right)\mathrm{h}\right)\right|d\mu\right].\end{aligned}$$

Now, using the s-convexity of $|\mathcal{Y}'|$ in the second sense, we have

$$
\begin{aligned}
&\left|\tfrac{1}{90}\left[7\mathcal{Y}(\mathrm{g})+32\mathcal{Y}\left(\tfrac{3\mathrm{g}+\mathrm{h}}{4}\right)+32\mathcal{Y}\left(\tfrac{\mathrm{g}+3\mathrm{h}}{4}\right)+7\mathcal{Y}(\mathrm{h})+12\mathcal{Y}\left(\tfrac{\mathrm{g}+\mathrm{h}}{2}\right)\right]-\tfrac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}}\mathcal{Y}(\omega)d\omega\right|\\
&\leq\tfrac{\mathrm{h}-\mathrm{g}}{16}\left[\int_0^1\left|\mu-\tfrac{28}{90}\right|\left(\tfrac{4-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{g})|+\int_0^1\left|\mu-\tfrac{28}{90}\right|\left(1-\tfrac{4-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{h})|\right.\\
&+\int_0^1\left|\mu-\tfrac{44}{60}\right|\left(\tfrac{3-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{g})|+\int_0^1\left|\mu-\tfrac{44}{60}\right|\left(1-\tfrac{3-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{h})|\\
&+\int_0^1\left|\mu-\tfrac{24}{90}\right|\left(\tfrac{2-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{g})|+\int_0^1\left|\mu-\tfrac{24}{90}\right|\left(1-\tfrac{2-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{h})|\\
&\left.+\int_0^1\left|\mu-\tfrac{62}{90}\right|\left(\tfrac{1-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{g})|+\int_0^1\left|\mu-\tfrac{62}{90}\right|\left(1-\tfrac{1-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{h})|\right]\\
&=\tfrac{\mathrm{h}-\mathrm{g}}{16}\left[\left(\int_0^1\left|\mu-\tfrac{28}{90}\right|\left(\tfrac{4-\mu}{4}\right)^s d\mu+\int_0^1\left|\mu-\tfrac{44}{60}\right|\left(\tfrac{3-\mu}{4}\right)^s d\mu\right.\right.\\
&+\int_0^1\left|\mu-\tfrac{24}{90}\right|\left(\tfrac{2-\mu}{4}\right)^s d\mu+\int_0^1\left|\mu-\tfrac{62}{90}\right|\left(\tfrac{1-\mu}{4}\right)^s d\mu\Big)|\mathcal{Y}'(\mathrm{g})|\\
&+\Big(\int_0^1\left|\mu-\tfrac{28}{90}\right|\left(1-\tfrac{4-\mu}{4}\right)^s d\mu+\int_0^1\left|\mu-\tfrac{44}{60}\right|\left(1-\tfrac{3-\mu}{4}\right)^s d\mu\\
&\left.\left.+\int_0^1\left|\mu-\tfrac{24}{90}\right|\left(1-\tfrac{2-\mu}{4}\right)^s d\mu+\int_0^1\left|\mu-\tfrac{62}{90}\right|\left(1-\tfrac{1-\mu}{4}\right)^s d\mu\Big)|\mathcal{Y}'(\mathrm{h})|\right].
\end{aligned}
\tag{11}
$$

Here, we used the following equalities:

$$
\begin{aligned}
E_1(s) &= \int_0^1\left|\mu-\frac{28}{90}\right|\left(\frac{4-\mu}{4}\right)^s d\mu\\
&= \int_0^{\frac{28}{90}}\left(\frac{28}{90}-\mu\right)\left(\frac{4-\mu}{4}\right)^s d\mu+\int_{\frac{28}{90}}^1\left(\mu-\frac{28}{90}\right)\left(\frac{4-\mu}{4}\right)^s d\mu\\
&= \frac{4^{-s}45^{-s-2}\left(45^{s+1}\left(2^{2s+3}(7s-76)-3^{s+1}(31s+197)\right)+2^{s+3}83^{s+2}\right)}{(s+1)(s+2)},\\
E_2(s) &= \int_0^1\left|\mu-\frac{62}{90}\right|\left(1-\frac{1-\mu}{4}\right)^s d\mu\\
&= \frac{4^{-s}45^{-s-2}\left(45^{s+1}\left(2^{2s+3}(7s-76)-3^{s+1}(31s+197)\right)+2^{s+3}83^{s+2}\right)}{(s+1)(s+2)},\\
E_3(s) &= \int_0^1\left|\mu-\frac{44}{60}\right|\left(\frac{3-\mu}{4}\right)^s d\mu\\
&= \frac{4^{-s}15^{-s-2}\left(2^{s+3}17^{s+2}-15^{s+1}\left(2^{s+2}(2s+19)+3^{s+1}(23-11s)\right)\right)}{(s+1)(s+2)},\\
E_4(s) &= \int_0^1\left|\mu-\frac{24}{90}\right|\left(1-\frac{2-\mu}{4}\right)^s d\mu\\
&= \frac{4^{-s}15^{-s-2}\left(2^{s+3}17^{s+2}-15^{s+1}\left(2^{s+2}(2s+19)+3^{s+1}(23-11s)\right)\right)}{(s+1)(s+2)},\\
E_5(s) &= \int_0^1\left|\mu-\frac{24}{90}\right|\left(\frac{2-\mu}{4}\right)^s d\mu\\
&= \frac{4^{-s}15^{-s-2}\left(15^{s+1}\left(-11s+2^{s+2}(2s-11)-37\right)+2^{s+3}13^{s+2}\right)}{(s+1)(s+2)},\\
E_6(s) &= \int_0^1\left|\mu-\frac{44}{60}\right|\left(1-\frac{3-\mu}{4}\right)^s d\mu\\
&= \frac{4^{-s}15^{-s-2}\left(15^{s+1}\left(-11s+2^{s+2}(2s-11)-37\right)+2^{s+3}13^{s+2}\right)}{(s+1)(s+2)},\\
E_7(s) &= \int_0^1\left|\mu-\frac{62}{90}\right|\left(\frac{1-\mu}{4}\right)^s d\mu\\
&= \frac{4^{-s}45^{-s-2}\left(45^{s+1}(31s+17)+2^{s+3}7^{s+2}\right)}{(s+1)(s+2)},\\
E_8(s) &= \int_0^1\left|\mu-\frac{28}{90}\right|\left(1-\frac{4-\mu}{4}\right)^s d\mu\\
&= \frac{4^{-s}45^{-s-2}\left(45^{s+1}(31s+17)+2^{s+3}7^{s+2}\right)}{(s+1)(s+2)}.
\end{aligned}
$$

Putting the above calculations into inequality (11), we then obtain

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathcal{Y}(\mathrm{g})+32\mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{4}\right)+12\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right)+32\mathcal{Y}\left(\frac{\mathrm{g}+3\mathrm{h}}{4}\right)+7\mathcal{Y}(\mathrm{h})\right]-\frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}}\mathcal{Y}(\omega)d\omega\right|\\&\leq\frac{(\mathrm{h}-\mathrm{g})}{16}\frac{2^{1-2s}}{(45)^{s+2}(s+1)(s+2)}\Big(45^{s+1}\Big(3^{s+1}+7\,4^{s+1}-1\Big)s-19\,4^{s+2}45^{s+1}-47\,45^{s+1}\\&-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\Big)\left[\left|\mathcal{Y}'(\mathrm{g})\right|+\left|\mathcal{Y}'(\mathrm{h})\right|\right].\end{aligned}$$

Thus, the proof of Theorem 5 has been finalized. □

Corollary 1. *By setting $s=1$ in Theorem 5, the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{90}\left[7\mathcal{Y}(\mathrm{g})+32\mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{4}\right)+12\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right)+32\mathcal{Y}\left(\frac{\mathrm{g}+3\mathrm{h}}{4}\right)+7\mathcal{Y}(\mathrm{h})\right]-\frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}}\mathcal{Y}(\omega)d\omega\right|\\&\leq\frac{239(\mathrm{h}-\mathrm{g})}{6480}\left[\left|\mathcal{Y}'(\mathrm{g})\right|+\left|\mathcal{Y}'(\mathrm{h})\right|\right].\end{aligned}$$

Corollary 2. *In Theorem 5, if $\mathcal{Y}(\mathrm{g})=\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right)=\mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{4}\right)=\mathcal{Y}\left(\frac{\mathrm{g}+3\mathrm{h}}{4}\right)=\mathcal{Y}(\mathrm{h})$, then we have the following new midpoint-type inequality for the s-convex function:*

$$\begin{aligned}&\left|\mathcal{Y}\left(\tfrac{\mathrm{g}+\mathrm{h}}{2}\right)-\tfrac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}}\mathcal{Y}(\omega)d\omega\right|\\&\leq\tfrac{(\mathrm{h}-\mathrm{g})}{16}\tfrac{2^{1-2s}}{(45)^{s+2}(s+1)(s+2)}\big(45^{s+1}\big(3^{s+1}+7\,4^{s+1}-1\big)s-19\,4^{s+2}45^{s+1}-47\,45^{s+1}\\&-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\big)\left[\left|\mathcal{Y}'(\mathrm{g})\right|+\left|\mathcal{Y}'(\mathrm{h})\right|\right].\end{aligned}\tag{12}$$

Corollary 3. *By setting $s=1$ in Corollary 2, we obtain*

$$\left|\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right)-\frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}}\mathcal{Y}(\omega)d\omega\right|\leq\frac{239(\mathrm{h}-\mathrm{g})}{6480}\left[\left|\mathcal{Y}'(\mathrm{g})\right|+\left|\mathcal{Y}'(\mathrm{h})\right|\right].\tag{13}$$

Remark 1. *We observe that the inequality (13) gave better bounds for the midpoint compared to the inequality (3).*

Theorem 6. *Assume that the premises established in Lemma 1 are fulfilled. If $|\mathcal{Y}'|^q$ is s-convex on $[\mathrm{g},\mathrm{h}]$, for some fixed $s\in(0,1]$ and $p,q>1$, then the following inequality holds:*

$$\begin{aligned}&\left|\tfrac{1}{90}\left[7\mathcal{Y}(\mathrm{g})+32\mathcal{Y}\left(\tfrac{3\mathrm{g}+\mathrm{h}}{4}\right)+12\mathcal{Y}\left(\tfrac{\mathrm{g}+\mathrm{h}}{2}\right)+32\mathcal{Y}\left(\tfrac{\mathrm{g}+3\mathrm{h}}{4}\right)+7\mathcal{Y}(\mathrm{h})\right]-\tfrac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}}\mathcal{Y}(\omega)d\omega\right|\\&\leq\tfrac{\mathrm{h}-\mathrm{g}}{16}\Bigg[\left(\tfrac{\left(\frac{14}{45}\right)^{p+1}+\left(\frac{31}{45}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\tfrac{4-3^{1+s}4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{g})|^q+\tfrac{4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}}\\&+\left(\tfrac{\left(\frac{4}{15}\right)^{p+1}+\left(\frac{11}{15}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\tfrac{4^{-s}\left(3^{1+s}-2^{1+s}\right)}{1+s}|\mathcal{Y}'(\mathrm{g})|^q+\tfrac{4^{-s}\left(2^{1+s}-1\right)}{1+s}|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}}\\&+\left(\tfrac{\left(\frac{4}{15}\right)^{p+1}+\left(\frac{11}{15}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\tfrac{4^{-s}\left(2^{1+s}-1\right)}{1+s}|\mathcal{Y}'(\mathrm{g})|^q+\tfrac{4^{-s}\left(3^{1+s}-2^{1+s}\right)}{1+s}|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}}\\&+\left(\tfrac{\left(\frac{14}{45}\right)^{p+1}+\left(\frac{31}{45}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\tfrac{4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{g})|^q+\tfrac{4-3^{1+s}4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}}\Bigg],\end{aligned}\tag{14}$$

where $p^{-1}+q^{-1}=1$.

Proof. From Lemma 1, and using Hölder's inequality, we have

$$\begin{aligned}
&\left|\tfrac{1}{90}\left[7\mathcal{Y}(g)+32\mathcal{Y}\left(\tfrac{3g+h}{4}\right)+12\mathcal{Y}\left(\tfrac{g+h}{2}\right)+32\mathcal{Y}\left(\tfrac{g+3h}{4}\right)+7\mathcal{Y}(h)\right]-\tfrac{1}{h-g}\int_g^h \mathcal{Y}(\omega)d\omega\right| \\
&\leq \tfrac{h-g}{16}\left[\int_0^1\left|\mu-\tfrac{28}{90}\right|\left|\mathcal{Y}'\left(\tfrac{4-\mu}{4}g+\tfrac{\mu}{4}h\right)\right|d\mu+\int_0^1\left|\mu-\tfrac{44}{60}\right|\left|\mathcal{Y}'\left(\tfrac{3-\mu}{4}g+\tfrac{1+\mu}{4}h\right)\right|d\mu\right. \\
&\left.+\int_0^1\left|\mu-\tfrac{24}{90}\right|\left|\mathcal{Y}'\left(\tfrac{2-\mu}{4}g+\tfrac{2+\mu}{4}h\right)\right|d\mu+\int_0^1\left|\mu-\tfrac{62}{90}\right|\left|\mathcal{Y}'\left(\tfrac{1-\mu}{4}g+\tfrac{3+\mu}{4}h\right)\right|d\mu\right] \\
&\leq \tfrac{h-g}{16}\left[\left(\int_0^1\left|\mu-\tfrac{28}{90}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{Y}'\left(\tfrac{4-\mu}{4}g+\tfrac{\mu}{4}h\right)\right|^q d\mu\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^1\left|\mu-\tfrac{44}{60}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{Y}'\left(\tfrac{3-\mu}{4}g+\tfrac{1+\mu}{4}h\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\left|\mu-\tfrac{24}{90}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{Y}'\left(\tfrac{2-\mu}{4}g+\tfrac{2+\mu}{4}h\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1\left|\mu-\tfrac{62}{90}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{Y}'\left(\tfrac{1-\mu}{4}g+\tfrac{3+\mu}{4}h\right)\right|^q d\mu\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{15}$$

Because $|\mathcal{Y}'|^q$ is s-convex, we have

$$\begin{aligned}
\int_0^1\left|\mathcal{Y}'\left(\tfrac{4-\mu}{4}g+\tfrac{\mu}{4}h\right)\right|^q d\mu &\leq \int_0^1\left(\tfrac{4-\mu}{4}\right)^s d\mu|\mathcal{Y}'(g)|^q+\int_0^1\left(\tfrac{\mu}{4}\right)^s d\mu|\mathcal{Y}'(h)|^q \\
&\leq \tfrac{4-3^{1+s}4^{-s}}{1+s}|\mathcal{Y}'(g)|^q+\tfrac{4^{-s}}{1+s}|\mathcal{Y}'(h)|^q.
\end{aligned} \tag{16}$$

Similarly,

$$\int_0^1\left|\mathcal{Y}'\left(\frac{3-\mu}{4}g+\frac{1+\mu}{4}h\right)\right|^q d\mu \leq \frac{4^{-s}\left(3^{1+s}-2^{1+s}\right)}{1+s}|\mathcal{Y}'(g)|^q+\frac{4^{-s}\left(2^{1+s}-1\right)}{1+s}|\mathcal{Y}'(h)|^q, \tag{17}$$

$$\int_0^1\left|\mathcal{Y}'\left(\frac{2-\mu}{4}g+\frac{2+\mu}{4}h\right)\right|^q d\mu \leq \frac{4^{-s}\left(2^{1+s}-1\right)}{1+s}|\mathcal{Y}'(g)|^q+\frac{4^{-s}\left(3^{1+s}-2^{1+s}\right)}{1+s}|\mathcal{Y}'(h)|^q, \tag{18}$$

and

$$\int_0^1\left|\mathcal{Y}'\left(\frac{1-\mu}{4}g+\frac{3+\mu}{4}h\right)\right|^q d\mu \leq \frac{4^{-s}}{1+s}|\mathcal{Y}'(g)|^q+\frac{4-3^{1+s}4^{-s}}{1+s}|\mathcal{Y}'(h)|^q. \tag{19}$$

By a simple computation, we obtain

$$\int_0^1\left|\mu-\frac{28}{90}\right|^p d\mu \leq \int_0^{\frac{28}{90}}\left(\frac{28}{90}-\mu\right)^p d\mu+\int_{\frac{28}{90}}^1\left(\mu-\frac{28}{90}\right)^p d\mu=\frac{\left[\left(\frac{14}{45}\right)^{1+p}+\left(\frac{31}{45}\right)^{1+p}\right]}{1+p}, \tag{20}$$

$$\int_0^1\left|\mu-\frac{62}{90}\right|^p d\mu=\frac{\left[\left(\frac{14}{45}\right)^{1+p}+\left(\frac{31}{45}\right)^{1+p}\right]}{1+p}, \tag{21}$$

$$\int_0^1\left|\mu-\frac{44}{60}\right|^p d\mu=\frac{(15)^{-1-p}\left[(4)^{1+p}+(11)^{1+p}\right]}{1+p}, \tag{22}$$

$$\int_0^1\left|\mu-\frac{24}{90}\right|^p d\mu=\frac{(15)^{-1-p}\left[(4)^{1+p}+(11)^{1+p}\right]}{1+p}. \tag{23}$$

Substituting (16)–(23) into (15), we then obtain

$$\left|\frac{1}{90}\left[7\mathcal{Y}(\mathrm{g})+32\mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{4}\right)+12\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right)+32\mathcal{Y}\left(\frac{\mathrm{g}+3\mathrm{h}}{4}\right)+7\mathcal{Y}(\mathrm{h})\right]-\frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}}\mathcal{Y}(\omega)d\omega\right|$$
$$\leq\frac{\mathrm{h}-\mathrm{g}}{16}\left[\left(\frac{\left(\frac{14}{45}\right)^{p+1}+\left(\frac{31}{45}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{4-3^{1+s}4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{g})|^{q}+\frac{4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{h})|^{q}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{\left(\frac{4}{15}\right)^{p+1}+\left(\frac{11}{15}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{4^{-s}\left(3^{1+s}-2^{1+s}\right)}{1+s}|\mathcal{Y}'(\mathrm{g})|^{q}+\frac{4^{-s}\left(2^{1+s}-1\right)}{1+s}|\mathcal{Y}'(\mathrm{h})|^{q}\right)^{\frac{1}{q}}$$
$$+\left(\frac{\left(\frac{4}{15}\right)^{p+1}+\left(\frac{11}{15}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{4^{-s}\left(2^{1+s}-1\right)}{1+s}|\mathcal{Y}'(\mathrm{g})|^{q}+\frac{4^{-s}\left(3^{1+s}-2^{1+s}\right)}{1+s}|\mathcal{Y}'(\mathrm{h})|^{q}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{\left(\frac{14}{45}\right)^{p+1}+\left(\frac{31}{45}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{g})|^{q}+\frac{4-3^{1+s}4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{h})|^{q}\right)^{\frac{1}{q}}\right].$$

The demonstration of Theorem 6 has then been finalized. □

Corollary 4. *Setting $s=1$ in Theorem 6, we have*

$$\left|\frac{1}{90}\left[7\mathcal{Y}(\mathrm{g})+32\mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{4}\right)+12\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right)+32\mathcal{Y}\left(\frac{\mathrm{g}+3\mathrm{h}}{4}\right)+7\mathcal{Y}(\mathrm{h})\right]-\frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}}\mathcal{Y}(\omega)d\omega\right|$$
$$\leq\frac{\mathrm{h}-\mathrm{g}}{16}\left(\frac{1}{8}\right)^{\frac{1}{q}}\left[\left(\frac{\left(\frac{14}{45}\right)^{p+1}+\left(\frac{31}{45}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(7|\mathcal{Y}'(\mathrm{g})|^{q}+|\mathcal{Y}'(\mathrm{h})|^{q}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{\left(\frac{4}{15}\right)^{p+1}+\left(\frac{11}{15}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(5|\mathcal{Y}'(\mathrm{g})|^{q}+3|\mathcal{Y}'(\mathrm{h})|^{q}\right)^{\frac{1}{q}}$$
$$+\left(\frac{\left(\frac{4}{15}\right)^{p+1}+\left(\frac{11}{15}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(3|\mathcal{Y}'(\mathrm{g})|^{q}+5|\mathcal{Y}'(\mathrm{h})|^{q}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{\left(\frac{14}{45}\right)^{p+1}+\left(\frac{31}{45}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(|\mathcal{Y}'(\mathrm{g})|^{q}+7|\mathcal{Y}'(\mathrm{h})|^{q}\right)^{\frac{1}{q}}\right].$$

The new midpoint-type inequality in terms of the first derivative is observed in the following result:

Corollary 5. *In Theorem 6, if* $\mathcal{Y}(\mathrm{g}) = \mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right) = \mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{4}\right) = \mathcal{Y}\left(\frac{\mathrm{g}+3\mathrm{h}}{4}\right) = \mathcal{Y}(\mathrm{h})$, *then we have the following midpoint-type inequality for the s-convex function:*

$$\begin{aligned}
&\left|\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right) - \frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega)d\omega\right| \\
&\leq \frac{\mathrm{h}-\mathrm{g}}{16}\left[\left(\frac{\left(\frac{14}{45}\right)^{p+1} + \left(\frac{31}{45}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{4-3^{1+s}4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{g})|^q + \frac{4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}}\right. \\
&+\left(\frac{\left(\frac{4}{15}\right)^{p+1} + \left(\frac{11}{15}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{4^{-s}(3^{1+s}-2^{1+s})}{1+s}|\mathcal{Y}'(\mathrm{g})|^q + \frac{4^{-s}(2^{1+s}-1)}{1+s}|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}} \\
&+\left(\frac{\left(\frac{4}{15}\right)^{p+1} + \left(\frac{11}{15}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{4^{-s}(2^{1+s}-1)}{1+s}|\mathcal{Y}'(\mathrm{g})|^q + \frac{4^{-s}(3^{1+s}-2^{1+s})}{1+s}|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}} \\
&\left.+\left(\frac{\left(\frac{14}{45}\right)^{p+1} + \left(\frac{31}{45}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{g})|^q + \frac{4-3^{1+s}4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 6. *In Corollary 5, setting* $s = 1$, *we have*

$$\begin{aligned}
&\left|\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right) - \frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega)d\omega\right| \\
&\leq \frac{\mathrm{h}-\mathrm{g}}{16}\left(\frac{1}{8}\right)^{\frac{1}{q}}\left[\left(\frac{\left(\frac{14}{45}\right)^{p+1} + \left(\frac{31}{45}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(7|\mathcal{Y}'(\mathrm{g})|^q + |\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}}\right. \\
&+\left(\frac{\left(\frac{4}{15}\right)^{p+1} + \left(\frac{11}{15}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(5|\mathcal{Y}'(\mathrm{g})|^q + 3|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}} \\
&+\left(\frac{\left(\frac{4}{15}\right)^{p+1} + \left(\frac{11}{15}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(3|\mathcal{Y}'(\mathrm{g})|^q + 5|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}} \\
&\left.+\left(\frac{\left(\frac{14}{45}\right)^{p+1} + \left(\frac{31}{45}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(|\mathcal{Y}'(\mathrm{g})|^q + 7|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 7. *Assume that the premises established in Lemma 1 are fulfilled. If* $|\mathcal{Y}'|^q$ *is s-convex on* $[\mathrm{g}, \mathrm{h}]$, *for some fixed* $s \in (0, 1]$ *and* $q \geq 1$, *then the following inequality holds:*

$$\begin{aligned}
&\left|\tfrac{1}{90}\left[7\mathcal{Y}(\mathrm{g}) + 32\mathcal{Y}\left(\tfrac{3\mathrm{g}+\mathrm{h}}{4}\right) + 12\mathcal{Y}\left(\tfrac{\mathrm{g}+\mathrm{h}}{2}\right) + 32\mathcal{Y}\left(\tfrac{\mathrm{g}+3\mathrm{h}}{4}\right) + 7\mathcal{Y}(\mathrm{h})\right] - \tfrac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega)d\omega\right| \\
&\leq \tfrac{\mathrm{h}-\mathrm{g}}{16}\left[\left(\tfrac{1157}{4050}\right)^{1-\frac{1}{q}}\left[E_1(s)|\mathcal{Y}'(\mathrm{g})|^q + E_8(s)|\mathcal{Y}'(\mathrm{h})|^q\right]^{\frac{1}{q}}\right. \\
&+\left(\tfrac{137}{450}\right)^{1-\frac{1}{q}}\left[E_3(s)|\mathcal{Y}'(\mathrm{g})|^q + E_6(s)|\mathcal{Y}'(\mathrm{h})|^q\right]^{\frac{1}{q}} + \left(\tfrac{137}{450}\right)^{1-\frac{1}{q}}\left[E_5(s)|\mathcal{Y}'(\mathrm{g})|^q + E_4(s)|\mathcal{Y}'(\mathrm{h})|^q\right]^{\frac{1}{q}} \\
&\left.+\left(\tfrac{1157}{4050}\right)^{1-\frac{1}{q}}\left[E_7(s)|\mathcal{Y}'(\mathrm{g})|^q + E_2(s)|\mathcal{Y}'(\mathrm{h})|^q\right]^{\frac{1}{q}}\right].
\end{aligned} \tag{24}$$

Proof. From Lemma 1, and using power mean inequality, we have

$$\begin{aligned}
&\left|\tfrac{1}{90}\left[7\mathcal{Y}(\mathrm{g})+32\mathcal{Y}\left(\tfrac{3\mathrm{g}+\mathrm{h}}{4}\right)+12\mathcal{Y}\left(\tfrac{\mathrm{g}+\mathrm{h}}{2}\right)+32\mathcal{Y}\left(\tfrac{\mathrm{g}+3\mathrm{h}}{4}\right)+7\mathcal{Y}(\mathrm{h})\right]-\tfrac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}}\mathcal{Y}(\omega)d\omega\right| \\
&\leq \tfrac{\mathrm{h}-\mathrm{g}}{16}\left[\int_0^1\left|\mu-\tfrac{28}{90}\right|\left|\mathcal{Y}'\left(\tfrac{4-\mu}{4}\mathrm{g}+\tfrac{\mu}{4}\mathrm{h}\right)\right|d\mu+\int_0^1\left|\mu-\tfrac{44}{60}\right|\left|\mathcal{Y}'\left(\tfrac{3-\mu}{4}\mathrm{g}+\tfrac{1+\mu}{4}\mathrm{h}\right)\right|d\mu\right. \\
&\left.+\int_0^1\left|\mu-\tfrac{24}{90}\right|\left|\mathcal{Y}'\left(\tfrac{2-\mu}{4}\mathrm{g}+\tfrac{2+\mu}{4}\mathrm{h}\right)\right|d\mu+\int_0^1\left|\mu-\tfrac{62}{90}\right|\left|\mathcal{Y}'\left(\tfrac{1-\mu}{4}\mathrm{g}+\tfrac{3+\mu}{4}\mathrm{h}\right)\right|d\mu\right] \\
&\leq \tfrac{\mathrm{h}-\mathrm{g}}{16}\left[\left(\int_0^1\left|\mu-\tfrac{28}{90}\right|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\mu-\tfrac{28}{90}\right|\left|\mathcal{Y}'\left(\tfrac{4-\mu}{4}\mathrm{g}+\tfrac{\mu}{4}\mathrm{h}\right)\right|^q d\mu\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^1\left|\mu-\tfrac{44}{60}\right|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\mu-\tfrac{44}{60}\right|\left|\mathcal{Y}'\left(\tfrac{3-\mu}{4}\mathrm{g}+\tfrac{1+\mu}{4}\mathrm{h}\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\left|\mu-\tfrac{24}{90}\right|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\mu-\tfrac{24}{90}\right|\left|\mathcal{Y}'\left(\tfrac{2-\mu}{4}\mathrm{g}+\tfrac{2+\mu}{4}\mathrm{h}\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1\left|\mu-\tfrac{62}{90}\right|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\mu-\tfrac{62}{90}\right|\left|\mathcal{Y}'\left(\tfrac{1-\mu}{4}\mathrm{g}+\tfrac{3+\mu}{4}\mathrm{h}\right)\right|^q d\mu\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{25}$$

Since $|\mathcal{Y}'|^q$ is s-convex on the interval $[\mathrm{g},\mathrm{h}]$, we have

$$\begin{aligned}
&\int_0^1\left|\mu-\tfrac{28}{90}\right|\left|\mathcal{Y}'\left(\tfrac{4-\mu}{4}\mathrm{g}+\tfrac{\mu}{4}\mathrm{h}\right)\right|^q d\mu \\
&\leq \int_0^1\left|\mu-\tfrac{28}{90}\right|\left(\tfrac{4-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{g})|^q+\int_0^1\left|\mu-\tfrac{28}{90}\right|\left(1-\tfrac{4-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{h})|^q \\
&= E_1(s)|\mathcal{Y}'(\mathrm{g})|^q+E_8(s)|\mathcal{Y}'(\mathrm{h})|^q,
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\int_0^1\left|\mu-\tfrac{44}{90}\right|\left|\mathcal{Y}'\left(\tfrac{3-\mu}{4}\mathrm{g}+\tfrac{1+\mu}{4}\mathrm{h}\right)\right|^q d\mu \\
&\leq \int_0^1\left|\mu-\tfrac{44}{60}\right|\left(\tfrac{3-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{g})|^q+\int_0^1\left|\mu-\tfrac{44}{60}\right|\left(1-\tfrac{3-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{h})|^q \\
&= E_3(s)|\mathcal{Y}'(\mathrm{g})|^q+E_6(s)|\mathcal{Y}'(\mathrm{h})|^q,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\int_0^1\left|\mu-\tfrac{24}{90}\right|\left|\mathcal{Y}'\left(\tfrac{2-\mu}{4}\mathrm{g}+\tfrac{2+\mu}{4}\mathrm{h}\right)\right|^q d\mu \\
&\leq \int_0^1\left|\mu-\tfrac{24}{90}\right|\left(\tfrac{2-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{g})|^q+\int_0^1\left|\mu-\tfrac{24}{90}\right|\left(1-\tfrac{2-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{h})|^q \\
&= E_5(s)|\mathcal{Y}'(\mathrm{g})|^q+E_4(s)|\mathcal{Y}'(\mathrm{h})|^q,
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&\int_0^1\left|\mu-\tfrac{62}{90}\right|\left|\mathcal{Y}'\left(\tfrac{1-\mu}{4}\mathrm{g}+\tfrac{3+\mu}{4}\mathrm{h}\right)\right|^q d\mu \\
&\leq \int_0^1\left|\mu-\tfrac{62}{90}\right|\left(\tfrac{1-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{g})|^q+\int_0^1\left|\mu-\tfrac{62}{90}\right|\left(1-\tfrac{1-\mu}{4}\right)^s d\mu|\mathcal{Y}'(\mathrm{h})|^q \\
&= E_7(s)|\mathcal{Y}'(\mathrm{g})|^q+E_2(s)|\mathcal{Y}'(\mathrm{h})|^q.
\end{aligned} \tag{29}$$

Here, we used the following equalities:

$$\int_0^1\left|\mu-\frac{28}{90}\right|d\mu=\frac{1157}{4050}=\int_0^1\left|\mu-\frac{62}{90}\right|d\mu,\quad\text{and}\quad\int_0^1\left|\mu-\frac{44}{60}\right|d\mu=\frac{137}{450}=\int_0^1\left|\mu-\frac{24}{90}\right|d\mu.$$

Using (26) to (29) in (25), then we obtain the required result. Thus, the proof of Theorem 7 is completed. □

3. Applications

This section presents several applications of the newly established results to numerical quadrature formulas.

3.1. Application to Quadrature Formulas

Consider a partition Y of the interval $[g, h]$, defined by $\mathrm{Y} : g = \omega_0 < \omega_1 < \ldots < \omega_n = h$, and let $h_j = \omega_{j+1} - \omega_j$. The associated Boole's quadrature formula is given by

$$\begin{aligned} S(\mathcal{Y}, \mathrm{Y}) \quad &= \sum_{j=0}^{n-1} \frac{h_j}{90} \Big[7\mathcal{Y}(\omega_j) + 12\mathcal{Y}\Big(\tfrac{\omega_j+\omega_{j+1}}{2}\Big) + 32\mathcal{Y}\Big(\tfrac{3\omega_j+\omega_{j+1}}{4}\Big) \\ &\quad + 32\mathcal{Y}\Big(\tfrac{\omega_j+3\omega_{j+1}}{4}\Big) + 7\mathcal{Y}(\omega_{j+1}) \Big]. \end{aligned} \tag{30}$$

The classical error analysis for this quadrature rule states that if $\mathcal{Y} \in C^6[g, h]$, then the integral can be expressed as

$$I = \int_g^h \mathcal{Y}(\mu) d\mu = S(\mathcal{Y}, \mathrm{Y}) + \mathcal{R}_B(\mathcal{Y}, \mathrm{Y}), \tag{31}$$

with the remainder term satisfying the error bound

$$|\mathcal{R}_B(\mathcal{Y}, \mathrm{Y})| \leq \frac{239}{6480} \|\mathcal{Y}^{(6)}\|_\infty \sum_{j=0}^{n-1} h_j^7, \tag{32}$$

where $\|\mathcal{Y}^{(6)}\|_\infty = \max_{\omega \in [g,h]} |\mathcal{Y}^{(6)}(\omega)|$. We note that the bound (32) is proportional to $\|\mathcal{Y}^{(6)}\|_\infty$, as required for the dimensional consistency and linearity of the error functional.

However, this classical error estimate is inapplicable if $\mathcal{Y}$ lacks a sixth derivative or if $\|\mathcal{Y}^{(6)}\|_\infty$ is unbounded. The following propositions provide alternative error bounds for the remainder term $\mathcal{R}_B(\mathcal{Y}, \mathrm{Y})$ based on properties of the first derivative.

Proposition 1. *Assume the conditions of Lemma 1 hold. If $|\mathcal{Y}'|$ is convex on* [g, h]*, then for any partition* Y *of* [g, h]*, the following inequality holds:*

$$|\mathcal{R}_B(\mathcal{Y}, \mathrm{Y})| \leq \frac{239}{6480} \sum_{j=0}^{n-1} (\omega_{j+1} - \omega_j)^2 \big[|\mathcal{Y}'(\omega_j)| + |\mathcal{Y}'(\omega_{j+1})| \big]. \tag{33}$$

Proof. Applying Corollary 1 to each subinterval $[\omega_j, \omega_{j+1}]$ for $j = 0, 1, 2, \ldots, n-1$ yields:

$$\begin{aligned} &\Big| \tfrac{\omega_{j+1}-\omega_j}{90} \Big[7\mathcal{Y}(\omega_j) + 32\mathcal{Y}\Big(\tfrac{3\omega_j+\omega_{j+1}}{4}\Big) + 12\mathcal{Y}\Big(\tfrac{\omega_j+\omega_{j+1}}{2}\Big) \\ &+ 32\mathcal{Y}\Big(\tfrac{\omega_j+3\omega_{j+1}}{4}\Big) + 7\mathcal{Y}(\omega_{j+1}) \Big] - \textstyle\int_{\omega_j}^{\omega_{j+1}} \mathcal{Y}(\omega) d\omega \Big| \\ &\leq \tfrac{239(\omega_{j+1}-\omega_j)^2}{6480} \big[|\mathcal{Y}'(\omega_j)| + |\mathcal{Y}'(\omega_{j+1})| \big]. \end{aligned} \tag{34}$$

Summing the inequality (34) over all subintervals and employing the triangle inequality, together with the convexity of $|\mathcal{Y}'|$, we obtain

$$\left| S(\mathcal{Y}, \mathrm{Y}) - \int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\mu) d\mu \right| \leq \frac{239}{6480} \sum_{j=0}^{n-1} (\omega_{j+1} - \omega_j)^2 \big[|\mathcal{Y}'(\omega_j)| + |\mathcal{Y}'(\omega_{j+1})| \big],$$

which completes the proof. □

Remark 2. *One can similarly derive an approximation based on Theorems 6 and 7, respectively; the details are omitted for brevity and are left to the interested reader.*

3.2. Application to the Midpoint Formula

Consider a partition Y of the interval $[\mathrm{g}, \mathrm{h}]$, defined by $\mathrm{Y} : \mathrm{g} = \omega_0 < \omega_1 < \omega_2 < \ldots < \omega_n = \mathrm{h}$. The composite midpoint rule for numerical integration is given by

$$M(\mathcal{Y}, \mathrm{Y}) = \sum_{j=0}^{n-1} (\omega_{j+1} - \omega_j) \mathcal{Y}\left(\frac{\omega_j + \omega_{j+1}}{2}\right). \tag{35}$$

A standard result states that if $\mathcal{Y} : [\mathrm{g}, \mathrm{h}] \to \mathbb{R}$ is a twice-differentiable function, such that its second derivative $\mathcal{Y}''$ exists on (g, h) and is bounded, i.e., $K = \max_{\omega \in (\mathrm{g}, \mathrm{h})} |\mathcal{Y}''(\omega)| < \infty$, then the integral can be expressed as

$$I = \int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\mu) d\mu = M(\mathcal{Y}, \mathrm{Y}) + \mathcal{R}_M(\mathcal{Y}, \mathrm{Y}), \tag{36}$$

where the error term $\mathcal{R}_M(\mathcal{Y}, \mathrm{Y})$ satisfies the bound

$$|\mathcal{R}_M(\mathcal{Y}, \mathrm{Y})| \leq \frac{K}{24} \sum_{j=0}^{n-1} (\omega_{j+1} - \omega_j)^3. \tag{37}$$

This classical error estimate requires the existence and boundedness of the second derivative. The following result provides an alternative error bound for the remainder term $\mathcal{R}_M(\mathcal{Y}, \mathrm{Y})$ in terms of the first derivative, offering an improvement over previous results in certain contexts.

Proposition 2. *Assume that the conditions of Lemma 1 are satisfied. If $|\mathcal{Y}'|$ is a convex function on $[\mathrm{g}, \mathrm{h}]$, then for any partition Y of $[\mathrm{g}, \mathrm{h}]$, the following inequality holds:*

$$|\mathcal{R}_M(\mathcal{Y}, \mathrm{Y})| \leq \frac{1}{8} \sum_{j=0}^{n-1} (\omega_{j+1} - \omega_j)^2 \left[|\mathcal{Y}'(\omega_j)| + |\mathcal{Y}'(\omega_{j+1})| \right]. \tag{38}$$

Proof. Applying Corollary 3 to each subinterval $[\omega_j, \omega_{j+1}]$ for $j = 0, 1, 2, \ldots, n-1$ yields the following local error estimate:

$$\left| (\omega_{j+1} - \omega_j) \mathcal{Y}\left(\frac{\omega_j + \omega_{j+1}}{2}\right) - \int_{\omega_j}^{\omega_{j+1}} \mathcal{Y}(\omega) d\omega \right| \leq \frac{1}{8} (\omega_{j+1} - \omega_j)^2 \left[|\mathcal{Y}'(\omega_j)| + |\mathcal{Y}'(\omega_{j+1})| \right]. \tag{39}$$

Summing the inequality (39) over all subintervals and applying the triangle inequality, we obtain the global error bound:

$$\begin{aligned} |\mathcal{R}_M(\mathcal{Y}, \mathrm{Y})| &= \left| \sum_{j=0}^{n-1} \left((\omega_{j+1} - \omega_j) \mathcal{Y}\left(\frac{\omega_j + \omega_{j+1}}{2}\right) - \int_{\omega_j}^{\omega_{j+1}} \mathcal{Y}(\omega) d\omega \right) \right| \\ &\leq \sum_{j=0}^{n-1} \left| (\omega_{j+1} - \omega_j) \mathcal{Y}\left(\frac{\omega_j + \omega_{j+1}}{2}\right) - \int_{\omega_j}^{\omega_{j+1}} \mathcal{Y}(\omega) d\omega \right| \\ &\leq \frac{1}{8} \sum_{j=0}^{n-1} (\omega_{j+1} - \omega_j)^2 \left[|\mathcal{Y}'(\omega_j)| + |\mathcal{Y}'(\omega_{j+1})| \right], \end{aligned}$$

which completes the proof. □

Remark 3. *The estimations for midpoint-type inequality given in this work provided better bounds compared to the estimation proved in [8].*

3.3. Application to Special Means

Let $s \in (0,1]$ and $u_1, u_2, u_3 \in \mathbb{R}$. We define a function $\mathcal{Y} : [0,\infty) \to \mathbb{R}$ as

$$\mathcal{Y}(\mu) = \begin{cases} u_1, & \mu = 0; \\ u_2\mu^s + u_3, & \mu > 0\,. \end{cases}$$

If $u_2 \geq 0$ and $0 \leq u_3 \leq u_1$, then $\mathcal{Y} \in \mathcal{K}_s^2$ (see [22]). Hence, for $u_1 = u_3 = 0$ and $u_2 = 1$, we have a function $\mathcal{Y} : [\mathrm{g}, \mathrm{h}] \to \mathbb{R}$ defined by $\mathcal{Y}(u) = u^s$, which belongs to $\mathcal{K}_s^2$.

In [22], the following result is given. Let $\mathcal{Y} : I_1 \to \mathbb{R}^+$ be a non-decreasing and s-convex function on I_1, and let $\mathcal{G} : J \to I_2 \subseteq I_1$ be a non-negative convex function on J. Then the composition $\mathcal{Y} \circ \mathcal{G}$ is s-convex on J.

A direct result of the previous finding can be stated as follows:

Corollary 7. *Suppose $\mathcal{G} : I \to I_1 \subset [0,\infty)$ is a non-negative convex function on I, then $\mathcal{G}^s(t)$ is s-convex on $[0,\infty)$, $0 < s < 1$.*

We shall consider the following special means. For arbitrary real numbers $\mathrm{g}, \mathrm{h}, \ldots, \mathrm{h}_n$ ($\mathrm{g} \neq \mathrm{h}$), we have the following:

The arithmetic mean: $\mathcal{A} = \mathcal{A}(\mathrm{g}, \mathrm{h}, \ldots, \mathrm{h}_n) = \frac{\mathrm{g}+\mathrm{h}+\ldots+\mathrm{h}_n}{n}$.

The harmonic mean: $\mathcal{H} = \mathcal{H}(\mathrm{g}, \mathrm{h}) = \frac{2}{\frac{1}{\mathrm{g}}+\frac{1}{\mathrm{h}}}, \quad \mathrm{g}, \mathrm{h} > 0.$

The logarithm mean: $\mathcal{L} = \mathcal{L}(\mathrm{g}, \mathrm{h}) = \frac{\mathrm{h}-\mathrm{g}}{\ln \mathrm{h} - \ln \mathrm{g}}, \qquad \mathrm{g} \neq \mathrm{h}, \quad \mathrm{g}, \mathrm{h} \in \mathbb{R}^+.$

The p-logarithmic mean: $\mathcal{L}_p(\mathrm{g}, \mathrm{h}) = \left(\frac{\mathrm{h}^{p+1}+\mathrm{g}^{p+1}}{(p+1)(\mathrm{h}-\mathrm{g})}\right)^{\frac{1}{p}}, \quad \mathrm{g}, \mathrm{h} > 0, \quad \mathrm{g} \neq \mathrm{h}$ and $p \in \mathbb{R}\backslash\{-1, 0\}$. It is well-known that $\mathcal{L}_p$ is monotonic non-decreasing over $p \subset \mathbb{R}$, with $\mathcal{L}_{-1} := \mathcal{L}$ and $\mathcal{L}_0 := \mathcal{I}$. In particular, we have the following inequality $\mathcal{L} \leq \mathcal{A}$.

Based on the outcomes presented in Section 2, we establish new inequalities for the means mentioned above.

Proposition 3. *Consider $\mathcal{Y} : [\mathrm{g}, \mathrm{h}] \to \mathbb{R}$, $(0 < \mathrm{g} < \mathrm{h})$, $\mathcal{Y}(t) = t^s$, $s(0,1]$. Then,*

$$\begin{aligned} \frac{1}{\mathrm{h}-\mathrm{g}} \int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega) d\omega &= \mathcal{L}_s^s(\mathrm{g}, \mathrm{h}), \\ \frac{\mathcal{Y}(\mathrm{g}) + \mathcal{Y}(\mathrm{h})}{2} &= \mathcal{A}(\mathrm{g}^s, \mathrm{h}^s), \\ \mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right) &= \mathcal{A}^s(\mathrm{g}, \mathrm{h}), \\ \mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{2}\right) &= \mathcal{A}^s(\mathrm{g}, \mathrm{g}, \mathrm{g}, \mathrm{h}), \\ \mathcal{Y}\left(\frac{\mathrm{g}+3\mathrm{h}}{2}\right) &= \mathcal{A}^s(\mathrm{g}, \mathrm{h}, \mathrm{h}, \mathrm{h}). \end{aligned}$$

(1) *Using the inequality (10), we obtain*

$$\begin{aligned} &\left|\tfrac{1}{90}[14\mathcal{A}(\mathrm{g}^s, \mathrm{h}^s) + 32\mathcal{A}^s(\mathrm{g}, \mathrm{g}, \mathrm{g}, \mathrm{h}) + 12\mathcal{A}^s(\mathrm{g}, \mathrm{h}) + 32\mathcal{A}^s(\mathrm{g}, \mathrm{h}, \mathrm{h}, \mathrm{h})] - \mathcal{L}_s^s(\mathrm{g}, \mathrm{h})\right| \\ &\leq \tfrac{s(\mathrm{h}-\mathrm{g})}{16} \tfrac{2^{1-2s}}{(45)^{s+2}(s+1)(s+2)} \big(45^{s+1}\big(3^{s+1} + 7\,4^{s+1} - 1\big)s - 19\,4^{s+2}45^{s+1} - 47\,45^{s+1} \\ &-133\,135^{s+1} + 14^{s+2} + 78^{s+2} - 90^{s+2} + 102^{s+2} + 166^{s+2}\big)\Big[|\mathrm{g}|^{s-1} + |\mathrm{h}|^{s-1}\Big]. \end{aligned} \tag{40}$$

For instance, if $s = 1$ in (40), then we have

$$\left|\frac{1}{90}[26\mathcal{A}(\mathrm{g}, \mathrm{h}) + 32\mathcal{A}(\mathrm{g}, \mathrm{g}, \mathrm{g}, \mathrm{h}) + 32\mathcal{A}(\mathrm{g}, \mathrm{h}, \mathrm{h}, \mathrm{h})] - \mathcal{L}(\mathrm{g}, \mathrm{h})\right| \leq \frac{239(\mathrm{h}-\mathrm{g})}{6480}.$$

(2) *Using the inequality (12), we obtain*

$$\begin{aligned}|\mathcal{A}^s(g,h)-\mathcal{L}_s^s(g,h)| \leq \tfrac{s(h-g)}{16}\tfrac{2^{1-2s}}{(45)^{s+2}(s+1)(s+2)}\big(45^{s+1}\big(3^{s+1}+7\,4^{s+1}-1\big)s\\ -19\,4^{s+2}45^{s+1}-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}\\ -90^{s+2}+102^{s+2}+166^{s+2}\big)\Big[|g|^{s-1}+|h|^{s-1}\Big].\end{aligned} \tag{41}$$

For instance, if $s=1$ in (41), then we have

$$|\mathcal{A}(g,h)-\mathcal{L}(g,h)| \leq \frac{239(h-g)}{6480}.$$

4. Numerical Examples

In this section, a comprehensive numerical investigation is carried out to confirm the validity and efficiency of the newly established results. Several computational experiments are presented to highlight the applicability of the proposed inequalities, especially for the approximation of integrals involving differentiable s-convex functions. To better illustrate the outcomes, three-dimensional graphical models are employed, providing clear insights into the numerical behavior of the derived inequalities. These visual demonstrations play a crucial role in verifying the accuracy and practical significance of the theoretical contributions.

Example 1. *Let $\mathcal{Y}:[g,h]=[1,2]\to\mathbb{R}$ be a function defined by $\mathcal{Y}(\mu)=\mu^{5s+1}$. Then by applying inequality (10) to the function $\mathcal{Y}(\mu)=\mu^{5s+1}$, the left-hand side of inequality (10) for $s=0.2$ is*

$$\begin{aligned}&\Big|\tfrac{1}{90}\Big[7\mathcal{Y}(g)+32\mathcal{Y}\Big(\tfrac{3g+h}{4}\Big)+12\mathcal{Y}\Big(\tfrac{g+h}{2}\Big)+32\mathcal{Y}\Big(\tfrac{g+3h}{4}\Big)+7\mathcal{Y}(h)\Big]-\tfrac{1}{h-g}\int_g^h\mathcal{Y}(\omega)d\omega\Big|\\ &=9\times10^{-9},\end{aligned} \tag{42}$$

and the right-hand side of inequality (10) for $s=0.2$ is

$$\begin{aligned}&\tfrac{(h-g)}{16}\tfrac{2^{1-2s}}{(45)^{s+2}(s+1)(s+2)}\big(45^{s+1}\big(3^{s+1}+7\,4^{s+1}-1\big)s-19\,4^{s+2}45^{s+1}-47\,45^{s+1}\\ &-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\big)[|\mathcal{Y}'(g)|+|\mathcal{Y}'(h)|]=3.7177\times10^{-1}.\end{aligned} \tag{43}$$

From Equations (42) and (43), it is clear that the left-hand side is less than the right-hand side

$$9\times10^{-9}<3.7177\times10^{-1}.$$

This demonstrates that the inequality (10) is valid. A visual confirmation of this result is provided in Figure 1.

Example 2. *Let $\mathcal{Y}:[g,h]=[1,2]\to\mathbb{R}$ be a function defined by $\mathcal{Y}(\mu)=e^{s\mu}$. Then by applying inequality (10) to the function $\mathcal{Y}(\mu)=e^{s\mu}$, the left-hand side of inequality (10) for $s=0.2$ is*

$$\begin{aligned}&\Big|\tfrac{1}{90}\Big[7\mathcal{Y}(g)+32\mathcal{Y}\Big(\tfrac{3g+h}{4}\Big)+12\mathcal{Y}\Big(\tfrac{g+h}{2}\Big)+32\mathcal{Y}\Big(\tfrac{g+3h}{4}\Big)+7\mathcal{Y}(h)\Big]-\tfrac{1}{h-g}\int_g^h\mathcal{Y}(\omega)d\omega\Big|\\ &=4.4653\times10^{-11},\end{aligned} \tag{44}$$

and the right-hand side of inequality (10) for $s=0.2$ is

$$\begin{aligned}&\tfrac{(h-g)}{16}\tfrac{2^{1-2s}}{(45)^{s+2}(s+1)(s+2)}\big(45^{s+1}\big(3^{s+1}+7\,4^{s+1}-1\big)s-19\,4^{s+2}45^{s+1}-47\,45^{s+1}\\ &-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\big)[|\mathcal{Y}'(g)|+|\mathcal{Y}'(h)|]=0.03362.\end{aligned} \tag{45}$$

From Equations (44) and (45), it is clear that the left-hand side is less than the right-hand side

$$4.4653 \times 10^{-11} < 0.03362.$$

This observation serves as a numerical verification of inequality (10). A visual confirmation of this result is provided in Figure 2.

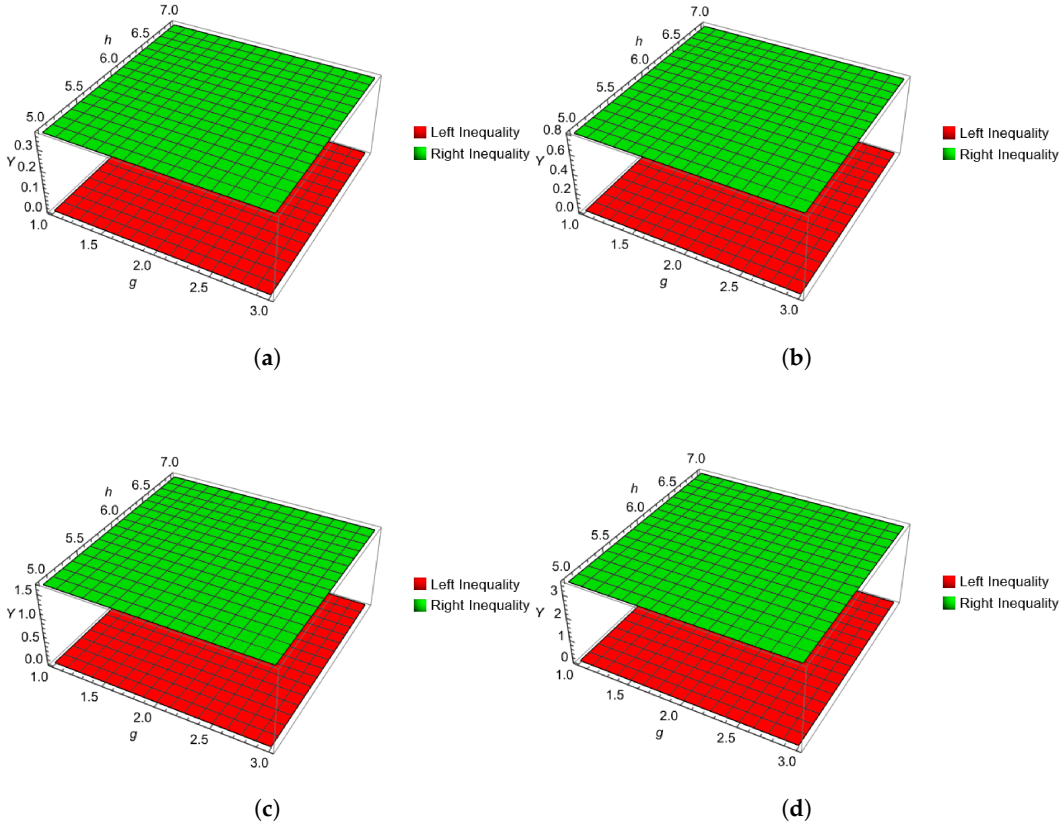


Figure 1. The graphical evidence presented in Figures (**a**–**d**) demonstrates that for every $s \in (0,1)$, the value of the left-hand side of inequality (10) is strictly bounded below by its right-hand side. This observation provides empirical confirmation of the result stated in Theorem 5.

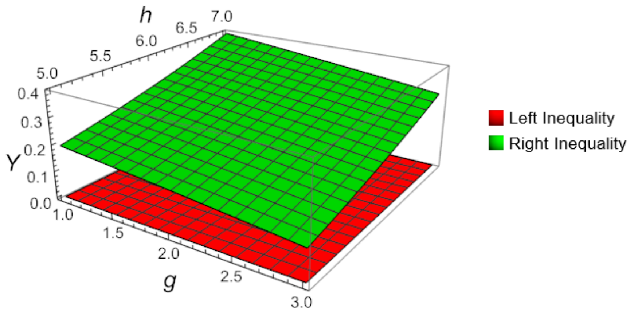


Figure 2. Three-dimensional plot for Theorem 5 in Example 2, when $g \in [1,3]$, $h \in [5,7]$, and $s = 0.2$, computed and plotted with Mathematica.

Example 3. *Let* $\mathcal{Y} : [g,h] = \left[0, \frac{\Pi}{2}\right] \to \mathbb{R}$ *be a function defined by* $\mathcal{Y}(\mu) = 1 - \cos(s\mu)$. *Then by applying inequality (10) to the function* $\mathcal{Y}(\mu) = 1 - \cos(s\mu)$, *the left-hand side of inequality (10) for* $s = \frac{1}{2}$ *is*

$$\left|\frac{1}{90}\left[7\mathcal{Y}(g) + 32\mathcal{Y}\left(\frac{3g+h}{4}\right) + 12\mathcal{Y}\left(\frac{g+h}{2}\right) + 32\mathcal{Y}\left(\frac{g+3h}{4}\right) + 7\mathcal{Y}(h)\right] - \frac{1}{h-g}\int_g^h \mathcal{Y}(\omega)d\omega\right| = 1.1143459 \times 10^{-7}, \quad (46)$$

and the right-hand side of inequality (10) for $s = 1/2$ is

$$\frac{(\mathrm{h}-\mathrm{g})}{16}\frac{2^{1-2s}}{(45)^{s+2}(s+1)(s+2)}\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s-19\,4^{s+2}45^{s+1}-47\,45^{s+1}\right.$$
$$\left.-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)[|\mathcal{Y}'(\mathrm{g})|+|\mathcal{Y}'(\mathrm{h})|]=0.027579. \tag{47}$$

From Equations (46) and (47), it is clear that the left-hand side is less than the right-hand side

$$1.1143459\times10^{-7}<0.027579.$$

This demonstrates that the inequality (10) is valid. A visual confirmation of this result is provided in Figure 3.

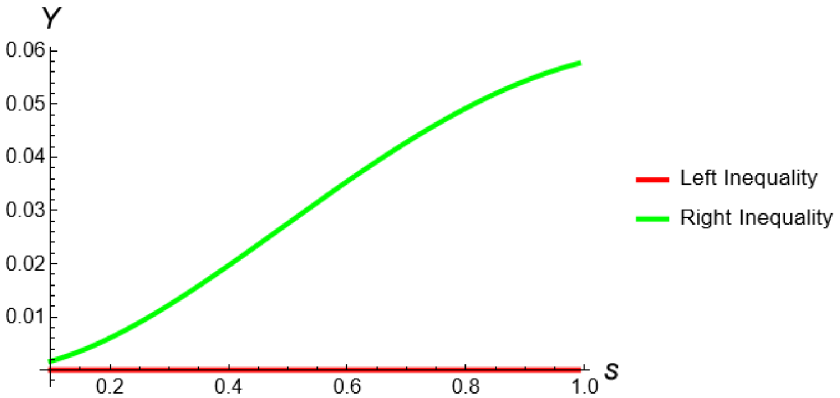


Figure 3. Two-dimensional plot for Theorem 5 in Example 3, when $[g,h]\in\left[0,\frac{\Pi}{2}\right]$, and $s\in(0,1]$, computed and plotted with Mathematica.

Example 4. *Let $\mathcal{Y}:[\mathrm{g},\mathrm{h}]=[1,2]\to\mathbb{R}$ be a function defined by $\mathcal{Y}(\mu)=\mu^{5s+1}$ and $p=q=2$. Then by applying inequality (14) to the function $\mathcal{Y}(\mu)=\mu^{5s+1}$, the right-hand side of inequality (14) for $s=0.2$ is*

$$\begin{aligned}&\frac{\mathrm{h}-\mathrm{g}}{16}\left[\left(\frac{\left(\frac{14}{45}\right)^{p+1}+\left(\frac{31}{45}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{4-3^{1+s}4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{g})|^q+\frac{4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}}\right.\\&+\left(\frac{\left(\frac{4}{15}\right)^{p+1}+\left(\frac{11}{15}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{4^{-s}\left(3^{1+s}-2^{1+s}\right)}{1+s}|\mathcal{Y}'(\mathrm{g})|^q+\frac{4^{-s}\left(2^{1+s}-1\right)}{1+s}|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}}\\&+\left(\frac{\left(\frac{4}{15}\right)^{p+1}+\left(\frac{11}{15}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{4^{-s}\left(2^{1+s}-1\right)}{1+s}|\mathcal{Y}'(\mathrm{g})|^q+\frac{4^{-s}\left(3^{1+s}-2^{1+s}\right)}{1+s}|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}}\\&\left.+\left(\frac{\left(\frac{14}{45}\right)^{p+1}+\left(\frac{31}{45}\right)^{p+1}}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{g})|^q+\frac{4-3^{1+s}4^{-s}}{1+s}|\mathcal{Y}'(\mathrm{h})|^q\right)^{\frac{1}{q}}\right]\\&=3.6528\times10^{-1}.\end{aligned} \tag{48}$$

From Equations (42) and (48), it is clear that the left-hand side is less than the right-hand side

$$9\times10^{-9}<3.6528\times10^{-1}.$$

This observation serves as a numerical verification of inequality (14). A visual confirmation of this result is provided in Figure 4.

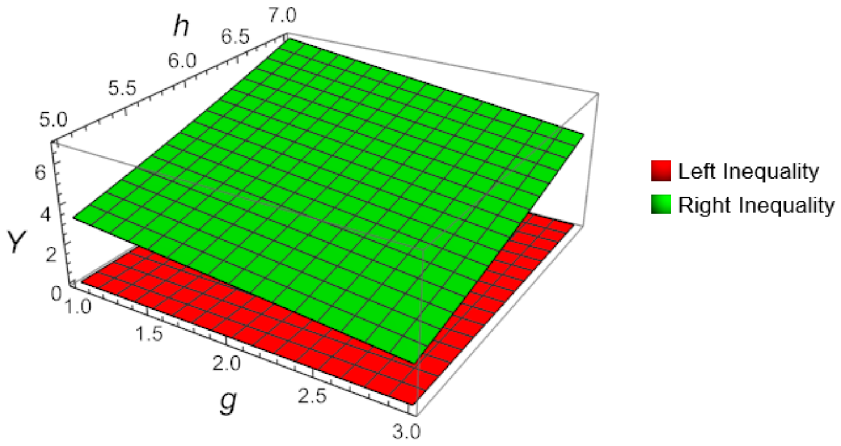


Figure 4. Three-dimensional plot for Theorem 6 in Example 4, when $g \in [1,3]$, $h \in [5,7]$, and $s = 0.2$, computed and plotted with Mathematica.

Example 5. *Let* $\mathcal{Y} : [\mathrm{g}, \mathrm{h}] = [1,2] \to \mathbb{R}$ *be a function defined by* $\mathcal{Y}(\mu) = (\mu+1)^{5s+1}$ *and* $q = 2$. *Then by applying inequality (24) to the function* $\mathcal{Y}(\mu) = (\mu+1)^{5s+1}$, *then the left-hand side of inequality (24) for* $s = \frac{1}{2}$ *is*

$$\left|\frac{1}{90}\left[7\mathcal{Y}(\mathrm{g}) + 32\mathcal{Y}\left(\frac{3\mathrm{g}+\mathrm{h}}{4}\right) + 12\mathcal{Y}\left(\frac{\mathrm{g}+\mathrm{h}}{2}\right) + 32\mathcal{Y}\left(\frac{\mathrm{g}+3\mathrm{h}}{4}\right) + 7\mathcal{Y}(\mathrm{h})\right] - \frac{1}{\mathrm{h}-\mathrm{g}}\int_{\mathrm{g}}^{\mathrm{h}} \mathcal{Y}(\omega)d\omega\right| = 2.6058 \times 10^{-7}, \tag{49}$$

and the right-hand side of inequality (24) for $s = \frac{1}{2}$ *is*

$$\frac{\mathrm{h}-\mathrm{g}}{16}\left[\left(\frac{1157}{4050}\right)^{1-\frac{1}{q}}[E_1(s)|\mathcal{Y}'(\mathrm{g})|^q + E_8(s)|\mathcal{Y}'(\mathrm{h})|^q]^{\frac{1}{q}} + \left(\frac{137}{450}\right)^{1-\frac{1}{q}}[E_3(s)|\mathcal{Y}'(\mathrm{g})|^q + E_6(s)|\mathcal{Y}'(\mathrm{h})|^q]^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{137}{450}\right)^{1-\frac{1}{q}}[E_5(s)|\mathcal{Y}'(\mathrm{g})|^q + E_4(s)|\mathcal{Y}'(\mathrm{h})|^q]^{\frac{1}{q}} + \left(\frac{1157}{4050}\right)^{1-\frac{1}{q}}[E_7(s)|\mathcal{Y}'(\mathrm{g})|^q + E_2(s)|\mathcal{Y}'(\mathrm{h})|^q]^{\frac{1}{q}}\right] = 3.4874. \tag{50}$$

From Equations (49) and (50), it is clear that the left-hand side is less than the right-hand side

$$2.6058 \times 10^{-7} < 3.4874.$$

This observation serves as a numerical verification of inequality (24).

5. Conclusions

The main focus of this study was to establish new Boole's formula-type inequalities for single-time differentiable s-convex functions in the second sense. By employing an integral identity, several inequalities were derived that extend the classical framework and provide improved bounds for both Boole's and midpoint formulas. The findings indicate that the proposed error estimates outperform some existing results, with s-convexity offering tighter approximations compared to standard convexity. In particular, the analysis demonstrates that as the parameter s varies, the obtained bounds surpass those of the classical case ($s = 1$), while the error tends to diminish as $s \to 0^+$. The study further validates its results through applications to quadrature formulas and special means of real numbers, supported by numerical examples, computational analysis, and graphical models. Evidence from Tables 1 and 2 confirms the effectiveness of the proposed approach for generalized convex functions. These contributions enrich approximation theory, real analysis, and the study of functional inequalities by providing more precise error estimates for polynomial and functional approximations.

Table 1. Computational analysis between the left-hand and the right-hand sides of inequality (10) for discretization of "s" in Example 1.

s	Left Part	Right Part
0.1	4.6039×10^{-7}	0.2440
0.2	9.0000×10^{-9}	0.3717
0.3	4.5617×10^{-7}	0.5484
0.4	1.7000×10^{-8}	0.7984
0.5	9.7969×10^{-7}	1.1570
0.6	3.3000×10^{-8}	1.6741
0.7	4.1696×10^{-6}	2.4212
0.8	6.5000×10^{-8}	3.5005
0.9	6.8820×10^{-5}	5.0581
1	3.7200×10^{-4}	7.3028

Table 2. Computational analysis between the left-hand and the right-hand sides of inequality (10) for discretization of "s" in Example 2.

s	Left Part	Right Part
0.1	6.0151×10^{-13}	0.01567
0.2	4.4653×10^{-11}	0.03362
0.3	5.9121×10^{-10}	0.05451
0.4	3.8618×10^{-9}	0.07915
0.5	1.7129×10^{-8}	0.10843
0.6	5.9483×10^{-8}	0.14348
0.7	1.7446×10^{-7}	0.18563
0.8	4.5226×10^{-7}	0.23651
0.9	1.0668×10^{-6}	0.29809
1	2.3363×10^{-6}	0.37279

Looking ahead, this research can be extended by incorporating q-calculus, fractional calculus, higher-order derivatives, multidimensional frameworks, and alternative convexity structures. The outcomes presented here highlight the significance of Boole's formula-type inequalities as powerful tools in the advancement of integral inequality theory.

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