

Review

High-Dimensional Numerical Methods for Nonlocal Models

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Abstract

Nonlocal models offer a unified framework for describing long-range spatial interactions and temporal memory effects. The review briefly outlines several representative physical problems, including anomalous diffusion, material fracture, viscoelastic wave propagation, and electromagnetic scattering, to illustrate the broad applicability of nonlocal systems. However, the intrinsic global coupling and historical dependence of these models introduce significant computational challenges, particularly in high-dimensional settings. From the perspective of algorithmic strategies, the review systematically summarizes high-dimensional numerical methods applicable to nonlocal equations, emphasizing core approaches for overcoming the curse of dimensionality, such as structured solution frameworks based on FFT, spectral methods, probabilistic sampling, physics-informed neural networks, and asymptotically compatible schemes. By integrating recent advances and common computational principles, the review establishes a dual “problem review + method review” structure that provides a systematic perspective and valuable reference for the modeling and high-dimensional numerical simulation of nonlocal systems.

Keywords: nonlocal models; high-dimensional computation; curse of dimensionality**MSC:** 31B10; 35K57; 35R11; 60J60; 65R20; 74A70

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1. Introduction

Multiscale interactions and temporal memory effects are pervasive in natural and engineered systems. Nonlocal models provide a theoretical framework for characterizing long-range effects and historical dependence through mathematical mechanisms, including integral operators and fractional derivatives. Compared to classical partial differential equations (PDEs), they exhibit superior physical consistency and can more naturally describe complex phenomena such as crack initiation, hysteresis, and structural heterogeneity. (see Figure 1 for a conceptual overview).

Nonlocal models demonstrate significant advantages in several typical problems. In porous media flow and groundwater transport, the spatiotemporal fractional diffusion equations more accurately characterize particle retention, jumps, and long-range migration, providing a more precise physical interpretation than classical diffusion models [1–11]. In problems involving seismic wave propagation, shear waves in biological tissues, and multi-scale structural vibrations, nonlocal constitutive relations capture the creep and relaxation of viscoelastic materials, significantly improving the prediction of dynamic behaviors in complex geological media and composite materials [12–22]. In structural failure analysis, nonlocal gradient damage models [23–28] effectively describe the cooperative evolution

of micro-defects and size effects, while peridynamics (PD) overcomes the limitations of classical continuum mechanics by naturally simulating the dynamic processes of crack initiation, growth, and branching [29–34]. Meanwhile, Limkatanyu et al. [35] developed a nanobeam–substrate system model based on the mixture stress-driven nonlocal theory, which provides a unified description of long-range nonlocal interactions and surface energy effects. This approach reveals the nonlocal coupling between size effects and structural stiffness at the micro- and nanoscales. In addition, the inherent spatial dispersion and nonlocal polarization responses in electromagnetic scattering can be more accurately represented by nonlocal models, enabling effective simulation of frequency-dependent resonances and near-field coupling in subwavelength structures [36–39]. Collectively, these examples illustrate some central advantages of nonlocal models:

- **Spatial long-range coupling:** Nonlocal operators explicitly account for interactions beyond an infinitesimal neighborhood, enabling the modeling of particle transport or stress transmission across finite distances.
- **Temporal memory:** They provide a systematic framework to capture delayed responses of a system to its past states, which is particularly relevant for viscoelastic relaxation and anomalous diffusion processes with retardation behavior.
- **Enhanced physical consistency:** Nonlocal models naturally avoid stress singularities at crack tips, allow for spontaneous crack initiation and propagation, and better accommodate multiscale mechanical behavior across heterogeneous materials.

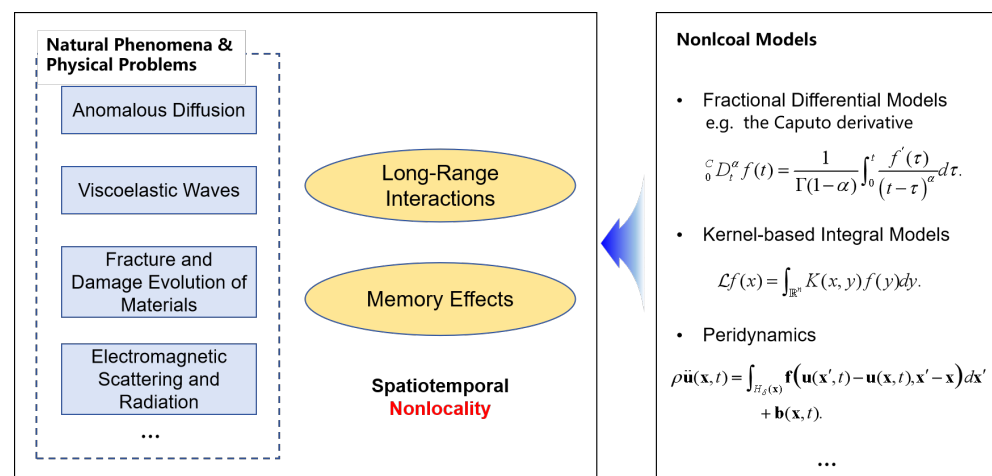


Figure 1. Conceptual framework of spatiotemporal nonlocality: from physical problems to modeling approaches.

In recent years, some progress has been made in the development of numerical methods for nonlocal models. Researchers have proposed a series of improvements in finite difference [40–45], finite element [46–48], and spectral methods [49–51]. Chen and Deng developed finite difference schemes employing weighted and shifted Lubich operators, which demonstrated high-order convergence in solving space-fractional diffusion equations [52]. Building upon the block-by-block strategy, Xu et al. [49,53] developed high-order numerical schemes that markedly improve computational accuracy in the time direction while maintaining stability. The Karniadakis group systematically developed a Petrov–Galerkin spectral method based on poly-fractionomials, which not only enables high-accuracy resolution of singular solutions but also demonstrates spectral convergence for tempered fractional Sturm–Liouville eigenvalue problems [54,55]. These approaches have shown favorable stability and convergence behavior when applied to low-dimensional problems.

However, when extending to three or higher spatial dimensions, nonlocality introduces severe computational challenges due to global coupling, dense matrix structures, and

memory-intensive history terms. This results in the well-known curse of dimensionality, manifested in:

- **Densification of system matrices:** Spatial coupling leads to fully populated or high-bandwidth matrices, dramatically increasing memory and storage demands.
- **Memory accumulation in time-fractional models:** Historical convolution terms scale poorly with time steps, with per-step costs growing linearly or quadratically.
- **Degraded convergence of iterative solvers:** High condition numbers hinder the efficiency of Krylov subspace methods in nonlocal settings.

This review provides a comprehensive and systematic synthesis of research advances in nonlocal modeling and high-dimensional numerical computation. It reviews four representative classes of nonlocal problems, highlighting the broad applicability and modeling diversity of nonlocal systems. Unlike traditional reviews that establish a one-to-one correspondence between problems and algorithms, this work conducts a systematic discussion along two parallel dimensions—modeling and computation. In the algorithm section, this paper organizes high-dimensional numerical methods based on core strategies, emphasizing key pathways to overcome the curse of dimensionality. By integrating recent advancements and general computational principles, this study offers a dual perspective that both examines problem essence and outlines methodological strategies. It provides a reference for researchers to understand, compare, and advance high-dimensional numerical computations for nonlocal models.

The structure of this review is as follows. In Section 2, four representative categories of nonlocal physical problems and their corresponding models are introduced. Section 3 systematically discusses numerical computation methods for high-dimensional nonlocal models from an algorithmic strategy perspective. These methods are not limited to the four problem classes addressed in Section 2. Section 4 concludes the review and outlines potential directions for future exploration.

2. Several Typical Types of Nonlocal Problems

Nonlocal operators are fundamental mathematical tools in the construction of nonlocal models. These operators, typically represented by fractional derivatives or integral operators, are used to capture long-range interactions and memory effects within a system. Common forms of fractional derivatives include the Caputo, Riemann-Liouville, and Grünwald-Letnikov types [56,57]. The Caputo operator is widely used in engineering and physical problems due to its clear physical meaning with respect to initial conditions, defined as

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1, \quad (1)$$

where Γ is the Gamma function. And α represents the fractional order, reflecting the intensity of the time memory effect. By switching the order of integration and differentiation, the Riemann-Liouville definition can be obtained. When transformed into a discrete form with a step size Δt , the Grünwald-Letnikov definition is derived,

$${}_0^G D_t^\alpha f(t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t - k\Delta t), \quad \alpha > 0. \quad (2)$$

For spatially non-local problems, the Riesz fractional Laplace operator is a commonly used mathematical tool. In addition to the Riesz definition, fractional Laplacian operators also have spectral and directional definitions, each with its advantages in different application scenarios.

Moreover, nonlocal integral operators capture nonlocal effects in space or time through integral forms, utilizing kernel functions to describe the interaction strength and range between a point and other points. The specific form is as follows

$$\mathcal{L}f(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy. \quad (3)$$

With the development of the theory, the nonlocal vector calculus framework has been proposed, extending the traditional scalar field nonlocal effects. This framework is capable of describing long-range interactions between the components of vector fields, providing a unified mathematical foundation for modeling complex systems [58].

The following section will delve into several typical non-local problems and their latest research developments.

2.1. Anomalous Diffusion

Anomalous diffusion phenomena are ubiquitous in natural environments and complex engineering systems, commonly observed in heterogeneous settings such as porous media, fracture networks, and groundwater systems. Unlike the linear diffusion behavior predicted by classical Fick's law, these processes exhibit distinctive features including non-Gaussian distributions, extended concentration tailing, and breakthrough curve hysteresis. The root cause lies in the strong memory effects and long-range spatial correlations present during particle migration, which are typically associated with the structural complexity of the medium, particle retention behavior, or long-distance hopping.

To characterize such phenomena, the Caputo fractional advection-diffusion equations (FADE) is widely adopted, typically expressed as

$${}_0^C D_t^\alpha u(x, t) = -v \frac{\partial u(x, t)}{\partial x} + k \frac{\partial^2 u(x, t)}{\partial x^2}, \quad (4)$$

where $u(x, t)$ denotes the solute concentration, v is the flow velocity, and k is the non-negative diffusion coefficient. The model incorporates a convolution kernel function to express the time memory effect, and its order α can regulate the migration lag and release behavior of particles [4]. Qiao et al. [6] further confirmed through controlled experiments on chloride ion migration in vertical fractures that the FADE exhibits higher fitting accuracy than traditional convection-diffusion equations in simulating non-Fickian diffusion behavior, particularly in reproducing the power-law tail of penetration curves and hysteresis response, as shown in Figure 2. Considering the complexity introduced by medium-scale heterogeneity and fracture-matrix coupling, Lei et al. [11] proposed a scale-amplification modeling method based on fractional derivatives, effectively capturing non-normal behavior in solute transport processes and providing important support for simulating migration processes over long time scales and large spatial regions.

In addition to temporal nonlocality, a key mechanism underlying anomalous diffusion lies in the non-Gaussian characteristics of spatial jump processes, such as Lévy flights. The spatial fractional diffusion equation (SFDE) has emerged as a important tool for describing such behavior, with its mathematical form as follows

$$\frac{\partial u(x, t)}{\partial t} = k D_x^\alpha u(x, t) + f(x, t). \quad (5)$$

Sun et al. [8] developed a generalized finite difference method (GFDM) for the three-dimensional SFDE to achieve numerical solutions, enabling accurate characterization of long-range jumps and boundary responses in irregular grids and heterogeneous media. Wang et al. [10] further extended this method to three-dimensional time-space fractional

convection-diffusion models and constructed a numerical framework suitable for complex boundary conditions and discontinuous problems.

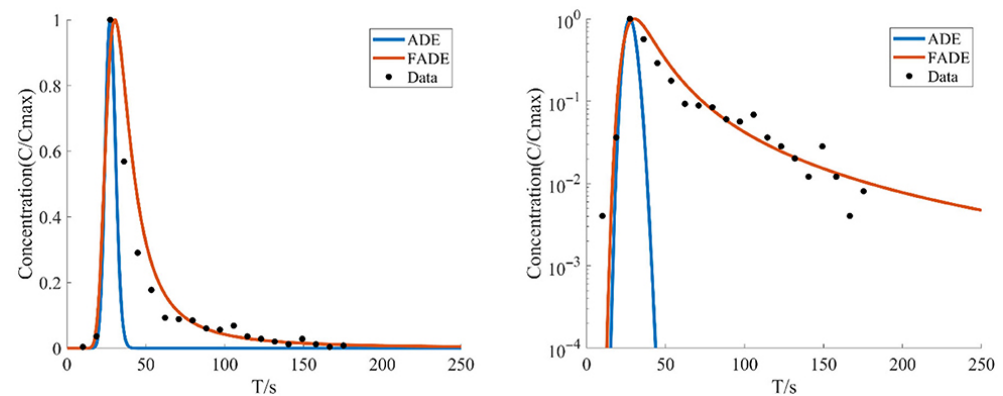


Figure 2. Measured data (symbols) and numerical simulation results (lines) [6]. The orange line denotes the solution from the FADE model. Copyright © 2020 Qiao, Xu, Zhao, Qian, Wu and Sun.

In discontinuous media or physical systems with fractal structures, the Hausdorff derivative can also be used to describe the modulation effect of the internal structural complexity of the medium on diffusion behavior, which is defined as

$${}^H D_t^\alpha f(t) = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t^\alpha) - f(t)}{\Delta t^\alpha}. \quad (6)$$

Hao et al. [7] proposed a time-dependent fractal model based on the Hausdorff derivative in their study of the migration process of inorganic arsenic in porous media. This model exhibits high sensitivity to characteristics such as the tail extension of the solute penetration curve and the main peak hysteresis. Furthermore, Tang et al. [9] developed a high-precision meshfree numerical solution method for surface diffusion problems in complex geometric structures by combining the Hausdorff derivative and the generalized finite difference method, which applies to problems involving complex structures.

Overall, the Caputo-FADE, SFDE, and Hausdorff derivative models respectively characterize the key mechanisms of anomalous diffusion from different dimensions. The FADE emphasizes temporal nonlocality, capturing memory-driven hysteresis and release behavior in heterogeneous media. The SFDE highlights spatial nonlocality, describing long-range jumps and superdiffusion in discontinuous structures. The Hausdorff derivative model extends nonlocality to fractal media, explicitly incorporating structural complexity within its equations.

Based on these spatio-temporal and structural generalizations, Li et al. [59] proposed a spatio-temporal interface-coupled fractional derivative model applicable to single-fissure systems, which unifies subdiffusive, superdiffusive, and non-Fickian interface mass exchange phenomena. This model excels in reproducing the heavy-tailed breakthrough curves observed in laboratory tracer experiments, highlighting the advantage of fractional modeling in capturing multiscale complex nonlocal transport phenomena.

2.2. Viscoelastic Waves

In seismic wave propagation and structural vibration engineering, the involved media often exhibit both elastic and viscous characteristics [12,14]. The viscoelasticity of materials results in a nonlocal relationship between stress and strain, meaning that the stress response at a given point is influenced not only by the local strain but also by the strain history and deformations occurring at remote locations.

To more accurately capture nonlocal response mechanisms, fractional-order viscoelastic models incorporating spatiotemporal nonlocal characteristics have emerged as a mainstream approach in contemporary material modeling. Among these, the Kelvin–Voigt fractional derivative (KVFD) model is a representative example, defined by the following stress–strain relationship

$$\left(1 + \ell_\sigma^\alpha (-\Delta)^{\alpha/2}\right) \sigma_{xx} = E \left(1 + \tau_\varepsilon^\beta D_t^\beta\right) \varepsilon_{xx}. \quad (7)$$

This model effectively captures both the long-range spatial interactions and temporal memory effects of materials, making it suitable for modeling media with microstructural defects, multiscale structural systems, and shear wave propagation in biological tissues [14,15]. Javadi [16] employed the KVFD model to simulate the viscoelastic behavior of beams in their study on nonlinear vibration and demonstrated that fractional-order parameters significantly influence the amplitude-response and frequency-response characteristics of the system. However, the KVFD model often performs poorly in simulating relaxation behavior.

In the study of the vibrational behavior of viscoelastic nanobeams, Martin introduced the fractional Zener model, whose constitutive equation is expressed as

$$\sigma(x, t) - \mu \frac{\partial^2 \sigma(x, t)}{\partial x^2} = G(0) \varepsilon(x, t) - \int_0^t D_\tau^\alpha G(t - \tau) \varepsilon(x, \tau) d\tau, \quad 0 < \alpha < 1, \quad (8)$$

enabling the simultaneous prediction of material creep and relaxation responses. The relaxation modulus of this model is defined as

$$G(t) = \frac{k_1 k_2}{k_1 + k_2} \left(1 + \frac{k_1}{k_2} E_\alpha\left(-\frac{t^\alpha}{\tau_\alpha^\alpha}\right)\right), \quad (9)$$

where E_α denotes the Mittag-Leffler function, as detailed in [13]. The research demonstrates that by replacing integer-order derivatives with Riemann–Liouville or Caputo fractional derivatives, the fractional Zener model significantly enhances the ability to characterize wave propagation and material relaxation phenomena. Bazhlekova et al. [18] further systematically investigated the generalized fractional Zener model by incorporating generalized convolution derivatives, proposing and proving the complete monotonicity and subordination principle under this model, thereby enriching the internal framework of fractional constitutive theory.

The development of nonlocal models has continued to advance. Wang et al. [17] constructed a fractional Laplacian wave equation to address seismic wave propagation in anisotropic attenuating media, simultaneously accounting for anisotropy in both velocity and attenuation. This model effectively captures the energy propagation characteristics within complex geological structures and is well-suited for full waveform inversion and crustal-scale seismic simulations. In addition, the work of Wang and Guo [60] further demonstrated the significant applicability of nonlocal models in geological exploration. As shown in Figure 3, the viscoelastic wave equation incorporating nonlocal effects exhibits a stronger attenuation behavior when simulating seismic wave propagation in realistic geological media. Shi et al. [22] integrated fractional viscoelastic constitutive theory with stress wave theory, establishing a viscoelastic stress wave propagation model based on fractional derivatives, more accurately characterizing the frequency-dependent propagation behavior of stress waves in viscoelastic rods.

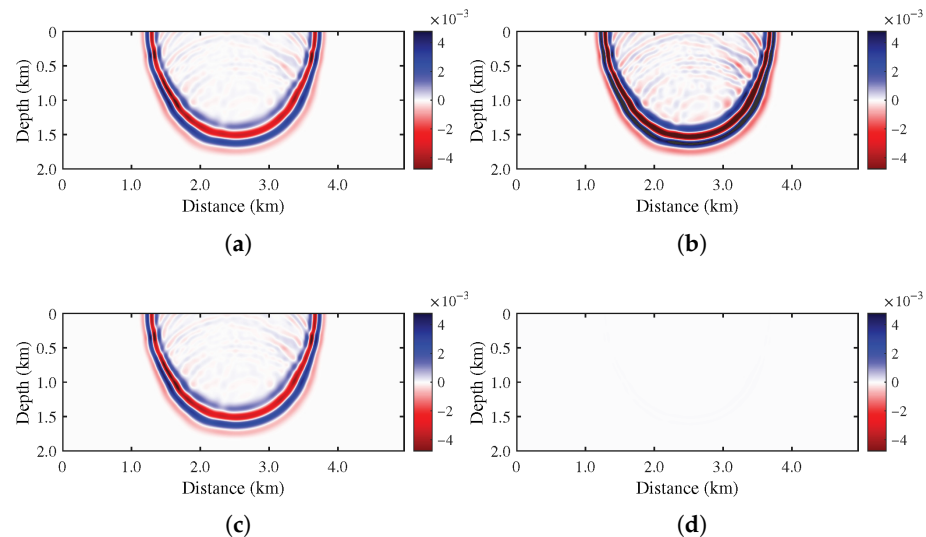


Figure 3. Snapshots of the wavefield at 600 ms [60]: (a) Reference. (b) Simulation results based on the elastic wave equation. (c) Simulation results based on the time-fractional viscoacoustic wave equation. (d) Residual.

It is thus evident that fractional-order derivative models exhibit strong physical consistency and mathematical flexibility in modeling viscoelastic wave propagation, establishing them as the core theoretical framework in current research. However, beyond fractional-order models, some researchers have conducted related studies by combining traditional viscoelastic constitutive relations with nonlocal elasticity theory, providing new insights into understanding viscoelastic wave propagation behavior. A representative example is the work by Shariati et al. [61], who analyzed the vibration stability of function-gradient viscoelastic nanobeams undergoing axial motion. They employed the Kelvin–Voigt constitutive equations combined with Eringen’s nonlocal elasticity theory to formulate the following governing equations:

$$\sigma(x, t) - (\mu L)^2 \frac{\partial^2 \sigma}{\partial x^2} = E(x) \varepsilon(x, t) + \beta \frac{\partial \varepsilon}{\partial t}. \quad (10)$$

The model incorporates both the nonlocal length scale μ and the material viscosity coefficient β , while also accounting for the continuous axial gradient distribution of the elastic modulus $E(x)$. It effectively reveals the complex coupling between viscous dissipation and nonlocal softening effects.

In summary, the KVFD captures the long-range effects and time memory of viscoelastic materials but has limitations in describing stress relaxation. The fractional-order Zener model incorporates both creep and relaxation responses, making it suitable for broadband viscoelastic materials and nanostructures. The fractional Laplacian wave equation effectively characterizes anisotropic attenuation and frequency-dependent propagation in complex geological media, while the nonlocal elasticity–viscoelastic coupling framework reveals the interplay between viscous dissipation and nonlocal softening.

2.3. Fracture and Damage Evolution of Materials

Fracture and damage evolution of materials constitute fundamental physical mechanisms underlying solid failure. These processes involve complex phenomena such as long-range stress interactions, indeterminate crack path propagation, cooperative evolution of microdefect clusters, and pronounced scale effects.

On the one hand, to effectively characterize the cooperative evolution of micro-defects and scale effects, researchers have developed nonlocal gradient damage models, which are fundamentally defined as

$$\sigma = \mathbf{C} : (\varepsilon + l_c^2 \nabla^2 \varepsilon), \quad (11)$$

where σ denotes the Cauchy stress tensor, $\mathbf{C}$ represents the fourth-order elastic stiffness tensor, ε is the infinitesimal strain tensor, and l_c denotes the intrinsic material length parameter. The model quantifies the spatial correlation of damage through the second-order gradient term $\nabla^2 \varepsilon$ and employs l_c to account for scale effects, thereby addressing the issue of damage localization in classical models.

Building upon this foundation, Haim et al. [23] introduced a permeability gradient to develop a nonlocal damage-poroelastic model, which effectively eliminates spurious oscillations observed in local models and mitigates mesh dependency. Ren et al. [27] proposed an extended gradient damage model to characterize fracture behavior, wherein the irreversibility of damage evolution was ensured by decoupling the damage evolution from the cohesive law. Wu and his collaborators [28] constructed a strictly monotonically convergent gradient damage model to describe the damage and fracture behavior of various quasi-brittle materials. By solving a first-kind Volterra integral equation, they naturally incorporated arbitrary forms of cohesive laws into the modeling framework.

On the other hand, the PD method proposed by Silling [29] has been widely employed in the study of crack path uncertainty and dynamic fracture processes. It is formulated as

$$\rho \ddot{\mathbf{u}}(\mathbf{x}, t) = \int_{H_\delta(\mathbf{x})} \mathbf{f}(\mathbf{u}(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t), \mathbf{x}' - \mathbf{x}) d\mathbf{x}' + \mathbf{b}(\mathbf{x}, t). \quad (12)$$

Here, $\mathbf{u}$ denotes the displacement vector, $\mathbf{f}$ is the pairwise force function, $\mathbf{b}$ is some prescribed loading force density, and $H_\delta(\mathbf{x})$ refers to a spherical neighborhood centered at $\mathbf{x}$ with radius δ , as illustrated in Figure 4. By introducing nonlocal terms such as spatial integrals of pairwise forces and damage criteria, the PD framework overcomes the limitations of classical continuum mechanics based on partial differential equations, which require spatial derivatives and assume continuity of the displacement field. This approach allows for the simulation of crack initiation, propagation, and branching without the need to pre-specifying the crack path.

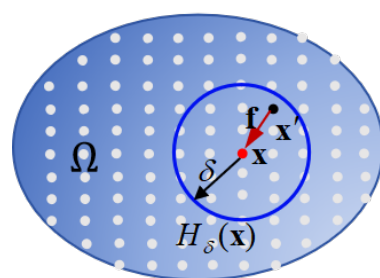


Figure 4. Interaction point and its neighborhood $H_\delta(\mathbf{x})$ within the continuous domain Ω .

Zhou et al. [62] proposed an improved bond-based PD model, introducing shear bonds alongside traditional tensile bonds. Numerical results revealed that the diagonal shear crack bands formed in rock-like materials under compression are not merely the outcome of localized failure, but rather the manifestation of accumulated long-range interactions within the horizon. Ma et al. [33] integrated a strain energy density softening criterion into a bond-based PD model to construct a damage evolution function accounting for material softening and heterogeneity. This approach directly reflects the structural fracture process of rock materials, revealing the critical role of nonlocal interactions in the synergistic evolution of multiple cracks. Abdoh et al. developed a three-dimensional PD model to reveal

fiber interactions and damage accumulation pathways within ballistic materials under high-speed impact loading [34]. Collectively, these studies demonstrate that long-range interactions are not only a fundamental physical origin of crack initiation and propagation, but also a core element for accurately characterizing the damage mechanisms of complex materials under multi-crack cooperative evolution and extreme loading conditions.

2.4. Electromagnetic Scattering and Radiation

At the nanoscale or within strongly coupled media, electromagnetic scattering and radiation exhibit pronounced nonlocal characteristics. The response of a material to an external electromagnetic field depends not only on the local electric field but is also strongly coupled to the field distribution in the surrounding spatial domain. Similarly, the electromagnetic field intensity at a given point is no longer determined solely by local source terms or material parameters, but is instead associated with the current and charge distributions over the entire scatterer or its boundaries.

Nonlocal electromagnetic responses primarily arise from collective electronic behavior and spatial dispersion effects, which are particularly pronounced in systems such as plasmas and metallic nanostructures. To characterize this effect, the constitutive relation between the electric displacement field $\mathbf{D}$ and the electric field $\mathbf{E}$ can be expressed in a spatial convolution form:

$$\mathbf{D}(\mathbf{r}, \omega) = \varepsilon_0 \int \varepsilon(\mathbf{r}, \mathbf{r}'; \omega) \mathbf{E}(\mathbf{r}', \omega) d\mathbf{r}. \quad (13)$$

Here, the nonlocal dielectric kernel function $\varepsilon(\mathbf{r}, \mathbf{r}'; \omega)$ characterizes the long-range coupling of fields between different spatial locations. Previous studies have demonstrated that incorporating nonlocal responses can account for physical phenomena [63–65] that local models, such as size-dependent blue shifts of plasmonic resonances, spectral line broadening, and the saturation of hotspot field intensities in sub-nanometer gaps, cannot capture.

In terms of modeling approaches, the integral equation framework inherently embodies nonlocality. The integral representation constructed via Green's functions directly characterizes the global coupling between field points and source points. The surface integral equations (SIE), derived from the surface equivalence principle, is particularly well-suited for scattering problems involving perfect electric conductors (PECs) and can be expressed as

$$-ik\eta \iint_{\partial\Omega} \left(\mathbf{J}(\mathbf{r}') + \frac{1}{k^2} \nabla(\nabla \cdot \mathbf{J}(\mathbf{r}')) \right) G(\mathbf{r}, \mathbf{r}') d\mathbf{r}' = \mathbf{E}_{\text{inc}}(\mathbf{r}), \quad (14)$$

$$\frac{1}{2} \mathbf{J}(\mathbf{r}') - \hat{\mathbf{n}} \times \text{P.V.} \iint_{\partial\Omega} \mathbf{J}(\mathbf{r}') \times \nabla G(\mathbf{r}, \mathbf{r}') d\mathbf{r}' = \hat{\mathbf{n}} \times \mathbf{H}_{\text{inc}}(\mathbf{r}). \quad (15)$$

where, $\mathbf{E}_{\text{inc}}$ and $\mathbf{H}_{\text{inc}}$ denote the incident electric and magnetic fields; G represents the free-space Green's function; $\mathbf{J}(\mathbf{r})$ is the induced surface current density within the domain; and $\hat{\mathbf{n}}$ denotes the unit normal vector to the surface of the scatterer, as detailed in [37]. The SIE requires only surface discretization to efficiently capture nonlocal interactions at boundaries. For structures with complex medium distributions or subwavelength nanoscale features, the volume integral equation (VIE) is more appropriate, generally defined as

$$\frac{1}{1 - 1/\varepsilon_\gamma} \frac{\mathbf{J}(\mathbf{r})}{j\omega\varepsilon(\mathbf{r})} + j\omega\mu_0 \int_V G\mathbf{J}(\mathbf{r}') dV' + \frac{j}{\omega\varepsilon_0} \nabla \nabla \cdot \int_V G\mathbf{J}(\mathbf{r}') dV' = \mathbf{E}_{\text{inc}}(\mathbf{r}). \quad (16)$$

The VIE characterizes the electromagnetic field distribution inside the medium through discretization of the volume integral, allowing it to handle internal inhomogeneities and dispersive properties. However, this approach incurs higher computational costs.

To further enhance the physical fidelity of the model, Bagci et al. [38] have coupled the VIE with the hydrodynamic equation (HDE) to capture the collective dynamical response of free electrons in metals. This approach demonstrates precise analytical capabilities for nonlocal plasmonic modes, longitudinal field components, and energy dissipation mechanisms in metal–dielectric–metal structures.

In summary, the SIE is suitable for boundary-dominated scattering problems, while the VIE is more appropriate for medium-dominated structural responses. The coupled models excel in capturing metallic electron dynamics and strong nonlocal interactions. For specific application scenarios, the choice of modeling strategy should comprehensively consider factors such as structural scale and material properties. For ease of comparison, Table 1 provides a systematic summary of the application scenarios, advantages, and limitations of the representative methods.

Table 1. Comparison of Nonlocal Electromagnetic Modeling Methods for Scattering and Radiation Problems.

Method	Applicable Scenarios	Advantages	Limitations
Convolutional Dielectric Kernel	Metal nanostructures, plasmonic media	Captures longitudinal plasmon modes; explains resonance blueshift, spectral broadening, and hot-spot saturation	Requires specification of nonlocal kernels; parameterization may be model dependent and complex
SIE	PECs, smooth boundaries	Reduces dimensionality; efficient for large-scale scattering; satisfies radiation condition naturally	Limited to simple boundary conditions and non-dispersive media
VIE	Inhomogeneous dielectrics, subwavelength composites	Suitable for dielectric-metal hybrids and multilayers	High computational cost due to volumetric discretization; challenging to handle sharp resonances
VIE–HDE	Metal–dielectric–metal systems	Enables joint treatment of electromagnetic fields and nonlocal electron dynamics; accurately reproduces nonlocal resonances and charge spill-out effects	Computationally demanding

The aforementioned classes of representative nonlocal problems are predominantly modeled within a constant-order fractional framework. However, the assumption of a fixed fractional order in both space and time limits the capability of such models to capture the dynamic variation of nonlocal interaction strength with spatial location, temporal evolution, or environmental conditions in real physical systems [66–68]. To address these limitations, researchers have proposed variable-order fractional nonlocal models, whose core concept is to incorporate spatiotemporal dependence into the fractional derivative by defining the order α as a function of spatial position, time, or other state variables, i.e., $\alpha = \alpha(x, t, \dots)$. This formulation explicitly reflects the dynamic influence of medium properties, environmental conditions, or loading history in the model.

The flexibility of variable-order formulations allows the fractional order to vary spatially or temporally, capturing heterogeneous diffusion or evolving memory effects. Figures 5 and 6 illustrate nonlocal models with variable order exhibit greater physical consistency than conventional models in addressing phenomena such as anomalous transport and viscoelastic behavior of materials.

- A representative variable-order diffusion model can be expressed as

$$D_t^{\alpha(\mathbf{x},t)} u(\mathbf{x}, t) = K_{\beta(\mathbf{x},t)} (-\Delta)^{\beta(\mathbf{x},t)/2} u(\mathbf{x}, t), \quad (17)$$

which enables the simulation of local diffusion mechanisms in heterogeneous media that adapt to variations in position and time [68,69]. The choice of fractional orders is governed by the characteristics of the medium and the diffusion state. In general, the time-fractional order $\alpha(\mathbf{x}, t)$ is influenced by the evolution of retention behavior across multiple scales, whereas the space-fractional order $\beta(\mathbf{x}, t)$ is associated with the structural non-stationarity of the medium.

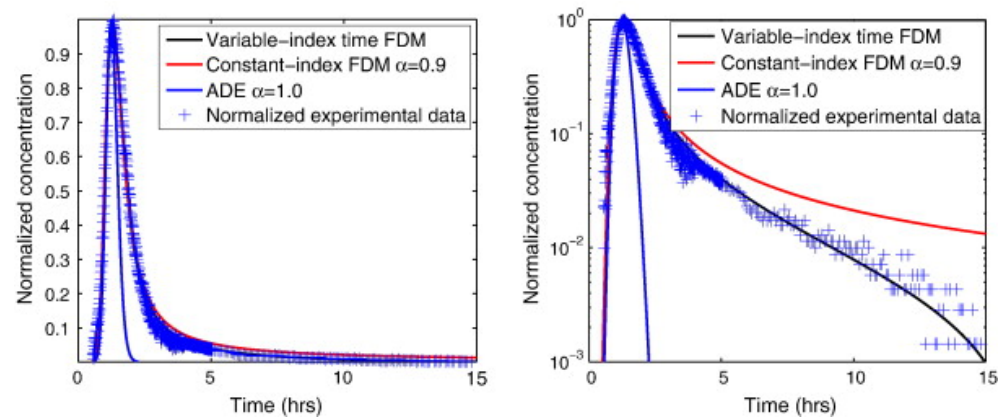
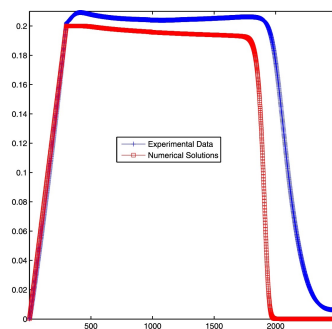


Figure 5. Comparison between constant-order and variable-order fractional diffusion models. The variable-order formulation more accurately captures anomalous diffusion behaviors [70].

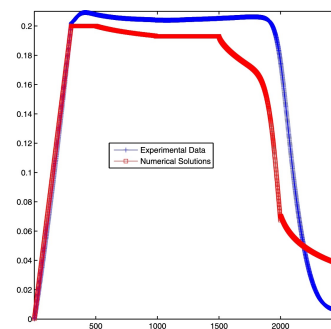
- The variable-order viscoelastic constitutive model

$$\sigma(t) + \tau_{\sigma}^{\alpha(t)} D_t^{\alpha(t)} \sigma(t) = E \left[\varepsilon(t) + \tau_{\varepsilon}^{\alpha(t)} D_t^{\alpha(t)} \varepsilon(t) \right], \quad (18)$$

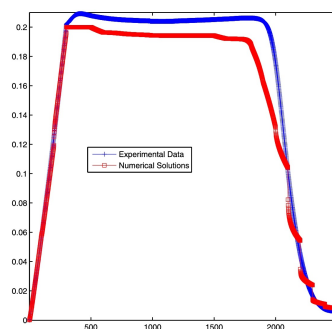
where the order $\alpha(t)$ varies with strain, temperature, or loading rate, thereby allowing the model to accurately characterize the dynamic adjustment of the material's relaxation mechanism under different loading stages [66,71].



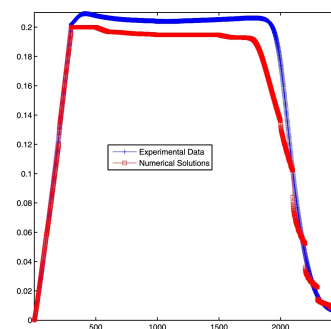
(a) constant-order FDE model



(b) variable-order FDE model, $N = 5$



(c) variable-order FDE model, $N = 25$



(d) variable-order FDE model, $N = 125$

Figure 6. Comparison between variable-order and constant-order FDE models in fitting material response: (a) constant-order FDE model. (b) variable-order FDE model, $N = 5$. (c) variable-order FDE model, $N = 25$. (d) variable-order FDE model, $N = 125$. The variable-order fractional differential equation models more accurately describe the memory behavior of shape memory polymers [72].

3. Numerical Computation for High-Dimensional Nonlocal Models

Nonlocal models offer distinct advantages for characterizing material fracture, anomalous diffusion, and long-range interactions; however, their high-dimensional numerical realization faces formidable challenges, including rapidly escalating memory demands, bottlenecks in computational efficiency, limited geometric adaptability, and difficulties in ensuring stability over long-time simulations. Here, high-dimensional in this paper refers not only to spatial dimensions of three or more, but also to extended models incorporating time, parameters, or random variables, all of which contribute to the exponential growth in computational complexity. Although numerous efficient numerical methods have been developed for the one-dimensional setting, their extension to higher dimensions is often not directly applicable. The techniques discussed below emphasize general algorithmic strategies, rather than being restricted to specific physical problems.

3.1. Computational Challenges

The curse of dimensionality constitutes the primary obstacle in high-dimensional numerical computation. Due to the involvement of long-range interactions, nonlocal models typically generate dense or even fully dense and structurally complex matrices after discretization, leading to a substantial increase in computational resource requirements, as shown in Table 2. In the one-dimensional case, employing Gaussian elimination to directly solve the discretized system at each time step generally requires $O(N^2)$ storage and $O(N^3)$ computational cost. As the spatial dimension increases, the size of the discretized matrix grows exponentially. For a three-dimensional nonlocal problem, i.e., dimension $d = 3$, assuming a uniform grid of $n = 200$ points per dimension within a unit cube, the number of unknowns reaches $N = 8 \times 10^6$. Explicitly storing the dense stiffness matrix in this case would occupy approximately 512 TB of memory, rendering direct methods practically infeasible.

Although iterative solvers such as the Conjugate Gradient (CG) method and Generalized Minimal Residual (GMRES) can reduce overall computational complexity, the matrix condition number significantly deteriorates markedly with mesh refinement, resulting in poor convergence performance. Moreover, each iteration entails dense or high-bandwidth matrix–vector multiplications, resulting in a substantial per-iteration cost.

More critically, many real-world problems are inherently high-dimensional, and their physical and geometric complexity further exacerbates the computational burden associated with increasing dimensionality. For example, in geomechanical hydraulic fracturing, fracture propagation and fluid flow constitute a strongly coupled multiphysics system; dynamically simulating long-term crack evolution therefore imposes stringent demands on numerical stability, memory consumption, and computational resources.

Table 2. Memory and computational complexity estimates for typical nonlocal models under increasing spatial dimensions with $n = 256$.

d	N	Matrix Memory	MatVec Cost per Step	Feasibility Assessment
1	2.56×10^2	0.51 MB	$O(10^4)$	Feasible
2	6.55×10^4	2.67×10^1 GB	$O(10^9)$	Feasible
3	1.68×10^7	2.05×10^3 TB	$O(10^{14})$	Intractable
4	4.29×10^9	1.34×10^8 TB	$O(10^{19})$	Intractable

3.2. Efficient Implementation of High-Dimensional Numerical Computation

In addressing the core challenges inherent in high-dimensional numerical computation for nonlocal models, the past decade has seen a proliferation of innovative methodologies aimed at improving computational efficiency, numerical stability, and scalability. These can be categorized as follows: (i) Probabilistic sampling methods, which reconstruct non-

local operators via stochastic integration to alleviate computational burdens from dense coupling; (ii) Structure-utilizing methods, which leverage matrix structural patterns for efficient computation; (iii) Spectral methods and high-order discretization techniques, which enhance global accuracy while maintaining fewer degrees of freedom; (iv) Neural network approximations, integrating data-driven learning with physical constraints; (v) Asymptotically compatible schemes ensure numerical stability when degenerating to the classical local limit. This section focuses on these representative methods and systematically reviews their research developments, analyzing their core mechanisms, key challenge mitigation strategies, and practical computational performance in depth, to provide theoretical and methodological support for the high-dimensional numerical computation of nonlocal models.

To mitigate the curse of dimensionality, randomized probabilistic frameworks have shown significant potential. By employing Monte Carlo techniques or stochastic process representations, these methods reformulate high-dimensional nonlocal integral operators as expectation estimates, thereby circumventing the discretization overhead of traditional mesh-based schemes. Agrawal et al. [73] proposed a random feature expansion approach transmitting information in feature space to bypass costly spatial discretization, achieving notable results in state spaces of up to hundreds of dimensions. Yang et al. [74] developed a Feynman–Kac-based method, leveraging the intrinsic adaptability of stochastic path integrals to complex boundaries and avoiding dense linear solvers. Recognizing the adaptability of stochastic particles in avoiding unnecessary storage, Lei et al. [75] introduced a stochastic particle method based on the Lawson–Euler weak formulation, which achieves both accuracy preservation and computational efficiency for moderately high-dimensional problems. These randomized probabilistic frameworks provide mesh-free and dimension-independent solutions for high-dimensional nonlocal problems. Their reliance on sampling rather than explicit discretization endows them with exceptional scalability and parallelizability, particularly when handling integral operators and long-range interactions. However, their performance is highly dependent on the variance of the estimator. Without effective variance reduction techniques or favorable kernel regularity, the required sample size or number of features may grow exponentially, significantly reducing computational efficiency in extreme settings.

Moreover, researchers have also devoted efforts to leveraging intrinsic structures such as multilevel hierarchies and low-rank properties to develop acceleration strategies and construct efficient reduced-order models, thereby alleviating the high computational and memory costs [76–81]. A representative example is the work of Wang’s team [82,83]. In the context of fractional diffusion equations, they employed a matrix-splitting framework that leverages the Toeplitz structure of coefficient matrices (see Figure 7). By integrating FFT-accelerated matrix–vector products, directional decomposition for dimensionality reduction, and structure-preserving boundary treatments, they reduced computational complexity from the conventional $O(N^3)$ to $O(N \log N)$ and memory requirements to $O(N)$, effectively overcoming long-standing bottlenecks in three-dimensional nonlocal simulations. In parallel, Vollmann et al. [84] further proposed a structured finite element method that exploits the translation and reflection invariance of the kernel function to construct symmetric multilevel Toeplitz stiffness matrices. This design drastically reduces storage requirements and facilitates efficient algorithmic acceleration. Guan et al. [85] combined a greedy reduced-order method with a sparse grid strategy, reformulating the solution of large-scale linear systems into a set of independent small-scale problems, thereby reducing the overall computational cost. These strategies achieve significant computational efficiency gains without sacrificing accuracy, and are well-suited for large-scale deterministic simulations and uncertainty quantification. However, this efficiency typically relies on

strong structural assumptions, such as regular grids and symmetric kernels. In irregular domains or under strongly anisotropic interactions, the matrix structure may break down, rendering fast transforms ineffective. Despite these limitations, structure-utilizing methods remain one of the most scalable and accurate solver categories for solving high-dimensional nonlocal problems.

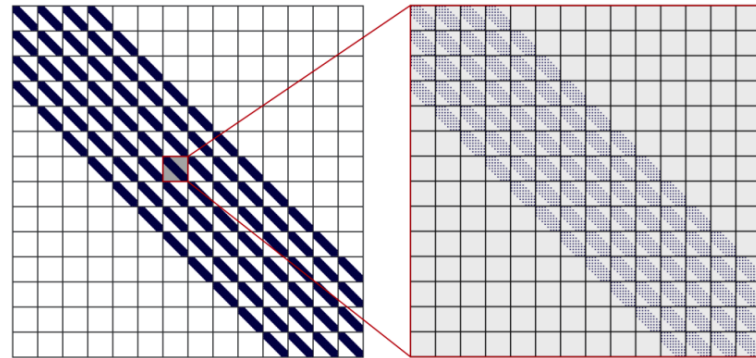


Figure 7. Block-Topplits-Topplits-Block Matrix Structure.

Spectral methods [86–91], renowned for their superconvergence in handling smooth problems, serve as a powerful high-accuracy discretization technique. In bounded domains, Chen et al. [92] combined the Caffarelli–Silvestre extension with a Laguerre-enriched approach, leveraging spectral diagonalization in the extended dimension to reduce high-dimensional singular problems to low-dimensional Poisson subproblems. In unbounded domains, Sheng et al. [93] employed a mapped Chebyshev fast spectral method, utilizing the Dunford–Taylor representation for operator diagonalization in high-dimensional integral-type nonlocal problems. For typical nonlocal diffusion problems with power-law slow decay, Tang et al. [94] developed a rational spectrum method that exhibits superior approximation performance when dealing with slow decay tails and algebraic nonlocal kernels. Polynomial-based high-dimensional collocation schemes have likewise demonstrated robustness and exponential convergence. Abdelkawy et al. [95] employed a shifted Legendre spectral collocation method to solve high-dimensional nonlinear fractional wave equations involving the Riesz operator, achieving exponential convergence. Moussa et al. [96] further introduced a spectral collocation method based on non-smoothly mapped Legendre polynomials for solving time-fractional diffusion–wave equations, which retains spectral convergence in high-dimensional computations and outperforms conventional spectral methods in both accuracy and stability. In short, spectral methods effectively handle challenges such as singular weights, unbounded domains, and fractional operators. In moderate dimensions (2D–5D), when combined with tensor product structures, matrix diagonalization, or FFT fast algorithms, spectral methods still maintain acceptable computational efficiency. However, as dimensionality increases, both computational cost and memory requirements grow polynomially or even exponentially. Additionally, spectral methods generate dense global matrices and are sensitive to mapping parameters, with reduced flexibility for irregular geometries or low-regularity solutions.

With the rapid advancement of emerging technologies, deep learning offers powerful tools for solving high-dimensional partial differential equations [97–101]. Their mesh-free property and ability to capture global solutions exhibited when dealing with nonlocal operators have attracted wide attention. Pang et al. [102] developed nonlocal physics-informed neural networks (nPINNs) that combine Sobol sequence sampling with composite Gaussian quadrature, enabling stable training and accurate nonlocal parameter identification in three dimensions. Shen et al. [103] introduced MC-Nonlocal-PINNs, reformulating generalized nonlocal integral operators into a Beta-distribution-based sampleable form. The

approach enhances adaptability to singular kernels, fractional derivatives, and variable integration domains, removes dependence on automatic differentiation, and achieves nearly dimension-independent complexity. Karniadakis et al. [104] employed Monte Carlo and quasi-Monte Carlo sampling with sample deduplication and classification to replace high-dimensional grid integration. The generalized Monte Carlo PINNs (GMC-PINNs) reduced computational time by about 50% compared with conventional fPINNs over 300,000 iterations. However, applying PINNs to high-dimensional nonlocal problems remains highly challenging. The evaluation of nonlocal operators during training introduces heavy computational overhead, as each iteration involves high-dimensional integrations that require costly Monte Carlo or quadrature approximations. Singular kernels and sharp solution features further complicate optimization, often causing gradient instability and loss imbalance. Long-memory effects in time-fractional or history-dependent operators dramatically increase the number of training constraints, while standard architectures suffer from spectral bias—favoring low-frequency components and failing to capture oscillatory or singular dynamics. Although recent advances, such as importance sampling, auxiliary variables, and Fourier feature embeddings, have partially alleviated these issues, their scalability and reliability for complex nonlocal operators remain limited. Addressing these challenges demands both algorithmic innovation and a deeper theoretical understanding of neural representations of nonlocality.

Beyond the curse of dimensionality, a common challenge lies in maintaining numerical stability when degenerating to the classical local limit. To address this issue, Du and his collaborators [105–108] have conducted systematic research on constructing asymptotically compatible schemes for nonlocal models. By employing structure-preserving discretization, boundary-layer adjustments, and mapping techniques, they ensure that as the nonlocal scale tends to zero, the numerical solution correctly converges to its local limit, exhibiting high robustness—particularly suitable for complex problems such as crack propagation in peridynamics and multiphysics coupling. Del Teso et al. [109,110] unified local and nonlocal diffusion operators under the framework of Lévy-type nonlocal operators and developed a class of asymptotically compatible schemes that naturally maintain consistency in the nonlocal-to-local limit, further proving that the scheme preserves structural monotonicity and stability.

Research on nonlocal problems with complex boundaries in high dimensions has also made some progress [111–113]. A representative example is the work of Du and Ni [114,115] on high-dimensional Fisher–KPP nonlocal diffusion equations with free boundaries. Their multiscale matching method is coupled with the kernel structure of the fractional Laplace operator, enabling a refined characterization of the interplay between boundary propagation speed and nonlinear feedback of the solution. This approach ensures numerical accuracy while achieving smooth evolution and topological preservation of the free boundary, demonstrating strong adaptability in simulations of ecological invasion, financial liquidity fronts, and biological tissue propagation.

Overall, these algorithmic strategies embody multiple trade-offs between scalability, accuracy, geometric adaptability, and robustness. Their effectiveness often depends on the specific problem characteristics and available computational resources. Table 3 summarizes the representative complexity and convergence behavior of these methods to facilitate practical selection and further research development.

Table 3. Comparison of representative numerical methods for high-dimensional nonlocal models. In probabilistic methods, N denotes the number of Monte Carlo or particle samples, determining computational cost and variance convergence rate; in structure-exploiting methods, N represents the number of grid points or the matrix size (FFT complexity $\mathcal{O}(N \log N)$); in PINNs, N is the number of training points (collocation points). M denotes the number of random features or basis functions; E is the number of training epochs; Q denotes the number of quadrature or Monte Carlo sampling points used for temporal/spatial integration; p represents the order of convergence; and P is the number of trainable network parameters.

Method Category	Core Strategy	Computational Complexity	Convergence/Accuracy Rates	Main Advantages	Limitations
Probabilistic Methods [73–75]	Sampling-based estimation of nonlocal integrals	$\mathcal{O}(N) - \mathcal{O}(N \log N)$ for Monte Carlo/particle schemes; $\mathcal{O}(NM)$ for random-feature approximations	Typical MC rate $\mathcal{O}(N^{-1/2})$; with variance-reduction up to $\mathcal{O}(N^{-1})$; Random feature approximation $\mathcal{O}(M^{-1/2})$	Mesh-free, dimension-robust, naturally parallelizable; effective for integral operators and mean-field interactions	Statistical variance; slow convergence without variance reduction; requires large samples for singular kernels or strong nonlinearity
Structure-exploiting techniques [76–85]	Exploits block-Toeplitz/multilevel-Toeplitz matrices, FFT-based fast convolution, affine-decomposed operators for parametric reduction	$\mathcal{O}(N \log N)$ per iteration for FFT-based FDM/FEM	Same numerical order as the underlying discretization $\mathcal{O}(h^p)$	Memory-efficient, and scalable, preserves accuracy while reducing cost, interpretable	Requires regular grids and translation-invariant kernels to maintain Toeplitz structure, complex boundaries or irregular domains break efficiency
Spectral Methods [86–96]	Global high-order polynomial approximation	$\mathcal{O}(N^d)$ for rational or Laguerre spectral expansions; $\mathcal{O}(N_t N_s^d)$ for time-space spectral schemes	Spectral (exponential) convergence for smooth or weakly singular solutions; optimal error bounds in fractional Sobolev spaces	Spectral accuracy with relatively few degrees of freedom; suitable for singular-weight or unbounded-domain problems	Dense global matrices require memory optimization; parameter tuning for mapped/rational bases
PINNs [97–104]	Neural approximation of solutions, trained via physics-based loss	Typically $\mathcal{O}(NP)$ per iteration; total cost $\mathcal{O}(ENP)$ with E training epochs	$\mathcal{O}(N^{-1/2})$ for standard MC-PINNs; $\mathcal{O}(Q^{-p})$ for MC-tfPINN with Quadrature;	Mesh-free; flexible for irregular domains; scalable to extremely high dimensions (up to 100,000 D); compatible with inverse problems and noisy data	Sensitive to neural network architecture and optimization; convergence sensitive to sampling variance; performance drops for strong singularities or complex boundary layers

4. Discussion

Nonlocal models display theoretical advantages in describing long-range spatial interactions and temporal memory effects. This paper analyzes the subject from two parallel perspectives. On one hand, it reviews representative nonlocal physical problems and their modeling characteristics, demonstrating the applicability and advantages of nonlocal modeling across diverse physical contexts. On the other hand, it systematically summarizes high-dimensional numerical computation methods from an algorithmic strategy standpoint, with a focus on outlining core approaches to addressing the dimension catastrophe. By integrating recent research advances with common computational principles, this review provides a systematic research framework for understanding the theoretical characteristics of nonlocal modeling and its high-dimensional numerical implementation.

To more intuitively demonstrate the applicability of different numerical strategies in typical nonlocal problems, Table 4 summarizes the correspondence between the four categories of typical nonlocal problems and high-dimensional numerical methods discussed in this paper. Structured and spectral methods yield the best numerical simulation performance for anomalous diffusion problems. In viscoelastic wave propagation, long-memory kernels are well suited to FFT acceleration and spectral methods. For fracture and damage problems, random sampling and PINNs better capture interactions between discontinuous fields and microcracks. Although the spectral method is applicable to electromagnetic scattering, oscillations and dispersion kernels weaken its stability, necessitating integration with boundary element methods or fast multipole methods to enhance robustness. Overall, algorithm suitability exhibits problem-dependent characteristics, with no universal optimal solution existing.

Table 4. Mapping between representative nonlocal physical problems and the classes of high-dimensional numerical methods most compatible with them. The filled asterisks (*) qualitatively indicate the adaptability level: ***** excellent, **** good, *** moderate, ** limited. This summary bridges the physical models (Section 2) and computational strategies (Section 3).

Problem Type	Randomized Probabilistic Methods	Structure-Exploiting Methods	Spectral Methods	PINNs and Related Deep Learning Methods
Anomalous diffusion	***** Random trajectories (Lévy flights, subordinated Brownian motions) directly correspond to the physical process. Monte Carlo integration scales weakly with dimension and naturally captures long-range memory; efficiency is mainly limited by statistical variance.	***** The fractional Laplacian forms translation-invariant convolution kernels. FFT-based Toeplitz solvers achieve near-linear scaling, making this class ideal for high-dimensional homogeneous media.	***** Fractional Laplacian and variable-order operators are smooth and separable; spectral bases yield exponential convergence and efficient tensorization in moderate dimensions.	**** fPINNs and MC-fPINNs efficiently encode fractional operators through residual losses; limited by variance in integral evaluation and spectral bias near boundaries.
Viscoelastic wave propagation	*** Stochastic time-marching handles hereditary effects but demands variance control and hybrid coupling to maintain temporal accuracy in oscillatory regimes.	**** If the memory kernel is spatially stationary or separable, FFT-accelerated convolution and structured Krylov methods are equally effective for handling long-memory integrals.	**** Legendre or mapped Chebyshev bases capture oscillatory yet continuous fields. Efficiency decreases for highly coupled or high-frequency regimes.	*** BO-fPINNs and MC-fPINNs approximate fractional damping accurately, but long-memory convolution causes unbalanced gradients and training stiffness.
Fracture and damage evolution (Peridynamics)	*** Particle-based schemes capture microcrack statistics but are costly for strong nonlinearity.	** Crack initiation, propagation, and localization disrupt the global structural integrity of the matrix, leading to a significant decline in the efficiency of Toeplitz-type fast algorithms.	** Fields exhibit discontinuities and topology changes; global spectral bases lose exponential convergence.	*** nPINNs can infer peridynamic parameters, and MC-Nonlocal-PINNs approximate singular kernels, but training suffers from non-smooth losses and poor convergence around crack tips.
Electromagnetic scattering and radiation	** Useful for stochastic integral or kernel approximation; efficiency depends on kernel smoothness.	*** Periodic or layered structures preserve block-Toeplitz symmetry enabling FFT acceleration, whereas strongly coupled plasmonic systems break these structural advantages.	*** Fourier or rational spectral bases approximate spatial dispersion efficiently in smooth periodic media but yield dense systems in 3D complex geometries.	** MC-Nonlocal-PINNs can approximate nonlocal dielectric kernels; oscillatory fields exacerbate training instability; hybrid PINN-BEM strategies recommended.

From the perspective of algorithmic strategies, structural algorithms can significantly reduce computational complexity in problems with regular geometry. At the same time, spectral methods have a clear advantage in high-order approximations of smooth solutions. Probabilistic methods based on sampling have a natural adaptability to high-dimensional irregular domains and are suitable for modeling uncertainty propagation in multi-physics scenarios. PINNs offer a flexible, data-driven framework for solving complex geometry and parameter inversion problems; however, they still face challenges in addressing nonlocal models. The high cost of evaluating integral terms during training, training instability caused by singular kernels and long-range memory effects, and the inherent spectral bias of network structures. Asymptotically compatible schemes address numerical degradation issues in nonlocal models as they approach local limits, serving as a crucial bridge between nonlocal and local modeling frameworks.

Despite significant progress in non-local high-dimensional computation research in recent years, several areas requiring further exploration and open questions remain:

- Efficient algorithms for complex boundaries and irregular domains: Existing structured and spectral methods largely rely on regular geometry, while achieving high-order accuracy and stability under complex boundary conditions remains a bottleneck. Developing hybrid algorithms based on sparse grids, local spectral bases, and fast integration could provide a breakthrough.
- Integration mechanisms between deep learning and nonlocal operators: Future research should further explore the potential of emerging architectures like Fourier Neural Operators (FNO), Graph Neural Networks (GNN), and Transformers in approximating nonlocal kernels. FNO captures long-range correlations in the frequency domain, the GNNs are well-suited for modeling pointwise nonlocal dependencies,

while the Transformer frameworks with attention mechanisms show promise for improved generalization in spatio-temporal memory operators. Complementary training strategies such as importance sampling, physics-guided regularization, hierarchical loss balancing, and multiscale Fourier embedding are also critical for enhancing network scalability and physical consistency.

- High-performance computing and heterogeneous parallelization frameworks: Given the global coupling nature of nonlocal integrals, further development of GPU/CPU co-computing, distributed storage, and communication compression strategies is needed to overcome memory constraints in high-dimensional integral computation and gradient backpropagation.

Overall, nonlocal modeling and high-dimensional numerical computation are situated at a pivotal stage where theoretical refinement and practical implementation are beginning to converge. The research frontier is gradually shifting from foundational questions of computability to broader concerns of efficiency, generalizability, and scalability, which hold significant academic value and broad application prospects.

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Abbreviations

The following abbreviations are used in this manuscript:

CG	Conjugate Gradient
FADE	Fractional advection-diffusion equations
FDE	Fractional diffusion equations
FFT	Fast Fourier transform
GFDM	Generalized finite difference method
GMC-PINNs	Generalized Monte Carlo PINNs
GMRES	Generalized Minimal Residual
HDE	Hydrodynamic equation
KVFD	Kelvin–Voigt fractional derivative
nPINNs	Nonlocal physics-informed neural networks
PD	Peridynamics
PDEs	Partial differential equations
PVBF	Pulse vector basis functions
SFDE	Spatial fractional diffusion equation
VIE	Volume integral equation

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