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Research on the Spinors of Jacobsthal and Jacobsthal–Lucas Hybrid Number Polynomials

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Abstract

By drawing on the concepts of Jacobsthal polynomials, Jacobsthal–Lucas polynomials, and hybrid numbers, this paper constructs, for the first time, a novel class of mathematical objects with recursive properties—namely, the sequences of Jacobsthal and Jacobsthal–Lucas hybrid number polynomials. Meanwhile, within this framework, this paper further defines their hybrid spinor sequences and explores their core algebraic structures and properties, including Binet-like formulas, generating functions, and the Vajda identity. Finally, the matrix representations of the spinors corresponding to Jacobsthal and Jacobsthal–Lucas hybrid number polynomials are presented.

Keywords: hybrid number; Jacobsthal hybrid polynomial; Jacobsthal–Lucas hybrid number polynomial; hybrid spinor; matrix representation

MSC: 11B39

1. Introduction

The extension of number systems constitutes a core theme in the evolution of mathematical theories, whose developmental trajectory profoundly reflects humanity's endeavor to explore the completeness of algebraic structures. The expansion from real numbers to complex numbers not only fundamentally resolved the completeness issue inherent in the Fundamental Theorem of Algebra—for instance, addressing the unsolvability of the equation $x^2 + 1 = 0$ within the real number field—but also provided a powerful tool for describing rotations in two-dimensional spaces [1]. This attribute not only enhanced the algebraic closure of the number system but also laid a solid foundation for engineering applications, such as phase analysis in electromagnetism and Fourier transforms in signal processing.

Traditional algebraic descriptions of hybrid numbers often face two challenges: on the one hand, their multi-dimensional nature makes direct computation cumbersome (e.g., complex multiplication rules for multi-component hybrid numbers); on the other hand, it is difficult to establish an intuitive connection between hybrid number operations and geometric behaviors (e.g., how hybrid Jacobsthal polynomials act on vector spaces). Spinors, as a bridge between algebraic structures and geometric representations, solve these problems: they enable the natural embedding of four-dimensional hybrid numbers into the algebra of real matrices—this embedding not only converts abstract hybrid number operations into concrete matrix computations (greatly simplifying numerical and symbolic

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calculations), but also links the algebraic properties of hybrid numbers to the transformation of vectors in $\mathbb{R}^2$(laying the foundation for the geometric interpretation of subsequent polynomials). Theoretically, spinors fill the "gap" between the algebraic structure of hybrid numbers and geometric operations: without spinor representation, the relationship between hybrid Jacobsthal polynomials and vector space transformations would remain abstract and difficult to verify, making it hard to intuitively understand how these polynomials "act" in the hybrid plane. Practically, spinors are not just a "computational tool"—their relevance extends to broader fields (e.g., quantum mechanics, robotics, and differential geometry) where high-dimensional algebraic operations and geometric transformations need to be unified. By clarifying the role of spinors in this work, we also provide a potential link for extending the research results of hybrid Jacobsthal polynomials to other interdisciplinary scenarios (e.g., using hybrid polynomial-spinor frameworks to model low-dimensional vector rotation problems).

Prior studies on hybrid numbers (e.g., quaternions, octonions) have not integrated Jacobsthal and Jacobsthal–Lucas polynomials, resulting in a gap in the construction of polynomial-hybrid number systems. We further clarify how our work addresses these gaps: specifically, by constructing hybrid number polynomials with Jacobsthal-type polynomial bases and deriving their corresponding spinor representations, we fill the existing void in the unification of polynomial, hybrid number, and spinor theories. We contrast our construction with that of Özçevik et al. (2024) [2], who focused on hybrid numbers with Fibonacci polynomial bases: our work extends this to Jacobsthal-type polynomials, which have distinct recurrence relations and better applicability to quantum spin systems.

The main contributions of this research are as follows:

(1) Proposing the concepts of Jacobsthal and Jacobsthal–Lucas hybrid number polynomials, along with their corresponding spinor representations;
(2) Establishing a theoretical framework for the matrix representation of hybrid spinors. Investigating and deriving the core algebraic structures and properties—including Binet formulas, generating functions, and Vajda identities—of the spinors associated with Jacobsthal and Jacobsthal–Lucas hybrid number polynomials;
(3) Establishing a theoretical framework for the matrix representation of hybrid spinors.

2. Preliminaries

2.1. The Hybrid Numbers

Özdemir [3] introduced the set of hybrid numbers, denoted by $\mathbb{K}$, as defined by the set of hybrid numbers, as follows:

$$\mathbb{K} = \{a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1\} \quad (1)$$

This set of numbers can be regarded as a set of quadruplets. The real, complex, dual, and hyperbolic units are defined as

$$1 \leftrightarrow (1,0,0,0),\ \mathbf{i} \leftrightarrow (0,1,0,0),\ \boldsymbol{\varepsilon} \leftrightarrow (0,0,1,0),\ \mathbf{h} \leftrightarrow (0,0,0,1)$$

respectively. We refer to these units as hybrid units. For a hybrid number $Z = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h}$, the real number a is called the scalar part and denoted by S(Z); the part $b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h}$ is called the vector part and denoted by V(Z). It follows from Theorems 2.8 in Reference [3] that the set of hybrid numbers forms a non-commutative ring with respect to the operations of addition and multiplication.

It follows from Theorem 3.1 in Reference [3] that the hybrid number ring $\mathbb{K}$ is isomorphic to the ring of real 2×2 matrices $M_{2\times 2}$ and the matrix corresponding to the hybrid number Z is denoted as $\phi(Z) \in M_{2\times 2}(\mathbb{R})$.

By Theorem 3.1 in Reference [3], which focuses on the isomorphism between the hybrid number ring and the real 2×2 matrix ring, we obtain the corresponding matrix representations as follows:

$$\phi(1) \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \phi(\mathbf{i}) \leftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \phi(\boldsymbol{\varepsilon}) \leftrightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad \phi(\mathbf{h}) \leftrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Here is the multiplication table for the basis of hybrid numbers (Table 1):

Table 1. Multiplication rules.

·	1	$\boldsymbol{i}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{h}$
1	1	$\boldsymbol{i}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{h}$
$\boldsymbol{i}$	$\boldsymbol{i}$	-1	$1-\boldsymbol{h}$	$\boldsymbol{\varepsilon}+\boldsymbol{i}$
$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}$	$1+\boldsymbol{h}$	0	$-\boldsymbol{\varepsilon}$
$\boldsymbol{h}$	$\boldsymbol{h}$	$-\boldsymbol{\varepsilon}-\boldsymbol{i}$	$\boldsymbol{\varepsilon}$	1

Let $Z_1 = a_1 + b_1\mathbf{i} + c_1\boldsymbol{\varepsilon} + d_1\mathbf{h}$, $Z_2 = a_2 + b_2\mathbf{i} + c_2\boldsymbol{\varepsilon} + d_2\mathbf{h}$ be any two hybrid numbers. It is given in [3] that their algebraic operations can be defined.

(1) Equality: $Z_1 = Z_2 \Longleftrightarrow a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$.
(2) Addition: $Z_1 + Z_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\boldsymbol{\varepsilon} + (d_1 + d_2)\mathbf{h}$.
(3) Subtraction: $Z_1 - Z_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\boldsymbol{\varepsilon} + (d_1 - d_2)\mathbf{h}$.
(4) Scalar multiplication: $sZ_1 = sa_1 + sb_1\mathbf{i} + sc_1\boldsymbol{\varepsilon} + sd_1\mathbf{h} (s \in \mathbb{R})$.
(5) Multiplication: $Z_1 \times Z_2 = (a_1a_2 - b_1b_2 + b_1c_2 + b_2c_1 + d_1d_2) + (a_1b_2 + a_2b_1 + b_1d_2 - b_2d_1)\mathbf{i} + (a_1c_2 + a_2c_1 + b_1d_2 - c_1d_2 - b_2d_1 + c_2d_1)\boldsymbol{\varepsilon} + (a_1d_2 - b_1c_2 + b_2c_1 + a_2d_1)\mathbf{h}$.
(6) Conjugate: $\overline{Z}_1 = a_1 - b_1\mathbf{i} - c_1\boldsymbol{\varepsilon} - d_1\mathbf{h}$.

Further details on hybrid numbers can be found in Reference [3].

2.2. Jacobsthal and Jacobsthal–Lucas Polynomials

For $n \geq 0$, the n-th terms of Jacobsthal polynomials is defined by the following recurrence relations:

$$J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x), J_0(x) = 0, J_1(x) = 1, \tag{2}$$

For $n \geq 0$, the n-th terms of Jacobsthal–Lucas polynomials is defined by the following recurrence relations:

$$j_{n+2}(x) = j_{n+1}(x) + 2xj_n(x),\ j_0(x) = 2,\ j_1(x) = 1. \tag{3}$$

Notably, $J_n(1/2) = F_n$ and $j_n(1/2) = L_n$ correspond to the n-th Fibonacci and Lucas numbers, respectively. When $x = 1$, these polynomials reduce to Jacobsthal numbers and Jacobsthal–Lucas numbers. For the sequences $\{J_n(x)\}$ and $\{j_n(x)\}$, their characteristic equation is $t^2 - t - 2x = 0$, with two characteristic roots given by $\alpha(x) = \frac{1+\sqrt{8x+1}}{2}$ and $\beta(x) = \frac{1-\sqrt{8x+1}}{2}$. Accordingly, the following relationships hold: $\alpha(x) + \beta(x) = 1$, $\alpha(x)\beta(x) = -2x$, $\alpha(x) - \beta(x) = \sqrt{8x+1} = \Delta(x)$, $\alpha^2(x) + 2x = \Delta(x)\alpha(x)$, $\beta^2(x) + 2x = -\Delta(x)\beta(x)$. The Binet formulas and generating functions for $\{J_n(x)\}$ and $\{j_n(x)\}$ are, respectively:

$$J_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\Delta(x)},\ j_n(x) = \alpha^n(x) + \beta^n(x), \tag{4}$$

and

$$\sum_{i=1}^{\infty} J_i(x)t^{i-1} = (1-t-2xt^2)^{-1}, \sum_{i=1}^{\infty} j_i(x)t^{i-1} = (1+4xt)/(1-t-2xt^2). \tag{5}$$

Let the generating matrix be $Q = \begin{bmatrix} 1 & 2x \\ 1 & 0 \end{bmatrix}$. Then the recurrence relations (3) and (4) can be expressed as

$$\begin{bmatrix} J_n(x) \\ J_{n-1}(x) \end{bmatrix} = Q^{n-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} j_n(x) \\ j_{n-1}(x) \end{bmatrix} = Q^{n-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{6}$$

Using the Binet formula and properties of the generating functions of Jacobsthal polynomials, several new combinatorial identities can be derived. For instance,

$$\sum_{k=0}^{n} C_n^k J_k(x) = \frac{1}{\Delta(x)}[(1+\alpha(x))^n - (1+\beta(x))^n] \tag{7}$$

Such identities can be applied to quickly calculate the number of feasible solutions under constraints in combinatorial optimization problems (e.g., the knapsack problem). Additionally, Jacobsthal and Jacobsthal–Lucas polynomials find extensive applications in fields including dynamic cryptography, parametric geometric modeling, and polynomial interpolation. For further details, refer to the relevant literature [4].

2.3. Hybrid Spinor

Reference [2] points out that the correspondence between a hybrid number and its spinor can be expressed via the transformation σ as

$$\sigma : \mathbb{K} \to \mathbb{HS}$$

and

$$\sigma(a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h}) = \begin{bmatrix} a + c\mathbf{h} \\ (c-b) + d\mathbf{h} \end{bmatrix} \tag{8}$$

The spinor associated with a hybrid number is termed a hybrid spinor. According to Equation (8), the hybrid spinor corresponding to the conjugate$\overline{Z} = a - b\mathbf{i} - c\boldsymbol{\varepsilon} - d\mathbf{h}$ of Z is

$$\sigma(a - b\mathbf{i} - c\boldsymbol{\varepsilon} - d\mathbf{h}) = \begin{bmatrix} a - c\mathbf{h} \\ (b-c) - d\mathbf{h} \end{bmatrix} \tag{9}$$

We associate with the product of two hybrid numbers $Z_1 \times Z_2$, which is equal to the hybrid number-matrix product $L(Z_1)Z_2$, with a spinor matrix product as follows:

$$Z_1 \times Z_2 \to L(Z_1)Z_2 \to \hat{Q}\sigma(Z_2) \tag{10}$$

where $L(Z_1)$ denotes the left hybrid spinor matrix of Z_1, and

$$\hat{Q} = \begin{bmatrix} a_1 + c_1\mathbf{h} & (b_1 - c_1) + d_1\mathbf{h} \\ (c_1 - b_1) + d_1\mathbf{h} & a_1 - c_1\mathbf{h} \end{bmatrix}. \tag{11}$$

3. Spinors of Jacobsthal and Jacobsthal–Lucas Hybrid Number Polynomials

In this section, we construct Jacobsthal hybrid number polynomials and Jacobsthal–Lucas hybrid number polynomials, present the definition of their hybrid spinors,

and derive their core theorems and formulas. For non-negative integers $n \geq 0$, the symbols $J_n(x), j_n(x), \alpha(x), \beta(x), \Delta(x), HJ_n(x), Hj_n(x), HSJ_n(x)$, and $HSj_n(x)$ denote Jacobsthal polynomials, Jacobsthal–Lucas polynomials, characteristic roots, discriminants, Jacobsthal hybrid number polynomials, Jacobsthal–Lucas hybrid number polynomials, spinors of Jacobsthal hybrid number polynomials, and spinors of Jacobsthal–Lucas hybrid number polynomials, respectively. They are abbreviated as $J_n, j_n, \alpha, \beta, \Delta, HJ_n, Hj_n, HSJ_n, HSj_n$.

3.1. Hybrid Spinors of Jacobsthal Hybrid Number Polynomials

Definition 1. *A polynomial of the form*

$$HJ_n = J_n + J_{n+1}\mathbf{i} + J_{n+2}\boldsymbol{\varepsilon} + J_{n+3}\mathbf{h} \tag{12}$$

is called a Jacobsthal hybrid polynomial, where J_n denotes the n-th Jacobsthal polynomial, and $\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}$ as specified in Table 1.

Definition 2. *For a non-negative integer n, the n-th hybrid spinor HSJ_n of the Jacobsthal hybrid number polynomial HJ_n is defined as*

$$HSJ_n = \begin{bmatrix} J_n + J_{n+2}\mathbf{h} \\ (J_{n+2} - J_{n+1}) + J_{n+3}\mathbf{h} \end{bmatrix} \tag{13}$$

Using the definition of Jacobsthal polynomials and the definition of Jacobsthal hybrid number polynomials (12), the following corollary can be readily derived.

Corollary 1. *For any non-negative integer $n \geq 0$, the hybrid spinor sequence of Jacobsthal hybrid number polynomials satisfies the recurrence relation, as follows:*

$$HSJ_{n+2} = HSJ_{n+1} + 2xHSJ_n \tag{14}$$

where $HSJ_0 = \begin{bmatrix} J_0 + J_2\mathbf{h} \\ (J_2 - J_1) + J_3\mathbf{h} \end{bmatrix}$, $HSJ_1 = \begin{bmatrix} J_1 + J_3\mathbf{h} \\ (J_3 - J_2) + J_4\mathbf{h} \end{bmatrix}$.

Proof of Corollary 1. For any non-negative integer $n \geq 0$, by the recurrence relation (2), we have

$$\begin{aligned} HSJ_{n+1} + 2xHSJ_n &= \begin{bmatrix} J_{n+1} + J_{n+3}\mathbf{h} \\ (J_{n+3} - J_{n+2}) + J_{n+4}\mathbf{h} \end{bmatrix} + 2x\begin{bmatrix} J_n + J_{n+2}\mathbf{h} \\ (J_{n+2} - J_{n+1}) + J_{n+3}\mathbf{h} \end{bmatrix} \\ &= \begin{bmatrix} J_{n+1} + 2xJ_n + (J_{n+3} + 2xJ_{n+2})\mathbf{h} \\ (J_{n+3} + 2xJ_{n+2} - J_{n+2} - 2xJ_{n+1}) + (J_{n+4} + 2xJ_{n+3})\mathbf{h} \end{bmatrix} \\ &= \begin{bmatrix} J_{n+2} + J_{n+4}\mathbf{h} \\ (J_{n+4} - J_{n+3}) + J_{n+5}\mathbf{h} \end{bmatrix} = HSJ_{n+2}. \end{aligned}$$

□

Notably, the hybrid spinor sequence of Jacobsthal hybrid number polynomials inherits the recurrence property of the Jacobsthal hybrid number polynomial sequence. Thus, we can first define the hybrid spinor sequence $\{HSJ_n\}_{n\geq 0}$ of Jacobsthal hybrid number polynomials using (14), and then derive Formula (13).

Applying the correspondence σ between hybrid numbers and their spinors to Jacobsthal hybrid number polynomials, it is straightforward to establish a correspondence

between Jacobsthal hybrid number polynomials and their associated hybrid spinors, as follows:

$$\sigma : JHP \to \mathbb{HS}$$

and

$$\sigma(HJ_n) = \sigma(J_n + J_{n+1}\mathbf{i} + J_{n+2}\boldsymbol{\varepsilon} + J_{n+3}\mathbf{h}) = \begin{bmatrix} J_n + J_{n+2}\mathbf{h} \\ (J_{n+2} - J_{n+1}) + J_{n+3}\mathbf{h} \end{bmatrix} = HSJ_n \tag{15}$$

Example 1. *In quantum mechanics, the state of a spin-*$1/2$ *particle can be described by a spinor. The spinor representation of Jacobsthal hybrid number polynomials can naturally correspond to quantum states, as follows:*

Consider the spin state along the x-axis, whose standard spinor is

$$\chi_+^x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

A spin state modulated by a polynomial is constructed using $HJ_2(x)$*, as follows:*

$$\Psi(x) = HJ_2\chi_+^x = \frac{1}{\sqrt{2}} \begin{bmatrix} x & 2 \\ 0 & x \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x+2 \\ x \end{bmatrix}$$

The evolution of this state in a magnetic field can be described by the exponential mapping of the spinor matrix. When $x = 1$*,*

$$\Psi(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

and the square of its modulus is

$$\frac{3^2 + 1^2}{2} = 5,$$

which satisfies the normalization condition (and can be achieved by adjusting coefficients). This example demonstrates the application potential of hybrid number polynomial spinors in quantum state modulation, where polynomial coefficients can encode the spatial distribution of physical parameters.

Definition 3. *Let the conjugate of the Jacobsthal hybrid number polynomial* HJ_n *be* $\overline{HJ}_n = J_n - J_{n+1}\mathbf{i} - J_{n+2}\boldsymbol{\varepsilon} - J_{n+3}\mathbf{h}$*. From the correspondence (15), the hybrid spinor associated with* $\overline{HJ}_n$ *is defined as*

$$\sigma(\overline{HJ}_n) = \begin{bmatrix} J_n - J_{n+2}\mathbf{h} \\ (J_{n+1} - J_{n+2}) - J_{n+3}\mathbf{h} \end{bmatrix} = \overline{HSJ}_n \tag{16}$$

The hyperbolic hybrid conjugate of the hybrid spinor HSJ_n is defined as

$$HSJ_n^* = \begin{bmatrix} J_n - J_{n+2}\mathbf{h} \\ (J_{n+2} - J_{n+1}) - J_{n+3}\mathbf{h} \end{bmatrix} \tag{17}$$

The conjugate hybrid spinor of the hybrid spinor HSJ_n is defined as

$$\widetilde{HSJ}_n = \begin{bmatrix} -J_{n+2} + J_n\mathbf{h} \\ J_{n+3} + (J_{n+2} - J_{n+1})\mathbf{h} \end{bmatrix} = \mathbf{h}R\overline{HSJ}_n \tag{18}$$

The adjoint hybrid spinor of the hybrid spinor HSJ_n is defined as

$$\widehat{HSJ}_n = \begin{bmatrix} -J_n + J_{n+2}\mathbf{h} \\ (J_{n+1} - J_{n+2}) - J_{n+3}\mathbf{h} \end{bmatrix} = -R\overline{HSJ}_n \tag{19}$$

where $R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Corollary 2. *For the hybrid spinor HSJ_n of Jacobsthal hybrid polynomials and its various conjugates, the following relations hold:*

(i) $HSJ_n + \overline{HSJ}_n = 2\begin{bmatrix} J_n \\ 0 \end{bmatrix}$,

(ii) $HSJ_n + HSJ_n^* = 2J_n\begin{bmatrix} 1 \\ 2x \end{bmatrix}$,

(iii) $HSJ_n + \widehat{HSJ}_n = 2J_{n+2}\begin{bmatrix} \mathbf{h} \\ 0 \end{bmatrix}$,

(iv) $\mathbf{h}\widetilde{HSJ}_n = -\widehat{HSJ}_n$.

Proof of Corollary 2. This can be verified directly using the definitions of Jacobsthal hybrid number polynomials and their various hybrid spinors; details are omitted here. □

Theorem 1. *The Binet-like formula for the hybrid spinor sequence $\{HSJ_n(x)\}_{n\geq 0}$ is given by*

$$HSJ_n = \frac{1}{\Delta}\left[\alpha^n\begin{bmatrix} 1+\alpha^2\mathbf{h} \\ (\alpha^2-\alpha)+\alpha^3\mathbf{h} \end{bmatrix} - \beta^n\begin{bmatrix} 1+\beta^2\mathbf{h} \\ (\beta^2-\beta)+\beta^3\mathbf{h} \end{bmatrix}\right] \tag{20}$$

where α, β are the two characteristic roots of the recurrence relation (2) for Jacobsthal polynomials.

Proof of Theorem 1. Recall that the Binet formula for the Jacobsthal polynomial sequence $J_n = \frac{\alpha^n - \beta^n}{\Delta}$. For the hybrid spinor corresponding to the Jacobsthal hybrid number polynomial HSJ_n is

$$\begin{aligned} HSJ_n &= \begin{bmatrix} J_n + J_{n+2}\mathbf{h} \\ (J_{n+2} - J_{n+1}) + J_{n+3}\mathbf{h} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\alpha^n-\beta^n}{\Delta} + \frac{\alpha^{n+2}-\beta^{n+2}}{\Delta}\mathbf{h} \\ \left(\frac{\alpha^{n+2}-\beta^{n+2}}{\Delta} - \frac{\alpha^{n+1}-\beta^{n+1}}{\Delta}\right) + \frac{\alpha^{n+3}-\beta^{n+3}}{\Delta}\mathbf{h} \end{bmatrix} \\ &= \frac{1}{\Delta}\left[\alpha^n\begin{bmatrix} 1+\alpha^2\mathbf{h} \\ (\alpha^2-\alpha)+\alpha^3\mathbf{h} \end{bmatrix} - \beta^n\begin{bmatrix} 1+\beta^2\mathbf{h} \\ (\beta^2-\beta)+\beta^3\mathbf{h} \end{bmatrix}\right] \end{aligned}$$

□

Theorem 2. *The generating function for the hybrid spinor sequence $\{HSJ_n\}_{n\geq 0}$ corresponding to the Jacobsthal hybrid number polynomial sequence $\{HJ_n\}_{n\geq 0}$ is given by*

$$G_{HSJ_n}(t) = \frac{1}{1-t-2xt^2}\begin{bmatrix} \mathbf{h} + (1+2x\mathbf{h})t \\ (1+2x)\mathbf{h} + 2x(1+\mathbf{h})t \end{bmatrix} \tag{21}$$

Proof of Theorem 2. Let the generating function be defined as $G_{HSJ_n}(t) = \sum_{n=0}^{\infty}(HSJ_n)t^n$. Using the recurrence relation (14) established above, we can derive the explicit form of

this generating function. First, start with the recurrence relation (14) and sum over all non-negative integers

$$\sum_{n=0}^{\infty}(HSJ_{n+2})t^n = \sum_{n=0}^{\infty}(HSJ_{n+1})t^n + \sum_{n=0}^{\infty}(2xHSJ_n)t^n$$

Thus, we have

$$\sum_{n=2}^{\infty}(HSJ_n)t^{n-2} = \sum_{n=1}^{\infty}(HSJ_n)t^{n-1} + 2x\sum_{n=0}^{\infty}(HSJ_n)t^n$$

$$\frac{1}{t^2}(-HSJ_0 - HSJ_1 t + G_{HSJ_n}(t)) = \frac{1}{t}(-HSJ_0 + G_{HSJ_n}(t)) + 2xG_{HSJ_n}(t)$$

Substitute $HSJ_0 = \begin{bmatrix} \mathbf{h} \\ (1+2x)\mathbf{h} \end{bmatrix}$ and $HSJ_1 = \begin{bmatrix} 1+(1+2x)\mathbf{h} \\ 2x+(1+4x)\mathbf{h} \end{bmatrix}$ into the above formula and simplify and arrange; then we can obtain (21). □

Definition 4. *The hybrid spinor matrix of the Jacobsthal hybrid number polynomial HJ_n is defined as*

$$\widehat{Q}_n = \begin{bmatrix} J_n + J_{n+2}\mathbf{h} & (J_{n+1} - J_{n+2}) + J_{n+3}\mathbf{h} \\ (J_{n+2} - J_{n+1}) + J_{n+3}\mathbf{h} & J_n - J_{n+2}\mathbf{h} \end{bmatrix} \tag{22}$$

Theorem 3. *For $n \geq 1$, the Vajda identity for the hybrid spinor sequence $\{HSJ_n\}_{n\geq 0}$ is*

$$\begin{aligned} &\hat{Q}_{n+m}HSJ_{n+r} - \hat{Q}_n HSJ_{n+m+r} = \\ &\frac{(-2x)^n(\alpha^m - \beta^m)\alpha^r}{\Delta^2}\begin{bmatrix} 1-8x^3+(1+4x-2x(2x+1)\sqrt{1+8x})\mathbf{h} \\ 4x-4x^2\sqrt{1+8x}+(1+6x+2x\sqrt{1+8x})\mathbf{h} \end{bmatrix} + \\ &\frac{(-2x)^n(\alpha^m - \beta^m)\beta^r}{\Delta^2}\begin{bmatrix} 1-8x^3+(1+4x+2x(2x+1)\sqrt{1+8x})\mathbf{h} \\ 4x+4x^2\sqrt{1+8x}+(1+6x-2x\sqrt{1+8x})\mathbf{h} \end{bmatrix} \end{aligned} \tag{23}$$

Proof of Theorem 3. From reference [5], the Vajda identity for the Jacobsthal hybrid number polynomial sequence can be derived as follows:

$$HJ_{n+m}\cdot HJ_{n+r} - HJ_n \cdot HJ_{n+m+r} = \frac{(-2x)^n(\alpha^m - \beta^m)(\alpha^r\hat{\beta}\hat{\alpha} - \beta^r\hat{\alpha}\hat{\beta})}{\Delta^2}, \tag{24}$$

where $\hat{\alpha} = 1 + \alpha\mathbf{i} + \alpha^2\boldsymbol{\varepsilon} + \alpha^3\mathbf{h}$, and $\hat{\beta} = 1 + \beta\mathbf{i} + \beta^2\boldsymbol{\varepsilon} + \beta^3\mathbf{h}$. Substitute the characteristic roots $\alpha = \frac{1+\sqrt{8x+1}}{2}$ and $\beta = \frac{1-\sqrt{8x+1}}{2}$ into $\hat{\alpha}$ and $\hat{\beta}$. Simple algebraic manipulations yield

$$\hat{\alpha}\hat{\beta} = 1 - 8x^3 + (1+2x\sqrt{1+8x})\mathbf{i} + \left(1+4x+2x(2x+1)\sqrt{1+8x}\right)\boldsymbol{\varepsilon} + \left(1+6x-2x\sqrt{1+8x}\right)\mathbf{h},$$
$$\hat{\beta}\hat{\alpha} = 1 - 8x^3 + (1-2x\sqrt{1+8x})\mathbf{i} + \left(1+4x-2x(2x+1)\sqrt{1+8x}\right)\boldsymbol{\varepsilon} + \left(1+6x-2x\sqrt{1+8x}\right)\mathbf{h}.$$

From Formula (10), we know $\sigma(HJ_{n+m}\cdot HJ_{n+r}) = \hat{Q}_{n+m}\cdot HSJ_{n+r}$, where $\hat{Q}_{n+m}$ and HSJ_{n+r} are defined in Formulas (22) and (13), respectively. By the linearity of the transformation σ, it follows that

$$\sigma(HJ_{n+m}\cdot HJ_{n+r} - HJ_n\cdot HJ_{n+m+r}) = \hat{Q}_{n+m}\cdot HSJ_{n+r} - \hat{Q}_n\cdot HSJ_{n+m+r}.$$

Substituting the Vajda identity (24) for Jacobsthal hybrid number polynomials into the left-hand side, and applying the transformation σ to the left-hand side of (24), we complete the theorem. □

The Vajda identity characterizes a symmetry within the "hybrid polynomial space"—specifically, it encodes the transformation behavior of Jacobsthal hybrid polynomials under reflection operations (analogous to the root symmetries observed in classical polynomial systems). In the context of quantum mechanics, this identity corresponds to a conservation law governing the probabilities of spinors: when a spinor undergoes evolution through a unitary transformation, the Vajda identity guarantees that the total probability defined as the sum of the squares of the polynomial coefficients remains constant.

In Theorem 3, setting $m = -r$ yields the Catalan identity for the hybrid spinor sequence, as follows:

$$\begin{aligned}&\hat{Q}_{n-r}HSJ_{n+r} - \hat{Q}_nHSJ_n = \\ &\frac{(-2x)^n(1-(\alpha/\beta)^r)}{\Delta^2}\begin{bmatrix}1-8x^3+(1+4x-2x(2x+1)\sqrt{1+8x})\mathbf{h}\\ 4x-4x^2\sqrt{1+8x}+(1+6x+2x\sqrt{1+8x})\mathbf{h}\end{bmatrix}+\\ &\frac{(-2x)^n((\beta/\alpha)^r-1)}{\Delta^2}\begin{bmatrix}1-8x^3+(1+4x+2x(2x+1)\sqrt{1+8x})\mathbf{h}\\ 4x+4x^2\sqrt{1+8x}+(1+6x-2x\sqrt{1+8x})\mathbf{h}\end{bmatrix}\end{aligned}$$

Setting $r = -m = 1$ yields the Cassini identity, as follows:

$$\begin{aligned}&\hat{Q}_{n-1}HSJ_{n+1} - \hat{Q}_nHSJ_n = \\ &\frac{(-2x)^n}{\beta^2+2x}\begin{bmatrix}1-8x^3+(1+4x-2x(2x+1)\sqrt{1+8x})\mathbf{h}\\ 4x-4x^2\sqrt{1+8x}+(1+6x+2x\sqrt{1+8x})\mathbf{h}\end{bmatrix}-\\ &\frac{(-2x)^n}{\alpha^2+2x}\begin{bmatrix}1-8x^3+(1+4x+2x(2x+1)\sqrt{1+8x})\mathbf{h}\\ 4x+4x^2\sqrt{1+8x}+(1+6x-2x\sqrt{1+8x})\mathbf{h}\end{bmatrix}\end{aligned}$$

Setting $r = t - n$ and $m = 1$ yields the d'Ocagne identity, as follows:

$$\begin{aligned}&\hat{Q}_{n+1}HSJ_t - \hat{Q}_nHSJ_{t+1} = \\ &\frac{(-2x)^n\alpha^{t-n}}{\Delta}\begin{bmatrix}1-8x^3+(1+4x-2x(2x+1)\sqrt{1+8x})\mathbf{h}\\ 4x-4x^2\sqrt{1+8x}+(1+6x+2x\sqrt{1+8x})\mathbf{h}\end{bmatrix}+\\ &\frac{(-2x)^n\beta^{t-n}}{\Delta}\begin{bmatrix}1-8x^3+(1+4x+2x(2x+1)\sqrt{1+8x})\mathbf{h}\\ 4x+4x^2\sqrt{1+8x}+(1+6x-2x\sqrt{1+8x})\mathbf{h}\end{bmatrix}\end{aligned}$$

3.2. Hybrid Spinors of Jacobsthal–Lucas Hybrid Number Polynomials

By applying the correspondence σ between hybrid numbers and their spinors to Jacobsthal–Lucas hybrid number polynomials $Hj_n = j_n + j_{n+1}\mathbf{i} + j_{n+2}\boldsymbol{\varepsilon} + j_{n+3}\mathbf{h}$, the hybrid spinor of Jacobsthal–Lucas hybrid number polynomials is defined as

$$HSj_n = \begin{bmatrix}j_n + j_{n+2}\mathbf{h}\\ (j_{n+2}-j_{n+1}) + j_{n+3}\mathbf{h}\end{bmatrix} \tag{25}$$

Notably, the hybrid spinor sequence of Jacobsthal–Lucas hybrid number polynomials also satisfies the recurrence relation $HSj_{n+2} = HSj_{n+1} + 2xHSj_n$, and its properties are completely analogous to those presented in Section 3.1. Thus, we only provide the following conclusions, with proofs omitted.

Theorem 4. *The Binet-like formula for the hybrid spinor sequence* $\{HSj_n\}_{n\geq 0}$ *of Jacobsthal–Lucas hybrid number polynomials is*

$$HSj_n = \alpha^n \begin{bmatrix} 1+\alpha^2\mathbf{h} \\ (\alpha^2-\alpha)+\alpha^3\mathbf{h} \end{bmatrix} + \beta^n \begin{bmatrix} 1+\beta^2\mathbf{h} \\ (\beta^2-\beta)+\beta^3\mathbf{h} \end{bmatrix}, \tag{26}$$

where α, β *are the two characteristic roots of the recurrence relation (4) for the Jacobsthal–Lucas polynomial sequence.*

Theorem 5. *The generating function for the hybrid spinor sequence* $\{HSj_n\}_{n\geq 0}$ *of Jacobsthal–Lucas hybrid number polynomials is*

$$G_{HSj_n}(t) = \frac{1}{1-t-2xt^2} \begin{bmatrix} 2+(1+4x)\mathbf{h}-t+2xt\mathbf{h} \\ 4x+(1+6x)\mathbf{h}-2xt+2x(1+4x)t\mathbf{h} \end{bmatrix} \tag{27}$$

Definition 5. *The hybrid spinor matrix of the Jacobsthal–Lucas hybrid number polynomial* Hj_n *is defined as*

$$\hat{R}_n = \begin{bmatrix} j_n + j_{n+2}\mathbf{h} & (j_{n+1}-j_{n+2})+j_{n+3}\mathbf{h} \\ (j_{n+2}-j_{n+1})+j_{n+3}\mathbf{h} & j_n - j_{n+2}\mathbf{h} \end{bmatrix} \tag{28}$$

Theorem 6. *For* $n \geq 1$, *the Vajda identity for the hybrid spinor sequence* $\{HSj_n\}_{n\geq 0}$ *is*

$$\begin{aligned} &\hat{R}_{n+m}HSj_{n+r} - \hat{R}_nHSj_{n+m+r} = \\ &(-2x)^n(\alpha^m-\beta^m)\beta^r \begin{bmatrix} 1-8x^3+(1+4x+2x(2x+1)\sqrt{1+8x})\mathbf{h} \\ 4x+4x^2\sqrt{1+8x}+(1+6x-2x\sqrt{1+8x})\mathbf{h} \end{bmatrix} + \\ &(-2x)^n(\alpha^m-\beta^m)\alpha^r \begin{bmatrix} 1-8x^3+(1+4x-2x(2x+1)\sqrt{1+8x})\mathbf{h} \\ 4x-4x^2\sqrt{1+8x}+(1+6x+2x\sqrt{1+8x})\mathbf{h} \end{bmatrix}. \end{aligned} \tag{29}$$

In Theorem 6, setting $m = -r$ yields the Catalan identity for the hybrid spinor sequence $\{HSj_n\}_{n\geq 0}$, as follows:

$$\begin{aligned} &\hat{R}_{n-r}HSj_{n+r} - \hat{R}_nHSj_n = \\ &(-2x)^n\left(\left(\frac{\beta}{\alpha}\right)^r - 1\right) \begin{bmatrix} 1-8x^3+(1+4x+2x(2x+1)\sqrt{1+8x})\mathbf{h} \\ 4x+4x^2\sqrt{1+8x}+(1+6x-2x\sqrt{1+8x})\mathbf{h} \end{bmatrix} \\ &\quad + (-2x)^n\left(1-\left(\frac{\alpha}{\beta}\right)^r\right) \begin{bmatrix} 1-8x^3+(1+4x-2x(2x+1)\sqrt{1+8x})\mathbf{h} \\ 4x-4x^2\sqrt{1+8x}+(1+6x+2x\sqrt{1+8x})\mathbf{h} \end{bmatrix} \end{aligned}$$

Setting $r = -m = 1$ yields the Cassini identity for the sequence $\{HSj_n\}_{n\geq 0}$, as follows:

$$\begin{aligned} &\hat{R}_{n-1}HSj_{n+1} - \hat{R}_nHSj_n = \\ &(-2x)^n\left(\frac{\beta}{\alpha} - 1\right) \begin{bmatrix} 1-8x^3+\big(1+4x+2x(2x+1)\sqrt{1+8x}\big)\mathbf{h} \\ 4x+4x^2\sqrt{1+8x}+\big(1+6x-2x\sqrt{1+8x}\big)\mathbf{h} \end{bmatrix} \\ &\quad + (-2x)^n\left(1-\frac{\alpha}{\beta}\right) \begin{bmatrix} 1-8x^3+\big(1+4x-2x(2x+1)\sqrt{1+8x}\big)\mathbf{h} \\ 4x-4x^2\sqrt{1+8x}+\big(1+6x+2x\sqrt{1+8x}\big)\mathbf{h} \end{bmatrix}. \end{aligned}$$

Setting $r = t - n$ and $m = 1$ yields the d'Ocagne identity, as follows:

$$\begin{aligned}
&\hat{R}_{n+1}HSj_t - \hat{R}_n HSj_{t+1} = \\
&(-2x)^n \beta^{t-n} \Delta \begin{bmatrix} 1 - 8x^3 + \left(1 + 4x + 2x(2x+1)\sqrt{1+8x}\right)\mathbf{h} \\ 4x + 4x^2\sqrt{1+8x} + \left(1 + 6x - 2x\sqrt{1+8x}\right)\mathbf{h} \end{bmatrix} \\
&\quad + (-2x)^n \alpha^{t-n} \Delta \begin{bmatrix} 1 - 8x^3 + \left(1 + 4x - 2x(2x+1)\sqrt{1+8x}\right)\mathbf{h} \\ 4x - 4x^2\sqrt{1+8x} + \left(1 + 6x + 2x\sqrt{1+8x}\right)\mathbf{h} \end{bmatrix}
\end{aligned}$$

4. Matrix Representations of the Hybrid Spinor Sequences $\{HSJ_n\}_{n\geq 0}$ and $\{HSj_n\}_{n\geq 0}$

For $n \geq 1$, define

$$S_n = \begin{bmatrix} HSJ_{n+1} & HSJ_n \\ HSJ_n & HSJ_{n-1} \end{bmatrix}, \quad L_n = \begin{bmatrix} HSj_{n+1} & HSj_n \\ HSj_n & HSj_{n-1} \end{bmatrix}$$

where HSJ_n denotes the hybrid spinor of Jacobsthal polynomials, and HSj_n denotes the hybrid spinor of Jacobsthal–Lucas polynomials. S_n and L_n are referred to as the hybrid spinor matrices of Jacobsthal polynomials and Jacobsthal–Lucas polynomials, respectively.

Let the 4th-order generating matrix be $\Phi = \begin{bmatrix} I_2 & 2xI_2 \\ I_2 & \mathbf{0} \end{bmatrix}$, where I_2 is the 2nd-order identity matrix. Then the following theorem holds, as follows:

Theorem 7. *Let $n \geq 1$. Then*

$$S_{n+1} = \Phi^n S_1 \tag{30}$$

$$L_{n+1} = \Phi^n L_1 \tag{31}$$

Proof of Theorem 7. Here, we only prove Formula (30). The proof of Formula (31) is analogous. In fact, when $n = 1$, the right-hand side of the equation is

$$\Phi S_1 = \begin{bmatrix} I_2 & 2xI_2 \\ I_2 & \mathbf{0} \end{bmatrix} \begin{bmatrix} HSJ_2 & HSJ_1 \\ HSJ_1 & HSJ_0 \end{bmatrix} = \begin{bmatrix} HSJ_3 & HSJ_2 \\ HSJ_2 & HSJ_1 \end{bmatrix} = S_2$$

Thus, the conclusion holds for $n = 1$. Assume that the conclusion holds for $n > 1$. For the case of $n + 1$, we have

$$\begin{aligned}
\Phi^{n+1} S_1 = \Phi^1 \Phi^n S_1 = \Phi^1 S_{n+1} &= \begin{bmatrix} I_2 & 2xI_2 \\ I_2 & \mathbf{0} \end{bmatrix} \begin{bmatrix} HSJ_{n+2} & HSJ_{n+1} \\ HSJ_{n+1} & HSJ_n \end{bmatrix} \\
&= \begin{bmatrix} HSJ_{n+3} & HSJ_{n+2} \\ HSJ_{n+2} & HSJ_{n+1} \end{bmatrix} \\
&= S_{n+2}
\end{aligned}$$

By the principle of mathematical induction, Formula (30) holds for all $n \geq 1$. □

5. Conclusions

Based on the concept of hybrid numbers, this paper constructs, for the first time, Jacobsthal hybrid number polynomials, Jacobsthal–Lucas hybrid number polynomials, and their corresponding hybrid spinor sequences, and systematically investigates their

algebraic structures and mathematical properties. The research content primarily includes the following three aspects:

- It innovatively proposes the mathematical definitions of Jacobsthal and Jacobsthal–Lucas hybrid number polynomials, as well as their hybrid spinors, laying a theoretical foundation for subsequent studies.
- Through rigorous mathematical derivations, it obtains core algebraic properties of these hybrid spinors, including Binet-like formulas, generating functions, and Vajda identities.
- It establishes a theoretical framework for the matrix representation of hybrid spinors, simplifying the process of relevant algebraic operations to a certain extent.

In terms of theoretical innovation, this research makes the following two prominent contributions:

- On one hand, it enriches the research system of hybrid number and spinor theory.
- On the other hand, it provides new research tools for fields such as mathematical physics and geometry. In particular, the proposed matrix representation method not only has theoretical value but also exhibits unique advantages in practical applications.

In terms of application prospects, the results of this research are expected to demonstrate potential value in multiple fields, as follows:

- In quantum mechanics, they can be applied to describe the characteristics of particle spin states.
- In dynamic system analysis, they facilitate the establishment of hierarchical structure models for complex networks.

Additionally, the theoretical framework established in this paper can be fully extended to other types of hybrid number polynomials and their spinors, thereby providing new research ideas for the development of related disciplines.

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