mathematics

MDPI

Article

Finite Fuzzy Topologies on Boolean Algebras

Brahim Chaourar * and Moussa Benoumhani

Department of Mathematics and Statistics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11623, Saudi Arabia; mumhani@imamu.edu.sa
* Correspondence: imchaourar@imamu.edu.sa

Abstract

We consider finite B-fuzzy topologies, that is, collections of mappings from a finite set to the Boolean algebra B, satisfying analogous conditions of classical topologies. We establish a one-to-one correspondence between the class of B-fuzzy topologies and the corresponding class of classical ones. Then many consequences follow. For example, we deduce the number of B-fuzzy topologies of small size.

Keywords: fuzzy topology; number of fuzzy topologies; boolean algebra; combinatorics

MSC: 54A40; 05B99; 03E72

1. Introduction

The concept of *fuzzy set theory* was initiated by Zadeh [1], who used the closed unit interval $[0,1]$ for the membership set. This was later generalized by Goguen [2] to arbitrary ordered structures, recommending a complete and distributive lattice. Fuzzy topology is a natural extension of general topology, where the latter is a special case. Chang [3] provided the first definition of a fuzzy topology, leading to extensive work by researchers including Gantner, Steinlage, and Warren [4], Schnare [5], Hutton and Reilly [6], Sarkar [7], Badarad and Conard [8,9], Höhle [10,11], and Rodabaugh [12], while Lowen [13,14] modified Chang's definition to obtain a fuzzy Tychonoff theorem.

Let $X_n = \{1, 2, \ldots, n\}$ be an $n-$element set and B_t the finite Boolean algebra of cardinality 2^t, where t is a positive integer. Let F be the fuzzy set of mappings from X_n into B_t ordered by the usual ordering of mappings, that is,

$$g \leq h \iff g(i) \leq h(i) \quad \text{for each } i \in X_n, and for any g, h \in F.$$

We write $g < h$ when $g \leq h$ and $g \neq h$, i.e., $g(i) \neq h(i)$ for some $i \in X_n$. We follow the notations in [15–17], and denote an element $g \in F$ by $g = (g(1), g(2), \ldots, g(n))$ or just by $g(1)g(2)\ldots g(n)$. The ordered set F is also a Boolean algebra where its least element is $0_F = (0, 0, \ldots, 0)$ and greatest element is $1_F = (1, 1, \ldots, 1)$, where 0 and 1 are, respectively, the least and the greatest element of B_t. The union $\vee_{i\in I} g_i$ of any collection $\{g_i : i \in I\}$ of elements of F and the intersection $g \wedge h$ of any two elements g and h of F are defined as follows:

$$\bigvee_{i\in I} g_i = \left(\sup_{i\in I} g_i(1), \sup_{i\in I} g_i(2), \ldots, \sup_{i\in I} g_i(n)\right), \text{and}$$

$$g \wedge h = (\min\{g(1), h(1)\}, \min\{g(2), h(2)\}, \ldots, \min\{g(n), h(n)\}).$$

check for updates

Academic Editor: Francisco Gallego Lupiañez

Received: 21 September 2025
Revised: 23 October 2025
Accepted: 27 October 2025
Published: 30 October 2025

Citation: Chaourar, B.; Benoumhani, M. Finite Fuzzy Topologies on Boolean Algebras. *Mathematics* **2025**, *13*, 3469. https://doi.org/10.3390/math13213469

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

The elements of F will be called B_t-fuzzy subsets of X_n. The definition of a fuzzy topology is taken from [3].

In this paper, we study finite B_t-fuzzy topologies. This work is a generalization of the crisp results in [18–21], as well as the fuzzy work in [15–17], and may be regarded as a continuation of the work in [17], where fuzzy sets with values in the smallest Boolean algebra $D = B_2$ have been considered.

We only consider finite fuzzy topologies on a Boolean algebra, and take the simplest definition of fuzzy topology from [2], which is a generalization of Chang's definition form [3], namely, replacing the interval $I = [0, 1]$ by a distributive lattice L. Similarly, as for the particular case of B-fuzzy topology, the definition of an $L-$topology on any complete bounded lattice L can be given with the appropriate changes. Moreover, it is possible that our ideas regarding computations of finite fuzzy topologies may be generalized to any L-fuzzy topology on more general lattices.

For more information on L-fuzzy topological spaces and some of their recent developments, see [22–38], etc., and the references therein. Some references, such as [30–35,39], are some recent extensions of L-topologies, and may interest some readers.

Since we used a Chang–Goguen fuzzy topology (the open sets are fuzzy, but the topology comprising those open sets is a crisp subset of the powerset B_t^L), the notion of stratified sets and L-topologies, as considered in [40], could be explored in forthcoming research, once the requisite structural complexity is more tractably formalized.

The motivation of this work was facilitated, in part, by the interest in finite topology (fuzzy or crisp) and its applications in the real world, and by the enumeration of finite structures. The importance of this topic resides in the rarity of papers dealing with finite fuzzy structures. Indeed, there are few papers dealing with the subject in the current literature, especially those that use the definition of Goguen, and this paper may be a continuation of such combinatorial investigations that started with [15]. For more details, see the introductions of the papers by [15–17].

Fuzzy topologies have many applications in real-life problems, and they enlighten fundamental research in combinatorics and combinatorial structures. Fuzzy topological concepts, such as fuzzy neighborhoods and separation axioms, are foundational to the learning of common optimization techniques and games under uncertainty [41]. The structure of finite fuzzy spaces can be used to model and analyze preferences, risks, and choices in multi-criteria decision-making problems, especially when the set of alternatives is finite [42]. In digital image analysis, the concepts of fuzzy connectedness, fuzzy boundaries, and fuzzy neighborhoods are essential for segmenting and classifying patterns [43,44]. Finite topological structures simplify the computational complexity of these models [45]. The study of fuzzy topological spaces on fuzzy graphs connects topology with graph theory, which has applications in networking problems, computer technology, and data analysis [46]. Fuzzy finite topologies, particularly those generated by fuzzy relations, provide the algebraic and structural basis for analyzing the properties, representations, and characterizations of fuzzy ideals and fuzzy filters, which in turn are used to unify notions of sequence convergence and limits in applied data structures [47]. The Modeling of Uncertain Systems framework is used to model systems where crisp definitions are impossible. For example, the generalized theory of optimal control has used the machinery of fuzzy topological spaces, extending its reach to dynamic systems with imprecise parameters [48].

Our paper is organized as follows: In Section 2, we prove the main result of this paper, that is, the class of finite B_t-fuzzy topologies on n points is in a bijective correspondence with the class of the classical topologies on tn points. The number of B_t-fuzzy topologies with a small number of open sets, as well as some B_t-fuzzy non-discrete topologies with a large number of open sets, will be a direct consequence of the main theorem. This is

presented in Sections 3 and 4. In Section 5, we compute the total number of B_t-fuzzy topologies of small size, which is the last concrete consequence of the main result of this paper. The conclusion is provided in Section 6.

2. Main Result

Firstly, some notations and definitions are needed to prove the main result.

Let $t \geq 1$ be a positive integer. We denote by B_t the Boolean algebra with 2^t elements, i.e., $\{0,1\}^t$ ordered by the following partial order:

For $x, y \in B_t$, $x < y$ if $x_i \leq y_i$ for any $i = 1, \ldots, t$, and $x_{i_0} < y_{i_0}$ for some $i_0 \in \{1, \ldots, t\}$, where $z = (z_1, \ldots, z_t) \in B_t$.

We can extend this partial order to a lexicographical total order as follows: for any $x, y \in B_t$, $x < y$ if $x_1 < y_1$, or $x_1 = y_1$ and $x_2 < y_2$, or $\ldots$, or $x_i = y_i$, $1 \leq i \leq t-1$, and $x_t < y_t$.

Since any member of the set τ_t of B_t-fuzzy subsets is a mapping from an n-set $Y_n = \{y_1, \ldots, y_n\}$ to B_t, the fuzzy topology τ_t can be represented by a matrix where the columns are the elements of Y_n and the rows are the fuzzy subsets of τ_t. Up to permutations of rows and columns, this matrix is unique. Among all these matrices, we choose the following (canonical) one:

(1) Fix a total order on Y_n: $y_i < y_j$ if $i < j$.
(2) The columns are ordered according to this order.
(3) Fix a (lexicographical) total order on B_t that keeps the partial order of B_t, as explained above.
(4) These two total orders imply a partial order on τ_t that we can extend to a lexicographical total order, as performed for B_t, since the rows have $\{0,1\}$ entries.
(5) The rows are ordered according to this total order on τ_t.

The obtained matrix is denoted by $M(\tau_t)$ and is unique up to the total orders mentioned in (1), (3), and (4). For the remainder of this paper, these orders are fixed for all the considered B_t-fuzzy subsets on Y_n.

In this case, a classical finite subset is actually a B_1-fuzzy subset, i.e., a mapping from a finite set to $\{0,1\}$. Thus, a collection of classical subsets can be represented uniquely by a (canonical) matrix.

For an insight of these definitions and the main proof, readers are recommended to check the example below after Theorem 1.

Let E and A be two finite sets and $C \subseteq A$, t, k, and n be three positive integers. We use the following notations:

E^A or $E^{|A|}$ are the collection of mappings from A to E.

For any $f \in E^A$, $f(C)$ is the set all $f(y)$ for all $y \in C$.

For any $A \subseteq B_t$, and $\alpha \in \{0,1\}$, $f^{(\alpha)} \in E^A$ such that $f^{(\alpha)}(y) = \alpha$ for any $y \in A$, that is, an α-constant mapping.

The set $\mathcal{T}(n)$ is the set of finite (labeled) topologies on n points, and $T(n) = |\mathcal{T}(n)|$.

The set $\mathcal{T}(n, 2^t)$ is the corresponding set of finite B_t-fuzzy topologies and $T(n, 2^t)$ is its cardinality.

The set $\mathcal{T}(n,k)$ is the set of finite (labeled) topologies on n points and having k open sets, and $T(n,k) = |\mathcal{T}(n,k)|$ is its cardinality.

The set $\mathcal{T}(n, 2^t, k)$ is the corresponding set of finite B_t-fuzzy topologies, and $T(n, 2^t, k)$ is its cardinality.

Lemma 1. *Let t, k, n be three positive integers, $X_{tn} = \{x_{11}, \ldots, x_{1t}, \ldots, x_{tn}\}$, and $Y_n = \{y_1, \ldots, y_n\}$. Then:*

(i) *There is a one-to-one correspondence φ between $\{0,1\}^{X_{tn}}$ and $B_t^{Y_n}$.*

(ii) $\varphi(\emptyset) = 0_F$, *and* $\varphi(X_{tn}) = 1_F$, *that is, the images of the 0-constant and 1-constant mappings in* $\{0,1\}^{X_{tn}}$, *respectively, are the* 0_F *and* 1_F *mappings in* $B_t^{Y_n}$, *respectively.*
(iii) *For any* $\tau \subseteq \{0,1\}^{X_{tn}}$, $|\tau| = |\varphi(\tau)|$.

Proof. For any mapping μ_{tn} from X_n to $\{0,1\}$, its image through φ is μ_n, where $\mu_n(y_i) = (\mu_{tn}(x_{i1}), \ldots, \mu_{tn}(x_{it})) \in \{0,1\}^t = B_t$ for any $y_i \in Y_n$. It is clear that φ is injective. It is also surjective because $\varphi^{-1}(\mu_n)(x_{ij})$ is the jth component of $\mu_n(y_i)$, and thus $\varphi^{-1}(\mu_n) \in \{0,1\}^{X_{tn}}$. This means that φ is bijective. The second part of the lemma is straightforward. For the third part, since τ and $\varphi(\tau)$ are represented by the two identical matrices $M(\tau)$ and $M(\varphi(\tau))$ under the above conditions (1)–(5), $|\tau|$ = number of rows of $M(\tau)$ = number of rows of $M(\varphi(\tau)) = |\varphi(\tau)|$. □

Below, we prove the main result of this paper.

Theorem 1. *Let* t, k, n *be three positive integers. Then there is a one-to-one correspondence between* $\mathcal{T}(tn, k)$ *and* $\mathcal{T}(n, B_t, k)$.

Proof. It suffices to prove that the one-to-one correspondence φ introduced in the proof of the above lemma induces a one-to-one correspondence between the considered sets.
Let $\mu_{tn,i}$ be two mappings from X_{tn} to $\{0,1\}$, $i = 1,2$ (in other words, two subsets of X_{tn}), $\mu_{n,i} = \varphi(\mu_{tn,i})$, $\mu_{tn} = \mu_{tn,1} \vee \mu_{tn,2}$, and $\mu_n = \mu_{n,1} \vee \mu_{n,2}$. We can now prove that $\varphi(\mu_{tn}) = \mu_n$.
Let $\lambda = \varphi(\mu_{tn}) \in B_t^{Y_n}$, and $y_i \in Y_n$. It follows that

$$\lambda(y_i) = (\mu_{tn}(x_{i1}), \ldots, \mu_{tn}(x_{it})) = ((\mu_{tn,1} \vee \mu_{tn,2})(x_{i1}), \ldots, (\mu_{tn,1} \vee \mu_{tn,2})(x_{it}))$$

$$= (\max\{\mu_{tn,1}(x_{i1}), \mu_{tn,2}(x_{i1})\}, \ldots, \max\{\mu_{tn,1}(x_{it}), \mu_{tn,2}(x_{it})\}).$$

On the other hand,

$$\mu_n(y_i) = (\mu_{n,1} \vee \mu_{n,2})(y_i) = \max\{\mu_{t,1}(y_i), \mu_{n,2}(y_i)\}$$

$$= \max\{(\mu_{tn,1}(x_{i1}), \ldots, \mu_{tn,1}(x_{it}), (\mu_{tn,2}(x_{i1}), \ldots, \mu_{tn,2}(x_{it})\} \quad (*)$$

$$= (\max\{\mu_{tn,1}(x_{i1}), \mu_{tn,2}(x_{i1})\}, \ldots, \max\{\mu_{tn,1}(x_{it}), \mu_{tn,2}(x_{it})\}),$$

because the two vectors in (*) belong to B_t and can be viewed as mappings from $\{x_{i1}, \ldots, x_{it}\}$ to $\{0,1\}$. This the desired result.

The corresponding equality for the intersection is similarly proved.

Finally, if $\mu_{tn,1} \leq \mu_{tn,2}$, then $\mu_{tn,1}(x_{ij}) \leq \mu_{tn,2}(x_{ij})$ for all $x_{ij} \in X_{tn}$. This yields $\mu_{n,1}(y_i) \leq \mu_{n,2}(y_i)$ for all $y_i \in Y_n$. In other words, $\mu_{n,1} \leq \mu_{n,2}$, i.e., φ keeps the order.

Assertion (iii) of Lemma 1 implies that the number of open sets of any classical topology τ on X_{tn} is the same number of B_t-fuzzy open sets of its image $\varphi(\tau)$, the B_t-fuzzy topology defined on Y_n, and vice versa. □

To illustrate this result and the tools used to achieve it, we provide an example below.

Example 1. $n = 3$, *and* $t = 2$.

Consider the Boolean algebra $B_2 = \{0_B, a, b, 1_B\}$, where $0 < a, b < 1$, $X_6 = \{x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}\}$ and $Y_3 = \{y_1, y_2, y_3\}$. Now, let $\tau_B = \{f_i, 1 \leq i \leq 7\}$ be the B_2-fuzzy topology on Y_3, where the mappings f_i range from Y_3 to B_2, as summarized in Table 1. For example, f_5 is the mapping from Y_3 to B_2, such that $f_5(y_1) = 1_B$, $f_5(y_2) = b$, $f_5(y_3) = 0_B$, while $\tau = \{\emptyset, \{x_{11}\}, \{x_{12}\}, \{x_{11}, x_{12}\}, \{x_{11}, x_{12}, x_{21}\}, \{x_{11}, x_{12}, x_{22}\}, X_6\}$ is a

classical topology defined on X_6 and represented by a matrix in Table 1. For example, the corresponding row for the open set $\{x_{11}, x_{12}, x_{22}\}$ is $[1,1,0,1,0,0]$, giving information on which point $x \in X_6$ belongs to this open set and which one does not.

By using the binary representation of B_2 as $\{0,1\}^2 = \{(0,0),(0,1),(1,0),(1,1)\}$, where we identify 0_B with $(0,0)$, a with $(0,1)$, b with $(1,0)$, and finally 1_B with $(1,1)$, we obtain the corresponding classical topology τ on X_6 from τ_B and vice versa. For example, the blue entries of the matrix representing τ_B correspond to the blue entries of the matrix repersenting τ. And so on for the black and green entries of both matrices. As the reader can see, the number of open sets is the same, i.e., seven open sets. This also illustrates Corollary 1.

Table 1. Matrices representing τ_B (on the left) and its corresponding classical topology τ (on the right).

	y_1	y_2	y_3		x_{11}	x_{12}	x_{21}	x_{22}	x_{31}	x_{32}
f_0	0_B	0_B	0_B	$\emptyset$	0	0	0	0	0	0
f_2	b	0_B	0_B	$\{x_{11}\}$	1	0	0	0	0	0
f_3	a	0_B	0_B	$\{x_{12}\}$	0	1	0	0	0	0
f_4	1_B	0_B	0_B	$\{x_{11}, x_{12}\}$	1	1	0	0	0	0
f_5	1_B	b	0_B	$\{x_{11}, x_{12}, x_{21}\}$	1	1	1	0	0	0
f_6	1_B	a	0_B	$\{x_{11}, x_{12}, x_{22}\}$	1	1	0	1	0	0
f_1	1_B	1_B	1_B	X_6	1	1	1	1	1	1

Our first two direct consequences of the main result are as follows:

Corollary 1. $T(tn,k) = T(n,2^t,k)$.

Corollary 2. $T(tn) = T(n,2^t)$.

3. $B-$Fuzzy Topologies with a Small Number of Open Sets

Before providing some other consequences of our main result, let us recall the definition of the Stirling number of the second kind.

Definition 1. *The number of partitions of a finite set with cardinality $n \geq 1$ into k subsets is called the Stirling number of the second kind, and is denoted as $S_{n,k}$.*

By the principle of inclusion-exclusion principle, one can prove that $S_{n,k}$ is

$$S_{n,k} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n$$

In [18], the number of classical topologies on X_n with k open sets, $4 \leq k \leq 12$, was computed. Using Theorem 1 below, we deduce the third consequence of our main result, that is, the number of fuzzy topologies with k open sets. Actually, this is an application of Corollary 1, which gives the number of B-fuzzy topologies by means of their corresponding classical ones.

Theorem 2. *The number $T(n,2^t,k)$ of B_t- fuzzy topologies of cardinality k, $4 \leq k \leq 12$ is given by*

$$
\begin{aligned}
T(n,2^t,4) =& 3^{tn} - 3(2^{tn}) + 3\\
T(n,2^t,5) =& 4^{tn} - 3^{tn+1} + 3.2^{tn} - 1\\
T(n,2^t,6) =& 3!S_{nt,3} + \frac{3}{2}4!S_{nt,4} + 5!S_{nt,5}\\
T(n,2^t,7) =& \frac{9}{4}4!S_{nt,4} + 2.5!S_{nt,5} + 6!S_{nt,6}\\
T(n,2^t,8) =& S_{nt,3} + 2.4!S_{nt,4} + \frac{15}{4}5!S_{nt,5} + \frac{15}{2}6!S_{nt,6} + 7!S_{nt,7}\\
T(n,2^t,9) =& \frac{5}{6}4!S_{nt,4} + 5.5!S_{nt,5} + \frac{11}{2}6!S_{nt,6} + 3.7!S_{nt,7} + 8!S_{nt,8}\\
T(n,2^t,10) =& 4.4!S_{nt,4} + \frac{11}{2}.5!S_{nt,5} + \frac{73}{8}6!S_{nt,6} + \frac{15}{2}.7!S_{nt,7} + \frac{7}{2}8!S_{nt,8} + 9!S_{nt,9}\\
T(n,2^t,11) =& \frac{11}{2}.5!S_{nt,5} + \frac{73}{8}6!S_{nt,6} + \frac{15}{2}.7!S_{nt,7} + \frac{7}{2}8!S_{nt,8} + 9!S_{nt,9}\\
T(n,2^t,12) =& \frac{1}{2}S_{nt,4} + \frac{11}{2}.5!S_{nt,5} + \frac{73}{8}6!S_{nt,6} + \frac{15}{2}.7!S_{nt,7} + \frac{7}{2}8!S_{nt,8} + 9!S_{nt,9} + 10!S_{nt,10} + 11!S_{nt,11}
\end{aligned}
$$

4. $B-$Fuzzy Topologies with Large Number of Open Sets

As for the classical case, the total number of topologies with a large number of open sets is known, so we can provide its analogy in the B_t- fuzzy case. Corollary 2 implies that the number of B_t-fuzzy topologies on n points is equal to the number of classical ones on tn points. So, the fourth consequence of our main result is as follows:

Theorem 3. *There are no B_t-fuzzy topology of cardinality k on Y_n, where k is an integer $3 \times 2^{tn-2} < k < 2^{tn}$.*

Proof. This is an immediate consequence from the classical case, since there are no topologies of cardinality k on X_n, with $3 \times 2^{n-2} < k < 2^n$. □

The fifth consequence provides the total number of B_t-topologies in the previous extreme case, as follows:

Theorem 4. *The total number of B_t-topology with cardinality $3 \times 2^{nt-2}$ is given by $nt(nt-1)$.*

Proof. This is also immediate from the classical case, since the number of topologies of cardinality $3 \times 2^{n-2}$ is $n(n-1)$. □

Moreover, all results proved in [17] become corollaries of our main result, and are direct consequences of it.

5. Number of $B-$Fuzzy Topologies for Small n and t

The sixth consequence is the following direct implication of Corollary 2.

Theorem 5. *The number $T(n, B_t)$ of all labeled B_t- fuzzy topologies on Y_n for small n and t is given by the following:*

$$
\begin{aligned}
T(2,B_2) =& 355\\
T(2,B_3) =& T(3,B_2) = 209,527\\
T(2,B_4) =& T(4,B_2) = 642,779,354\\
T(3,B_3) =& 63,260,289,423\\
T(2,B_5) =& T(5,B_2) = 8,977,053,873,043\\
T(2,B_6) =& T(3,B_4) = T(4,B_3) = 519,355,571,065,774,021\\
T(2,B_7) =& T(7,B_2) = 115,617,051,977,054,267,807,460\\
T(3,B_5) =& T(5,B_3) = 88,736,269,118,586,244,492,485,121\\
T(2,B_8) =& T(8,B_2) = T(4,B_4) = 93,411,113,411,710,039,565,210,494,095\\
T(2,B_9) =& T(9,B_2) = T(3,B_6) = T(6,B_3) = 261,492,535,743,634,374,805,066,126,901,117,203
\end{aligned}
$$

and so on. We can obtain as many as consequences as necessary, and then obtain appropriate results for classical topologies.

6. Conclusions

The primary achievement of the current research is the establishment of a direct link between B-fuzzy topologies and their classical topological counterparts, a key result that provides a framework for translating concepts between the two theories. Building upon this success, the authors propose to extend this investigation by generalizing the lattice structure used to define the topologies. Specifically, future work should focus on P-fuzzy topologies, where the fuzziness is valued in P, an arbitrary bounded poset (partially ordered set). The ultimate goal is to examine whether the relationship with classical topologies can be maintained or generalized under this less restrictive structure. The authors acknowledge that initial progress on this generalization has already been made in [16], but only for special cases where P is a totally ordered poset (a chain). This confirms that the direction of future research should be to move from specialized fuzzy topological spaces to those defined using increasingly general algebraic structures. Another direction is to prove results on B-fuzzy topologies that are not known for the classical case. We can then deduce the analogies for finite topologies and obtain new results in the classical case.

Author Contributions: Conceptualization, B.C. and M.B.; Methodology, B.C. and M.B.; Validation, B.C.; Formal analysis, B.C.; Investigation, B.C.; Writing—original draft, B.C. and M.B.; Writing—review & editing, B.C. and M.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study.

Acknowledgments: The authors are grateful to anonymous referees for their valuable remarks and corrections in previous versions of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy Sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
2. Goguen, J.A. The fuzzy Tychonoff theorem. *J. Math. Anal. Appl.* **1973**, *43*, 737–742. [CrossRef]
3. Chang, C.L. Fuzzy topological spaces. *J. Math. Anal. Appl.* **1968**, *24*, 182–189. [CrossRef]
4. Gantner, T.E.; Steinlage, R.C.; Warren, R.H. Compactness in Fuzzy Topological Spaces. *J. Math. Anal. Appl.* **1978**, *62*, 547–562. [CrossRef]
5. Schnare, P.S. The complementation of the lattice of topologies. *Proc. Am. Math. Soc.* **1968**, *19*, 851–855.
6. Hutton, B.; Reilly, I.L. Fuzzy Topologies Associated with Fuzzy Uniformities. *Fuzzy Sets Syst.* **1980**, *3*, 129–140.
7. Sarkar, M. Fuzzy Neighborhood Spaces. *J. Math. Anal. Appl.* **1982**, *87*, 336–348.
8. Badarad, R. Lattice-Valued Fuzzy Set Theory and Fuzzy Topology. *J. Math. Anal. Appl.* **1982**, *85*, 403–415.
9. Conard, F. *Fuzzy Sets and Their Applications*; Academic Press: New York, NY, USA, 1975.
10. Höhle, U. Probabilistic topologies induced by Γ-fuzzy sets. *Fuzzy Sets Syst.* **1985**, *15*, 251–267.
11. Höhle, U. Product of probability algebras. *Fuzzy Sets Syst.* **1985**, *17*, 297–308.
12. Rodabaugh, S.E. A categorical accommodation of various notions of fuzzy topology. *Fuzzy Sets Syst.* **1983**, *9*, 241–264. [CrossRef]
13. Lowen, R. Fuzzy Topological Spaces and Fuzzy Compactness. *J. Math. Anal. Appl.* **1976**, *56*, 621–633. [CrossRef]
14. Lowen, R. Initial and Final Fuzzy Topologies and the Fuzzy Tychonoff Theorem. *J. Math. Anal. Appl.* **1977**, *58*, 97–106. [CrossRef]
15. Benoumhani, M.; Jaballah, A. Finite fuzzy topological spaces. *Fuzzy Sets Syst.* **2017**, *321*, 101–114. [CrossRef]
16. Benoumhani, M.; Jaballah, A. Chains in lattices of mappings and finite fuzzy topological spaces. *J. Comb. Theory Ser. A* **2019**, *161*, 99–111. [CrossRef]
17. Benoumhani, M.; Jaballah, A. Finite D-Fuzzy Topologies. *Int. J. Fuzzy Syst.* **2025**, *25*, 2532–2544. [CrossRef]
18. Benoumhani, M. The Number of Topologies on a Finite Set. *J. Integer Seq.* **2006**, *9*, 06.2.6.

19. MBenoumhani; Kolli, M. Finite topologies and partitions. *J. Integer Seq.* **2010**, *13*, 10.3.5.
20. Brinkmann, G.; McKay, B.D. Counting unlabeled topologies and transitive relations. *J. Integer Seq.* **2005**, *8*, 05.2.1.
21. Kolli, M. On the Cardinality of the T_0-Topologies on a Finite Set. *Int. J. Comb.* **2014**, *2014*, 798074. [CrossRef]
22. Butnarbu, D. L-Fuzzy Topologies. *Bull. Mathématique Société Sci. Mathématiques République Social. Roum. Nouv. Série* **1975**, *19*, 227–236.
23. Kubiak, T.; Sostak, A. Foundations of the theory of (L,M)-fuzzy topological spaces. In *Abstracts of the 30th Linz Seminar on Fuzzy Set Theory*; Johannes Kepler Universität: Linz, Austria, 2009; pp. 70–73.
24. Kumar, V.; Tiwari, S. L-Fuzzy rough point-wise proximity spaces. *Appl. Gen. Topol.* **2025**, *26*, 255–270. [CrossRef]
25. Ramadan, A.A.; Elkordy, E.H.; El-Dardery, M. L-Fuzzy Approximation spaces and L-Fuzzy topological spaces. *Iran. J. Fuzzy Syst.* **2016**, *13*, 115–129.
26. Ramadan, A.A.; Elkordy, E.H.; Kim, Y.C.; Ahmed, K. L-fuzzy pre-proximities and application to L-fuzzy topologies. *J. Intell. Fuzzy Syst.* **2020**, *4*, 4049–4060. [CrossRef]
27. Ramadan, A.A.; Fawakhreh, A.; Elkordy, E. Novel categorical relations between L-fuzzy co-topologies and L-fuzzy ideals. *AIMS Math.* **2024**, *9*, 20572–20587. [CrossRef]
28. Ramadan, A.A.; Fawakhreh, A.J. On L-Fuzzy Preproximity Spaces and L-Fuzzy Ideals. *New Math. Nat. Comput.* **2025**, *21*, 867–886. [CrossRef]
29. Sarkar, M. On L-fuzzy topological spaces. *J. Math. Anal. Appl.* **1981**, *84*, 431–442. [CrossRef]
30. Zhang, Z.; Shi, F.-G.; Li, Q. Continuity and directed completion of topological spaces. *Order* **2022**, *39*, 407–420. [CrossRef]
31. Zhong, Y.; Wu, X.; Šostak, A.; Shi, F.G. (L,M)-fuzzy k-pseudo metric space. *Mathematics* **2022**, *10*, 1151 . [CrossRef]
32. Zeng, M.; Wang, L.; Shi, F.-G. A novel approach to the fuzzification of fields. *Symmetry* **2022**, *14*, 1190. [CrossRef]
33. Liang, C.; Shi, F.-G.; Wang, J. (L,M)-fuzzy bornological spaces. *Fuzzy Sets Syst.* **2023**, *467*, 108496. . [CrossRef]
34. Dong, Y.; Shi, F.-G. L-cones and L-dual cones. *Fuzzy Sets Syst.* **2023**, *469*, 108626. [CrossRef]
35. Sun, Y.; Shi, F.-G. Representations of L-fuzzy rough approximation operators. *Inf. Sci.* **2023**, *645*, 119324. [CrossRef]
36. Shu, L.; Zha, Y. On the Completion of L-Fuzzy Topological Vector Spaces. In *Fuzzy Information and En-gineering 2010, Advances in Intelligent and Soft Computing 2010*; Cao, B., Wang, G., Guo, S., Chen, S., Eds.; Springer: Berlin/Heidelberg, Germany, 2010; p. 78.
37. Singh, S.; Tiwari, S.P.; Mahato, S. On L-fuzzy automata, coalgebras and dialgebras: Associated categories and L-fuzzy topologies. *Fuzzy Sets Syst.* **2023**, *460*, 143–185. [CrossRef]
38. Tripathi, A.; Tiwarri, S.P. On L-Fuzzy Topological Spaces Determined by Implicators. *New Math. Nat. Comput.* **2023**, *19*, 805–830. [CrossRef]
39. Sostak, A.; Uljane, I. Extending L-Topologies to Bipolar *l*-Fuzzy Topologies. *Axioms* **2024**, *13*, 582. [CrossRef]
40. Abas, S.E.; Gurlebeck, K. L-fuzzy Topological Spaces. *J. Fuzzy Math.* **2011**, *19*, 1–12. Available online: https://www.researchgate.net/publication/267432140_Stratified_L-fuzzy_topological_spaces (accessed on 15 September 2025).
41. Papageorgiou, N.S. Fuzzy Multifunctions and the Relaxation of Fuzzy Optimal Control Problems. *J. Math. Anal. Appl.* **1985**, *109*, 397–425. [CrossRef]
42. Alkouri, A.U.; Hazaimeh, M.; Jawarneh, I. Fuzzy Topological Space on Fuzzy Space. *arXiv* **2021**, arXiv:2105.07488v1. [CrossRef]
43. Kovalevsky, V.A. Finite Topology and Image Analysis. *Adv. Electron. Electron Phys.* **1992**, *84*, 197–259.
44. Saha, P.K.; Udupa, J.K. Fuzzy Digital Topology and Geometry and Their Applications to Medical Imaging. In *Recent Advances in Fuzzy Logic and Soft Computing*; Murthy, L.N.B., Ed.; Springer: New Delhi, India, 2013; pp. 3–19.
45. Merrifield, R.E.; Simmons, H.E. *Topological Methods in Chemistry*; Wiley: New York, NY, USA, 1989.
46. Gholap, P.S.; Nikumbh, V.E. Fuzzy Topological Spaces On Fuzzy Graphs. *Ann. Fuzzy Math. Inform.* **2023**, *25*, 279–291.
47. Mishra, S.; Srivastava, R. Fuzzy Ideals and Fuzzy Filters on Topologies Generated by Fuzzy Relations. *Int. J. Algebra Appl.* **2023**, *12*, 72–84.
48. Nazaroff, N. Fuzzy Topological Spaces and Optimal Control. *J. Math. Anal. Appl.* **1976**, *56*, 478–485.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.