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Upper Bounds for the Numerical Radius of Off-Diagonal 2×2 Operator Matrices

Najla Altwaijry [1,*] and Silvestru Sever Dragomir [2,3]

[1] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
[2] Applied Mathematics Research Group, ISILC, Victoria University, P.O. Box 14428, Melbourne, VIC 8001, Australia; sever.dragomir@ajmaa.org
[3] Department of Mathematical and Physical Sciences, La Trobe University, Plenty Road, Bundoora, Melbourne, VIC 3086, Australia
* Correspondence: najla@ksu.edu.sa

Abstract

We establish multiple novel upper estimates for the numerical radius associated with off-diagonal operator matrices defined on a complex Hilbert space $\mathfrak{H}$. The operators considered have a specific structure, with zero diagonal entries and anti-diagonal entries given by a bounded linear operator $\mathcal{C}$ and the adjoint of another, $\mathcal{D}^*$. The primary contribution is a set of inequalities that connect the square of the numerical radius to expressions involving the norms of these constituent operators. As applications, we specialize our main results to obtain refined inequalities for two significant cases: when $\mathcal{D}$ is the adjoint of $\mathcal{C}$, where $\mathcal{C}$ and $\mathcal{D}$ represent the real and imaginary components of one operator $\mathcal{T}$.

Keywords: numerical radius; off-diagonal operator matrix; bounded linear operators; Hilbert spaces; real and imaginary parts of operators; adjoint operators

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1. Introduction

In various fields of mathematics, such as functional analysis, operator theory, and numerical analysis, mathematical inequalities are of foundational importance. They serve as indispensable instruments for analyzing the characteristics of mathematical structures and possess extensive applications in applied fields such as physics, engineering, and computer science. To illustrate, inequalities that concern operator norms and numerical radii are vital for establishing bounds and estimates within operator theory. Foundational work in this area has provided key numerical radius inequalities for various operator structures [1–3]. This field has continued to evolve with recent advances and refinements [4–6], including important generalizations to semi-Hilbert spaces and the study of the A-numerical radius [7–10]. Further related studies have explored properties of commuting operators and their joint spectral radii [11,12].

Recent progress in this domain has yielded enhanced bounds and more precise estimates, which has fostered significant theoretical advancements [13–17]. Researchers have investigated a variety of refinements and generalizations of classical inequalities, which has led to deeper insights and original results. For example, Bhunia et al. [18] examined improvements to numerical radius inequalities for operators and operator matrices. In a similar vein, Alahmari et al. [19] broadened these concepts to the a-numerical range in

C^*-algebras. Furthermore, Bourhim and Mabrouk [20] explored the a-numerical range on C^*-algebras, while the second author of this paper investigated inequalities pertaining to the Euclidean operator radius and their applications [21].

The examination of operator inequalities, especially those concerning the numerical radius, has attracted considerable interest in recent years. Spurred by their broad applicability, mathematicians have focused on extending and refining classical results. For instance, Kittaneh et al. [2,22,23] developed numerical radius inequalities for particular operator matrices, while Moslehian et al. [24] broadened Euclidean operator radius inequalities. Buzano's inequality [25] has also been a key tool in deriving power numerical radius inequalities, as demonstrated in recent studies [26,27]. Drawing inspiration from this body of work, we establish several new upper bounds for the numerical radius of an off-diagonal operator matrix. We will first recall some essential concepts and definitions.

Let $\mathfrak{H}$ represent a complex Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ along with the corresponding norm $\| \cdot \|$. The notation $\mathbb{L}(\mathfrak{H})$ refers to the C^*-algebra consisting of every bounded linear operator acting on $\mathfrak{H}$. For every operator $\mathcal{C} \in \mathbb{L}(\mathfrak{H})$, the adjoint is indicated by $\mathcal{C}^\star$, while the absolute value, expressed as the positive square root of $\mathcal{C}^\star\mathcal{C}$, is denoted as $|\mathcal{C}| = (\mathcal{C}^\star\mathcal{C})^{1/2}$. The real part and imaginary part of $\mathcal{C}$ are defined through $\mathscr{R}e(\mathcal{C}) = \frac{1}{2}(\mathcal{C} + \mathcal{C}^\star)$ and $\mathscr{I}m(\mathcal{C}) = \frac{1}{2i}(\mathcal{C} - \mathcal{C}^\star)$, respectively.

The numerical range of $\mathcal{C}$, denoted by $W(\mathcal{C})$, consists of all values $\{\langle \mathcal{C}\xi, \xi \rangle : \xi \in \mathfrak{H}, \|\xi\| = 1\}$. The operator norm $\|\mathcal{C}\|$ and numerical radius $\omega(\mathcal{C})$ are given by the following expressions:

$$\|\mathcal{C}\| = \sup\{|\langle \mathcal{C}\xi_1, \xi_2 \rangle| : \xi_1, \xi_2 \in \mathfrak{H}, \|\xi_1\| = \|\xi_2\| = 1\},$$

and

$$\omega(\mathcal{C}) = \sup\{|\langle \mathcal{C}\xi, \xi \rangle| : \xi \in \mathfrak{H}, \|\xi\| = 1\}.$$

It is widely recognized that the numerical radius $\omega(\cdot)$ forms a norm on $\mathbb{L}(\mathfrak{H})$ comparable to the operator norm $\| \cdot \|$. In particular, the following relations hold:

$$\frac{1}{2}\|\mathcal{C}\| \leq \omega(\mathcal{C}) \leq \|\mathcal{C}\|.$$

Such estimates are optimal: equality in the lower bound occurs when $\mathcal{C}^2 = 0$, whereas equality in the upper bound holds precisely when $\mathcal{C}$ is normal.

Kittaneh subsequently refined these estimates by proving that

$$\frac{1}{2}\sqrt{\| |\mathcal{C}|^2 + |\mathcal{C}^\star|^2 \|} \leq \omega(\mathcal{C}) \leq \frac{\sqrt{2}}{2}\sqrt{\| |\mathcal{C}|^2 + |\mathcal{C}^\star|^2 \|}.$$

Now, examine the off-diagonal segment of a 2×2 operator matrix $\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D} & 0 \end{bmatrix}$ operating on the Hilbert space $\mathfrak{H} \oplus \mathfrak{H}$. It has been shown that $\omega\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D} & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & \mathcal{D} \\ \mathcal{C} & 0 \end{bmatrix}\right)$, and $\omega\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D} & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & \mathcal{C} \\ e^{i\tau}\mathcal{D} & 0 \end{bmatrix}\right)$, for any $\tau \in \mathbb{R}$.

In 2011, Hirzallah, Kittaneh, and Shebrawi obtained the subsequent double inequality:

$$\frac{1}{2}\max\{\omega(\mathcal{C} + \mathcal{D}), \omega(\mathcal{C} - \mathcal{D})\} \leq \omega\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D} & 0 \end{bmatrix}\right) \leq \frac{1}{2}[\omega(\mathcal{C} + \mathcal{D}) + \omega(\mathcal{C} - \mathcal{D})]. \quad (1)$$

They also demonstrated that

$$\omega\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D} & 0 \end{bmatrix}\right) \leq \min\{\omega(\mathcal{C}), \omega(\mathcal{D})\} + \frac{1}{2}\min\{\|\mathcal{C}+\mathcal{D}\|, \|\mathcal{C}-\mathcal{D}\|\}. \tag{2}$$

More recently, in 2015, Kittaneh, Moslehian, and Yamazaki sharpened the triangle inequality for the numerical radius in the following manner:

$$\left\|\frac{\mathcal{C}+\mathcal{D}}{2}\right\| \leq \omega\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}\right) \leq \frac{\|\mathcal{C}\|+\|\mathcal{D}\|}{2}.$$

Motivated by these findings, this work seeks to establish multiple fresh upper estimates for the numerical radius of the off-diagonal operator matrix $\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}$, where $\mathcal{C}, \mathcal{D} \in \mathbb{L}(\mathfrak{H})$. We shall additionally provide comprehensive examples for particular situations, including cases where $\mathcal{D} = \mathcal{C}^\star$ and where $\mathcal{C} = \mathscr{R}e(\mathcal{T})$ and $\mathcal{D} = \mathscr{I}m(\mathcal{T})$, for $\mathcal{T} \in \mathbb{L}(\mathfrak{H})$.

2. Main Results

We commence this section through deriving multiple upper estimates concerning the off-diagonal operator matrix.

Theorem 1. *Let $\mathcal{C}, \mathcal{D} \in \mathbb{L}(\mathfrak{H})$. Then,*

$$\omega^2\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left\|\frac{|\mathcal{C}|^2+|\mathcal{D}|^2}{2}\right\| + \frac{1}{2} \times \begin{cases} (\|\mathscr{R}e(\mathcal{D}^\star\mathcal{C})\| + \|\mathscr{I}m(\mathcal{D}^\star\mathcal{C})\|); \\ \\ \beta_p\left(\|\mathscr{R}e(\mathcal{D}^\star\mathcal{C})\|^q + \|\mathscr{I}m(\mathcal{D}^\star\mathcal{C})\|^q\right)^{1/q} \\ \text{for } p,\ q > 1 \text{ with } \frac{1}{p}+\frac{1}{q} = 1; \\ \\ \sqrt{2}\max\{\|\mathscr{R}e(\mathcal{D}^\star\mathcal{C})\|, \|\mathscr{I}m(\mathcal{D}^\star\mathcal{C})\|\}, \end{cases} \tag{3}$$

where

$$\beta_p := \begin{cases} 2^{\frac{2-p}{2p}} \text{ if } p \in (1,2) \\ \\ 1 \text{ if } p \in [2,\infty). \end{cases} \tag{4}$$

Proof. The proof relies on a well-known representation for the numerical radius, which was used in [22]:

$$\omega\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}\right) = \frac{1}{2}\sup_{\tau\in\mathbb{R}}\left\|e^{i\tau}\mathcal{C} + e^{-i\tau}\mathcal{D}\right\| \tag{5}$$

for all $\mathcal{C}, \mathcal{D} \in \mathbb{L}(\mathfrak{H})$.

A straightforward calculation shows that

$$\begin{aligned} \left\|e^{i\tau}\mathcal{C}\xi + e^{-i\tau}\mathcal{D}\xi\right\|^2 &= \left\|e^{i\tau}\mathcal{C}\xi\right\|^2 + \left\|e^{-i\tau}\mathcal{D}\xi\right\|^2 + 2\mathscr{R}e\langle e^{i\tau}\mathcal{C}\xi, e^{-i\tau}\mathcal{D}\xi\rangle \\ &= \left\langle\left(|\mathcal{C}|^2+|\mathcal{D}|^2\right)\xi,\xi\right\rangle + 2\mathscr{R}e\left[e^{2i\tau}\langle\mathcal{D}^\star\mathcal{C}\xi,\xi\rangle\right] \\ &\leq \left\langle\left(|\mathcal{C}|^2+|\mathcal{D}|^2\right)\xi,\xi\right\rangle + 2\left|\mathscr{R}e\left[e^{2i\tau}\langle\mathcal{D}^\star\mathcal{C}\xi,\xi\rangle\right]\right|, \end{aligned}$$

from which we obtain the inequality

$$\left\|\frac{e^{i\tau}\mathcal{C}\xi + e^{-i\tau}\mathcal{D}\xi}{2}\right\|^2 \leq \frac{1}{2}\left[\left\langle\left(\frac{|\mathcal{C}|^2+|\mathcal{D}|^2}{2}\right)\xi,\xi\right\rangle + \left|\mathscr{R}e\left[e^{2i\tau}\langle\mathcal{D}^\star\mathcal{C}\xi,\xi\rangle\right]\right|\right] \tag{6}$$

for all $\tau \in \mathbb{R}$ and $\xi \in \mathfrak{H}$.

Furthermore, we can express the real part of the cross-term as

$$\begin{aligned}\mathscr{R}e\Big[e^{2i\tau}\langle \mathcal{C}\xi, \mathcal{D}\xi\rangle\Big] &= \mathscr{R}e\big[(\cos(2\tau) + i\sin(2\tau))\langle \mathscr{R}e(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle + i\langle \mathscr{I}m(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big] \\ &= \cos(2\tau)\langle \mathscr{R}e(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle - \sin(2\tau)\langle \mathscr{I}m(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\end{aligned}$$

for all $\tau \in \mathbb{R}$ and $\xi \in \mathfrak{H}$. An application of the elementary Hölder's inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, gives the following bound:

$$\begin{aligned}&\Big|\mathscr{R}e\Big[e^{2i\tau}\langle \mathcal{C}\xi, \mathcal{D}\xi\rangle\Big]\Big| \\ &\leq \begin{cases} \max\{|\cos(2\tau)|, |\sin(2\tau)|\}\big(\big|\langle \mathscr{R}e(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big| + \big|\langle \mathscr{I}m(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big|\big); \\ \big(|\cos(2\tau)|^{p} + |\sin(2\tau)|^{p}\big)^{1/p} \\ \times\Big(\big|\langle \mathscr{R}e(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big|^{q} + \big|\langle \mathscr{I}m(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big|^{q}\Big)^{1/q} \\ \text{for } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1; \\ (|\cos(2\tau)| + |\sin(2\tau)|)\max\big\{\big|\langle \mathscr{R}e(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big|, \big|\langle \mathscr{I}m(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big|\big\} \end{cases}\end{aligned}$$

for all $\tau \in \mathbb{R}$ and $\xi \in \mathfrak{H}$.

Substituting this back into inequality (6) yields

$$\begin{aligned}&\left\|\frac{e^{i\tau}\mathcal{C}\xi + e^{-i\tau}\mathcal{D}\xi}{2}\right\|^{2} \\ &\leq \frac{1}{2}\left\langle\left(\frac{|\mathcal{C}|^{2} + |\mathcal{D}|^{2}}{2}\right)\xi, \xi\right\rangle \\ &\quad + \frac{1}{2}\begin{cases} \max\{|\cos(2\tau)|, |\sin(2\tau)|\}\big(\big|\langle \mathscr{R}e(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big| + \big|\langle \mathscr{I}m(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big|\big); \\ \big(|\cos(2\tau)|^{p} + |\sin(2\tau)|^{p}\big)^{1/p} \\ \times\Big(\big|\langle \mathscr{R}e(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big|^{q} + \big|\langle \mathscr{I}m(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big|^{q}\Big)^{1/q} \\ \text{for } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1; \\ (|\cos(2\tau)| + |\sin(2\tau)|)\max\big\{\big|\langle \mathscr{R}e(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big|, \big|\langle \mathscr{I}m(\mathcal{D}^{\star}\mathcal{C})\xi, \xi\rangle\big|\big\} \end{cases}\end{aligned} \tag{7}$$

for all $\tau \in \mathbb{R}$ and $\xi \in \mathfrak{H}$.

By taking the supremum over all unit vectors $\xi \in \mathfrak{H}$, the preceding inequality becomes

$$\begin{aligned}&\left\|\frac{e^{i\tau}\mathcal{C} + e^{-i\tau}\mathcal{D}}{2}\right\|^{2} \\ &\leq \frac{1}{2}\left\|\frac{|\mathcal{C}|^{2} + |\mathcal{D}|^{2}}{2}\right\| \\ &\quad + \frac{1}{2}\begin{cases} \max\{|\cos(2\tau)|, |\sin(2\tau)|\}(\|\mathscr{R}e(\mathcal{D}^{\star}\mathcal{C})\| + \|\mathscr{I}m(\mathcal{D}^{\star}\mathcal{C})\|); \\ \big(|\cos(2\tau)|^{p} + |\sin(2\tau)|^{p}\big)^{1/p}\big(\|\mathscr{R}e(\mathcal{D}^{\star}\mathcal{C})\|^{q} + \|\mathscr{I}m(\mathcal{D}^{\star}\mathcal{C})\|^{q}\big)^{1/q} \\ \text{for } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1; \\ (|\cos(2\tau)| + |\sin(2\tau)|)\max\{\|\mathscr{R}e(\mathcal{D}^{\star}\mathcal{C})\|, \|\mathscr{I}m(\mathcal{D}^{\star}\mathcal{C})\|\}. \end{cases}\end{aligned} \tag{8}$$

To advance, we employ the following conventional trigonometric estimates:

$$\sup_{\tau\in\mathbb{R}} \max\{|\cos(2\tau)|, |\sin(2\tau)|\} = 1 \tag{9}$$

and for $p \geq 2$:

$$0 \leq |\cos(2\tau)|^p + |\sin(2\tau)|^p \leq \cos^2(2\tau) + \sin^2(2\tau) = 1,$$

yielding

$$\sup_{\tau\in\mathbb{R}} \big(|\cos(2\tau)|^p + |\sin(2\tau)|^p\big) = 1.$$

Let $p \in [1,2)$ and $t \in [0,\pi/2]$, and examine the function $g_p(t) = \cos^p(t) + \sin^p(t)$. For $t \in (0,\pi/2)$, the derivative becomes

$$g_p'(t) = p\cos t \sin^{p-1}(t) - p\sin t \cos^{p-1}(t) = p\sin t\cos t\Big[\sin^{p-2}(t) - \cos^{p-2}(t)\Big],$$

indicating that $g_p'(t) = 0$ solely at $t = \pi/4$. Since g_p' remains positive throughout $(0,\pi/4)$ and negative across $(\pi/4,\pi/2)$, we obtain

$$\sup_{t\in[0,\pi/2]} g_p(t) = g_p(\pi/4) = \cos^p(\pi/4) + \sin^p(\pi/4) = 2^{\frac{2-p}{2}} > 1.$$

Consequently,

$$\sup_{\tau\in\mathbb{R}} \big(|\cos(2\tau)|^p + |\sin(2\tau)|^p\big)^{1/p} = 2^{\frac{2-p}{2p}} \quad \text{for } p \in (1,2) \tag{10}$$

and

$$\sup_{\tau\in\mathbb{R}} (|\cos(2\tau)| + |\sin(2\tau)|) = \sqrt{2}. \tag{11}$$

Ultimately, computing the supremum over $\tau \in \mathbb{R}$ and utilizing representation (5) together with estimates (9)–(11) generates the required inequalities stated in (3). □

Remark 1. *It is noteworthy that for the case $p = q = 2$, inequality (7) simplifies to*

$$\left\|\frac{e^{i\tau}\mathcal{C}\xi + e^{-i\tau}\mathcal{D}\xi}{2}\right\|^2 \leq \frac{1}{2}\left\langle\left(\frac{|\mathcal{C}|^2 + |\mathcal{D}|^2}{2}\right)\xi,\xi\right\rangle + \frac{1}{2}\big|\langle(\mathcal{D}^\star\mathcal{C})\xi,\xi\rangle\big|$$

for all $\tau \in \mathbb{R}$ and $\xi \in \mathfrak{H}$. Taking the supremum over $\tau \in \mathbb{R}$ and all unit vectors $\xi \in \mathfrak{H}$ then gives

$$\omega^2\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left\|\frac{|\mathcal{C}|^2 + |\mathcal{D}|^2}{2}\right\| + \frac{1}{2}\omega(\mathcal{D}^\star\mathcal{C}). \tag{12}$$

This inequality is consistent with a result that was derived using a different approach by Bunia and Paul in 2022 (see [18], Lemma 2.4).

From an alternative viewpoint, we can also establish the following theorem:

Theorem 2. *Let $\mathcal{C}, \mathcal{D} \in \mathbb{L}(\mathfrak{H})$. Then,*

$$\omega^2\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}\right) \leq \frac{1}{4}\|\mathscr{I}m[(\mathcal{C}^\star - \mathcal{D}^\star)(\mathcal{C}+\mathcal{D})]\| + \frac{1}{4} \times \begin{cases} 2\left\||\mathcal{C}|^2 + |\mathcal{D}|^2\right\| \\ \left\||\mathcal{C}+\mathcal{D}|^{2q} + |\mathcal{C}-\mathcal{D}|^{2q}\right\|^{1/q}, \text{ for } q > 1 \\ \left\||\mathcal{C}|^2 + |\mathcal{D}|^2\right\| + \|\mathscr{R}e(\mathcal{D}^\star\mathcal{C})\|. \end{cases} \tag{13}$$

Proof. It can be observed that, for any $\tau \in \mathbb{R}$ and $\xi \in \mathfrak{H}$, the following holds:

$$\begin{aligned} &\left\|\frac{e^{i\tau}\mathcal{C}\xi + e^{-i\tau}\mathcal{D}\xi}{2}\right\|^2 \\ &= \left\|\frac{(\cos\tau + i\sin\tau)\mathcal{C}\xi + (\cos\tau - i\sin\tau)\mathcal{D}\xi}{2}\right\|^2 \\ &= \frac{1}{4}\|\cos\tau(\mathcal{C}+\mathcal{D})\xi + i\sin\tau(\mathcal{C}-\mathcal{D})\xi\|^2 \\ &= \frac{1}{4}\Big(|\cos\tau|^2\|(\mathcal{C}+\mathcal{D})\xi\|^2 + |\sin\tau|^2\|(\mathcal{C}-\mathcal{D})\xi\|^2\Big) \\ &- \frac{1}{2}\mathscr{R}e\big(i\sin\tau\cos\tau\langle(\mathcal{C}^\star - \mathcal{D}^\star)(\mathcal{C}+\mathcal{D})\xi, \xi\rangle\big) \\ &= \frac{1}{4}\Big(|\cos\tau|^2\|(\mathcal{C}+\mathcal{D})\xi\|^2 + |\sin\tau|^2\|(\mathcal{C}-\mathcal{D})\xi\|^2\Big) \\ &+ \frac{1}{2}\sin\tau\cos\tau\big\langle\mathscr{I}m[(\mathcal{C}^\star - \mathcal{D}^\star)(\mathcal{C}+\mathcal{D})]\xi, \xi\big\rangle. \end{aligned}$$

This leads to the following inequalities

$$\begin{aligned} &\left\|\frac{e^{i\tau}\mathcal{C}\xi + e^{-i\tau}\mathcal{D}\xi}{2}\right\|^2 \\ &\leq \frac{1}{4}\Big(|\cos\tau|^2\|(\mathcal{C}+\mathcal{D})\xi\|^2 + |\sin\tau|^2\|(\mathcal{C}-\mathcal{D})\xi\|^2\Big) \\ &+ \frac{1}{4}|\sin(2\tau)|\big|\big\langle\mathscr{I}m[(\mathcal{C}^\star - \mathcal{D}^\star)(\mathcal{C}+\mathcal{D})]\xi, \xi\big\rangle\big| \\ &\leq \frac{1}{4}\Big(|\cos\tau|^2\|(\mathcal{C}+\mathcal{D})\xi\|^2 + |\sin\tau|^2\|(\mathcal{C}-\mathcal{D})\xi\|^2\Big) \\ &+ \frac{1}{4}|\sin(2\tau)|\big|\big\langle(\mathcal{C}^\star - \mathcal{D}^\star)(\mathcal{C}+\mathcal{D})\xi, \xi\big\rangle\big| \end{aligned} \tag{14}$$

for all $\tau \in \mathbb{R}$ and $\xi \in \mathfrak{H}$.

Now, applying the basic Hölder's inequality for $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we deduce that

$$
\begin{aligned}
&|\cos\tau|^2\|(\mathcal{C}+\mathcal{D})\xi\|^2+|\sin\tau|^2\|(\mathcal{C}-\mathcal{D})\xi\|^2\\
&\leq\begin{cases}
\max\Big\{|\cos\tau|^2,|\sin\tau|^2\Big\}\Big(\|(\mathcal{C}+\mathcal{D})\xi\|^2+\|(\mathcal{C}-\mathcal{D})\xi\|^2\Big);\\
\Big(|\cos\tau|^{2p}+|\sin\tau|^{2p}\Big)^{1/p}\Big(\|(\mathcal{C}+\mathcal{D})\xi\|^{2q}+\|(\mathcal{C}-\mathcal{D})\xi\|^{2q}\Big)^{1/q};\\
\Big(|\cos\tau|^2+|\sin\tau|^2\Big)\max\Big\{\|(\mathcal{C}+\mathcal{D})\xi\|^2,\|(\mathcal{C}-\mathcal{D})\xi\|^2\Big\}
\end{cases}\\
&=\begin{cases}
2\max\Big\{|\cos\tau|^2,|\sin\tau|^2\Big\}\Big\langle\Big(|\mathcal{C}|^2+|\mathcal{D}|^2\Big)\xi,\xi\Big\rangle;\\
\Big(|\cos\tau|^{2p}+|\sin\tau|^{2p}\Big)^{1/p}\Big(\big\langle|\mathcal{C}+\mathcal{D}|^2\xi,\xi\big\rangle^q+\big\langle|\mathcal{C}-\mathcal{D}|^2\xi,\xi\big\rangle^q\Big)^{1/q};\\
\max\Big\{\big\langle|\mathcal{C}+\mathcal{D}|^2\xi,\xi\big\rangle,\big\langle|\mathcal{C}-\mathcal{D}|^2\xi,\xi\big\rangle\Big\}
\end{cases}
\end{aligned}
$$

for all $\tau\in\mathbb{R}$ and $\xi\in\mathfrak{H}$.

For a unit vector $\xi\in\mathfrak{H}$, by applying McCarthy's inequality for a positive operator P, which states $\langle P\xi,\xi\rangle^s\leq\langle P^s\xi,\xi\rangle$ for $s\geq 1$ (see [28], Theorem 1.2), we have

$$
\begin{aligned}
\big\langle|\mathcal{C}+\mathcal{D}|^2\xi,\xi\big\rangle^q+\big\langle|\mathcal{C}-\mathcal{D}|^2\xi,\xi\big\rangle^q&\leq\big\langle|\mathcal{C}+\mathcal{D}|^{2q}\xi,\xi\big\rangle+\big\langle|\mathcal{C}-\mathcal{D}|^{2q}\xi,\xi\big\rangle\\
&=\Big\langle\Big[|\mathcal{C}+\mathcal{D}|^{2q}+|\mathcal{C}-\mathcal{D}|^{2q}\Big]\xi,\xi\Big\rangle
\end{aligned}
$$

for $\xi\in\mathfrak{H}$ with $\|\xi\|=1$.

Furthermore, we have

$$
\begin{aligned}
&\max\Big\{\big\langle|\mathcal{C}+\mathcal{D}|^2\xi,\xi\big\rangle,\big\langle|\mathcal{C}-\mathcal{D}|^2\xi,\xi\big\rangle\Big\}\\
&=\frac{1}{2}\Big[\Big\langle\Big[|\mathcal{C}+\mathcal{D}|^2+|\mathcal{C}-\mathcal{D}|^2\Big]\xi,\xi\Big\rangle\Big]+\frac{1}{2}\Big|\Big\langle\Big[|\mathcal{C}+\mathcal{D}|^2-|\mathcal{C}-\mathcal{D}|^2\Big]\xi,\xi\Big\rangle\Big|
\end{aligned}
$$

for $\xi\in\mathfrak{H}$ with $\|\xi\|=1$.

We note that for all $\tau\in\mathbb{R}$,

$$
\max\Big\{|\cos\tau|^2,|\sin\tau|^2\Big\}\leq 1,
$$

$$
|\cos\tau|^{2p}+|\sin\tau|^{2p}\leq|\cos\tau|^2+|\sin\tau|^2=1,\ \text{for } p>1,
$$

which, by (14), gives

$$
\begin{aligned}
\left\|\frac{e^{i\tau}\mathcal{C}\xi+e^{-i\tau}\mathcal{D}\xi}{2}\right\|^2&\leq\frac{1}{4}\times\begin{cases}
2\Big\langle\Big(|\mathcal{C}|^2+|\mathcal{D}|^2\Big)\xi,\xi\Big\rangle;\\
\Big\langle\Big[|\mathcal{C}+\mathcal{D}|^{2q}+|\mathcal{C}-\mathcal{D}|^{2q}\Big]\xi,\xi\Big\rangle^{1/q},\ q>1;\\
\frac{1}{2}\Big[\Big\langle\Big[|\mathcal{C}+\mathcal{D}|^2+|\mathcal{C}-\mathcal{D}|^2\Big]\xi,\xi\Big\rangle\Big]\\
+\frac{1}{2}\Big|\Big\langle\Big[|\mathcal{C}+\mathcal{D}|^2-|\mathcal{C}-\mathcal{D}|^2\Big]\xi,\xi\Big\rangle\Big|
\end{cases}\\
&\quad+\frac{1}{4}\Big|\big\langle\mathscr{I}m[(\mathcal{C}^\star-\mathcal{D}^\star)(\mathcal{C}+\mathcal{D})]\xi,\xi\big\rangle\Big|
\end{aligned}
$$

for $\xi\in\mathfrak{H}$ satisfying $\|\xi\|=1$ and $\tau\in\mathbb{R}$.

Computing the supremum across every unit vector $\xi\in\mathfrak{H}$ yields the desired norm inequality:

$$\left\|\frac{e^{i\tau}\mathcal{C}+e^{-i\tau}\mathcal{D}}{2}\right\|^2 \leq \frac{1}{4}\times\begin{cases} 2\left\||\mathcal{C}|^2+|\mathcal{D}|^2\right\| \\ \left\||\mathcal{C}+\mathcal{D}|^{2q}+|\mathcal{C}-\mathcal{D}|^{2q}\right\|^{1/q},\ q>1 \\ \left\||\mathcal{C}|^2+|\mathcal{D}|^2\right\|+\|\mathscr{R}e(\mathcal{D}^\star\mathcal{C})\| \end{cases} \tag{15}$$
$$+\frac{1}{4}\|\mathscr{I}m[(\mathcal{C}^\star-\mathcal{D}^\star)(\mathcal{C}+\mathcal{D})]\|$$

for all $\tau\in\mathbb{R}$.

Ultimately, through computing the supremum across $\tau\in\mathbb{R}$ and applying representation (5), we obtain the target outcome (13). □

Theorem 3. *Let $\mathcal{C},\mathcal{D}\in\mathbb{L}(\mathfrak{H})$. Then,*

$$\omega\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}\right) \leq \begin{cases} \|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}\|+\left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{C},\mathcal{D}}\right\| \\ \beta_p\left[\|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}\|^q+\left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{C},\mathcal{D}}\right\|^q\right]^{1/q} \\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \sqrt{2}\max\left\{\|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}\|,\left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{C},\mathcal{D}}\right\|\right\}, \end{cases} \tag{16}$$

for all $\tau\in\mathbb{R}$, where β_p is defined by (4) and

$$\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}:=\frac{e^{i\tau}\mathcal{C}+e^{-i\tau}\mathcal{D}}{2}=\frac{1}{2}[\cos\tau(\mathcal{C}+\mathcal{D})+i\sin\tau(\mathcal{C}-\mathcal{D})]$$

for $\tau\in\mathbb{R}$.

Proof. It is evident that, for all $v,\tau\in\mathbb{R}$, we have

$$\begin{aligned}
&\cos v\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}+\sin v\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{C},\mathcal{D}} \\
&=\frac{1}{2}\cos v[\cos\tau(\mathcal{C}+\mathcal{D})+i\sin\tau(\mathcal{C}-\mathcal{D})] \\
&+\frac{1}{2}\sin v\left[\cos\left(\tau+\frac{\pi}{2}\right)(\mathcal{C}+\mathcal{D})+i\sin\left(\tau+\frac{\pi}{2}\right)(\mathcal{C}-\mathcal{D})\right] \\
&=\frac{1}{2}\cos v[\cos\tau(\mathcal{C}+\mathcal{D})+i\sin\tau(\mathcal{C}-\mathcal{D})] \\
&+\frac{1}{2}\sin v[-\sin\tau(\mathcal{C}+\mathcal{D})+i\cos\tau(\mathcal{C}-\mathcal{D})] \\
&=\frac{1}{2}(\cos v\cos\tau-\sin v\sin\tau)(\mathcal{C}+\mathcal{D}) \\
&+\frac{1}{2}i(\cos v\sin\tau+\sin v\cos\tau)(\mathcal{C}-\mathcal{D}) \\
&=\frac{1}{2}[\cos(\tau+v)(\mathcal{C}+\mathcal{D})+i\sin(\tau+v)(\mathcal{C}-\mathcal{D})]=\mathcal{K}_{\tau+v,\mathcal{C},\mathcal{D}}.
\end{aligned}$$

If we take the operator norm and apply a Hölder-type inequality, we obtain

$$\begin{aligned}\|\mathcal{K}_{\tau+v,\mathcal{C},\mathcal{D}}\| &= \left\|\cos v\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}} + \sin v\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{C},\mathcal{D}}\right\| \\ &\leq |\cos v|\|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}\| + |\sin v|\left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{C},\mathcal{D}}\right\| \\ &\leq \begin{cases} \max\{|\cos v|, |\sin v|\}\left[\|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}\| + \left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{C},\mathcal{D}}\right\|\right]; \\ \\ \left(|\cos v|^p + |\sin v|^p\right)^{1/p}\left[\|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}\|^q + \left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{C},\mathcal{D}}\right\|^q\right]^{1/q} \\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ [|\cos v| + |\sin v|]\max\left\{\|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}\|, \left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{C},\mathcal{D}}\right\|\right\}, \end{cases} \end{aligned} \tag{17}$$

for all $v, \tau \in \mathbb{R}$.

Given that, from the proof of Theorem 1,

$$\max\{|\cos v|, |\sin v|\} \leq 1,\ |\cos v| + |\sin v| \leq \sqrt{2}$$

and

$$\left(|\cos v|^p + |\sin v|^p\right)^{1/p} \leq \beta_p,$$

where β_p is given by (4), for all $v \in \mathbb{R}$. Then by (17), we obtain

$$\|\mathcal{K}_{\tau+v,\mathcal{C},\mathcal{D}}\| \leq \begin{cases} \|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}\| + \left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{C},\mathcal{D}}\right\|; \\ \\ \beta_p\left[\|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}\|^q + \left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{C},\mathcal{D}}\right\|^q\right]^{1/q} \\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sqrt{2}\max\left\{\|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}\|, \left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{C},\mathcal{D}}\right\|\right\}, \end{cases}$$

for all $v, \tau \in \mathbb{R}$.

By taking the supremum over $v \in \mathbb{R}$ and noting that

$$\sup_{v\in\mathbb{R}}\|\mathcal{K}_{\tau+v,\mathcal{C},\mathcal{D}}\| = \omega\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}\right),$$

we obtain the desired result (16). □

Corollary 1. *Let $\mathcal{C}, \mathcal{D} \in \mathbb{L}(\mathfrak{H})$. Then,*

$$\begin{aligned}&\omega\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}\right) \\ &\leq \begin{cases} \frac{\sqrt{2}}{4}(\|(1+i)\mathcal{C} + (1-i)\mathcal{D}\| + \|(1-i)\mathcal{C} + (1+i)\mathcal{D}\|); \\ \\ \delta_p\left[\|(1+i)\mathcal{C} + (1-i)\mathcal{D}\|^q + \|(1-i)\mathcal{C} + (1+i)\mathcal{D}\|^q\right]^{1/q} \\ \textit{for } p,q>1 \textit{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{2}\max\{\|(1+i)\mathcal{C} + (1-i)\mathcal{D}\|, \|(1-i)\mathcal{C} + (1+i)\mathcal{D}\|\}, \end{cases} \end{aligned} \tag{18}$$

where

$$\delta_p := \begin{cases} 2^{\frac{1-2p}{p}} \text{ if } p \in (1,2) \\ \\ \frac{\sqrt{2}}{4} \text{ if } p \in [2,\infty). \end{cases}$$

Proof. If we choose $\tau = \pi/4$, we obtain

$$\mathcal{K}_{\pi/4,\mathcal{C},\mathcal{D}} = \frac{\sqrt{2}}{4}[(\mathcal{C}+\mathcal{D}) + i(\mathcal{C}-\mathcal{D})] = \frac{\sqrt{2}}{4}[(1+i)\mathcal{C} + (1-i)\mathcal{D}]$$

and

$$\mathcal{K}_{\frac{3\pi}{4},\mathcal{C},\mathcal{D}} = \frac{\sqrt{2}}{4}[-(\mathcal{C}+\mathcal{D}) + i(\mathcal{C}-\mathcal{D})] = -\frac{\sqrt{2}}{4}[(1-i)\mathcal{C} + (1+i)\mathcal{D}]$$

and by making use of (16), we arrive at (18). □

It is also possible to state the following result:

Theorem 4. *Let $\mathcal{C}, \mathcal{D} \in \mathbb{L}(\mathfrak{H})$. Then,*

$$\omega^2\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left\| \frac{|\mathcal{C}|^2 + |\mathcal{D}|^2}{2} \right\| + \frac{1}{2} \times \begin{cases} \left[\left\| \frac{\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D}}{2} \right\| + \left\| \frac{\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D}}{2} \right\|\right]; \\ \\ \beta_p\left[\left\| \frac{\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D}}{2} \right\|^q + \left\| \frac{\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D}}{2} \right\|^q\right]^{1/q} \\ \text{for } p,q > 1 \text{ with } \frac{1}{p}+\frac{1}{q} = 1; \\ \\ \sqrt{2}\max\left\{\left\| \frac{\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D}}{2} \right\|, \left\| \frac{\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D}}{2} \right\|\right\}, \end{cases} \tag{19}$$

where β_p is defined in Theorem 1.

Proof. Let us consider

$$\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}} := \frac{e^{i\tau}\mathcal{C} + e^{-i\tau}\mathcal{D}}{2}, \text{ for } \tau \in \mathbb{R}.$$

We note that

$$\begin{aligned} 4|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}|^2 &= \left|e^{i\tau}\mathcal{C} + e^{-i\tau}\mathcal{D}\right|^2 = \left(e^{i\tau}\mathcal{C} + e^{-i\tau}\mathcal{D}\right)^\star\left(e^{i\tau}\mathcal{C} + e^{-i\tau}\mathcal{D}\right) \\ &= \left(e^{-i\tau}\mathcal{C}^\star + e^{i\tau}\mathcal{D}^\star\right)\left(e^{i\tau}\mathcal{C} + e^{-i\tau}\mathcal{D}\right) \\ &= \mathcal{C}^\star\mathcal{C} + \mathcal{D}^\star\mathcal{D} + e^{2i\tau}\mathcal{D}^\star\mathcal{C} + e^{-2i\tau}\mathcal{C}^\star\mathcal{D} \\ &= |\mathcal{C}|^2 + |\mathcal{D}|^2 + e^{2i\tau}\mathcal{D}^\star\mathcal{C} + e^{-2i\tau}\mathcal{C}^\star\mathcal{D}, \end{aligned}$$

which implies that

$$|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}|^2 = \frac{1}{2}\left(\frac{|\mathcal{C}|^2 + |\mathcal{D}|^2}{2} + \frac{e^{2i\tau}\mathcal{D}^\star\mathcal{C} + e^{-2i\tau}\mathcal{C}^\star\mathcal{D}}{2}\right) \tag{20}$$

for $\tau \in \mathbb{R}$.

In addition, we have

$$\begin{aligned}&\frac{e^{2i\tau}\mathcal{D}^\star\mathcal{C}+e^{-2i\tau}\mathcal{C}^\star\mathcal{D}}{2}\\&=\frac{[\cos(2\tau)+i\sin(2\tau)]\mathcal{D}^\star\mathcal{C}+[\cos(2\tau)-i\sin(2\tau)]\mathcal{C}^\star\mathcal{D}}{2}\\&=\frac{\cos(2\tau)(\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D})+i\sin(2\tau)(\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D})}{2}\end{aligned}$$

for $\tau\in\mathbb{R}$.

By taking the norm in (20), we obtain

$$\begin{aligned}\|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}\|^2&=\frac{1}{2}\left\|\frac{|\mathcal{C}|^2+|\mathcal{D}|^2}{2}+\frac{e^{2i\tau}\mathcal{D}^\star\mathcal{C}+e^{-2i\tau}\mathcal{C}^\star\mathcal{D}}{2}\right\|\\&\le\frac{1}{2}\left\|\frac{|\mathcal{C}|^2+|\mathcal{D}|^2}{2}\right\|+\frac{1}{2}\left\|\frac{e^{2i\tau}\mathcal{D}^\star\mathcal{C}+e^{-2i\tau}\mathcal{C}^\star\mathcal{D}}{2}\right\|\\&=\frac{1}{2}\left\|\frac{|\mathcal{C}|^2+|\mathcal{D}|^2}{2}\right\|\\&\quad+\frac{1}{4}\|\cos(2\tau)(\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D})+i\sin(2\tau)(\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D})\|\end{aligned}\tag{21}$$

for $\tau\in\mathbb{R}$.

By Hölder's inequality for norms, it follows that

$$\begin{aligned}&\|\cos(2\tau)(\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D})+i\sin(2\tau)(\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D})\|\\&\le\begin{cases}\max\{|\cos(2\tau)|,|\sin(2\tau)|\}[\|\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D}\|+\|\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D}\|];\\ \left(|\cos(2\tau)|^p+|\sin(2\tau)|^p\right)^{1/p}\left[\|\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D}\|^q+\|\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D}\|^q\right]^{1/q}\\ p,q>1\text{ with }\frac{1}{p}+\frac{1}{q}=1;\\ \{|\cos(2\tau)|+|\sin(2\tau)|\}\max\{\|\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D}\|,\|\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D}\|\};\end{cases}\\&\le\begin{cases}[\|\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D}\|+\|\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D}\|];\\ \beta_p\left[\|\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D}\|^q+\|\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D}\|^q\right]^{1/q}\\ p,q>1\text{ with }\frac{1}{p}+\frac{1}{q}=1;\\ \sqrt{2}\max\{\|\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D}\|,\|\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D}\|\}\end{cases}\end{aligned}$$

for $\tau\in\mathbb{R}$.

Consequently, from (21), we can deduce that

$$\begin{aligned}\|\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}\|^2&\le\frac{1}{2}\left\|\frac{|\mathcal{C}|^2+|\mathcal{D}|^2}{2}\right\|\\&\quad+\frac{1}{4}\times\begin{cases}[\|\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D}\|+\|\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D}\|];\\ \beta_p\left[\|\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D}\|^q+\|\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D}\|^q\right]^{1/q}\\ p,q>1\text{ with }\frac{1}{p}+\frac{1}{q}=1;\\ \sqrt{2}\max\{\|\mathcal{D}^\star\mathcal{C}+\mathcal{C}^\star\mathcal{D}\|,\|\mathcal{D}^\star\mathcal{C}-\mathcal{C}^\star\mathcal{D}\|\}\end{cases}\end{aligned}$$

for $\tau\in\mathbb{R}$.

Through computing the supremum across $\tau \in \mathbb{R}$, we attain the target outcome (19). □

Remark 2. *For the specific case $p = q = 2$ in (19), we obtain*

$$\omega^2\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left\|\frac{|\mathcal{C}|^2 + |\mathcal{D}|^2}{2}\right\| + \frac{1}{2}\left[\left\|\frac{\mathcal{D}^\star\mathcal{C} + \mathcal{C}^\star\mathcal{D}}{2}\right\|^2 + \left\|\frac{\mathcal{D}^\star\mathcal{C} - \mathcal{C}^\star\mathcal{D}}{2}\right\|^2\right]^{1/2}. \tag{22}$$

3. Applications to a Single Operator

Examine an operator $\mathcal{C} \in \mathbb{L}(\mathfrak{H})$. Substituting $\mathcal{D} = \mathcal{C}^\star$ into (3) yields the following upper estimates for the numerical radius of one operator:

$$\omega^2(\mathcal{C}) \leq \frac{1}{2}\left\|\frac{|\mathcal{C}|^2 + |\mathcal{C}^\star|^2}{2}\right\| + \frac{1}{2} \times \begin{cases} \left(\left\|\mathscr{R}e(\mathcal{C}^2)\right\| + \left\|\mathscr{I}m(\mathcal{C}^2)\right\|\right); \\ \beta_p\left(\left\|\mathscr{R}e(\mathcal{C}^2)\right\|^q + \left\|\mathscr{I}m(\mathcal{C}^2)\right\|^q\right)^{1/q} \\ \text{for } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1; \\ \sqrt{2}\max\left\{\left\|\mathscr{R}e(\mathcal{C}^2)\right\|, \left\|\mathscr{I}m(\mathcal{C}^2)\right\|\right\}. \end{cases}$$

Applying the substitution $\mathcal{D} = \mathcal{C}^\star$ to (12) gives

$$\omega^2(\mathcal{C}) \leq \frac{1}{2}\left\|\frac{|\mathcal{C}|^2 + |\mathcal{C}^\star|^2}{2}\right\| + \frac{1}{2}\omega\left(\mathcal{C}^2\right),$$

which corresponds to a result stated in [18] (Lemma 2.4).

Similarly, from (13) with $\mathcal{D} = \mathcal{C}^\star$, we find that

$$\omega^2(\mathcal{C}) \leq \frac{1}{4}\left\|\mathscr{I}m[(\mathcal{C}^\star - \mathcal{C})(\mathcal{C}^\star + \mathcal{C})]\right\| + \frac{1}{4} \times \begin{cases} 2\left\||\mathcal{C}|^2 + |\mathcal{C}^\star|^2\right\|; \\ \left\||\mathcal{C} + \mathcal{C}^\star|^{2q} + |\mathcal{C} - \mathcal{C}^\star|^{2q}\right\|^{1/q},\ q > 1; \\ \left\||\mathcal{C}|^2 + |\mathcal{C}^\star|^2\right\| + \left\|\mathscr{R}e(\mathcal{C}^2)\right\|. \end{cases}$$

Note that for $\mathcal{D} = \mathcal{C}^\star$, the operator $\mathcal{K}_{\tau,\mathcal{C},\mathcal{D}}$ becomes

$$\mathcal{K}_{\tau,\mathcal{C},\mathcal{C}^\star} := \frac{e^{i\tau}\mathcal{C} + e^{-i\tau}\mathcal{C}^\star}{2} = \frac{e^{i\tau}\mathcal{C} + \left(e^{i\tau}\mathcal{C}\right)^\star}{2} = \mathscr{R}e\left(e^{i\tau}\mathcal{C}\right)$$

for all $\tau \in \mathbb{R}$.

By utilizing (16), we then arrive at

$$\omega(\mathcal{C}) \leq \begin{cases} \|\mathscr{R}e(e^{i\tau}\mathcal{C})\| + \left\|\mathscr{R}e\left(e^{i\left(\tau+\frac{\pi}{2}\right)}\mathcal{C}\right)\right\|; \\ \\ \beta_p\left[\|\mathscr{R}e(e^{i\tau}\mathcal{C})\|^q + \left\|\mathscr{R}e\left(e^{i\left(\tau+\frac{\pi}{2}\right)}\mathcal{C}\right)\right\|^q\right]^{1/q} \\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sqrt{2}\max\left\{\|\mathscr{R}e(e^{i\tau}\mathcal{C})\|, \left\|\mathscr{R}e\left(e^{i\left(\tau+\frac{\pi}{2}\right)}\mathcal{C}\right)\right\|\right\} \end{cases}$$

for all $\tau \in \mathbb{R}$.

From (18), we can also obtain the inequality

$$\omega(\mathcal{C}) \leq \begin{cases} \frac{\sqrt{2}}{4}(\|(1+i)\mathcal{C}+(1-i)\mathcal{C}^\star\| + \|(1-i)\mathcal{C}+(1+i)\mathcal{C}^\star\|); \\ \\ \delta_p\left[\|(1+i)\mathcal{C}+(1-i)\mathcal{C}^\star\|^q + \|(1-i)\mathcal{C}+(1+i)\mathcal{C}^\star\|^q\right]^{1/q} \\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{2}\max\{\|(1+i)\mathcal{C}+(1-i)\mathcal{C}^\star\|, \|(1-i)\mathcal{C}+(1+i)\mathcal{C}^\star\|\}. \end{cases}$$

Moreover, if we apply (19) with $\mathcal{D} = \mathcal{C}^\star$, we obtain

$$\omega^2(\mathcal{C}) \leq \frac{1}{2}\left\|\frac{|\mathcal{C}|^2+|\mathcal{C}^\star|^2}{2}\right\| + \frac{1}{4}\times \begin{cases} \left[\left\|\mathcal{C}^2+(\mathcal{C}^\star)^2\right\| + \left\|\mathcal{C}^2-(\mathcal{C}^\star)^2\right\|\right]; \\ \\ \beta_p\left[\left\|\mathcal{C}^2+(\mathcal{C}^\star)^2\right\|^q + \left\|\mathcal{C}^2-(\mathcal{C}^\star)^2\right\|^q\right]^{1/q} \\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sqrt{2}\max\left\{\left\|\mathcal{C}^2+(\mathcal{C}^\star)^2\right\|, \left\|\mathcal{C}^2-(\mathcal{C}^\star)^2\right\|\right\}, \end{cases}$$

while from (22) it follows that

$$\omega^2(\mathcal{C}) \leq \frac{1}{2}\left\|\frac{|\mathcal{C}|^2+|\mathcal{C}^\star|^2}{2}\right\| + \frac{1}{2}\left[\left\|\frac{\mathcal{C}^2+(\mathcal{C}^\star)^2}{2}\right\|^2 + \left\|\frac{\mathcal{C}^2-(\mathcal{C}^\star)^2}{2}\right\|^2\right]^{1/2}.$$

For every operator $\mathcal{C} \in \mathbb{L}(\mathfrak{H})$, we define the functional

$$\Delta(\mathcal{C}) := \omega\left(\begin{bmatrix} 0 & \mathscr{R}e(\mathcal{C}) \\ \mathscr{I}m(\mathcal{C}) & 0 \end{bmatrix}\right) = \frac{1}{2}\sup_{\tau\in\mathbb{R}}\left\|e^{i\tau}\mathscr{R}e(\mathcal{C}) + e^{-i\tau}\mathscr{I}m(\mathcal{C})\right\|. \tag{23}$$

Evidently, $\Delta(\mathcal{C}) \geq 0$. When $\Delta(\mathcal{C}) = 0$, then $\mathcal{C} = 0$. Moreover, for $\mathcal{C}, \mathcal{D} \in \mathbb{L}(\mathfrak{H})$, it follows that

$$\begin{aligned}\Delta(\mathcal{C}+\mathcal{D}) &= \frac{1}{2}\sup_{\tau\in\mathbb{R}}\left\|e^{i\tau}\mathscr{R}e(\mathcal{C}+\mathcal{D})+e^{-i\tau}\mathscr{I}m(\mathcal{C}+\mathcal{D})\right\| \\ &= \frac{1}{2}\sup_{\tau\in\mathbb{R}}\left\|e^{i\tau}\mathscr{R}e(\mathcal{C})+e^{-i\tau}\mathscr{I}m(\mathcal{C})+e^{i\tau}\mathscr{R}e(\mathcal{D})+e^{-i\tau}\mathscr{I}m(\mathcal{D})\right\| \\ &\leq \frac{1}{2}\sup_{\tau\in\mathbb{R}}\left\|e^{i\tau}\mathscr{R}e(\mathcal{C})+e^{-i\tau}\mathscr{I}m(\mathcal{C})\right\| \\ &+ \frac{1}{2}\sup_{\tau\in\mathbb{R}}\left\|e^{i\tau}\mathscr{R}e(\mathcal{D})+e^{-i\tau}\mathscr{I}m(\mathcal{D})\right\| \\ &= \Delta(\mathcal{C})+\Delta(\mathcal{D}),\end{aligned}$$

and for any real scalar α,

$$\begin{aligned}\Delta(\alpha\mathcal{C}) &= \frac{1}{2}\sup_{\tau\in\mathbb{R}}\left\|e^{i\tau}\mathscr{R}e(\alpha\mathcal{C})+e^{-i\tau}\mathscr{I}m(\alpha\mathcal{C})\right\| \\ &= \frac{1}{2}|\alpha|\sup_{\tau\in\mathbb{R}}\left\|e^{i\tau}\mathscr{R}e(\mathcal{C})+e^{-i\tau}\mathscr{I}m(\mathcal{C})\right\| = |\alpha|\Delta(\mathcal{C}).\end{aligned}$$

Such characteristics confirm that $\Delta(\cdot)$ constitutes a *real norm* on $\mathbb{L}(\mathfrak{H})$.

From the definition of $\Delta(\cdot)$, it becomes apparent that

$$\Delta(\mathcal{C}) \leq \frac{1}{2}(\|\mathscr{R}e(\mathcal{C})\|+\|\mathscr{I}m(\mathcal{C})\|)$$

and

$$\frac{1}{2}\left\|e^{i\tau}\mathscr{R}e(\mathcal{C})+e^{-i\tau}\mathscr{I}m(\mathcal{C})\right\| \leq \Delta(\mathcal{C}) \quad \text{for all } \tau\in\mathbb{R}.$$

Consequently, this produces the relation

$$\frac{1}{2}\max\left\{\|\mathscr{R}e(\mathcal{C})\pm\mathscr{I}m(\mathcal{C})\|, \sqrt{2}\|(1+i)\mathscr{R}e(\mathcal{C})+(1-i)\mathscr{I}m(\mathcal{C})\|\right\} \leq \Delta(\mathcal{C})$$

for every $\mathcal{C}\in\mathbb{L}(\mathfrak{H})$.

From (1) and (2), by setting $\mathcal{C}=\mathscr{R}e(\mathcal{T})$ and $\mathcal{D}=\mathscr{I}m(\mathcal{T})$ for some $\mathcal{T}\in\mathbb{L}(\mathfrak{H})$, it also follows that

$$\begin{aligned}&\frac{1}{2}\max\{\|\mathscr{R}e(\mathcal{T})+\mathscr{I}m(\mathcal{T})\|, \|\mathscr{R}e(\mathcal{T})-\mathscr{I}m(\mathcal{T})\|\} \\ &\leq \Delta(\mathcal{T}) \leq \frac{1}{2}[\|\mathscr{R}e(\mathcal{T})+\mathscr{I}m(\mathcal{T})\|+\|\mathscr{R}e(\mathcal{T})-\mathscr{I}m(\mathcal{T})\|]\end{aligned}$$

and that

$$\Delta(\mathcal{T}) \leq \min\{\|\mathscr{R}e(\mathcal{T})\|, \|\mathscr{I}m(\mathcal{T})\|\}+\frac{1}{2}\min\{\|\mathscr{R}e(\mathcal{T})+\mathscr{I}m(\mathcal{T})\|, \|\mathscr{R}e(\mathcal{T})-\mathscr{I}m\mathcal{T}\|\}.$$

By using (3) with $\mathcal{C}=\mathscr{R}e(\mathcal{T})$ and $\mathcal{D}=\mathscr{I}m(\mathcal{T})$, we obtain

$$\begin{aligned}\Delta^2(\mathcal{T}) \leq \frac{1}{4}&\left\|\frac{|\mathcal{T}|^2+|\mathcal{T}^\star|^2}{2}\right\| \\ &+\frac{1}{2}\times\begin{cases}(\|\mathscr{R}e(\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T}))\|+\|\mathscr{I}m(\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T}))\|); \\ \beta_p\left(\|\mathscr{R}e(\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T}))\|^q+\|\mathscr{I}m(\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T}))\|^q\right)^{1/q} \\ \text{for } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \sqrt{2}\max\{\|\mathscr{R}e(\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T}))\|, \|\mathscr{I}m(\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T}))\|\},\end{cases}\end{aligned}$$

for all $\mathcal{T} \in \mathbb{L}(\mathfrak{H})$, while from (12) we can derive that

$$\Delta^2(\mathcal{T}) \leq \frac{1}{4}\left\|\frac{|\mathcal{T}|^2+|\mathcal{T}^\star|^2}{2}\right\| + \frac{1}{2}\omega(\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T})).$$

From (13), we also find for $\mathcal{C} = \mathscr{R}e(\mathcal{T})$ and $\mathcal{D} = \mathscr{I}m(\mathcal{T})$ that

$$\Delta^2(\mathcal{T}) \leq \frac{1}{4}\|\mathscr{I}m[(\mathscr{R}e(\mathcal{T}) - \mathscr{I}m(\mathcal{T}))(\mathscr{R}e(\mathcal{T}) + \mathscr{I}m(\mathcal{T}))]\| + \frac{1}{4} \times \begin{cases} \left\||\mathcal{T}|^2 + |\mathcal{T}^\star|^2\right\| \\ \left\||\mathscr{R}e(\mathcal{T}) + \mathscr{I}m(\mathcal{T})|^{2q} + |\mathscr{R}e(\mathcal{T}) - \mathscr{I}m(\mathcal{T})|^{2q}\right\|^{1/q}, \ q > 1 \\ \left\|\frac{|\mathcal{T}|^2+|\mathcal{T}^\star|^2}{2}\right\| + \|\mathscr{R}e(\mathscr{I}m(\mathcal{T})\mathscr{R}e\mathcal{T})\|. \end{cases}$$

For an operator $\mathcal{T} \in \mathbb{L}(\mathfrak{H})$ and a real number $\tau \in \mathbb{R}$, we may define

$$\mathcal{K}_{\tau,\mathcal{T}} := \mathcal{K}_{\tau,\mathscr{R}e(\mathcal{T}),\mathscr{I}m(\mathcal{T})} = \frac{e^{i\tau}\mathscr{R}e(\mathcal{T}) + e^{-i\tau}\mathscr{I}m(\mathcal{T})}{2}.$$

By inequality (16), we obtain

$$\Delta(\mathcal{T}) \leq \begin{cases} \|\mathcal{K}_{\tau,\mathcal{T}}\| + \left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{T}}\right\| \\ \beta_p\left[\|\mathcal{K}_{\tau,\mathcal{T}}\|^q + \left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{T}}\right\|^q\right]^{1/q} \\ \text{for } p,q > 1 \text{ with } \frac{1}{p}+\frac{1}{q} = 1; \\ \sqrt{2}\max\left\{\|\mathcal{K}_{\tau,\mathcal{T}}\|, \left\|\mathcal{K}_{\tau+\frac{\pi}{2},\mathcal{T}}\right\|\right\}, \end{cases}$$

for all $\tau \in \mathbb{R}$.

In particular, the following holds

$$\Delta(\mathcal{T}) \leq \begin{cases} \frac{\sqrt{2}}{4}(\|(1+i)\mathscr{R}e(\mathcal{T}) + (1-i)\mathscr{I}m(\mathcal{T})\| + \|(1-i)\mathscr{R}e\mathcal{T} + (1+i)\mathscr{I}m(\mathcal{T})\|); \\ \delta_p\left[\|(1+i)\mathscr{R}e(\mathcal{T}) + (1-i)\mathscr{I}m(\mathcal{T})\|^q + \|(1-i)\mathscr{R}e(\mathcal{T}) + (1+i)\mathscr{I}m(\mathcal{T})\|^q\right]^{1/q} \\ \text{for } p,q > 1 \text{ with } \frac{1}{p}+\frac{1}{q} = 1; \\ \frac{1}{2}\max\{\|(1+i)\mathscr{R}e(\mathcal{T}) + (1-i)\mathscr{I}m(\mathcal{T})\|, \|(1-i)\mathscr{R}e\mathcal{T} + (1+i)\mathscr{I}m(\mathcal{T})\|\}, \end{cases}$$

for all $\mathcal{T} \in \mathbb{L}(\mathfrak{H})$.

Finally, by making use of (19) for $\mathcal{C} = \mathscr{R}e(\mathcal{T})$ and $\mathcal{D} = \mathscr{I}m(\mathcal{T})$, we also obtain

$$\Delta^2(\mathcal{T}) \leq \frac{1}{4}\left\|\frac{|\mathcal{T}|^2+|\mathcal{T}^\star|^2}{2}\right\| + \frac{1}{4}\times\begin{cases} [\|\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T})+\mathscr{R}e(\mathcal{T})\mathscr{I}m(\mathcal{T})\|+\|\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T})-\mathscr{R}e(\mathcal{T})\mathscr{I}m(\mathcal{T})\|]; \\ \beta_p\left[\|\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T})+\mathscr{R}e(\mathcal{T})\mathscr{I}m(\mathcal{T})\|^q+\|\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T})-\mathscr{R}e(\mathcal{T})\mathscr{I}m(\mathcal{T})\|^q\right]^{1/q} \\ \text{for } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \sqrt{2}\max\{\|\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T})+\mathscr{R}e(\mathcal{T})\mathscr{I}m(\mathcal{T})\|,\|\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T})-\mathscr{R}e(\mathcal{T})\mathscr{I}m(\mathcal{T})\|\}, \end{cases}$$

and for the case $p=q=2$,

$$\Delta^2(\mathcal{T}) \leq \frac{1}{4}\left\|\frac{|\mathcal{T}|^2+|\mathcal{T}^\star|^2}{2}\right\| + \frac{1}{4}\left[\|\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T})+\mathscr{R}e(\mathcal{T})\mathscr{I}m(\mathcal{T})\|^2+\|\mathscr{I}m(\mathcal{T})\mathscr{R}e(\mathcal{T})-\mathscr{R}e(\mathcal{T})\mathscr{I}m(\mathcal{T})\|^2\right]^{1/2}.$$

4. Conclusions

In this paper, we have successfully established several new upper bounds for the numerical radius of off-diagonal operator matrices of the form $\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}$ on a complex Hilbert space $\mathfrak{H}$. Our approach has yielded a variety of inequalities that refine and generalize existing results in operator theory.

A primary achievement in our research involves creating estimates linking the numerical radius with norms of component operators alongside their algebraic structures. Key among our discoveries, we proved that

$$\omega^2\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left\|\frac{|\mathcal{C}|^2+|\mathcal{D}|^2}{2}\right\| + \frac{1}{2}(\|\mathscr{R}e(\mathcal{D}^\star\mathcal{C})\|+\|\mathscr{I}m(\mathcal{D}^\star\mathcal{C})\|).$$

Furthermore, we explored an alternative perspective, leading to results such as

$$\omega^2\left(\begin{bmatrix} 0 & \mathcal{C} \\ \mathcal{D}^\star & 0 \end{bmatrix}\right) \leq \frac{1}{4}\|\mathscr{I}m[(\mathcal{C}^\star-\mathcal{D}^\star)(\mathcal{C}+\mathcal{D})]\| + \frac{1}{2}\left\||\mathcal{C}|^2+|\mathcal{D}|^2\right\|.$$

The effectiveness of these broad inequalities became evident via concrete examples, such as obtaining fresh estimates for one operator $\mathcal{C}$ (via assigning $\mathcal{D}=\mathcal{C}^\star$) and for the functional $\Delta(\mathcal{T})$ based on real and imaginary components of operator $\mathcal{T}$.

The results featured in this document advance the continuing progress of operator inequalities. We anticipate that this study could act as an important foundation for additional exploration in this area. Subsequent studies might examine broadening these estimates to broader $n\times n$ operator matrices, assess their consequences across various operator algebras, or pursue possible improvements with extra conditions on operators $\mathcal{C}$ and $\mathcal{D}$.

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