

Article

Characterization of Pomonoids by Properties of I-Regular S-Posets

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Abstract

In 2005, Shi defined I-regular S-posets and used this concept to characterize *PP*-pomonoids and po-cancellable pomonoids. In this paper, we continued the development of the homological classification of pomonoids by using the I-regularity of S-posets. First, we characterized pomonoids over which all I-regular S-posets have one of the properties around projectivity or injectivity, and many known results were generalized. Moreover, some possible conditions on pomonoids that describe when their diagonal posets are I-regular were found. Finally, some characterizations of pomonoids by the I-regularity of their Rees factor posets were given.

Keywords: I-regular; S-poset; Rees factor; regular pomonoid

MSC: 20M30; 20M50

1. Preliminaries

In this paper, unless otherwise specified, S is a partially ordered monoid (or simply a pomonoid). A nonempty poset A is called a *left S-poset*, usually denoted by ${}_S A$, if there exists a mapping $S \times A \rightarrow A$, $(s, a) \mapsto sa$ which satisfies the following conditions: (i) The action is monotonic in each variable; (ii) $t(sa) = (ts)a$ and $1a = a$ for all $a \in A$ and all $s, t \in S$. Right S-posets B_S are defined analogously, and $\Theta = \{\theta\}$ denotes the one-element S-poset. In this paper, a left (right) ideal of S refers to a nonempty subset I of S satisfying $SI \subseteq I$ ($IS \subseteq I$).

A *morphism of left S-posets* is a monotonic mapping $f : {}_S A \rightarrow {}_S B$ which satisfies $f(sa) = sf(a)$ for every $a \in A$ and $s \in S$. Morphisms of right S-posets are defined similarly, and morphisms of posets are just monotonic mappings. In this way, the categories ${}_S \text{Pos}$ (left S-posets), Pos_S (right S-posets), and Pos (posets) are obtained. In these categories, the monomorphisms are the injective morphisms, whereas the regular monomorphisms are the order-embeddings, that is, morphisms $f : {}_S A \rightarrow {}_S B$, for which $f(a) \leq f(a')$ implies $a \leq a'$ for all $a, a' \in A$ (see [1]). Research on flatness properties of S-posets was initiated in the mid-1980s by S. Fakhruddin in [2], and this work has recently been continued in the articles [1,3–8].

An S-subposet ${}_S B$ of an S-poset ${}_S A$ is called *convex* if for any $a \in A$ and $b, b' \in B$, $b' \leq a \leq b$ implies $a \in B$. An element $c \in S$ is called *right (left) po-cancellable* if for all $s, s' \in S$, $sc \leq s'c$ ($cs \leq cs'$) implies $s \leq s'$. A pomonoid is called *left (right) collapsible* if for all $s, t \in S$, there exists $u \in S$ such that $us = ut$ ($su = tu$). A pomonoid is called *weakly right (left) reversible* if for all $s, t \in S$, there exist $u, v \in S$ such that $us \leq vt$ ($su \leq tv$). A left S-poset is called *simple* if it has no proper subposets and *completely reducible* if it is a coproduct of simple posets.



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An *order congruence* on an S-poset ${}_S A$ is an S-act congruence ρ such that the factor act A/ρ can be equipped with a compatible order, making the natural map $A \rightarrow A/\rho$ an S-poset morphism. A left S-poset ${}_S A$ is called *cyclic* if $A = Sa = \{sa \mid s \in S\}$ for some $a \in A$. In [9], an S-poset ${}_S A$ is cyclic if and only if there exists an order congruence λ on S such that $A \cong S/\lambda$. If K is a convex left ideal of a pomonoid S , then there exists an S-poset congruence where one of its classes is K and all the others are singletons. Moreover, the factor S-poset by this congruence is called the *Rees factor S-poset of S by K* and denoted S/K .

Various flatness properties of S-posets are defined in terms of tensor products. To define the tensor product $A \otimes_S B$ of a right S-poset A_S and a left S-poset ${}_S B$ (see [10]), we consider a preorder θ on the set $A \times B$, defined by $(a, b)\theta(a', b')$ if and only if

$$\begin{aligned} a &\leq a_1 s_1, \\ a_1 t_1 &\leq a_2 s_2, & s_1 b &\leq t_1 b_2, \\ a_2 t_2 &\leq a_3 s_3, & s_2 b_2 &\leq t_2 b_3, \\ &\vdots & &\vdots \\ a_n t_n &\leq a', & s_n b_n &\leq t_n b'. \end{aligned}$$

for some $n \in \mathbb{N}$, $a_1, \dots, a_n \in A$, $b_2, \dots, b_n \in B$ and $s_1, t_1, \dots, s_n, t_n \in S$. Then $\theta \cap \theta^{-1}$ is an equivalence relation on $A \times B$, and we denote the equivalence class of (a, b) by $a \otimes b$. The quotient set

$$A \otimes_S B := (A \times B) / (\theta \cap \theta^{-1}) = \{a \otimes b \mid a \in A, b \in B\}$$

is a poset with respect to the order

$$a \otimes b \leq a' \otimes b' \Leftrightarrow (a, b)\theta(a', b').$$

This poset $A \otimes_S B$ is called the *tensor product* of A_S and ${}_S B$. Note that $as \otimes b = a \otimes sb$ for every $a \in A$, $b \in B$ and $s \in S$. In a natural way, one obtains a functor $A_S \otimes -$ of tensor multiplication from ${}_S \text{Pos}$ to Pos .

In [4,9], the definitions of flatness, weak flatness, principally weak flatness, and weak torsion freeness are formulated as follows:

- A left S-poset ${}_S B$ is called *flat* if for every right S-poset A_S and all pairs $(a, b), (a', b')$ in $A \times B$, $a \otimes b = a' \otimes b'$ in $A \otimes_S B$ implies the same equality holds in $(aS \cup a'S) \otimes_S B$. Equivalently, the functor $- \otimes_S B$ takes embeddings in Pos_S to monomorphisms in Pos .
- A left S-poset ${}_S B$ is called (*principally*) *weakly flat* if the functor $- \otimes_S B$ maps embeddings of (principal) right ideals $I \subseteq S$ in Pos_S to monomorphisms in Pos .
- A left S-poset ${}_S B$ is called *weakly torsion free* if $cb = cb'$ implies $b = b'$ whenever $b, b' \in B$ and c is a left po-cancellable element.
- A left S-poset ${}_S B$ is said to satisfy Condition (P) if for all $b, b' \in B$, and $s, s' \in S$, $sb \leq s'b'$ implies $b = ub'', b' = vb''$ for some $b'' \in B$ and $u, v \in S$ with $su \leq s'v$.

For a more complete discussion of flatness properties of posets over pomonoids, the reader is referred to [4,11,12]. The following relations exist among flatness properties of S-posets:

$$\begin{array}{ccccccccc} \text{free} & \Rightarrow & \text{projective} & \Rightarrow & \text{strongly flat} & \Rightarrow & \text{(P)} & \Rightarrow & \text{(WP)} & \Rightarrow & \text{(PWP)} \\ & & & & & & \Downarrow & & \Downarrow & & \Downarrow \\ & & & & & & (P_w) & \Rightarrow & (WP)_w & \Rightarrow & (PWP)_w \\ & & & & & & \Downarrow & & \Downarrow & & \Downarrow \\ \text{po-flat} & \Rightarrow & \text{w.po-f.} & \Rightarrow & \text{p.w.po-f.} & \Rightarrow & \text{GP-po-f.} & \Rightarrow & \text{po-t.f.} \\ & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \text{flat} & \Rightarrow & \text{w.f.} & \Rightarrow & \text{p.w.f.} & \Rightarrow & \text{GP-f.} & \Rightarrow & \text{w.t.f.} \end{array}$$

We let ${}_S A$ be an S -poset. An element $a \in A$ is called *I-regular* if there exists an S -morphism $f : {}_S S a \rightarrow {}_S S$ such that $f(a)a = a$. An S -poset ${}_S A$ is called *I-regular* if all elements of A are *I-regular*.

Example 1. Let $S = \{1, 0\}$ be a pomonoid, where the multiplication operation is defined as $1 \cdot 1 = 1$ and $1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 0$ (i.e., 1 is the identity element and 0 is the zero element), and the partial order is $0 \leq 1$. We take the trivial S -poset $A = \{a\}$, the left action of this poset is defined such that $s \cdot a = a$ for all $s \in S$, and the partial order on it is trivial. Obviously, A is an *I-regular* S -poset.

It is clear that a regular pomonoid S was *I-regular* as a left S -poset, but the converse is not true. For example, if S is a right po-cancellable pomonoid, then S is an *I-regular* left S -poset without being a regular pomonoid. In the special case where S is an ordered group, S is not only a regular pomonoid but also an *I-regular* left S -poset.

In [10], Shi introduced the concept of *I-regular* S -posets and offered characterizations of two classes of pomonoids (left *PP*-pomonoid and right po-cancellable pomonoid) by the *I-regularity* of S -posets. However, the projectivity, injectivity, and the direct product properties of *I-regular* S -posets were not involved. In [13], the authors focused on the homological classification of pomonoids based on the flatness of Rees factor S -posets, but did not establish a connection with *I-regularity*. In this paper, we continue the development of *I-regular* S -posets. In Section 2, we characterize pomonoids over which all *I-regular* S -posets have one of the properties around projectivity or injectivity. In [10], characterizations of pomonoids over which all free (projective) S -posets are *I-regular* are given; in [9], the authors characterized pomonoids over which all strongly flat S -posets are *I-regular*. Consequently, we investigate pomonoids over which all left S -posets with one of the properties are *I-regular* in Section 3. In Section 4, we investigate the direct product of *I-regular* S -posets. Finally, we study the classification of pomonoids by the *I-regularity* property of right Rees factor S -posets and tabulate the results.

2. Characterizations of Pomonoids Where All *I-regular* Left S -Posets Possess Projectivity or Injectivity-Related Properties

In this section, we investigate pomonoids over which all *I-regular* left S -posets have one of the properties introduced in Section 1. To achieve this goal, we need the following lemmas.

Lemma 1 ([10], Proposition 4.2). *Let A be an S -poset and $a \in A$. The following assertions are equivalent:*

- (1) a is *I-regular*.
- (2) There exists an element $e \in E(S)$ such that $a = ea$ and $sa \leq ta$ implies $se \leq te$ for $s, t \in S$.
- (3) $Sa \cong Se$ in ${}_S \text{Pos}$ for some $e \in E(S)$.
- (4) Sa is projective.

Proof. (3) $\Rightarrow$ (4). For any surjective S -morphism $g : B \rightarrow C$ and any S -morphism $f : Se \rightarrow C$. Suppose $f(e) = c \in C$. Since g is surjective, there exists $b \in B$ with $g(b) = c$. We define $h(te) = tb$, then $g(h(te)) = g(tb) = tg(b) = tc = tf(e) = f(te)$, so $gh = f$. Thus, Se is projective, and $Sa \cong Se$ implies Sa is projective. $\square$

If $a \in A, e^2 = e \in S$ as in Lemma 1 (2), then we call $\{a, e\}$ an *I-regular pair* (in A).

From [10], a pomonoid S is called *left PP-pomonoid* if the S -subposet Sx is projective for all $x \in S$. (Note, however, that Sx may be an ideal of S in the ordered sense.) By Lemma 1, an S -poset is *I-regular* if and only if all cyclic S -subposets of A are projective. Thus, we have

Lemma 2 ([10], Lemma 4.7). *A pomonoid S is I-regular if and only if S is a left PP pomonoid.*

Lemma 3. *${}_S\Theta$ is I-regular if and only if S contains a right zero element.*

Proof. This follows from Theorem 1 of [4] and Lemma 1. $\square$

Lemma 4 ([10], Lemma 4.5). *All S -subposets of I-regular S -poset are I-regular, and coproducts of I-regular S -posets are I-regular.*

Lemma 5 ([4]). *Let S be a pomonoid, $a, a' \in A, b, b' \in B$. Then $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$ if and only if there exist $a_1, \dots, a_m \in A, b_2, \dots, b_m \in B, s_1, t_1, \dots, s_m, t_m \in S$ such that*

$$\begin{array}{ll} a \leq a_1 s_1, & \\ a_1 t_1 \leq a_2 s_2, & s_1 b \leq t_1 b_2, \\ a_2 t_2 \leq a_3 s_3, & s_2 b_2 \leq t_2 b_3, \\ \vdots & \vdots \\ a_m t_m \leq a', & s_m b_m \leq t_m b'. \end{array}$$

Similar to ([14], Lemma 3.2), we have

Lemma 6. *Let S be a pomonoid. If there exists an I-regular left S -poset, then there exist a largest I-regular left ideal of S .*

In the following, $T(S)$ is always used to represent the largest I-regular left ideal of S .

Theorem 1. *For any pomonoid S , all I-regular left S -posets are principally weakly flat if and only if for every idempotent $e \in T(S)$ and every element $s \in S$ the product se is a regular element in S .*

Proof. We let $s \in S$ and $e^2 = e \in T$. If $Sse = Se$, then there exists $t \in S$ such that $tse = e$, and then it follows that $setse = se$, hence se is a regular element. In another case, we have $Sse \neq Se$. We can construct an S -act M as follows:

$$M = \{(x, te) \mid te \in Se \setminus Sse\} \cup \{(y, te) \mid te \in Se \setminus Sse\} \cup \{(z, te) \mid te \in Sse\}$$

where x, y, z are three elements not belonging to S , and define a left S -action on M by

$$r(w, te) = \begin{cases} (w, rte), & \text{if } rte \in Se \setminus Sse, \\ (z, rte), & \text{if } rte \in Sse, \end{cases} \quad w \in \{x, y\},$$

$$r(z, te) = (z, rte).$$

The order on M is defined as

$$(w_1, s) \leq (w_2, t) \Leftrightarrow (w_1 = w_2 \text{ and } s \leq t) \text{ or } (w_1 \neq w_2, s \leq i \leq t \text{ for some } i \in Sse).$$

Then M is an S -poset according to the above definition. It is clear that there are isomorphisms $S(x, e) \simeq Se \simeq S(y, e)$. Since $Se \leq T(S)$, Se is I-regular by Lemma 4, thus $S(x, e)$ and $S(y, e)$ are also I-regular. From Lemma 4, it follows that $M = S(x, e) \cup S(y, e)$ is I-regular. By assumption, M is principally weakly flat. Clearly, $s(x, e) = (z, se) = s(y, e)$, then $se \otimes (x, e) = se \otimes (y, e)$ in $seS \otimes_S M$. By Lemma 5, there exist $s_1, \dots, s_n, u_1, v_1, \dots, u_n, v_n \in S, b_2, b_3, \dots, b_n \in seS, w_1, \dots, w_n \in \{x, y, z\}$ such that

$$\begin{aligned}
(x, e) &\leq u_1(w_1, s_1), \\
seu_1 &\leq b_2v_1, & v_1(w_1, s_1) &\leq u_2(w_2, s_2), \\
b_2u_2 &\leq b_3v_2, & v_2(w_2, s_2) &\leq u_3(w_3, s_3), \\
&\dots & &\dots \\
b_nu_n &\leq sev_n, & v_n(w_n, s_n) &\leq (y, e).
\end{aligned}$$

We denote 1 by u_0s_0 and $v_{n+1}s_{n+1}$; by the definition of M , there exist $k \in \{0, 1, \dots, n, n+1\}$ and $j \in Sse$ such that $v_k s_k \leq j \leq u_{k+1} s_{k+1}$. So we have $se \leq see \leq seu_1 s_1 \leq b_2 v_1 s_1 \leq b_2 u_2 s_2 \leq \dots \leq b_{k+1} v_k s_k \leq b_{k+1} j \leq b_{k+1} u_{k+1} s_{k+1} \leq \dots \leq b_n u_n s_n \leq b_n v_n s_n \leq sev_n s_n \leq se$, which implies that $se = b_{k+1} j$, hence $se \in seSse$. Now the result follows.

Conversely, we suppose ${}_S A$ is an I-regular left S -poset and for $a, a' \in A, s \in S, s \otimes a = s \otimes a'$ in $S \otimes_S A$. Then there exist $a_2, a_3, \dots, a_m, b_2, b_3, \dots, b_k \in A, u_1, v_1, \dots, u_m, v_m \in S, p_1, q_1, \dots, p_k, q_k \in S, s_1, \dots, s_m, r_1, \dots, r_k \in S$ such that

$$\begin{aligned}
s &\leq s_1 u_1, \\
s_1 v_1 &\leq s_2 u_2, & u_1 a &\leq v_1 a_2, \\
s_2 v_2 &\leq s_3 u_3, & u_2 a_2 &\leq v_2 a_3, \\
&\dots & &\dots \\
s_m v_m &\leq s, & u_m a_m &\leq v_m a', \\
s &\leq r_1 p_1, \\
r_1 q_1 &\leq r_2 p_2, & p_1 a' &\leq q_1 b_2, \\
r_2 q_2 &\leq r_3 p_3, & p_2 b_2 &\leq q_2 b_3, \\
&\dots & &\dots \\
r_k q_k &\leq s, & p_k b_k &\leq q_k a.
\end{aligned}$$

Because ${}_S A$ is I-regular, there exist $e, f \in E(S)$ such that $\{a, e\}, \{a', f\}$ are I-regular pairs and Se, Sf are I-regular left ideals, thus $e, f \in T(S)$. By hypothesis, there exist $x, y \in S$ such that $se = sexse$ and $sf = sfysf$. From $s \otimes a = s \otimes a'$, we obtain that $sa = sa'$. We can now calculate

$$\begin{aligned}
sexa' &\leq sexr_1 p_1 a' \leq sexr_1 q_1 b_2 \leq sexr_2 p_2 b_2 \\
&\leq \dots \leq sexr_k p_k b_k \leq sexr_k q_k a \\
&\leq sexsa = sexsea = sea = sa = sa'.
\end{aligned}$$

Thus, $sexf \leq sf$ (using the I-regular pair property). Hence

$$\begin{aligned}
s \otimes a &= s \otimes ea = se \otimes a = sexse \otimes a \\
&= sex \otimes sa = sex \otimes sa' = sex \otimes sfa' \\
&= sexsf \otimes a' \leq sf \otimes a' = s \otimes fa' \\
&= s \otimes a'
\end{aligned}$$

in $sS \otimes_S A$. Similarly, using the I-regular pair property, we can obtain $sfyse \leq se$. Therefore,

$$\begin{aligned}
s \otimes a' &= s \otimes fa' = sf \otimes a' = sfysf \otimes a' \\
&= sfy \otimes sfa' = sfy \otimes sa' = sfy \otimes sa \\
&= sfyse \otimes a \leq se \otimes a = s \otimes ea \\
&= s \otimes a
\end{aligned}$$

in $sS \otimes A$. So ${}_S A$ is principally weakly flat. $\square$

Next, we give an example to apply the above theorem.

Example 2. Let $S = \{1, e\}$ with multiplication defined as $1 \cdot 1 = 1, 1 \cdot e = e \cdot 1 = e, e \cdot e = e$ and partial order $e \leq 1$; here, the set of idempotents $E(S) = \{1, e\}$. Checking Theorem 1's condition, we find that for all $s \in S$ and $e \in E(S)$, the product se is regular. By Theorem 1, all I-regular left S -posets are principally weakly flat, which is verified by the I-regular S -poset $A = \{a, b\}$ (with action $1 \cdot a = a, 1 \cdot b = b, e \cdot a = b, e \cdot b = b$ and order $a \leq b$): for $s = e, e \cdot a = b \leq e \cdot b = b$ implies $a \leq b$, satisfying the principal weak flatness condition. In contrast, finite pomonoids violating Theorem 1's condition (i.e., containing an idempotent $e \in E(S)$ and an element $s \in S$ such that se is non-regular) are rare, especially when $E(S) = S$ (all elements are idempotent), as idempotents inherently make se regular; for such pomonoids, some I-regular left S -posets fail to be principally weakly flat, though constructing a finite example requires a pomonoid with non-regular products of idempotents and elements, which is non-trivial due to the regularity of idempotent-generated products.

Corollary 1. For any pomonoid S , the following statements are equivalent:

- (1) S is a regular pomonoid.
- (2) S is a left PP pomonoid and all I-regular left S -posets are principally weakly po-flat.
- (3) S is a left PP pomonoid and all I-regular left S -posets are principally weakly flat.

Proof. (1) $\Rightarrow$ (2). We let S be a regular pomonoid. Then S is left PP and all left S -posets are principally weakly po-flat by ([15], Theorem 2.3).

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). From Proposition 4.6 of [10], it follows that S is a left PP pomonoid if and only if S is an I-regular S -poset if and only if $T(S) = S$. So by Theorem 1, S is a regular pomonoid. $\square$

Theorem 2. Let S be a pomonoid. If all I-regular left S -posets are GP-flat, then for every idempotent $e \in T(S)$ and every element $s \in S$, there exist $n \in \mathbb{N}, x \in S$ such that $(se)^n = (se)^n xse$.

Proof. It is similar to that of Theorem 1. $\square$

Theorem 3. For any pomonoid S , the following statements are equivalent:

- (1) All I-regular left S -posets are weakly torsion free.
- (2) For every left po-cancellable element r and for every idempotent $e \in T(S)$, $re\mathcal{L}e$.

Proof. (1) $\Rightarrow$ (2). For left po-cancellable element $r \in S$ and $e^2 = e \in T(S)$, if $Sre \neq Se$, then M is an I-regular S -poset constructed in Theorem 1. By assumption, M must be weakly torsion free. But now from $r(x, e) = (z, re) = r(y, e)$, we get $(x, e) = (y, e)$, a contradiction. Hence $Sre = Se$, which means that $re\mathcal{L}e$.

(2) $\Rightarrow$ (1). We let ${}_sA$ be an I-regular S -poset and $ra = rb$ for any left po-cancellable element r , any $a, b \in A$. Since ${}_sA$ is I-regular, there exist $e, f \in E(S)$ such that $\{a, e\}, \{b, f\}$ are I-regular pairs. Thus, we have $rea = rfb$. Since $e \in T(S)$, we have $re\mathcal{L}e$, which implies there exists $t \in S$ such that $tre = e$. Therefore, $ea = trea = trfb$ and $rb = ra = rtrfb$, which implies $rf = rtrf$. Since r is a left po-cancellable element, then $f = trf$ and so $a = ea = trfb = fb = b$. Therefore, A is weakly torsion free. $\square$

Theorem 4. For any pomonoid S , the following statements are equivalent:

- (1) All I-regular left S -posets are projective.
- (2) All I-regular left S -posets are strongly flat.
- (3) All I-regular left S -posets satisfy Condition (P).

(4) Every idempotent of $T(S)$ generates a minimal left ideal.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). We let $e^2 = e \in T(S)$. Then Se is I-regular by Lemma 4. Suppose I is a left ideal of S such that $I \subseteq Se, I \neq Se$. We let M be an I-regular S -poset constructed in Theorem 1. By assumption, M satisfies Condition (P). By [9], Proposition 2.11, M must be a coproduct of cyclic S -subposets which is impossible because $S(x, e) \cap S(y, e) = \{(z, se)\}$. Hence, Se is a minimal left ideal.

(4) $\Rightarrow$ (1). We let ${}_S A$ be an I-regular left S -poset. For any $a \in A$, the cyclic subposet Sa is, by Lemma 1, isomorphic to some left ideal $Se, e \in T(S)$. By assumption, all such ideals Se are simple. Hence, ${}_S A$ is a coproduct of simple subposets each of which is isomorphic to a left ideal generated by an idempotent. By Theorem 3.4 of [10], ${}_S A$ is projective. $\square$

Recall from [16,17] that a left S -poset ${}_S A$ is called *regular-injective* if for any regular monomorphism $h : {}_S B \rightarrow {}_S C$ and morphism $f : {}_S B \rightarrow {}_S A$ there exists a morphism $g : {}_S C \rightarrow {}_S A$ such that $f = gh$. A left S -poset ${}_S A$ is called *regular-(principally) weakly injective* if for any regular monomorphism $i : I \rightarrow S$ where I is a (principally) left ideal of S and for any S -poset morphism $f : {}_S I \rightarrow {}_S A$ there exists an S -poset morphism $g : {}_S S \rightarrow {}_S A$ such that $f = gi$. A left S -poset A is called *regular-divisible* if ${}_S dA = {}_S A$ for every right po-cancellable element d of S .

Proposition 1. For any pomonoid S , all regular-principally weakly injective S -posets are regular-divisible.

Proof. We let ${}_S M$ be a regular-principally weakly injective S -poset and let d be any right po-cancellable element of S . We let $m \in M$. Since d is right po-cancellable, there exists an S -poset morphism $f : {}_S Sd \rightarrow {}_S M$ defined by $f(sd) = sm$ for all $s \in S$. Since ${}_S M$ is regular-principally weakly injective, there exists an S -poset morphism $g : {}_S S \rightarrow {}_S M$ such that $f = gi$ where i is the regular monomorphism of Sd into S . Now,

$$m = f(d) = (gi)(d) = g(d) = d(g(1)) \in dM.$$

Hence $M \subseteq dM$, and thus $dM = M$, which means that ${}_S M$ is regular-divisible. $\square$

From the previous definitions and proposition, we have following implications:

$$\text{regular-injective} \Rightarrow \text{regular-weakly injective} \Rightarrow$$

$$\text{regular-principally weakly injective} \Rightarrow \text{regular-divisible}.$$

Theorem 5. Let S be a pomonoid. All I-regular left S -posets are regular-divisible if and only if all left ideals $Se, e^2 = e \in T(S)$, are regular-divisible.

Proof. Necessity is obvious because Se is I-regular by Lemma 4.

Regarding sufficiency, we let ${}_S A$ be an arbitrary I-regular S -poset and let $a \in A$. Then by Lemma 1, Sa is isomorphic to $Se, e \in T(S)$. Since Se is regular-divisible, then for any right po-cancellable $d \in S$, we have $dSe = Se$ and thus $dSa = Sa$. But then

$$dA = d\left(\bigcup_{a \in A} Sa\right) = \bigcup_{a \in A} dSa = \bigcup_{a \in A} Sa = A$$

which shows that ${}_S A$ is regular-divisible. $\square$

For any $q \in S$, an element $p \in S$ is said to be q -po-cancellable if for any $s, t \in S$, $sp \leq tp$ always implies $sq \leq tq$.

Theorem 6. Let S be a pomonoid. If all I-regular left S -posets are regular-principally weakly injective, then the largest left ideal $T(S) \subseteq S$ is regular and if $p \in S \setminus T(S)$ is e -po-cancellable for $e^2 = e \in T(S)$, then $e \in pS$.

Proof. For any $t \in T(S)$, we have $St \subseteq T(S)$ and St is I-regular. By assumption, St is regular-principally weakly injective, and there exists a morphism $g : S \rightarrow St$ such that $gi = 1_{St}$ where $i : St \rightarrow S$ is the inclusion morphism. So $st = g(1)$ for some $s \in S$. Now, $tst = tg(1) = g(t) = t$. Thus, t is regular.

We let $p \in S \setminus T(S)$ be e -po-cancellable for $e^2 = e \in T(S)$. Then, by setting $f(p) = e$, we obtain an S -poset morphism f from Sp into Se . Since Se is I-regular, it is regular-principally weakly injective and there exists a morphism $g : S \rightarrow Se$ such that $f = gi$ where $i : Sp \rightarrow S$ is the inclusion homomorphism. Now, $e = f(p) = g(p) = pg(1) \in pS$. $\square$

Lemma 7. S is a regular pomonoid if and only if all left S -posets are regular-principally weakly injective.

Proof. We let ${}_S A$ be a left S -poset and $f : {}_S Ss \rightarrow {}_S A, s \in S$ be an S -homomorphism. If S is regular, then there exists $s' \in S$ such that $s = ss's$. We set $f(s's) = a$ and define a mapping $g : S \rightarrow A$ by $g(1) = a$. Then, g is well defined and $g(s) = sg(1) = sa = sf(s's) = f(s)$. Since g is the extension of f to S , ${}_S A$ is regular-principally weakly injective.

Conversely, suppose that all left S -posets are regular-principally weakly injective. Then, for every $s \in S$, the principal left ideal Ss of S is a regular-principally weakly injective left S -poset. Hence the identity map i of Ss to Ss can be extended to the S -homomorphism g of S onto Ss . We set $g(1) = s's$ for some $s' \in S$. Then, $s = i(s) = g(s) = sg(1) = ss's$, hence s is a regular element. $\square$

Theorem 7. A pomonoid S is an I-regular left S -poset and all I-regular left S -posets are regular-principally weakly injective if and only if S is a regular pomonoid.

Proof. Regarding necessity, suppose that ${}_S S$ is an I-regular left S -poset and all I-regular left S -posets are regular-principally weakly injective. Then $T(S) \subseteq S$ is regular by Theorem 6. We let $p \in S \setminus T(S)$. If ${}_S S$ is an I-regular S -poset, then there exists an idempotent $e \in S$ such that $h : {}_S Sp \rightarrow {}_S Se$ is an isomorphism. Since ${}_S Se$ is I-regular, it is regular-principally weakly injective and then there exists an S -homomorphism $g : {}_S S \rightarrow {}_S Se$ such that g is an extension of h . Hence $e = h(p) = g(p) = pg(1)$, and then $p = ep = pg(1)p$, so p is also regular. Thus, S is a regular pomonoid.

Regarding sufficiency, if S is a regular pomonoid, then S is a left PP -pomonoid and so ${}_S S$ is an I-regular left S -poset by Lemma 2. Using Lemma 7, we determine that all left S -posets are regular-principally weakly injective. So the result follows. $\square$

From [10], a left S -poset ${}_S A$ is called *faithful (strongly faithful)* if from $sa \leq ta, s, t \in S$, for all (some) $a \in A$ it follows that $s \leq t$.

Theorem 8. For any pomonoid S , the following statements are equivalent:

- (1) All I-regular left S -posets are faithful.
- (2) For any $e^2 = e \in T(S)$, Se is faithful.

Proof. The implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). We let ${}_S A$ be an I-regular left S -poset and $ux \leq vx$, $u, v \in S, x \in A$. We take $a \in A$; there exists $e^2 = e \in T(S)$ such that $\{a, e\}$ is an I-regular pair. For any $s \in S$, we have $usa \leq vsa$; it follows that $use \leq vse$. Since ${}_S Se$ is faithful, $u \leq v$. $\square$

Example 3. Let $S = \{1, e\}$ be the finite pomonoid, equipped with a discrete partial order (i.e., only reflexivity holds: $1 \leq 1$, $e \leq e$, and no other order relations exist). The set of idempotents of S is $E(S) = \{1, e\}$, and the largest I-regular left ideal of S is $T(S) = S$. First, we verify that the left S -poset Se is faithful for all $e \in T(S)$: When $e = 1$, $S \cdot 1 = S = \{1, e\}$. By the definition of a faithful poset (if $s \cdot b \leq t \cdot b$ for all $b \in B$, then $s \leq t$), if $s \cdot x \leq t \cdot x$ for all $x \in S$, taking $x = 1$ gives $s \cdot 1 = s \leq t = t \cdot 1$, so $S \cdot 1$ is faithful. When $e = e$, $S \cdot e = \{e\}$; if $s \cdot e \leq t \cdot e$ (which simplifies to $e \leq e$, a condition that always holds) for all $e \in S \cdot e$, we note that in the discrete partial order, $s \leq t$ if and only if $s = t$. Since $S \cdot e = \{e\}$ is a singleton set, the condition “for all $b \in B$ ” is trivial, so $S \cdot e$ is also faithful. Next, we present an example of a faithful I-regular left S -poset: We let $A = S \cdot 1 = \{1, e\}$, which is I-regular as shown in Example 1. For any $s, t \in S$, if $s \cdot a \leq t \cdot a$ for all $a \in A$, taking $a = 1$ yields $s \leq t$; in the discrete partial order, this implies $s = t$, so A is faithful.

Theorem 9. Let S be a pomonoid. Then all I-regular left S -posets are strongly faithful if and only if S is right po-cancellable.

Proof. We let all I-regular left S -posets be strongly faithful and for any $s, t, z \in S$, $sz \leq tz$. For $a \in A$, we have $s(za) \leq t(za)$. Since ${}_S A$ is strongly faithful, $s \leq t$ and so S is right po-cancellable.

Conversely, suppose that ${}_S A$ is an I-regular left S -poset and for $a \in A, s, t \in S$, $sa \leq ta$. Then there exists $e \in E(S)$ such that $ea = a$ and $se \leq te$. Since S is right po-cancellative, we have $s \leq t$ and so ${}_S A$ is strongly faithful as required. $\square$

3. Characterizations of Pomonoids Where All Left S -Posets with Specific Properties Are I-Regular

In [10], characterizations of pomonoids over which all free (projective) S -posets are I-regular are given. In [9], the authors characterize pomonoids over which all strongly flat S -posets are I-regular. In this section, we continue to investigate pomonoids over which all left S -posets with one of the properties are I-regular.

Proposition 2. For any pomonoid S , all strongly faithful S -posets are I-regular.

Proof. We let ${}_S A$ be a strongly faithful S -poset. Then for any $a \in A$, there exists a morphism $f : {}_S Sa \rightarrow {}_S S$ defined by $f(sa) = s$ which satisfies $f(a)a = a$. Therefore, ${}_S A$ is I-regular. $\square$

Theorem 10. For any pomonoid S , all completely reducible left S -posets are I-regular if and only if S contains a right zero element.

Proof. Regarding necessity, the one element left S -poset ${}_S \Theta$ is obviously completely reducible. Hence, by assumption, ${}_S \Theta$ is I-regular. By Lemma 1, ${}_S \Theta$ is projective, which implies that S contains a right zero element from ([4], Theorem 1).

Regarding sufficiency, from the existence of a right zero element, it follows that the only simple left S -poset is one-element S -poset. Obviously the one-element poset is projective and I-regular by Lemma 1. But then, by Lemma 3, every completely reducible left S -poset is I-regular. $\square$

Lemma 8. Let S be a left zero semigroup with 1 adjoined and ${}_S A$ a weakly po-flat left S -poset. Suppose that $a \in A$ and $s, t \in S \setminus \{1\}$ are such that $sa \leq ta$. Then, $s \leq t$.

Proof. From $sa \leq ta$, it follows that $s \otimes a \leq t \otimes a$ in $S \otimes_S A$. Thus, we have $s \otimes a \leq t \otimes a$ in $(sS \cup tS) \otimes_S A$, since ${}_S A$ is weakly po-flat. Therefore, by Lemma 5, there exist $u_1, v_1, \dots, u_n, v_n \in S, s_1, \dots, s_n \in (sS \cup tS), a_2, \dots, a_n \in A$ such that

$$\begin{aligned} s &\leq s_1 u_1, \\ s_1 v_1 &\leq s_2 u_2, & u_1 a &\leq v_1 a_2, \\ s_2 v_2 &\leq s_3 u_3, & u_2 a_2 &\leq v_2 a_3, \\ &\dots & &\dots \\ s_n v_n &\leq t, & u_n a_n &\leq v_n a. \end{aligned}$$

Since $s_1, \dots, s_n \in (sS \cup tS)$, where s, t are left zero elements, it is easy to show that $s_1, s_2, \dots, s_n$ are also left zero elements. Thus, $s \leq t$ and the result follows. $\square$

Theorem 11. Let S be a left zero semigroup with 1 adjoined. Then all weakly po-flat left S -posets are I -regular.

Proof. If $S = \{1\}$, then the result is clear. Now we let S be a left zero semigroup with 1 adjoined. Suppose that ${}_S A$ is a weakly po-flat left S -poset and $a \in A$. By Lemma 1, we show that ${}_S Sa$ is a projective left S -poset.

Suppose that $sa \neq a$ for any $s \in S \setminus \{1\}$. We define a mapping $f : {}_S Sa \rightarrow {}_S S$ as follows: $f(ta) = t, t \in S$.

Suppose $ta = t'a$. If $t, t' \in S \setminus \{1\}$, then $t = t'$ by Lemma 8. If $t \in S \setminus \{1\}$ and $t' = 1$, then $ta = a$, it is a contradiction. If $t' \in S \setminus \{1\}$ and $t = 1$, the result is similar. This means that f is well defined. It is clear that f is an isomorphism of left S -posets. Thus, ${}_S Sa$ is projective.

Now suppose that there exists an element $s \in S \setminus \{1\}$ such that $sa = a$. We define a mapping $f : {}_S Sa \rightarrow {}_S S$ as follows:

$$\begin{aligned} f(a) &= s, \\ f(ta) &= t, \quad t \in S \setminus \{1\}. \end{aligned}$$

By Lemma 8, it is easy to see that f is well defined. Clearly, $f : {}_S Sa \rightarrow {}_S S$ is an isomorphism of left S -posets. Thus, ${}_S Sa$ is projective since s is an idempotent of S . $\square$

Proposition 3. All weakly torsion free left S -posets are I -regular if and only if S is right po-cancellable.

Proof. Regarding necessity, suppose all weakly torsion free left S -posets are I -regular. We let $s, t, c \in S$ where c is left po-cancellable and $sc \leq tc$. Consider the left S -poset $A = S$, which is weakly torsion free. Since A is I -regular, for the element $c \in A$, there exists an idempotent $e \in E(S)$ such that:

1. $c = ec$;
2. For all $s, t \in S$, $sc \leq tc$ implies $se \leq te$.

By the given condition $sc \leq tc$, the second property yields $se \leq te$. If S were not right po-cancellable, there would exist $s \neq t \in S$ such that $sc \leq tc$ but $s \not\leq t$, contradicting the weak torsion freeness of A . Thus, S must be right po-cancellable.

Regarding sufficiency, we let S be right po-cancellable and let A be a weakly torsion free left S -poset. For any $a \in A$, we define a map $f : Sa \rightarrow S$ by $f(sa) = s$ for all $s \in S$.

First, we verify that f is well defined: If $sa = ta$ for some $s, t \in S$, then the weak torsion freeness of A (together with right po-cancellability of S) implies $s = t$. Thus, $f(sa) = f(ta)$, so f is well defined.

Next, since $a = 1 \cdot a$, we have $f(a) = f(1 \cdot a) = 1$. It follows that $f(a) \cdot a = 1 \cdot a = a$. By the definition of I -regularity, a is I -regular. Since a was arbitrary, A is I -regular. $\square$

Proposition 4. All flat left S -posets are I -regular if and only if S is a regular pomonoid.

Proof. Regarding necessity, suppose all flat left S -posets are I -regular. The pomonoid S itself is a flat left S -poset. Thus, S is I -regular as a left S -poset. By Lemma 2, a pomonoid S is left PP if and only if it is I -regular as a left S -poset. Hence, S is left PP. Now, we take any $s \in S$. Since S is flat and I -regular, Theorem 1 implies that for every idempotent $e \in T(S)$ and every $s \in S$, se is regular in S . Since $T(S) = S$ (as S is I -regular), $s = s \cdot 1$ must be regular. Thus, every element of S is regular, so S is a regular pomonoid.

Regarding sufficiency, we let S be a regular pomonoid and let A be a flat left S -poset. For any $a \in A$, we need to show a is I -regular.

Since S is regular, for every $s \in S$, there exists $s' \in S$ such that $s = ss's$. We define a map $f : Sa \rightarrow S$ by $f(sa) = ss'$ for all $s \in S$.

First, we verify f is well defined: If $sa = ta$ for $s, t \in S$, the flatness of A implies there exists $u \in S$ such that $su = tu$ and $ua = a$. Then, $ss' = ss'(su)u' = (ss's)uu' = suu'$, and similarly $tt' = tuu'$. Since $su = tu$, we have $ss' = tt'$, so $f(sa) = f(ta)$. Thus, f is well defined.

Next, we compute $f(a)$: Since $a = 1 \cdot a$, $f(a) = f(1 \cdot a) = 1 \cdot 1' = 1'$ (where $1'$ is the inverse of 1 in S , which is 1 itself, since $1 = 1 \cdot 1 \cdot 1$). Thus, $f(a) = 1$, and $f(a) \cdot a = 1 \cdot a = a$.

Alternatively, we can use regularity directly. For $a \in A$, $a = 1 \cdot a = (1 \cdot 1' \cdot 1) \cdot a = 1' \cdot (1 \cdot a) = 1' \cdot a$. Thus, $f(a) = 1'$ and $f(a) \cdot a = a$, so a is I -regular. Since a was arbitrary, A is I -regular. $\square$

4. Direct Product of I -Regular S -Posets

In the following, we first investigate I -regularity of $D(S)$. First, we remind the reader of some preliminaries.

First, notice that for $a \in A$, $\rho_a : {}_S A \rightarrow {}_S A$ denotes the right translation map defined by $\rho_a(s) = sa$ and $\overrightarrow{\ker} \rho_a = \{(s, t) | sa \leq ta\}$. We let ${}_S A$ and ${}_S B$ be left S -posets over a pomonoid S . It is known that ${}_S A$ is I -regular if and only if for every $a \in A$ there exists an idempotent $e \in S$ such that $\overrightarrow{\ker} \rho_a = \overrightarrow{\ker} \rho_e$. It is also known that a pomonoid S is left PP if and only if for each $s \in S$, $\overrightarrow{\ker} \rho_s = \overrightarrow{\ker} \rho_e$ for some idempotent $e \in S$. Therefore, each left PP-pomonoid as a left S -poset is I -regular. Furthermore, if we denote by $\text{Con}_S A$ the set of all congruences on the poset ${}_S A$, the order relation on $\text{Con}_S A$ is defined by $\rho \leq \lambda$ if and only if $\rho \cap \lambda = \rho$. Then clearly $(\text{Con}_S A, \cap)$ is a pomonoid with identity $({}_S A \times {}_S A)$. It can be routinely verified that for $(a, b) \in A \times B$, $\overrightarrow{\ker} \rho_{(a,b)} = \overrightarrow{\ker} \rho_a \cap \overrightarrow{\ker} \rho_b$.

The next theorem gives a characterization of pomonoids over which $D(S)$ is I -regular.

Theorem 12. Let S be a pomonoid. The diagonal S -poset $D(S)$ is I -regular if and only if

- (1) S is a left PP-pomonoid.
- (2) The set $R = \{\overrightarrow{\ker} \rho_e | e \in E(S)\} \cup (S \times S)$ is a subpomonoid of $T = (\text{Con}_S S, \cap)$.

Proof. Regarding necessity, we take $s \in S$. Since $D(S)$ is I -regular, $\overrightarrow{\ker} \rho_s = \overrightarrow{\ker} \rho_{(s,s)} = \overrightarrow{\ker} \rho_e$ for some idempotent $e \in S$. Thus, S is a left PP pomonoid. On the other hand, by assumption, for each pair of idempotents $e, f \in S$, $\overrightarrow{\ker} \rho_e \cap \overrightarrow{\ker} \rho_f = \overrightarrow{\ker} \rho_{(e,f)} = \overrightarrow{\ker} \rho_h$ for some idempotent $h \in S$, which complete the proof of necessity.

Regarding sufficiency, we let $(s, t) \in D(S)$ for $s, t \in S$. Since S is a left PP pomonoid, S is I -regular by Lemma 2. Thus, there exist idempotents e, f in S such that $\overrightarrow{\ker} \rho_s = \overrightarrow{\ker} \rho_e$

and $\overrightarrow{\ker\rho_t} = \overrightarrow{\ker\rho_f}$. Since R is a subpomonoid of T , there exists an idempotent $h \in S$ such that $\overrightarrow{\ker\rho_e} \cap \overrightarrow{\ker\rho_f} = \overrightarrow{\ker\rho_h}$. Now, we get $\overrightarrow{\ker\rho_{(s,t)}} = \overrightarrow{\ker\rho_s} \cap \overrightarrow{\ker\rho_t} = \overrightarrow{\ker\rho_e} \cap \overrightarrow{\ker\rho_f} = \overrightarrow{\ker\rho_h}$. Hence, $D(S)$ is I-regular. $\square$

Theorem 13. The following are equivalent for a left PP-pomonoid S :

- (1) Every finite product of I-regular S -posets is I-regular.
- (2) S^n is I-regular for every $n \in \mathbb{N}$.
- (3) The diagonal S -poset $D(S)$ is I-regular.

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (1). We let ${}_S A$ and ${}_S B$ be two I-regular posets. We take $(a, b) \in A \times B$. Suppose that $\overrightarrow{\ker\rho_a} = \overrightarrow{\ker\rho_e}$ and $\overrightarrow{\ker\rho_b} = \overrightarrow{\ker\rho_f}$ for some idempotents $e, f \in S$. By Theorem 12, we have $\overrightarrow{\ker\rho_{(a,b)}} = \overrightarrow{\ker\rho_a} \cap \overrightarrow{\ker\rho_b} = \overrightarrow{\ker\rho_e} \cap \overrightarrow{\ker\rho_f} = \overrightarrow{\ker\rho_h}$ for some idempotent $h \in S$. Now, by induction, we obtain the desired result. $\square$

Theorem 14. Let S be a pomonoid, $E(S)$ the set of idempotents of S . Then, the following conditions are equivalent:

- (1) The diagonal S -poset $D(S)$ is I-regular and $|E(S)| = 1$.
- (2) S is right po-cancellable.

Proof. (1) $\Rightarrow$ (2). We let $su \leq tu$, for $u, s, t \in S$. Then $(s, t) \in \overrightarrow{\ker\rho_u} = \overrightarrow{\ker\rho_{(u,u)}}$. Since $D(S)$ is I-regular, there exists $e \in E(S)$ such that $\overrightarrow{\ker\rho_{(u,u)}} = \overrightarrow{\ker\rho_e}$. But $|E(S)| = 1$, that is $e = 1$, then $s \leq t$. Hence, S is right po-cancellable.

(2) $\Rightarrow$ (1). If S is right po-cancellable, then $|E(S)| = 1$ and S is a left PP pomonoid. For $(s, t) \in D(S)$, it is clear that $\overrightarrow{\ker\rho_s} = \overrightarrow{\ker\rho_1}$ and $\overrightarrow{\ker\rho_t} = \overrightarrow{\ker\rho_1}$. So $\overrightarrow{\ker\rho_{(s,t)}} = \overrightarrow{\ker\rho_s} \cap \overrightarrow{\ker\rho_t} = \overrightarrow{\ker\rho_1} \cap \overrightarrow{\ker\rho_1} = \overrightarrow{\ker\rho_1}$. Hence, (s, t) is an I-regular element of $D(S)$ and it is I-regular. $\square$

Proposition 5. For a right collapsible pomonoid S , if $\prod_{i \in I} A_i$ is I-regular, then for every $i \in I$, A_i is I-regular.

Proof. We let $sa_i \leq ta_i$, for $a_i \in A_i$. Since S is right collapsible, there exists $r \in S$ such that $sr = tr$. We consider the fixed element $a_j \in A_j$, for $j \neq i$ and take

$$d_k = \begin{cases} a_i, & k = i \\ ra_j, & k \neq i. \end{cases}$$

Then $s(d_k)_I \leq t(d_k)_I$. Since $\prod_{i \in I} A_i$ is I-regular, by Lemma 1 there exists $e \in E(S)$ such that $(d_k)_I = e(d_k)_I$, $se \leq te$. So $a_i = ea_i$ and A_i is I-regular. $\square$

Proposition 6. For a left PP-pomonoid S , the following are equivalent:

- (1) If $\prod_{i \in I} A_i$ is I-regular, then A_i is I-regular.
- (2) ${}_S \Theta$ is I-regular.
- (3) S has a right zero element.

Proof. (1) $\Rightarrow$ (2). We let S be a left PP-pomonoid. From Lemma 2, it follows that S is I-regular as a left S -poset. Since $S \cong S \times {}_S \Theta$, we have, by assumption, that ${}_S \Theta$ is I-regular.

(2) $\Rightarrow$ (3). By Lemma 3, it is obvious.

(3) $\Rightarrow$ (1). If S has a right zero element, then S is right collapsible. By Proposition 5, the result follows. $\square$

5. Classification of Pomonoids by I-Regularity Property of Right Rees Factor S -Posets

In this section, we give a classification of pomonoids by I-regularity property of their right Rees factor S -posets.

Lemma 9 ([13], Lemma 1.8). *Let S be a pomonoid and K a convex, proper right ideal of S . The following assertions are equivalent:*

- (1) S/K is free.
- (2) S/K is projective.
- (3) S/K is strongly flat.
- (4) S/K satisfies condition (P).
- (5) $|K| = 1$.

Lemma 10 ([4], Theorem 1). *Let S be a pomonoid. Then:*

- (1) Θ_S is free if and only if $|S| = 1$.
- (2) Θ_S is projective if and only if S has a left zero element.
- (3) Θ_S satisfies condition (E) if and only if S is left collapsible.
- (4) The following assertions are equivalent:
 - (a) Θ_S satisfies condition (P);
 - (b) Θ_S satisfies condition (P_w) ;
 - (c) Θ_S is po-flat;
 - (d) Θ_S is flat;
 - (e) Θ_S is weakly po-flat;
 - (f) Θ_S is weakly flat;
 - (g) S is weakly right reversible.
- (5) Θ_S is (always) principally weakly (po-)flat and (po-)torsion free.

Theorem 15. *Let S be a pomonoid and K a convex, right ideal of S . Then S/K is I-regular if and only if $|K| = 1$ and S is right PP, or $K_S = S$ and S contains a left zero element.*

Proof. Suppose that S/K is I-regular for the convex right ideal K_S of S . Then there are two cases as follows:

Case 1. $K_S = S$. Then $S/K \cong \Theta_S$ is I-regular and so by Lemma 3 S contains a left zero element.

Case 2. K_S is a convex proper right ideal of S . Since S/K is I-regular, S/K is projective. Thus, by Lemma 9, $|K| = 1$ and so $S/K \cong S_S$. Since S/K is I-regular, S_S is I-regular and so by Lemma 2, S is right PP as required.

Conversely, suppose $|K| = 1$ and S is right PP. Then $S/K \cong S_S$ and so by Lemma 2, S/K is I-regular.

If $K_S = S$ and S contains a left zero element, then $S/K \cong \Theta_S$ and by Lemma 3, S/K is I-regular. $\square$

Theorem 16. *Let S be a pomonoid. The following statements are equivalent:*

- (1) All projective right Rees factor S -posets are I-regular.
- (2) All free right Rees factor S -posets are I-regular.
- (3) If S has a left zero element, then S is right PP.

Proof. Implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Suppose that S contains a left zero element z . If $K_S = zS = \{z\}$, then $S/K_S \cong S_S$ and so S/K_S is free, since S_S is free. Thus, by assumption, S/K_S is I-regular and by Lemma 2, S is right PP.

(3) $\Rightarrow$ (1). Suppose that S/K is projective for the convex right ideal K_S of S . Then there are two cases as follows:

Case 1. $K_S = S$. Then $S/K \cong \Theta_S$ is projective and so by Lemma 3, S/K is I-regular.

Case 2. K_S is a convex proper right ideal of S . Since S/K is projective, by Lemma 9, $|K| = 1$, and so $K_S = zS = \{z\}$ for some $z \in S$. Thus, z is a left zero element and so by assumption, S is right PP. From Lemma 2, it follows that $S/K \cong S_S$ is I-regular. $\square$

Theorem 17. Let S be a pomonoid. The following statements are equivalent:

- (1) All strongly flat right Rees factor S -posets are I-regular.
- (2) If S is left collapsible, then S contains a left zero element and S is right PP.

Proof. (1) $\Rightarrow$ (2). If S is left collapsible, then by Lemma 10, Θ_S satisfies Condition (E) and so it is strongly flat. Thus, by assumption, Θ_S is I-regular and so by Lemma 3 S contains a left zero element. Thus, S is right PP by Theorem 16.

(2) $\Rightarrow$ (1). Suppose that S/K is strongly flat for the convex right ideal K_S of S . Then there are two cases as follows:

Case 1. $K_S = S$. Then $S/K \cong \Theta_S$ is strongly flat and so by Lemma 10, S is left collapsible. By assumption, S contains a left zero element; thus, by Lemma 3, S/K is I-regular.

Case 2. K_S is a convex proper right ideal of S . Since S/K is strongly flat, by Lemma 9, $|K| = 1$ and so $K_S = zS = \{z\}$ for some $z \in S$. Thus, z is a left zero element and so S is left collapsible. By assumption, S is right PP. Hence, $S/K \cong S_S$ is I-regular by Lemma 2. $\square$

Theorem 18. Let S be a pomonoid. The following statements are equivalent:

- (1) All right Rees factor S -posets satisfying Condition (P) are I-regular.
- (2) If S is weakly right reversible, then S contains a left zero element and S is right PP.

Proof. It is similar to that of Theorem 17. $\square$

Recall from [13] that we let K be convex, proper right ideal of pomonoid S , S/K is principally weakly flat if and only if K is left stabilizing. S/K is principally po-flat if and only if K is strongly left stabilizing.

Theorem 19. Let S be a pomonoid. The following statements are equivalent:

- (1) All weakly flat right Rees factor S -posets are I-regular.
- (2) If S is weakly right reversible, then S contains a left zero element and S is right PP, and S has no proper, left stabilizing convex right ideal K with $|K| > 1$.

Proof. (1) $\Rightarrow$ (2). If S is weakly right reversible, then S contains a left zero element and also S is right PP by Theorem 18. If S has a proper, left stabilizing convex right ideal K with $|K| > 1$, then S/K is weakly flat and by assumption $|K| = 1$, a contradiction is obtained.

(2) $\Rightarrow$ (1). We let K be a convex right ideal of pomonoid S and S/K is weakly flat. If K is convex, proper right ideal of S , by [13], Lemma 1.7, S is weakly right reversible and K is a proper, left stabilizing convex right ideal of S . By assumption, $|K| = 1$ and S is right PP; then, $S/K \cong S$ is I-regular. But if $K = S$, $S/K \cong \Theta$ is weakly flat and by Lemma 10, S is weakly right reversible. By assumption, S has a left zero element and by Lemma 3, S/K is I-regular. $\square$

The following theorem can be proven by a similar argument of the proof of Theorem 19.

Theorem 20. Let S be a pomonoid. The following statements are equivalent:

- (1) All weakly po-flat right Rees factor S -posets are I-regular.

(2) If S is weakly right reversible, then S contains a left zero and S is right PP, and S has no proper, strongly left stabilizing convex right ideal K with $|K| > 1$.

Theorem 21. Let S be a pomonoid. The following statements are equivalent:

- (1) All principally weakly flat right Rees factor S -posets are I-regular.
- (2) S has a left zero element and S is right PP, and S has no proper, left stabilizing convex right ideal K with $|K| > 1$.

Proof. (1) $\Rightarrow$ (2). Since Θ is principally weakly flat, by assumption, Θ is I-regular. Using Lemma 3, we determine that S has a left zero element and also S is right PP by Theorem 16. Suppose that S has proper, left stabilizing convex right ideal K with $|K| > 1$. Then S/K is principally weakly flat and by assumption that $|K| = 1$, a contradiction is obtained.

(2) $\Rightarrow$ (1). We let K be a convex right ideal of pomonoid S and S/K be principally weakly flat. If K is convex, proper right ideal of S , then by [13], Lemma 1.7, K is a proper, left stabilizing convex right ideal of S . By assumption, $|K| = 1$ and S is right PP; hence, $S/K \cong S$ is I-regular. But if $K = S$, then $S/K \cong \Theta$ is principally weakly flat. Since S has a left zero element and by Lemma 3, S/K is I-regular. $\square$

Similarly, one can prove the following theorem.

Theorem 22. Let S be a pomonoid. The following statements are equivalent:

- (1) All principally weakly po-flat right Rees factor S -posets are I-regular.
- (2) S has a left zero element and S is right PP, and S has no proper, strongly left stabilizing convex right ideal K with $|K| > 1$.

Theorem 23. Let S be a pomonoid. The following statements are equivalent:

- (1) All I-regular right Rees factor S -posets are free.
- (2) If S has a left zero element, then $S = \{1\}$.

Proof. (1) $\Rightarrow$ (2). Suppose that S has a left zero element. From Lemma 3, it follows that Θ is I-regular. By assumption that Θ_S is free, we determine that $S = \{1\}$ by Lemma 10.

(2) $\Rightarrow$ (1). Suppose that S/K is I-regular for the convex right ideal K of S . Then there are two cases as follows:

Case 1. $K = S$. Then $S/K \cong \Theta$ and so by Lemma 3, S contains a left zero element. Hence, by assumption, $S = \{1\}$ and so $S/K \cong \Theta$ is free by Lemma 10.

Case 2. K is a proper, convex right ideal of S . Since S/K is projective, by Lemma 9 we have $|K| = 1$. Thus $S/K \cong S_S$ is free, since S_S is free. $\square$

Theorem 24. Let S be a pomonoid. Then all I-regular right Rees factor S -posets are projective.

Proof. It follows from Lemma 1. $\square$

Remark 1. If the order of S is discrete (as an S -act), then by the main results in this paper, we can easily obtain all the characterization of monoids by properties of regular (Rees factor) S -acts.

Below, we tabulate the results.

$\Rightarrow$	I-regular
free	$\exists \text{l.zero} \Rightarrow rpp$
projective	$\exists \text{l.zero} \Rightarrow rpp$
strongly flat	$\text{l.c.} \Rightarrow \exists \text{l.zero} + rpp$
(P)	$\text{r.r.} \Rightarrow \exists \text{l.zero} + rpp$
w.f	$\text{w.r.r.} \Rightarrow \exists \text{l.zero} + rpp + \text{no proper l.s. convex right ideal } K \text{ with } K > 1$
w.po-f.	$\text{w.r.r.} \Rightarrow \exists \text{l.zero} + rpp + \text{no proper s.l.s. convex right ideal } K \text{ with } K > 1$
p.w.f.	$\exists \text{l.zero} + rpp + \text{no proper l.s. convex right ideal } K \text{ with } K > 1$
p.w.po-f.	$\exists \text{l.zero} + rpp + \text{no proper s.l.s. convex right ideal } K \text{ with } K > 1$

Abbreviations: l.c. = left collapsible; l.zero = left zero; w.r.r. = weakly right reversible; r.r. = right reversible; rpp = right PP; s.l.s. = strongly left stabilizing; l.s. = left stabilizing.

6. Conclusions and Limitations

6.1. Conclusions

This paper extends the homological classification of pomonoids using I-regular S-posets, with three key contributions:

1. It characterizes pomonoids where all I-regular left S-posets possess projectivity (injectivity)-related properties (e.g., principally weakly flat, weakly torsion free), generalizing Shi's 2005 results on PP-pomonoids and po-cancellable pomonoids.
2. It establishes conditions for the I-regularity of diagonal posets $D(S)$ and Rees factor posets, linking I-regularity to idempotent behavior and pomonoid structure (e.g., right PP, left collapsible).
3. It supplements results on I-regularity of posets with specific properties (e.g., strongly faithful, completely reducible), providing a more comprehensive framework for pomonoid classification.

6.2. Limitations and Future Directions

Despite these contributions, the work has several limitations that can guide future research:

1. Restriction to Left S-posets: All results focus on left S-posets, with no analogous analysis for right S-posets. Future work could extend I-regularity to right S-posets and explore left-right duality in pomonoid classification.
2. Finite Pomonoid Bias: While examples use finite pomonoids, the paper does not explicitly address infinite pomonoids (e.g., pomonoids over infinite semigroups). Infinite pomonoids may exhibit distinct I-regularity behavior (e.g., non-finite idempotent sets), requiring new techniques.
3. Connection to Other Regularity Notions: I-regularity is not compared to other regularity concepts in S-poset theory (e.g., GP-regularity, strongly regular). Future work could explore hierarchies between these notions and their impact on pomonoid classification.

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