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Note on an Identity of Ramanujan

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Abstract

We extend a summation identity involving the Pochhammer symbol published by Ramanujan in 1915 in the problem section of the "Journal of the Indian Mathematical Society". Moreover, we offer some applications of our theorem. Among others, we present a new series representation for $1/\pi$.

Keywords: summation identity; Pochhammer symbol; falling factorial; series representation; harmonic number; π

MSC: 05A10; 05A19

1. Introduction and Statement of the Main Result

In 1915 Ramanujan [1], [2] (p. 331) published the following problem in the "Journal of the Indian Mathematical Society":

Question 700. *Sum the series*

$$(a+b+1)\Big(\frac{a}{b}\Big)^2+(a+b+3)\Big\{\frac{a(a+1)}{b(b+1)}\Big\}^2+(a+b+5)\Big\{\frac{a(a+1)(a+2)}{b(b+1)(b+2)}\Big\}^2+\ldots$$

to n terms.

Berndt et al. [3] remarked that the sum with n terms can be written as

$$\sum_{k=1}^{n}(a+b+2k-1)\Big(\frac{(a)_k}{(b)_k}\Big)^2,$$

where

$$(c)_k=c(c+1)(c+2)\cdots(c+k-1),\quad c\in\mathbb{C},$$

denotes the Pochhammer symbol. The editors published two solutions given by Rama Aiyar and Appukuttan Brady [4], who used Euler's identity

$$\sum_{k=0}^{n}(1-a_{k+1})a_1a_2\cdots a_k=1-a_1a_2\cdots a_{n+1}$$

and the telescope trick to show that the sum of the series is equal to

$$\frac{1}{b-a-1}\Big(a^2-\Big(\frac{(a)_{n+1}}{(b)_n}\Big)^2\Big).$$

We note that the formula given in [3] (p. 42) is stated incorrectly; $\big((a)_{n+1}/(b)_n\big)^2$ is replaced by $(a)_{n+1}/(b)_n$. The aim of this paper is to present an extension of Ramanujan's identity.

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We use induction to prove the following formula, which involves the variables a, b, the parameter λ, and a sequence, $C(k)$.

Theorem 1. *Let $n \geq 1$ be an integer and let $\lambda, a, b, C(1), ..., C(n+1)$ be complex numbers with $-b \notin \{0, 1, ..., (n-1)\}$. Then, we have*

$$\sum_{k=1}^{n} \Big[(a+k)^{\lambda} C(k+1) - (b+k-1)^{\lambda} C(k)\Big] \left(\frac{(a)_k}{(b)_k}\right)^{\lambda} = \left(\frac{(a)_{n+1}}{(b)_n}\right)^{\lambda} C(n+1) - a^{\lambda} C(1). \quad (1)$$

In the next section, we offer a short proof of Equation (1) and we conclude with a few remarks. Among others, we show that Equation (1) can be applied to find a new series representation for $\pi^{-\lambda/2}$.

2. Proof and Remarks

Proof of Theorem 1. Let $L(n)$ and $R(n)$ be the expressions on the left-hand side and on the right-hand side of Equation (1), respectively. We have

$$L(1) = \Big[(a+1)^{\lambda} C(2) - b^{\lambda} C(1)\Big] \left(\frac{a}{b}\right)^{\lambda} = \left(\frac{(a)_2}{(b)_1}\right)^{\lambda} C(2) - a^{\lambda} C(1) = R(1).$$

Using the induction hypothesis and $(x)_{n+1} = (x)_n (x+n)$ gives

$$\begin{aligned}
L(n+1) &= L(n) + \Big[(a+n+1)^{\lambda} C(n+2) - (b+n)^{\lambda} C(n+1)\Big] \left(\frac{(a)_{n+1}}{(b)_{n+1}}\right)^{\lambda} \\
&= R(n) + \Big[(a+n+1)^{\lambda} C(n+2) - (b+n)^{\lambda} C(n+1)\Big] \left(\frac{(a)_{n+1}}{(b)_{n+1}}\right)^{\lambda} \\
&= \left(\frac{(a)_{n+1}}{(b)_n}\right)^{\lambda} C(n+1) - a^{\lambda} C(1) + \Big[(a+n+1)^{\lambda} C(n+2) - (b+n)^{\lambda} C(n+1)\Big] \left(\frac{(a)_{n+1}}{(b)_n (b+n)}\right)^{\lambda} \\
&= -a^{\lambda} C(1) + \left(\frac{a+n+1}{b+n} \frac{(a)_{n+1}}{(b)_n}\right)^{\lambda} C(n+2) \\
&= -a^{\lambda} C(1) + \left(\frac{(a)_{n+2}}{(b)_{n+1}}\right)^{\lambda} C(n+2) \\
&= R(n+1).
\end{aligned}$$

This completes the proof. □

Remark 1. *(I) Equation (1) can also be written in terms of the rising factorial $x^{\overline{n}} = (x)_n$. The falling factorial is defined by*

$$x^{\underline{n}} = x(x-1) \cdots (x-n+1), \quad x \in \mathbb{C}.$$

Using induction on n we obtain the following counterpart of Equation (1):

$$\sum_{k=1}^{n} \Big[(a-k)^{\lambda} C(k+1) - (b-k+1)^{\lambda} C(k)\Big] \left(\frac{a^{\underline{k}}}{b^{\underline{k}}}\right)^{\lambda} = \left(\frac{a^{\underline{n+1}}}{b^{\underline{n}}}\right)^{\lambda} C(n+1) - a^{\lambda} C(1).$$

(II) If we set

$$C(k) = \frac{1}{(a+b+k-1)^{\mu}}$$

in Equation (1), then we obtain

$$\sum_{k=1}^{n}\left[\frac{(a+k)^{\lambda}}{(a+b+k)^{\mu}}-\frac{(b+k-1)^{\lambda}}{(a+b+k-1)^{\mu}}\right]\left(\frac{(a)_k}{(b)_k}\right)^{\lambda}=\left(\frac{(a)_{n+1}}{(b)_n}\right)^{\lambda}\frac{1}{(a+b+n)^{\mu}}-\frac{a^{\lambda}}{(a+b)^{\mu}}. \tag{2}$$

Since

$$(a+k)^2-(b+k-1)^2=(a-b+1)(a+b+2k-1),$$

we conclude that Equation (2) with $\lambda=2$ and $\mu=0$ leads to Ramanujan's formula.

(III) We apply Equation (1) with

$$a=n,\quad b=n+\frac{1}{2},\quad C(k)=\Big(\frac{k}{(k+\alpha)(k+\beta)}\Big)^{\lambda},\quad \alpha,\beta>0.$$

Using the representation

$$\left(\frac{(a)_{n+1}}{(b)_n}\right)^{\lambda}C(n+1)=\frac{1}{2^{\lambda/2}}\left[\frac{(2n+1/2)(n+1)}{(n+\alpha+1)(n+\beta+1)}\right]^{\lambda}\left[(2n)^{1/2}\frac{\Gamma(2n+1)}{\Gamma(2n+3/2)}\cdot n^{-1/2}\frac{\Gamma(n+1/2)}{\Gamma(n)}\right]^{\lambda}$$

and the limit formula

$$\lim_{n\to\infty}n^{b-a}\frac{\Gamma(n+a)}{\Gamma(n+b)}=1 \tag{3}$$

gives for $\lambda>0$:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\left\{\left[\frac{(k+1)(k+n)}{(k+\alpha+1)(k+\beta+1)}\right]^{\lambda}-\left[\frac{k(k+n-1/2)}{(k+\alpha)(k+\beta)}\right]^{\lambda}\right\}\left(\frac{(n)_k}{(n+1/2)_k}\right)^{\lambda}=2^{\lambda/2}.$$

(IV) We set in Equation (1)

$$a=1/2,\quad b=1,\quad C(k)=k^{-\lambda/2}$$

and apply

$$\frac{(a)_n}{(b)_n}=\frac{\Gamma(b)}{\Gamma(a)}\frac{\Gamma(n+a)}{\Gamma(n+b)},\quad \frac{(1/2)_n}{(1)_n}=\binom{n-1/2}{n},\quad \Gamma(1/2)=\sqrt{\pi}.$$

Then,

$$\sum_{k=1}^{n}\left(\Big(\frac{k+1/2}{\sqrt{k+1}}\Big)^{\lambda}-k^{\lambda/2}\right)\binom{k-1/2}{k}^{\lambda}=\frac{1}{\pi^{\lambda/2}}\left(n^{1/2}\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\right)^{\lambda}\left(\frac{n+1/2}{\sqrt{n(n+1)}}\right)^{\lambda}-\Big(\frac{1}{2}\Big)^{\lambda}.$$

Next, we let $n\to\infty$ and use Equation (3). This gives the series representation

$$\frac{1}{\pi^{\lambda/2}}=\frac{1}{2^{\lambda}}+\sum_{k=1}^{\infty}\left[\Big(\frac{k+1/2}{\sqrt{k+1}}\Big)^{\lambda}-k^{\lambda/2}\right]\binom{k-1/2}{k}^{\lambda}.$$

(V) Let $a,b>0$ and $\lambda(b-a-1)>0$. We denote by H_n the n-th harmonic number,

$$H_n=\sum_{k=1}^{n}\frac{1}{k}.$$

From Equation (1) with $C(k)=H_k$, we obtain

$$\sum_{k=0}^{n}\frac{1}{k+1}\left(\frac{(a)_{k+1}}{(b)_k}\right)^{\lambda}+\sum_{k=1}^{n}\Big[(a+k)^{\lambda}-(b+k-1)^{\lambda}\Big]H_k\left(\frac{(a)_k}{(b)_k}\right)^{\lambda}=\left(\frac{(a)_{n+1}}{(b)_n}\right)^{\lambda}H_{n+1}. \tag{4}$$

We have

$$\left(\frac{(a)_{n+1}}{(b)_n}\right)^{\lambda} H_{n+1} = \left[\frac{\Gamma(b)}{\Gamma(a)} \cdot n^{b-a-1} \frac{\Gamma(n+a+1)}{\Gamma(n+b)}\right]^{\lambda} \frac{H_{n+1}}{n^{\lambda(b-a-1)}}.$$

Since

$$H_n \sim \log(n) \quad (n \to \infty)$$

and $\lambda(b-a-1) > 0$, *we conclude from Equation (3) that*

$$\lim_{n\to\infty} \left(\frac{(a)_{n+1}}{(b)_n}\right)^{\lambda} H_{n+1} = 0.$$

Moreover, there exists a constant $c^* > 0$, *such that we obtain for large k*

$$0 < \frac{1}{k+1}\left(\frac{(a)_{k+1}}{(b)_k}\right)^{\lambda} \leq \frac{c^*}{k^{1+\lambda(b-a-1)}}.$$

It follows that the series

$$\sum_{k=0}^{\infty} \frac{1}{k+1}\left(\frac{(a)_{k+1}}{(b)_k}\right)^{\lambda}$$

is convergent and we obtain from Equation (4):

$$\sum_{k=0}^{\infty} \frac{1}{k+1}\left(\frac{(a)_{k+1}}{(b)_k}\right)^{\lambda} = \sum_{k=1}^{\infty} \Big[(b+k-1)^{\lambda} - (a+k)^{\lambda}\Big] H_k \left(\frac{(a)_k}{(b)_k}\right)^{\lambda}.$$

3. Conclusions

We have generalized a summation formula of Ramanujan and we have applied our result to deduce a limit theorem and two series formulas. The new series representation for $1/\pi$ might be of special interest and leads to the following question: Is it possible to apply Equation (1) to find remarkable summation formulas for other mathematical constants?

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