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Some Properties of Meromorphic Functions Defined by the Hurwitz–Lerch Zeta Function

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Abstract

The findings of this study are connected with geometric function theory and were acquired using subordination-based techniques in conjunction with the Hurwitz–Lerch Zeta function. We used the Hurwitz–Lerch Zeta function to investigate certain properties of multivalent meromorphic functions. The primary objective of this study is to provide an investigation on the argument properties of multivalent meromorphic functions in a punctured open unit disc and to obtain some results for its subclass.

Keywords: meromorphic function; Hadamard (convolution) product; Hurwitz–Lerch zeta function

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1. Introduction

Geometric function theory (GFT) is an amalgamation of geometry along with analysis. Analytic univalent and multivalent functions are two of the most important aspects of the GFT. Investigations in these areas constitute an ancient topic in mathematics, especially in complex analysis, that has atrracted quite a few scholars due to the absolute elegance of their geometrical properties as well as numerous study opportunities. Inarguably, the most significant field of complex analysis concerning using one or many variables is the investigation of univalent functions. Univalent functions are covered in detail in the standard works of Duren [1] and Goodman [2]. The idea of analytic function subordination was first introduced by Littlewood [3], and Rogosinski [4] developed the phrase and established the fundamental results utilizing subordination. The concept of subordination was recently employed by Srivastava and Owa [5] to explore a number of fascinating properties of the generalized hypergeometric function. A generalization of differential inequalities, known as differential subordinations, was the subject of an article by Miller and Mocanu [6]. A meromorphic function is a complex-valued function that is holomorphic (analytic) on a domain except for an isolated set of poles. The article presented investigations on certain differential subordination as well as superordination that leads to a specific class of univalent and multivalent meromorphic functions within the open unit disk. There have been numerous fascinating properties of the GFT that have been studied and investigated by several authors; see [7,8].

Let class $\Sigma_{p,n}$ be a meromorphic multivalent function given by

$$\mathfrak{f}(\xi) = \frac{1}{\xi^p} + \sum_{\kappa=n}^{\infty} a_\kappa \xi^\kappa \qquad (p \in \mathbb{N} = \{1, 2, \dots\},\ n > -p,\ \xi \in \mathfrak{D}^*), \tag{1}$$

which is analytic in the punctured unit disc $\mathfrak{D}^* = \{\xi : \xi \in \mathbb{C} \quad \text{and} \quad 0 < |\xi| < 1\}$. We note that $\Sigma_{p,1-p} = \Sigma_p$ is the class of p-valent meromorphic functions and $\Sigma_1 = \Sigma$.

For two functions $\mathfrak{f}_\iota(\xi) \in \Sigma_{p,n}$ $(\iota = 1, 2)$ given by

$$\mathfrak{f}_\iota(\xi) = \frac{1}{\xi^p} + \sum_{\kappa=n}^{\infty} a_{\kappa,\iota} \xi^\kappa,$$

we define the convolution of $\mathfrak{f}_1(\xi)$ and $\mathfrak{f}_2(\xi)$ as

$$(\mathfrak{f}_1 * \mathfrak{f}_2)(\xi) = \frac{1}{\xi^p} + \sum_{\kappa=n}^{\infty} a_{\kappa,1} a_{\kappa,2} \xi^\kappa = (\mathfrak{f}_2 * \mathfrak{f}_1)(\xi).$$

A function $\mathfrak{f}(\xi) \in \Sigma_{p,n}$ is said to be in class $MS_p^*(\alpha)$ (the class of p-valent meromorphically star-like functions of order $\alpha (0 \le \alpha < p)$ in $\mathfrak{D}^*$ iff

$$-\operatorname{Re}\left\{\frac{\xi \mathfrak{f}'(\xi)}{\mathfrak{f}(\xi)}\right\} > \alpha \quad (\xi \in \mathfrak{D}^*).$$

Let $\mathfrak{f}$ and $\mathfrak{l}$ be analytic in $\mathfrak{D}$. We say that $\mathfrak{f}$ is subordinate to $\mathfrak{l}$, denoted as $\mathfrak{f}(\xi) \prec \mathfrak{l}(\xi)$, if there exists an analytic function ϖ, with $\varpi(0) = 0$ and $|\varpi(\xi)| < 1$, for all $\xi \in \mathfrak{D}$, such that $\mathfrak{f}(\xi) = \mathfrak{l}(\varpi(\xi))$, $\xi \in \mathfrak{D}$. If the function $\mathfrak{l}$ is univalent in $\mathfrak{D}$, $\mathfrak{f}(\xi) \prec \mathfrak{l}(\xi)$ as

$$\mathfrak{f}(0) = \mathfrak{l}(0) \text{ and } \mathfrak{f}(\mathfrak{D}) \subset \mathfrak{l}(\mathfrak{D}).$$

Let $\Phi(r, s; \xi) : \mathbb{C}^2 \times \mathfrak{D} \to \mathbb{C}$ and $\mathfrak{h}$ be univalent in $\mathfrak{D}$. If $\lambda(\xi)$ is analytic in $\mathfrak{D}$ and satisfies the first differential subordination

$$\Phi(\lambda(\xi), \xi\lambda'(\xi); \xi) \prec \mathfrak{h}(\xi). \tag{2}$$

we call $\mathfrak{V}$ a dominant solution of the differential subordination in (2) if $\lambda(\xi) \prec \mathfrak{V}(\xi)$ for satisfying all λ (2). A dominant $\widetilde{\mathfrak{V}}$ is called the best dominant of (2) if $\widetilde{\mathfrak{V}}(\xi) \prec \mathfrak{V}(\xi)$ for all dominant $\mathfrak{V}$ (see [9]).

El-Ashwah and Bulboaca [10] defined the operator $\mathcal{L}^{\varrho}_{\varkappa,p}$ using the Hurwitz–Lerch Zeta function as follows:

$$\begin{aligned} \mathcal{L}^{\varrho}_{\varkappa,p}\mathfrak{f}(\xi) &= \frac{1}{\xi^p} + \sum_{\kappa=n}^{\infty} \left(\frac{\varkappa}{\kappa + \varkappa + p}\right)^{\varrho} a_\kappa \xi^\kappa \\ (\varkappa &\in \mathbb{C}\backslash\mathbb{Z}_0^- = \{0, -1, -2, \dots\}; \varrho \in \mathbb{C}, \xi \in \mathfrak{D}^*). \end{aligned} \tag{3}$$

Moreover, we could easily check that for all $\mathfrak{f}(\xi) \in \Sigma_{p,n}$, $\xi, t_i \in \mathfrak{D}^* (i = 1, 2, 3, \dots.., \kappa)$, $\kappa \in \mathbb{N}$, $\kappa > 1$ and $\varkappa \in \mathbb{C}\backslash\mathbb{Z}_0^-$, we have

$\mathcal{L}^0_{1,p}\mathfrak{f}(\xi) = \mathfrak{f}(\xi)$ and $\mathcal{L}^0_{\varkappa,p}\mathfrak{f}(\xi) = \mathfrak{f}(\xi)$;

$\mathcal{L}^1_{1,p}\mathfrak{f}(\xi) = \frac{1}{\xi^{p+1}} \int_0^\xi t_1^p \mathfrak{f}(t_1) dt_1$ $(\mathfrak{f} \in \Sigma_{p,n};\ \xi \in \mathfrak{D}^*)$;

$\mathcal{L}^2_{1,p}\mathfrak{f}(\xi) = \frac{1}{\xi^{p+1}} \int_0^\xi \frac{1}{t_1} \int_0^{t_1} t_2^p \mathfrak{f}(t_2) dt_2 dt_1$ $(\mathfrak{f} \in \Sigma_{p,n};\ \xi \in \mathfrak{D}^*)$;

$\vdots$

$\mathcal{L}_{1,p}^{\kappa}\mathfrak{f}(\xi) = \frac{1}{\xi^{p+1}}\int_0^{\xi}\frac{1}{t_1}\int_0^{t_1}\frac{1}{t_2}\int_0^{t_2}\cdots\frac{1}{t_{\kappa-1}}\int_0^{t_{\kappa-1}} t_{\kappa}^{p}\mathfrak{f}(t_{\kappa})dt_{\kappa}dt_{\kappa-1}\ldots.dt_2dt_1 \ \ (\mathfrak{f}\in\Sigma_{p,n};\ \xi\in\mathfrak{D}^*)$ and

$\mathcal{L}_{\varkappa,p}^{1}\mathfrak{f}(\xi) = \frac{\varkappa}{\xi^{\varkappa+p}}\int_0^{\xi} t_1^{\varkappa+p-1}\mathfrak{f}(t_1)dt_1 \ \ (\mathfrak{f}\in\Sigma_{p,n};\ \xi\in\mathfrak{D}^*);$

$\mathcal{L}_{\varkappa,p}^{2}\mathfrak{f}(\xi) = \frac{\varkappa^2}{\xi^{\varkappa+p}}\int_0^{\xi}\frac{1}{t_1}\int_0^{t_1} t_2^{\varkappa+p-1}\mathfrak{f}(t_2)dt_2dt_1 \ \ (\mathfrak{f}\in\Sigma_{p,n};\ \xi\in\mathfrak{D}^*);$

$\vdots$

$\mathcal{L}_{\varkappa,p}^{\kappa}\mathfrak{f}(\xi) = \frac{\varkappa^{\kappa}}{\xi^{\varkappa+p}}\int_0^{\xi}\frac{1}{t_1}\int_0^{t_1}\frac{1}{t_2}\int_0^{t_2}\cdots\frac{1}{t_{\kappa-1}}\int_0^{t_{\kappa-1}} t_{\kappa}^{\varkappa+p-1}\mathfrak{f}(t_{\kappa})dt_{\kappa}dt_{\kappa-1}\ldots.dt_2dt_1 \ \ (\mathfrak{f}\in\Sigma_{p,n};\ \xi\in\mathfrak{D}^*).$

Also, we can demonstrate that

$$\mathcal{L}_{\varkappa,p}^{\varrho+1}\mathfrak{f}(\xi) = \frac{\varkappa}{\xi^{\varkappa+p}}\int_0^{\xi} t^{\varkappa+p-1}\mathcal{L}_{\varkappa,p}^{\varrho}\mathfrak{f}(t)dt \ \ (\mathfrak{f}\in\Sigma_{p,n};\ \xi\in\mathfrak{D}^*).$$

Let us define the function

$$\Psi_p(v,\mathfrak{s};\xi) = \frac{1}{\xi^p} + \sum_{\kappa=1-p}^{\infty}\frac{(v)_{\kappa+p}}{(\mathfrak{s})_{\kappa+p}}\xi^{\kappa} \ \ (v\in\mathbb{C}^*=\mathbb{C}\backslash\{0\};\ \mathfrak{s}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\xi\in\mathfrak{D}^*), \tag{4}$$

where $(\zeta)_{\kappa}$ is the Pochhammer symbol defined as follows:

$$(\zeta)_{\kappa} = \frac{\Gamma(\zeta+\kappa)}{\Gamma(\zeta)} = \begin{cases} 1 & (\kappa=0) \\ \zeta(\zeta+1)\ldots(\zeta+\kappa+1) & (\kappa\in\mathbb{N}). \end{cases}$$

We see that

$$\Psi_p(v,\mathfrak{s};\xi) = \frac{1}{\xi^p}\,{}_2F_1(v,1;\mathfrak{s};\xi),$$

where

$${}_2F_1(v,\tau;\mathfrak{s};\xi) = \sum_{\kappa=0}^{\infty}\frac{(v)_{\kappa}(\tau)_{\kappa}}{(\mathfrak{s})_{\kappa}(1)_{\kappa}}\xi^{\kappa} \ \ (v,\tau,\mathfrak{s}\in\mathbb{C}\ and\ \ \mathfrak{s}\notin\mathbb{Z}_0^-;\xi\in\mathfrak{D}),$$

is the Gaussian hypergeometric function.

Using (3) and (4), we define the linear operator $\mathfrak{k}_{\varkappa,p}^{\varrho}(v,\mathfrak{s};\xi):\Sigma_{p,n}\to\Sigma_{p,n}$ as follows:

$$\mathfrak{k}_{\varkappa,p}^{\varrho}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi) = \Psi_p(v,\mathfrak{s};\xi) * \mathcal{L}_{\varkappa,p}^{\varrho}\mathfrak{f}(\xi) \ \ (\xi\in\mathfrak{D}^*). \tag{5}$$

whose series expansion for $\varkappa,\mathfrak{s}\in\mathbb{C}\backslash\mathbb{Z}_0^-;v\in\mathbb{C}^*;\varrho\in\mathbb{C},\xi\in\mathfrak{D}^*$ and for $\mathfrak{f}\in\Sigma_{p,n}$ is given by

$$\mathfrak{k}_{\varkappa,p}^{\varrho}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi) = \frac{1}{\xi^p} + \sum_{\kappa=n}^{\infty}\left(\frac{\varkappa}{\kappa+\varkappa+p}\right)^{\varrho}\frac{(v)_{\kappa+p}}{(\mathfrak{s})_{\kappa+p}}a_{\kappa}\xi^{\kappa}, \tag{6}$$

Operator $\mathfrak{k}_{\varkappa,p}^{\varrho}(v,\mathfrak{s};\xi)$, satisfies

$$\xi\left(\mathfrak{k}_{\varkappa,p}^{\varrho+1}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)\right)' = \varkappa\mathfrak{k}_{\varkappa,p}^{\varrho}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi) - (\varkappa+p)\mathfrak{k}_{\varkappa,p}^{\varrho+1}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi), \tag{7}$$

and

$$\xi\left(\mathfrak{k}_{\varkappa,p}^{\varrho}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)\right)' = v\mathfrak{k}_{\varkappa,p}^{\varrho}(v+1,\mathfrak{s};\xi)\mathfrak{f}(\xi) - (v+p)\mathfrak{k}_{\varkappa,p}^{\varrho}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi) \ \ (v\in\mathbb{C}\backslash\{-1\}). \tag{8}$$

We note that

(i) $\mathfrak{k}_{\varkappa,p}^{1}(1,1;\xi)\mathfrak{f}(\xi) = F_{\varkappa}\mathfrak{f}(\xi) = \frac{\varkappa}{\xi^{\varkappa+p}}\int_0^{\xi} t^{\varkappa+p-1}\mathfrak{f}(t)dt$, $(\varkappa>0)$ (see Miller and Mocanu ([9], p. 389);

(ii) $\mathfrak{k}_{1,p}^{\varrho}(1,1;\xi)\mathfrak{f}(\xi) = P^{\varrho}\mathfrak{f}(\xi) = \frac{1}{\xi^p\Gamma(\varrho)}\int_0^{\xi}\left(\log\frac{\xi}{t}\right)^{\varrho-1}t^p\mathfrak{f}(t)dt$, $(\varrho>0)$ (see Aqlan et al. [11]);

(iii) $\mathfrak{k}^{\varrho}_{\varkappa,p}(1,1;\xi)\mathfrak{f}(\xi) = J^{\varrho}_{\varkappa,p}\mathfrak{f}(\xi) = \frac{\alpha^{\varrho}}{\xi^{\varkappa+p}\Gamma(\varrho)}\int_0^{\xi}\left(\log\frac{\xi}{t}\right)^{\varrho-1}t^{\varkappa+p-1}\mathfrak{f}(t)dt$, $(\varkappa,\varrho>0)$ (see El-Ashwah and Aouf [12]);

(iv) $\mathfrak{k}^{\varrho}_{\varkappa,1}(1,1;\xi)\mathfrak{f}(\xi) = \mathcal{L}^{\varrho}_{\varkappa}\mathfrak{f}(\xi) = \frac{1}{\xi} + \sum_{\kappa=0}^{\infty}\left(\frac{\varkappa}{\kappa+1+\varkappa}\right)^{\varrho}a_{\kappa}\xi^{\kappa}$, $(\varkappa\in\mathbb{C}^{*},\varrho\in\mathbb{C})$ (see El-Ashwah [13]).

The applications of the Hurwitz–Lerch Zeta function in defining new meaningful operators and special classes of analytic functions are numerous in geometric function theory in recent investigations involving subordination techniques. For example, consideration is given to a novel convolution complex operator defined on meromorphic functions associated with the Kummer functions and Hurwitz–Lerch Zeta-type functions in [14], and extensions to the well-known star-likeness and convexity properties are developed for this operator. A comprehensive study on a new class of meromorphic functions introduced using the Hurwitz–Lerch Zeta function can be seen in [15]. In [16], investigations were conducted into the geometric properties of a subclass of meromorphic functions as they relate to a complex linear operator pertaining to the Hurwitz–Lerch Zeta and Kummer functions. New classes of analytic functions are defined in [17] by applying the Hurwitz–Lerch Zeta function, for which typical geometric properties are established. Many authors have investigated the Hurwitz–Lerch Zeta function; see [18–20].

This research study aims to establish an interesting connection between some argument and subordination results of multivalent meromorphic functions defined using the linear operator $\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)$ and to obtain some results for the subclass of $\Sigma_{p,n}$.

2. Preliminaries

The next lemmas are necessary to demonstrate our findings.

Lemma 1 ([21]). *Suppose that* $\mathfrak{p}(\xi)$ *is analytic in* $\mathfrak{D}$, *with* $\mathfrak{p}(0)=1$ *and* $\mathfrak{p}(\xi)\neq 0$ $(\xi\in\mathfrak{D})$, *and assume that*

$$\left|\arg(\mathfrak{p}(\xi)+\mathrm{b}\xi\mathfrak{p}'(\xi))\right| < \frac{\pi}{2}(\mathrm{a}+\frac{2}{\pi}\arctan(\mathrm{ba})) \quad (\mathrm{a,b}>0),$$

Then,

$$|\arg(\mathfrak{p}(\xi))| < \frac{\pi}{2}\mathrm{a} \quad (\xi\in\mathfrak{D}). \tag{9}$$

Lemma 2 ([22]; see also ([9], Theorem 3.1.6, p.71)). *Assume that* $\mathfrak{h}(\xi)$ *is a convex (univalent) function in* $\mathfrak{D}$ *with* $\mathfrak{h}(0)=1$, *and let*

$$\mathfrak{p}(\xi) = 1 + c_1\xi + \dots \tag{10}$$

be analytic in $\mathfrak{D}$. *If*

$$\mathfrak{p}(\xi) + \frac{\xi\mathfrak{p}'(\xi)}{\gamma} \prec \mathfrak{h}(\xi), \qquad \xi\in\mathfrak{D},$$

where $\gamma\neq 0$ *and* $\mathrm{Re}(\gamma)\geq 0$, *then*

$$\mathfrak{p}(\xi) \prec \mathfrak{g}(\xi) = \frac{\gamma}{\xi^{\gamma}}\int_0^{\xi}\mathfrak{h}(t)t^{\gamma-1}dt, \qquad \xi\in\mathfrak{D}, \tag{11}$$

and $\mathfrak{g}(\xi)$ *is the best dominant of (11).*

For v, τ, *and* $\mathfrak{s}$ $(\mathfrak{s} \notin \mathbb{Z}_0^-)$, real or complex numbers in the Gaussian hypergeometric function are given by

$$\begin{aligned} {}_2F_1(v,\tau;\mathfrak{s};\xi) &= \sum_{\kappa=0}^{\infty} \frac{(v)_\kappa(\tau)_\kappa}{(\mathfrak{s})_\kappa(1)_\kappa}\xi^\kappa \\ &= 1+\frac{v\tau}{\mathfrak{s}}\cdot\frac{\xi}{1!}+\frac{v(v+1)\tau(\tau+1)}{\mathfrak{s}(\mathfrak{s}+1)}\cdot\frac{\xi^2}{2!}+\cdots. \end{aligned}$$

The previous series totally converges for $\xi \in \mathfrak{D}$ to a function analytical in $\mathfrak{D}$ (for details, see ([23], Chapter 14)); see also [9].

Lemma 3. *For* v, τ, *and* $\mathfrak{s}$ $(\mathfrak{s} \notin \mathbb{Z}_0^-)$, *which are real or complex parameters,*

$$\int_0^1 t^{\tau-1}(1-t)^{\mathfrak{s}-\tau-1}(1-\xi t)^{-v}dt = \frac{\Gamma(\tau)\Gamma(\mathfrak{s}-v)}{\Gamma(\mathfrak{s})}\,{}_2F_1(v,\tau;\mathfrak{s};\xi) \quad (\mathrm{Re}(\mathfrak{s})>\mathrm{Re}(\tau)>0); \tag{12}$$

$${}_2F_1(v,\tau;\mathfrak{s};\xi) = {}_2F_1(\tau,v;\mathfrak{s};\xi); \tag{13}$$

$${}_2F_1(v,\tau;\mathfrak{s};\xi) = (1-\xi)^{-v}\,{}_2F_1(v,\mathfrak{s}-\tau;\mathfrak{s};\frac{\xi}{\xi-1}); \tag{14}$$

$${}_2F_1(1,1;2;\frac{v\xi}{v\xi+1}) = \frac{(1+v\xi)\ln(1+v\xi)}{v\xi}. \tag{15}$$

3. Some Arguments and Subordinate Results

In this paper, we assume that $\varkappa, \mathfrak{s} \in \mathbb{R}\backslash\mathbb{Z}_0^-; v \in \mathbb{R}^*; \varrho \in \mathbb{R}, \xi \in \mathfrak{D}^*,\ -1 \le B < A \le 1$, and the powers are perceived as fundamental values.

Theorem 1. *For* $0 < \delta \le 1$, *let* $\hbar \in \Sigma_{p,n}$ *and* $\mathfrak{f} \in \Sigma_{p,n}$ *satisfy*

$$\left|\arg\left\{\left(\frac{\mathfrak{k}^\varrho_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)}{\mathfrak{k}^\varrho_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{h}(\xi)}\right)^\mu\left[1+\delta\left(\frac{\mathfrak{k}^{\varrho-1}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)}{\mathfrak{k}^\varrho_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)}-\frac{\mathfrak{k}^{\varrho-1}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{h}(\xi)}{\mathfrak{k}^\varrho_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{h}(\xi)}\right)\right]\right\}\right| < \frac{\pi}{2}(\gamma+\frac{2}{\pi}\arctan(\frac{\delta}{\mu\varkappa}\gamma)). \tag{16}$$

Then,

$$\left|\arg\left(\frac{\mathfrak{k}^\varrho_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)}{\mathfrak{k}^\varrho_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{h}(\xi)}\right)^\mu\right| < \frac{\pi}{2}\gamma \quad (\xi\in\mathfrak{D}). \tag{17}$$

Proof. Define a function

$$\mathfrak{p}(\xi) = \left(\frac{\mathfrak{k}^\varrho_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)}{\mathfrak{k}^\varrho_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{h}(\xi)}\right)^\mu, \quad (\mu\neq 0). \tag{18}$$

Then, $\mathfrak{p}(\xi) = 1 + c_1\xi + \ldots$, is analytic in $\mathfrak{D}$ and $\mathfrak{p}(0) = 1$ and $\mathfrak{p}'(0) \neq 0$. Differentiating (18) logarithmically with respect to ξ, we have

$$\frac{1}{\mu}\frac{\xi\mathfrak{p}'(\xi)}{\mathfrak{p}(\xi)} = \left[\frac{\xi(\mathfrak{k}^\varrho_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi))'}{\mathfrak{k}^\varrho_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)} - \frac{\xi(\mathfrak{k}^\varrho_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{h}(\xi))'}{\mathfrak{k}^\varrho_{\varkappa,p}(v,\mathfrak{s};\xi)\ \mathfrak{h}(\xi)}\right].$$

Using (7), we have

$$\mathfrak{p}(\xi)+\frac{\delta}{\mu\varkappa}\xi\mathfrak{p}'(\xi) = \left(\frac{\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)}{\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{h}(\xi)}\right)^{\mu}\left\{1+\delta\left(\frac{\mathfrak{k}^{\varrho-1}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)}{\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)}-\frac{\mathfrak{k}^{\varrho-1}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{h}(\xi)}{\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{h}(\xi)}\right)\right\}.$$

since

$$\left|\arg\left(\mathfrak{p}(\xi)+\frac{\delta}{\mu\varkappa}\xi\mathfrak{p}'(\xi)\right)\right| = \left|\arg\left\{\left(\frac{\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)}{\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{h}(\xi)}\right)^{\mu}\left[1+\delta\left(\frac{\mathfrak{k}^{\varrho-1}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)}{\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)}-\frac{\mathfrak{k}^{\varrho-1}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{h}(\xi)}{\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{h}(\xi)}\right)\right]\right\}\right|.$$

By (16) and Lemma 1, the proof is complete. □

If we set $\mu=1$ and $\hbar(\xi)=\xi^{-p}$ in Theorem 1, we obtain the following.

Corollary 1. *If* $\mathfrak{f}\in\Sigma_{p,n}$ *satisfies*

$$\begin{aligned}&\left|\arg\left[(1-\delta)\xi^{p}\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)+\delta\xi^{p}\mathfrak{k}^{\varrho-1}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)\right]\right|\\ <\ &\frac{\pi}{2}(\gamma+\frac{2}{\pi}\arctan(\frac{\delta}{\varkappa}\gamma)),\end{aligned}$$

then

$$\left|\arg\left(\xi^{p}\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)\right)\right|<\frac{\pi}{2}\gamma\quad(\xi\in\mathfrak{D}).$$

If $p=1$ in Corollary 1, we obtain the following.

Corollary 2. *If* $\mathfrak{f}\in\Sigma$ *satisfies*

$$\begin{aligned}&\left|\arg\left[(1-\delta)\xi\mathfrak{k}^{\varrho}_{\varkappa,1}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)+\delta\xi\mathfrak{k}^{\varrho-1}_{\varkappa,1}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)\right]\right|\\ <\ &\frac{\pi}{2}(\gamma+\frac{2}{\pi}\arctan(\frac{\delta}{\varkappa}\gamma)),\end{aligned}$$

then

$$\left|\arg\left(\xi\mathfrak{k}^{\varrho}_{\varkappa,1}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)\right)\right|<\frac{\pi}{2}\gamma\quad(\xi\in\mathfrak{D}).$$

If we set $\upsilon,\mathfrak{s}=\varkappa=\delta=1$, $\varrho=0$, and $\hbar(\xi)=\xi^{-p}$, in Theorem 1, we obtain the following.

Example 1. *If* $\mathfrak{f}\in\Sigma_{p,n}$ *satisfies*

$$\left|\arg\left([\xi^{p}\mathfrak{f}(\xi)]^{\mu}\left(\frac{\xi\mathfrak{f}'(\xi)}{\mathfrak{f}(\xi)}+(1+p)\right)\right)\right|<\frac{\pi}{2}(\gamma+\frac{2}{\pi}\arctan(\frac{\gamma}{\mu})).$$

then

$$\left|\arg[\xi^{p}\mathfrak{f}(\xi)]^{\mu}\right|<\frac{\pi}{2}\gamma\quad(\xi\in\mathfrak{D}).$$

If we set $\mu=1$ and $\mathfrak{f}(\xi)=\xi^{-p}$ in Theorem 1, we obtain the following.

Corollary 3. *Let* $\frac{1}{\xi^p \mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{h}(\xi)} \neq 0$, $\mathfrak{h} \in \Sigma_{p,n}$, *and* $0 < \delta \leq 1$. *Suppose that*

$$\begin{aligned} & \left| \arg \left[(1+\delta) \frac{1}{\xi^p \mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{h}(\xi)} - \right. \right. \\ & \left. \left. \delta \frac{\mathfrak{k}^{\varrho-1}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{h}(\xi)}{\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{h}(\xi)} \left(\frac{1}{\xi^p \mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{h}(\xi)} \right) \right] \right| \\ < & \frac{\pi}{2}(\gamma + \frac{2}{\pi} \arctan(\frac{\delta}{\varkappa}\gamma)). \end{aligned}$$

Then,

$$\left| \arg \left(\xi^p \mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{h}(\xi) \right) \right| < \frac{\pi}{2}\gamma \quad (\xi \in \mathfrak{D}).$$

If we set $\upsilon, \mathfrak{s} = \mu = p = 1$, $\varrho = 0$, and $\hbar(\xi) = \xi^{-1}$ in Theorem 1, we obtain the following.

Example 2. *If* $\mathfrak{f} \in \Sigma$ *satisfies*

$$\left| \arg \left((1-\delta)\xi\mathfrak{f}(\xi) + \delta \left(\xi^2 \mathfrak{f}(\xi) \right)' \right) \right| < \frac{\pi}{2}(\gamma + \frac{2}{\pi} \arctan(\delta\gamma)).$$

then

$$|\arg \xi\mathfrak{f}(\xi)| < \frac{\pi}{2}\gamma \quad (\xi \in \mathfrak{D}).$$

Theorem 2. *Let* $0 < \delta \leq 1$. *Assume that* $\mathfrak{f} \in \Sigma_{p,n}$ *satisfies*

$$\left| \arg \left(\xi^p \mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};t)\mathfrak{f}(t) \right)^{\mu} \right| < \frac{\pi}{2} \left(\gamma + \frac{2}{\pi} \arctan \left[\frac{\delta}{\mu\varkappa}\gamma \right] \right) \quad (\xi \in \mathfrak{D}). \tag{19}$$

Then,

$$\left| \arg \left(\frac{\mu\varkappa}{\delta} \xi^{-\frac{\mu\varkappa}{\delta}} \int_0^{\xi} t^{\frac{\mu\varkappa-\delta}{\delta}} \left(t^p \mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};t)\mathfrak{f}(t) \right)^{\mu} dt \right) \right| < \frac{\pi}{2}\gamma. \tag{20}$$

Proof. Let

$$\mathfrak{p}(\xi) = \frac{\mu\varkappa}{\delta} \xi^{-\frac{\mu\varkappa}{\delta}} \int_0^{\xi} t^{\frac{\mu\varkappa-\delta}{\delta}} \left(t^p \mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};t)\mathfrak{f}(t) \right)^{\mu} dt \quad (\xi \in \mathfrak{D}). \tag{21}$$

Then, $\mathfrak{p}(\xi) = 1 + c_1\xi + \ldots$, is analytic in $\mathfrak{D}$ and $\mathfrak{p}(0) = 1$ and $\mathfrak{p}'(0) \neq 0$. Differentiating (21) with respect to ξ, we get

$$\mathfrak{p}(\xi) + \frac{\delta}{\mu\varkappa}\xi\mathfrak{p}'(\xi) = \left(\xi^p \mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi) \right)^{\mu}.$$

From Lemma 1, the proof is complete. □

If we set $\upsilon, \mathfrak{s} = 1$, $\varrho = 0$, and $p = \mu = \delta = 1$ in Theorem 2, we obtain the following.

Corollary 4. *If* $\mathfrak{f} \in \Sigma$ *satisfies*

$$|\arg(\xi\mathfrak{f}(\xi))| < \frac{\pi}{2}(\gamma + \frac{2}{\pi} \arctan(\frac{\gamma}{\varkappa})).$$

Then

$$\left| \arg \left(\frac{\varkappa}{\xi^{\chi}} \int_0^{\xi} t^{\varkappa} \mathfrak{f}(t) dt \right) \right| < \frac{\pi}{2}\gamma \quad (\xi \in \mathfrak{D}).$$

If we set $\varkappa = 1$ in Corollary 4, we obtain the following.

Corollary 5. *If* $\mathfrak{f} \in \Sigma$ *satisfies*

$$|\arg(\xi \mathfrak{f}(\xi))| < \frac{\pi}{2}(\gamma + \frac{2}{\pi}\arctan\gamma).$$

then

$$\left|\arg\left(\frac{1}{\xi}\int_0^{\xi} t\mathfrak{f}(t)dt\right)\right| < \frac{\pi}{2}\gamma \quad (\xi \in \mathfrak{D}).$$

Remark 1. *If we set* $\varkappa = 1$ *and* $\varrho = m$ *in Theorem 1 and 2, we obtain the results investigated by El-Ashwah ([24] at* $\lambda = 1$*).*

Theorem 3. *Let* $\mathfrak{f} \in \Sigma_{p,n}$ *and* $0 \leq \rho < 1$*. Suppose that*

$$\sum_{\kappa=n}^{\infty} d_\kappa |a_\kappa| \leq 1, \tag{22}$$

where

$$d_\kappa = \frac{1-B}{A-B}.\frac{\varkappa^{\varrho-1}[\varkappa + (1-\rho)(\kappa+p)]}{(\kappa+\varkappa+p)^{\varrho}}\frac{(\upsilon)_{\kappa+p}}{(\mathfrak{s})_{\kappa+p}}. \tag{23}$$

(i) If $-1 \leq B \leq 0$*, then for* $\xi \in \mathfrak{D}$*, we obtain*

$$(1-\rho)\xi^p \mathfrak{k}^{\varrho-1}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi) + \rho\xi^p \mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi) \prec \frac{1+A\xi}{1+B\xi}, \tag{24}$$

(ii) If $-1 \leq B \leq 0$*, and* $\sigma \geq 1$*, then for* $\xi \in \mathfrak{D}$*, we obtain*

$$\mathrm{Re}\left\{\left(\xi^p \mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)\right)^{\frac{1}{\sigma}}\right\} > \left\{\frac{\varkappa}{(1-\rho)}\int_0^1 t^{\frac{\varkappa}{(1-\rho)}-1}\left(\frac{1-At}{1-Bt}\right)dt\right\}^{\frac{1}{\sigma}}. \tag{25}$$

The result is sharp.

Proof. (i) Let

$$\mathfrak{g}(\xi) = (1-\rho)\xi^p \mathfrak{k}^{\varrho-1}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi) + \rho\xi^p \mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi) \tag{26}$$

Then,

$$\mathfrak{g}(\xi) = 1 + \sum_{\kappa=n}^{\infty}\frac{\varkappa^{\varrho-1}[\varkappa+(1-\rho)(\kappa+p)]}{(\kappa+\varkappa+p)^{\varrho}}\frac{(\upsilon)_{\kappa+p}}{(\mathfrak{s})_{\kappa+p}}a_\kappa\xi^{\kappa+p}. \tag{27}$$

Using (22) for $-1 \leq B \leq 0$, and $\xi \in \mathfrak{D}$, we have

$$\begin{aligned}\left|\frac{\mathfrak{g}(\xi)-1}{A-B\mathfrak{g}(\xi)}\right| &= \left|\frac{\sum_{\kappa=n}^{\infty}\frac{\varkappa^{\varrho-1}[\varkappa+(1-\rho)(\kappa+p)]}{(\kappa+\varkappa+p)^{\varrho}}\frac{(\upsilon)_{\kappa+p}}{(\mathfrak{s})_{\kappa+p}}a_\kappa\xi^{\kappa+p}}{A-B-B\sum_{\kappa=n}^{\infty}\frac{\varkappa^{\varrho-1}[\varkappa+(1-\rho)(\kappa+p)]}{(\kappa+\varkappa+p)^{\varrho}}\frac{(\upsilon)_{\kappa+p}}{(\mathfrak{s})_{\kappa+p}}a_\kappa\xi^{\kappa+p}}\right| \\ &\leq \frac{\sum_{\kappa=n}^{\infty} d_\kappa|a_\kappa|}{1-B+B\sum_{\kappa=n}^{\infty} d_\kappa|a_\kappa|} \\ &< 1,\end{aligned}$$

which proves (i) of Theorem 3.

(ii) Set

$$\psi(\xi) = \xi^p \mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi). \tag{28}$$

Then, the function $\psi(\xi)$ of the form (28) is analytic in $\mathfrak{D}$. Differentiating (28) with respect to ξ and using (7), we obtain

$$\begin{aligned}\xi\psi'(\xi) &= p\xi^p\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)+\xi^{p+1}\left(\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)\right)'\\ \frac{\xi\psi'(\xi)}{\varkappa} &= \xi^p\mathfrak{k}^{\varrho-1}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)-\xi^p\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)\\ \psi(\xi)+\frac{(1-\rho)}{\varkappa}\xi\psi'(\xi) &= (1-\rho)\xi^p\mathfrak{k}^{\varrho-1}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)+\rho\xi^p\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)\\ &\prec \frac{1+A\xi}{1+B\xi}.\end{aligned} \tag{29}$$

Application of Lemma 2 gives

$$\psi(\xi)\prec\frac{\varkappa}{(1-\rho)}\xi^{-\frac{\varkappa}{(1-\rho)}}\int_0^{\xi}t^{\frac{\varkappa}{(1-\rho)}-1}\frac{1+At}{1+Bt}dt$$

which is equivalent to

$$\xi^p\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)=\frac{\varkappa}{(1-\rho)}\int_0^1 u^{\frac{\varkappa}{(1-\rho)}-1}\left(\frac{1+Auw(\xi)}{1+Buw(\xi)}\right)du, \tag{30}$$

where $w(\xi)$ is analytic in $\mathfrak{D}$ with $w(0)=0$ and $|w(\xi)|<1$ $(\xi\in\mathfrak{D})$.

It follows from (30) that

$$\mathrm{Re}\left\{\xi^p\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)\right\}>\frac{\varkappa}{(1-\rho)}\int_0^1 u^{\frac{\varkappa}{(1-\rho)}-1}\left(\frac{1-Au}{1-Bu}\right)du>0.$$

Therefore, with the elementary inequality $\mathrm{Re}(w^{\frac{1}{\sigma}})\geq(\mathrm{Re}(w))^{\frac{1}{\sigma}}$ for $\mathrm{Re}(w)>0$ and $\sigma\in\mathbb{N}$, inequality (25) follows immediately.

To show the sharpness of (25), we take $\mathfrak{f}\in\Sigma_{p,n}$ defined by

$$\xi^p\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)=\frac{\varkappa}{(1-\rho)}\int_0^1 u^{\frac{\varkappa}{(1-\rho)}-1}\left(\frac{1+Au\xi}{1+Bu\xi}\right)du.$$

For this function, we find that

$$(1-\rho)\xi^p\mathfrak{k}^{\varrho-1}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)+\rho\xi^p\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)=\frac{1+A\xi}{1+B\xi},$$

and

$$\xi^p\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)\to\frac{\varkappa}{(1-\rho)}\int_0^1 u^{\frac{\varkappa}{(1-\rho)}-1}\left(\frac{1-Au}{1-Bu}\right)du\quad as\quad \xi\to e^{i\pi}.$$

Hence, the proof of Theorem 3 is complete. □

If we set $\sigma=1$ in Theorem 3, we obtain the following

Corollary 6. *Let* $\mathfrak{f}\in\Sigma_{p,n}$ *and* $0\leq\rho<1$. *Suppose that*

$$\sum_{\kappa=n}^{\infty}d_\kappa|a_\kappa|\leq 1,$$

where

$$d_\kappa=\frac{1-B}{A-B}.\frac{\varkappa^{\varrho-1}[\varkappa+(1-\rho)(\kappa+p)]}{(\kappa+\varkappa+p)^{\varrho}}\frac{(v)_{\kappa+p}}{(\mathfrak{s})_{\kappa+p}}.$$

(i) If $-1\leq B\leq 0$, *then for* $\xi\in\mathfrak{D}$, *we obtain*

$$(1-\rho)\xi^p\mathfrak{k}^{\varrho-1}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)+\rho\xi^p\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)\prec\frac{1+A\xi}{1+B\xi}, \tag{31}$$

(ii) *If* $-1 \leq B \leq 0$, *then for* $\xi \in \mathfrak{D}$, *we obtain*

$$\mathrm{Re}\left\{\xi^{p}\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)\right\} > \frac{\varkappa}{(1-\rho)}\int_{0}^{1} t^{\frac{\varkappa}{(1-\rho)}-1}\left(\frac{1-At}{1-Bt}\right)dt. \tag{32}$$

The result is sharp.

If we set $A = 1$ and $B = -1$ in Corollary 6, we obtain the following.

Corollary 7. *Let* $\mathfrak{f} \in \Sigma_{p,n}$ *and* $0 \leq \rho < 1$. *Suppose that*

$$\sum_{\kappa=n}^{\infty} d_{\kappa}|a_{\kappa}| \leq 1,$$

where

$$d_{\kappa} = \frac{\varkappa^{\varrho-1}[\varkappa + (1-\rho)(\kappa+p)]}{(\kappa+\varkappa+p)^{\varrho}}\frac{(v)_{\kappa+p}}{(\mathfrak{s})_{\kappa+p}}$$

Then, for $\xi \in \mathfrak{D}$, *we obtain*

$$(1-\rho)\xi^{p}\mathfrak{k}^{\varrho-1}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi) + \rho\xi^{p}\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi) \prec \frac{1+\xi}{1-\xi}$$

and

$$\mathrm{Re}\left\{\xi^{p}\mathfrak{k}^{\varrho}_{\varkappa,p}(v,\mathfrak{s};\xi)\mathfrak{f}(\xi)\right\} > \frac{\varkappa}{(1-\rho)}\int_{0}^{1} t^{\frac{\varkappa}{(1-\rho)}-1}\left(\frac{1-t}{1+t}\right)dt.$$

The result is sharp.

Example 3. *If we set* $\varrho = 1, \rho = 0$, *and* $\varkappa = 1$ *in Corollary 7, we obtain*

$$\begin{aligned}\mathrm{Re}\{\xi^{p}\mathfrak{f}(\xi)\} &> -1 +_{2} F_{1}(1,1,2;\frac{1}{2})\\ &> -1 + 2\ln 2 \approx 0.3863 \quad \textit{for } \mathfrak{f} \in \Sigma_{p,n}.\end{aligned}$$

Remark 2. *Similar results can be obtained if we used inequality (8).*

4. Partial Sums for Subclass $\Sigma_{p,n}$

The real component of the quotients between the normalized star-like or convex functions and their sequences of partial sums has recently been found to have sharp lower bounds by Silverman [25]. Additionally, Li and Owa [26] determined the radius at which starlikeness is implied by the partial sums of the well-known Libera integral operator [27] and the normalized univalent functions in $\mathfrak{D}$. Additionally, the works of Brickman et al. [28], Sheil-Small [29], Silvia [30], Singh and Singh [31], Yang and Owa [32], and others are cited for a number of other intriguing advancements pertaining to the partial sums of analytic univalent functions.

Theorem 4. *Let* $\mathfrak{f} \in \Sigma_{p,n}$ *be given by (1) and*

$$d_{\kappa} \geq \begin{cases} 1, & \kappa = n, n+1, \ldots.\varsigma \\ d_{\varsigma+1,} & \kappa = \varsigma+1, \varsigma+2, \ldots, \end{cases}$$

where d_{κ} *is given by (23) with condition (22), and let it define partial sums* $\vartheta_{1}(\xi)$ *and* $\vartheta_{\varsigma}(\xi)$ *as follows:*

$$\vartheta_{1}(\xi) = \xi^{-p}$$

and

$$\vartheta_\varsigma(\xi) = \frac{1}{\xi^p} + \sum_{\kappa=n}^{\varsigma} |a_\kappa| \xi^\kappa \qquad (\varsigma \in \mathbb{N},\ \varsigma > n), \tag{33}$$

Then, we have (i)

$$\mathrm{Re}\left\{ \frac{\mathfrak{f}(\xi)}{\vartheta_\varsigma(\xi)} \right\} > 1 - \frac{1}{d_{\varsigma+1,}} \quad (\xi \in \mathfrak{D}; \varsigma \in \mathbb{N},\ \varsigma > n), \tag{34}$$

and (ii)

$$\mathrm{Re}\left\{ \frac{\vartheta_\varsigma(\xi)}{\mathfrak{f}(\xi)} \right\} > 1 - \frac{1}{1 + d_{\varsigma+1,}} \quad (\xi \in \mathfrak{D}; \varsigma \in \mathbb{N},\ \varsigma > n). \tag{35}$$

The estimates in (34) and (35) are sharp for $\varsigma \in \mathbb{N},\ \varsigma > n$.

Proof. (i) Under the hypothesis of Theorem 4, we can see from (22) that

$$\sum_{\kappa=n}^{\varsigma} |a_\kappa| + d_{\varsigma+1} \sum_{\kappa=\varsigma+1}^{\infty} |a_\kappa| \leq \sum_{\kappa=n}^{\infty} d_\kappa |a_\kappa| \leq 1. \tag{36}$$

By setting

$$\begin{aligned} \mathfrak{g}_1(\xi) &= d_{\varsigma+1} \left\{ \frac{\mathfrak{f}(\xi)}{\vartheta_\varsigma(\xi)} - (1 - \frac{1}{d_{\varsigma+1}}) \right\} \\ &= 1 + \frac{d_{\varsigma+1} \sum_{\kappa=\varsigma+1}^{\infty} a_\kappa \xi^{\kappa+p}}{1 + \sum_{\kappa=n}^{\varsigma} a_\kappa \xi^{\kappa+p}}, \end{aligned} \tag{37}$$

and applying (36), we find that

$$\left| \frac{\mathfrak{g}_1(\xi) - 1}{\mathfrak{g}_1(\xi) + 1} \right| \leq \frac{d_{\varsigma+1} \sum_{\kappa=\varsigma+1}^{\infty} |a_\kappa|}{2 - 2\sum_{\kappa=n}^{\varsigma} |a_\kappa| - d_{\varsigma+1} \sum_{\kappa=\varsigma+1}^{\infty} |a_\kappa|} \leq 1, \tag{38}$$

which readily yields the assertion (34) of Theorem 4. If we take

$$\mathfrak{f}(\xi) = \xi^{-p} + \frac{\xi^{\varsigma+1}}{d_{\varsigma+1}}, \tag{39}$$

with $\xi = re^{\frac{i\pi}{\varsigma+p+1}}$ and let $r \to 1^-$, we obtain

$$\frac{\mathfrak{f}(\xi)}{\vartheta_\varsigma(\xi)} = 1 + \frac{\xi^{\varsigma+p+1}}{d_{\varsigma+1}} \to 1 - \frac{1}{d_{\varsigma+1}},$$

which shows that the bound in (34) is the most possible for each $\varsigma \in \mathbb{N},\ \varsigma > n$.

(ii) Similarly, if we set

$$\begin{aligned} \mathfrak{g}_2(\xi) &= (1 + d_{\varsigma+1}) \left\{ \frac{\vartheta_\varsigma(\xi)}{\mathfrak{f}(\xi)} - \frac{d_{\varsigma+1}}{1 + d_{\varsigma+1}} \right\} \\ &= 1 - \frac{(1 + d_{\varsigma+1}) \sum_{\kappa=\varsigma+1}^{\infty} |a_\kappa| \xi^{\kappa+p}}{1 + \sum_{\kappa=n}^{\varsigma} |a_\kappa| \xi^{\kappa+p}}, \end{aligned}$$

and make use of (36), we can deduce that

$$\left| \frac{\mathfrak{g}_2(\xi) - 1}{\mathfrak{g}_2(\xi) + 1} \right| \leq \frac{(1 + d_{\varsigma+1}) \sum_{\kappa=\varsigma+1}^{\infty} |a_\kappa|}{2 - 2\sum_{\kappa=n}^{\varsigma} |a_\kappa| - (1 - d_{\varsigma+1}) \sum_{\kappa=\varsigma+1}^{\infty} |a_\kappa|} \leq 1, \tag{40}$$

which yields inequality (35) of Theorem 4. The bound in (35) is sharp for each $\varsigma \in \mathbb{N}$, $\varsigma > n$, with the extremal function $\mathfrak{f}(\xi)$ given by (39). The proof of Theorem 4 is now complete. □

Theorem 5. *Let $\mathfrak{f} \in \Sigma_{p,n}$ be given by (1) and*

$$d_\kappa \geq \begin{cases} \frac{\kappa}{p}, & \kappa = n, n+1, \ldots . \varsigma \\ \frac{d_{\varsigma+1}}{\varsigma+1}(\kappa/p) & \kappa = \varsigma+1, \varsigma+2, \ldots , \end{cases}$$

where d_κ is given by (23) and satisfies condition (22). Then, we have (i)

$$\mathrm{Re}\left\{\frac{\mathfrak{f}'(\xi)}{\vartheta_\varsigma'(\xi)}\right\} > 1 - \frac{\varsigma+1}{d_{\varsigma+1,}} \quad (\xi \in \mathfrak{D}; \varsigma \in \mathbb{N},\ \varsigma > n), \tag{41}$$

and (ii)

$$\mathrm{Re}\left\{\frac{\vartheta_\varsigma'(\xi)}{\mathfrak{f}'(\xi)}\right\} > 1 - \frac{\varsigma+1}{\varsigma+1+d_{\varsigma+1,}} \quad (\xi \in \mathfrak{D}; \varsigma \in \mathbb{N},\ \varsigma > n). \tag{42}$$

The estimates in (41) and (42) are sharp for $\varsigma \in \mathbb{N}$, $\varsigma > n$. The results are sharp with the function $f(\xi)$ given by (39).

Proof. By setting

$$\begin{aligned} \mathfrak{g}(\xi) &= \frac{d_{\varsigma+1}}{\varsigma+1}\left\{\frac{\mathfrak{f}'(\xi)}{\vartheta_\varsigma'(\xi)} - (1 - \frac{\varsigma+1}{d_{\varsigma+1}})\right\} \\ &= \frac{1 + \frac{d_{\varsigma+1}}{\varsigma+1}\sum_{\kappa=\varsigma+1}^{\infty}\frac{\kappa}{p}a_\kappa\xi^{\kappa+p} + \sum_{\kappa=n}^{\varsigma}\frac{\kappa}{p}a_\kappa\xi^{\kappa+p}}{1 + \sum_{\kappa=n}^{\varsigma}\frac{\kappa}{p}a_\kappa\xi^{\kappa+p}}. \end{aligned} \tag{43}$$

we have

$$\left|\frac{\mathfrak{g}(\xi)-1}{\mathfrak{g}(\xi)+1}\right| \leq \frac{\frac{d_{\varsigma+1}}{\varsigma+1}\sum_{\kappa=\varsigma+1}^{\infty}\frac{\kappa}{p}|a_\kappa|}{2 - 2\sum_{\kappa=n}^{\varsigma}\frac{\kappa}{p}|a_\kappa| - \frac{d_{\varsigma+1}}{\varsigma+1}\sum_{\kappa=\varsigma+1}^{\infty}\frac{\kappa}{p}|a_\kappa|}. \tag{44}$$

Now,

$$\left|\frac{\mathfrak{g}(\xi)-1}{\mathfrak{g}(\xi)+1}\right| \leq 1,$$

if

$$\sum_{\kappa=n}^{\varsigma}\frac{\kappa}{p}|a_\kappa| + \frac{d_{\varsigma+1}}{\varsigma+1}\sum_{\kappa=\varsigma+1}^{\infty}\frac{\kappa}{p}|a_\kappa| \leq 1, \tag{45}$$

since the left-hand side of (41) is bounded above by $\sum_{\kappa=n}^{\infty} d_\kappa|a_\kappa|$ if

$$\sum_{\kappa=n}^{\varsigma}(d_\kappa - \frac{\kappa}{p})|a_\kappa| + \sum_{\kappa=\varsigma+1}^{\infty}\left(d_\kappa - \frac{d_{\varsigma+1}}{\varsigma+1}\frac{\kappa}{p}\right)|a_\kappa| \geq 0, \tag{46}$$

and the proof of (41) is then complete.

To prove result (42), we define the function $\mathfrak{h}(\xi)$ by

$$\begin{aligned} \mathfrak{h}(\xi) &= \left(\frac{\varsigma+1+d_{\varsigma+1}}{\varsigma+1}\right)\left\{\frac{\vartheta_\varsigma'(\xi)}{\mathfrak{f}'(\xi)} - \frac{d_{\varsigma+1}}{\varsigma+1+d_{\varsigma+1}}\right\} \\ &= 1 - \frac{(1 + \frac{d_{\varsigma+1}}{\varsigma+1})\sum_{\kappa=\varsigma+1}^{\infty}\frac{\kappa}{p}a_\kappa\xi^{\kappa+p}}{1 + \sum_{\kappa=n}^{\varsigma}\frac{\kappa}{p}a_\kappa\xi^{\kappa+p}}, \end{aligned}$$

and make use of (46). We can then deduce that

$$\left|\frac{\mathfrak{h}(\xi)-1}{\mathfrak{h}(\xi)+1}\right| \leq \frac{(1+\frac{d_{\varsigma+1}}{p+1})\sum_{\kappa=\varsigma+1}^{\infty}\frac{\kappa}{p}|a_\kappa|}{2-2\sum_{\kappa=n}^{\varsigma}\frac{\kappa}{p}|a_\kappa|-(1+\frac{d_{\varsigma+1}}{\varsigma+1})\sum_{\kappa=\varsigma+1}^{\infty}\frac{\kappa}{p}|a_\kappa|} \leq 1,$$

which leads us immediately to assertion (42) of Theorem 5. □

Remark 3. *Through the use of the specialization of parameters ϱ and $\varkappa$, we obtain different new results corresponding to the operators noted in the Introduction.*

5. Concluding Remarks and Observations

In this study, we investigate the argument properties of multivalent meromorphic functions in a punctured open unit disc $\mathfrak{D}^*$, defined by the linear operator $\mathfrak{k}^{\varrho}_{\varkappa,p}(\upsilon,\mathfrak{s};\xi)\mathfrak{f}(\xi)$. Also, we deduce some corollaries and particular cases. Moreover, we draw readers' attention to the possibility of further research based on other meromorphic function classes, such as those examined in several recent works.

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