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Green's Functions for Neumann Boundary Conditions

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Abstract

Green's functions for Neumann boundary conditions have been considered in Math, Physics, and Electromagnetism textbooks, but often with mistakes of omission and commission. Special constraints and other properties required for Neumann boundary conditions have generally not been noticed or treated correctly. In this paper, we derive appropriate Neumann Green's functions with these properties properly incorporated.

Keywords: Green's function; Neumann boundary conditions; electromagnetism

MSC: 35F15

1. Introduction

The Green's function method for solving Sturm–Liouville problems of the following form:

$$\mathcal{L}_{\mathbf{r}} f(\mathbf{r}) = \nabla \cdot [p(\mathbf{r}) \nabla f(\mathbf{r})] + q(\mathbf{r}) f(\mathbf{r}) = \rho(\mathbf{r}) \tag{1}$$

is described in many textbooks, for instance, references [1–8]. In this paper, we treat and contrast Green's functions for Dirichlet and Neumann boundary conditions. Textbooks generally discuss the Dirichlet case in detail, but do much less with the Green's function for the Neumann boundary condition, and what is said about the Neumann case often has mistakes of omission and commission.

The motivation for using a Green's function, $G(\mathbf{r}, \mathbf{r}')$, is that it satisfies homogeneous boundary conditions that makes it easier to solve for than the original problem, which has inhomogeneous boundary conditions. The application of Green's second theorem (for simplicity, we have taken $q(\mathbf{r}) = 0$. Our general conclusions also apply to the case with $q(\mathbf{r})$, but some of the equations would be more complicated)

$$\int [f(\mathbf{r}') \mathcal{L}_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') \mathcal{L}_{\mathbf{r}'} f(\mathbf{r}')] d^3 r' = \int \mathbf{dS}' \cdot [f(\mathbf{r}') p(\mathbf{r}') \nabla' G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') p(\mathbf{r}') \nabla'(\mathbf{r}')] \tag{2}$$

provides the solution to the original problem if $G(\mathbf{r}, \mathbf{r}')$ satisfies

$$\mathcal{L}_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \tag{3}$$

In this case, implementing Equations (1) and (3) leads to

$$f(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}')] d^3 r' + \int \mathbf{dS}' \cdot [f(\mathbf{r}') p(\mathbf{r}') \nabla' G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') p(\mathbf{r}') \nabla' f(\mathbf{r}')]. \tag{4}$$

The two simplest boundary conditions for which the Green's function method is applicable are the Dirichlet boundary condition for which the solution $f(\mathbf{r})$ is given on all bounding surfaces, and the Neumann boundary condition for which its normal derivative $\hat{\mathbf{n}} \cdot \nabla f(\mathbf{r})$ is given.

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For the Dirichlet boundary condition, the Green's function satisfies the homogeneous boundary condition

$$G_D(\mathbf{r}, \mathbf{r}') = 0 \tag{5}$$

for $\mathbf{r}'$ on all bounding surfaces. This reduces Equation (4) to

$$f_D(\mathbf{r}) = \int G_D(\mathbf{r}, \mathbf{r}')p(\mathbf{r}')\rho(\mathbf{r}')]]d^3r' + \int \mathbf{dS}' \cdot [f(\mathbf{r}')p(\mathbf{r}')\nabla' G_D(\mathbf{r}, \mathbf{r}')], \tag{6}$$

which is the solution to the Dirichlet problem once $G_D(\mathbf{r}, \mathbf{r}')$ is known.

The Neumann Green's function satisfies the boundary condition,

$$\hat{\mathbf{n}} \cdot \nabla = 0, \tag{7}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the surface of integration. This reduces Equation (4) to

$$f_N(\mathbf{r}) = \int G_D(\mathbf{r}, \mathbf{r}')p(\mathbf{r}')\rho(\mathbf{r}')]]d^3r' - \int \mathbf{dS}' \cdot [G_N(\mathbf{r}, \mathbf{r}')p(\mathbf{r}')\nabla'(\mathbf{r}')], \tag{8}$$

which is the solution to the Neumann problem once $G_N(\mathbf{r}, \mathbf{r}')$ is known.

The Neumann Green's function differs from the Dirichlet in that the Neumann boundary condition for the solution $f(\mathbf{r})$ must satisfy the constraint

$$\int \mathbf{dS} \cdot p(\mathbf{r})\nabla f(\mathbf{r}) = \int \rho d^3r, \tag{9}$$

which follows from applying the divergence theorem to Equation (1). This constraint applies to Neumann boundary conditions where $\hat{\mathbf{n}} \cdot \boldsymbol{\nabla} f(\mathbf{r})$ is specified. It does not apply to a Dirichlet boundary condition where $f(\mathbf{r})$ is specified on the surface, and the integral in Equation (9) automatically satisfies the constraint. Most texts do not mention this important constraint on the Neumann boundary condition.

There are cases where the boundary condition is Neumann on some surfaces and Dirichlet on others. In those cases, the normal derivative of f on the Dirichlet surfaces will adjust to satisfy the constraint surface integral. But for pure Neumann boundary conditions, the constraint must be satisfied or no solution exists.

We review the Dirichlet Green's function in the next section, the one-dimensional Neumann Green's function in Section 3, the three dimensional Neumann Green's function in Section 4, and a comparison of the three dimensional Dirichlet and Neumann Green's functions in Section 5. We summarize our conclusions in Section 6.

2. One-Dimensional Dirichlet Green's Function

We first review how a Dirac delta function arises when a function $f(x)$, defined in the finite range $0 \leq x \leq L$, is expanded in orthonormal Dirichlet eigenfunctions $u_n(x)$ of a Sturm–Liouville operator $\mathcal{L}_x$. The eigenfunctions satisfy the differential equation

$$\mathcal{L}_x u_n(x) = \frac{d}{dx}\left[p(x)\frac{du_n}{dx}\right] = \lambda_n u_n(x). \tag{10}$$

with homogeneous Dirichlet boundary conditions

$$u_n(0) = 0, \quad u_n(L) = 0. \tag{11}$$

A function $f(x)$ can be expanded in the Dirichlet eigenfunctions, $u_n(x)$, as

$$f(x) = \sum_{n=1}^{\infty} b_n u_n(x), \tag{12}$$

with the expansion coefficients b_n given by

$$b_n = \int_0^L u_n^*(x) f(x) dx. \tag{13}$$

If Equation (13) for the expansion coefficients is substituted into Equation (12) for $f(x)$, and the sum executed before the integral, we get

$$f(x) = \int_0^L dx' \sum_{n=1}^{\infty} u_n^*(x') u_n(x) f(x'). \tag{14}$$

From the definition of the Dirac delta function by its sifting property,

$$f(x) = \int_0^L dx' \delta(x - x') f(x') \tag{15}$$

for x and x' in the range $[0, L]$, we see that the delta function can be represented by a sum over Dirichlet eigenfunctions as

$$\delta(x - x') = \sum_{n=1}^{\infty} u_n^*(x') u_n(x). \tag{16}$$

For example, for the simple case

$$\mathcal{L}_x u_n = u_n'' = \lambda_n u_n, \quad u_n = \sqrt{\frac{2}{L}} \sin(n\pi x/L), \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2, \tag{17}$$

the delta function is represented by

$$\delta(x - x') = \frac{2}{L} \sum_{n=1}^{\infty} \sin(n\pi x'/L) \sin(n\pi x/L) \tag{18}$$

A Dirichlet Green's function that satisfies the differential equation

$$\mathcal{L}_{x'} G_D(x, x') = \delta(x - x'), \tag{19}$$

and satisfies the homogeneous Dirichlet boundary conditions in the variable x' can be formed from the Dirichlet eigenfunctions as

$$G_D(x, x') = \sum_{n=1}^{\infty} \frac{u_n^*(x') u_n(x)}{\lambda_n} = \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \sin(n\pi x'/L) \sin(n\pi x/L)/n^2. \tag{20}$$

Acting on this Green's function with the Sturm–Liouville operator $\mathcal{L}_{x'}$ removes the denominator in the sum, leaving the delta function of Equation (19). This shows that this is the appropriate Dirichlet Green's function.

3. One-Dimensional Neumann Green's Function

The above straightforward derivation for homogeneous Dirichlet boundary conditions is given in most texts, but a corresponding derivation for homogeneous Neumann boundary

conditions is generally absent. Neumann eigenfunctions, $v_n(x)$, satisfy the same differential Equation (10) as the Dirichlet eigenfunctions, but have the boundary conditions

$$v_n'(0) = 0, \quad v_n'(L) = 0. \tag{21}$$

The expansion of the function, $f(x)$, in Neumann eigenfunctions has a constant term corresponding to $n = 0$, so the expansion is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n v_n(x), \tag{22}$$

The Neumann expansion coefficients a_n are given by the integrals

$$a_n = \int_0^L v_n^*(x) f(x) dx, \quad n \geq 1, \tag{23}$$

and

$$a_0 = \frac{1}{L} \int_0^L f(x) dx = < f >, \tag{24}$$

where $< f >$ represents the average value of the function $f(x)$ over the interval $[0, L]$.

Now, putting Equations (23) and (24) into the expansion Equation (22) results in

$$f(x) = \int_0^L dx' \left[\frac{1}{L} + \sum_{n=1}^{\infty} u_n^*(x') u_n(x) \right] f(x'), \tag{25}$$

so the representation of the delta function in terms of Neumann eigenfunctions is

$$\delta(x - x') = \frac{1}{L} + \sum_{n=1}^{\infty} v_n^*(x') v_n(x). \tag{26}$$

For the simple case of $\mathcal{L}_x v_n = v_n'' = \lambda_n$ with the eigenfunctions satisfying the homogeneous Neumann boundary conditions of Equation (21) the delta function is represented by

$$\delta(x - x') = \frac{1}{L} + \frac{2}{L} \sum_{n=1}^{\infty} \cos(n\pi x'/L) \cos(n\pi x/L). \tag{27}$$

A Neumann's Green function can be formed using Neumann eigenfunctions of the operator $\mathcal{L}_x$ as the sum

$$G_N(x, x') = \sum_{n=1}^{\infty} \frac{v_n^*(x') v_n(x)}{\lambda_n} = \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \cos(n\pi x'/L) \cos(n\pi x/L)/n^2. \tag{28}$$

This Green's function satisfies the homogeneous Neumann boundary conditions

$$\partial_{x'} G_N(x, x')|_{(x'=0)} = 0, \quad \partial_{x'} G_N(x, x')|_{(x'=L)} = 0. \tag{29}$$

However, because of the constant term $1/L$ in Equation (27), the operation on $G_N(x, x')$ by $\mathcal{L}_{x'}$ is

$$\mathcal{L}_{x'} G_N(x, x') = \delta(x - x') - 1/L, \tag{30}$$

so the Neumann Green's function satisfies a different differential equation than the Dirichlet Green's function. This difference is not generally recognized in textbooks.

We now use the Green's function $G_N(x,x')$ to find the solution of the differential equation

$$\mathcal{L}_x f(x) = \frac{d}{dx}\left[p(x)\frac{df}{dx}\right] = \rho(x), \tag{31}$$

with the inhomogeneous Neumann boundary conditions

$$f'(0) = f'_0, \quad f'(L) = f'_L. \tag{32}$$

The boundary values must satisfy the constraint

$$p(L)f'_L - p(0)f'_0 = \int_0^L \rho(x)dx, \tag{33}$$

which follows by using Equations (1) and (31) to integrate Equation (31) once.

Green's theorem in one dimension (or integration by parts) for this differential equation leads to

$$\int_0^L [f(x')\mathcal{L}_{x'}G_N(x,x') - G_N(x,x')\mathcal{L}_{x'}f(x')]dx' = -G_N(x,L)p(L)f'_L + G_N(x,0)p(0)f'_0. \tag{34}$$

We used the boundary conditions (29) to eliminate terms containing $\partial_{x'}G(x,x')$ at the endpoints. Then, using Equation (30), we get

$$f(x) = < f > + \int_0^L G_N(x,x')\rho(x')dx' - G_N(x,L)p(L)f'_L + G_N(x,0)p(0)f'_0, \tag{35}$$

which constitutes the solution to the Sturm–Liouville problem for Neumann boundary conditions. The constant $< f >$, the average value of $f(x)$, arises from the term $1/L$ in Equation (30).

Although we have used an expansion in eigenfunctions to give an heuristic derivation of the Green's function for Neumann boundary conditions, any function satisfying the defining Equation (30) with the boundary conditions of Equation (29) will be a suitable Green's function $G_N(x.x')$. For example, a Neumann Green's function for the problem

$$f''(x) = x, \quad f'(0) = 0, \quad f'(L) = \int_0^L x dx = L^2/2, [\text{using Equation (33)}] \tag{36}$$

is given by

$$0 \le x' \le x: \quad G_{N1}(x,x') = -\frac{x'^2}{2L} + x \tag{37}$$

$$x \le x' \le L: \quad G_{N2}(x,x') = -\frac{x'^2}{2L} + x'. \tag{38}$$

The term $-x'^2/2L$ in G_{N1} and G_{N2} provides the $-1/L$ term in $\partial^2_{x'}G_N(x,x')$, and also satisfies the homogeneous Neumann boundary conditions at $x' = 0$ and L. The terms x and x' term in G_{N2} provide a unit step in G_N at $x' = x$ so that the next derivative will give the delta function in $\partial^2_{x'}G_N(x,x')$.

We note that the Green's function in Equations (37) and (38) is not symmetric with respect to x and x'. To show that a Neumann Green's function need not be symmetric, we repeat the usual proof of symmetry here. Applying Green's theorem to two Green's functions, $G(x_1,x')$ and $G(x_2,x')$, of a Sturm–Liouville operator $\mathcal{L}$ gives

$$\begin{aligned}\int_0^L [G(x_1,x')L_{x'}G(x_2,x') - G(x_2,x')\mathcal{L}_{x'}G(x_1,x')] \\ = [G(x_1,x')p(x')\partial_{x'}G(x_2,x') - G(x_2,x')p(x')\partial_{x'}G(x_1,x')]^{x'=L}_{x'=0}.\end{aligned} \tag{39}$$

The right-hand side of Equation (39) vanishes for either Dirichlet or Neumann homogeneous boundary conditions.

For a Dirichlet Green's function, with $\mathcal{L}_{x'}G_D(x,x') = \delta(x-x')$, Equation (39) reduces to

$$G_D(x_1,x_2) - G_D(x_2,x_1) = 0, \tag{40}$$

so it must be symmetric. However, for a Neumann Green's function, $\mathcal{L}_{x'}G_D(x,x') = -1/L + \delta(x-x')$, and Equation (39) reduces to

$$G_N(x_1,x_2) - G_N(x_2,x_1) = \frac{1}{L}\int_0^L [G_N(x_1,x') - G_N(x_2,x')]dx', \tag{41}$$

so a Neumann Green's function is not required to be symmetric.

We do know, however, from the eigenfunction expansion in Equation (28) that any Neumann Green's function can be made symmetric. For instance, adding the term $-x^2/2L$ to the Green's functions Equations (37) and (38) will make that Neumann Green's function symmetric without changing any of its actions. There is no need to do this, however, since the non-symmetric Green's function is simpler, and either form will solve the original differential equation.

4. Three-Dimensional Neumann Green's Function

In this section, we extend the one-dimensional results of the previous section to three dimensions. In three dimensions, we seek the solution of the differential equation

$$\mathcal{L}_{\mathbf{r}}f(\mathbf{r}) = \nabla\cdot[p(\mathbf{r})\nabla f(\mathbf{r})] = \rho(\mathbf{r}) \tag{42}$$

with the inhomogeneous Neumann boundary conditions on any bounding surface

$$\hat{\mathbf{n}}\cdot\nabla f(\mathbf{r}) = f(\mathbf{r}_S), \tag{43}$$

where $f(\mathbf{r}_S)$ is an almost arbitrary function specified on all surfaces. A solution to Equation (42) exists only if the boundary conditions satisfy the constraint

$$\int \mathbf{dS}\cdot p(\mathbf{r})\nabla f(\mathbf{r}) = \oint \rho(\mathbf{r})]d^3r. \tag{44}$$

This constraint follows by applying the divergence theorem to Equation (42).

The Neumann Green's function for this problem satisfies the differential equation

$$\mathcal{L}_{\mathbf{r}'}G_N(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') - 1/V, \tag{45}$$

with the homogeneous boundary condition

$$\hat{\mathbf{n}}'\cdot\nabla' G_N(\mathbf{r},\mathbf{r}') = 0 \tag{46}$$

on all surfaces. This Green's function automatically satisfies the constraint

$$\int \mathbf{dS}'\cdot p(\mathbf{r}')\nabla' G_N(\mathbf{r},\mathbf{r}') = 0, \tag{47}$$

which follows by applying the divergence theorem to Equation (45).

The extra term $1/V$ in Equation (45) is the 3D equivalent of the term $1/L$ in Equation (30), and also arises due to the constant term in any expansion using Neumann eigenfunctions.

The solution to Equation (42) is given by

$$f(\mathbf{r}) =< f > + \int G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')]]d^3r' - \int \mathbf{dS}' \cdot G(\mathbf{r},\mathbf{r}')p(\mathbf{r}')\nabla' f(\mathbf{r}'), \tag{48}$$

which follows from Green's theorem, and Equations (45) and (46) for $G_N(\mathbf{r},\mathbf{r}')$. As in the 1D case, $< f >$ is an arbitrary constant that equals the average value of f in the volume. It will be removed with the mixed boundary conditions we discuss below.

5. Three-Dimensional Green's Function Examples

In this section, we compare a three dimensional Green's function with Neumann boundary conditions with a corresponding Dirichlet Green's function.

We first consider the Dirichlet case of a conducting cube of dimensions, $(L \times L \times L)$ with a constant potential, V_0, on the top face, with the other five sides grounded. For simplicity, we let $p(\mathbf{r}) = 0$. This case is treated in a number of textbooks (see, for instance, Section 3.12 of [2]).

The solution starts with the representation of the delta function in terms of three dimensional eigenfuctions of the operator, $\mathcal{L} = \nabla^2$,

$$\begin{array}{l}\delta(x,y,z;x',y',z') \\ = \frac{8}{L^3}\sum_{lmn}\left[\sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{l\pi x'}{L}\right)\sin\left(\frac{m\pi y'}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)\right].\end{array} \tag{49}$$

Then, the equation (we are using Gaussian units, which just affects the coefficients of the sums below),

$$\nabla^2 G_D((x,y,z'\ x',y',z') = -4\pi\delta(x,y,z'x',y',z'), \tag{50}$$

has the solution

$$G_D(x,y,z;x',y',z') = \frac{8}{\pi^2 L}\sum_{lmn}\left[\frac{\sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{l\pi x'}{L}\right)\sin\left(\frac{m\pi y'}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)}{(l^2+m^2+n^2)}\right], \tag{51}$$

where $-(\frac{\pi^2}{L^2})(l^2+m^2+n^2)$ is the eigenfunction of ∇^2.

The Dirichlet potential, $\phi(\mathbf{r})$, inside the cube for no charge within the cube is

$$\begin{array}{l}\phi_D(\mathbf{r}) = \int \mathbf{dS}' \cdot [\phi(\mathbf{r}')\nabla' G_D(\mathbf{r},\mathbf{r}')] \\ = -\frac{8V_0}{\pi L^2}\int_0^L dx' \int_0^L dy' \sum_{lmn}\left[\frac{(-1)^n n\sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{l\pi x'}{L}\right)\sin\left(\frac{m\pi y'}{L}\right)}{(l^2+m^2+n^2)}\right].\end{array} \tag{52}$$

We now compare the Dirichlet case with a corresponding Neumann case for a uniform charge, Q_0, on the top face, with the other five sides grounded.

$$G_N(x,y,z;x',y',z') = \frac{32}{\pi L}\sum_{lmn}\left[\frac{\sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)\cos\left(\frac{n\pi z}{L}\right)\sin\left(\frac{l\pi x'}{L}\right)\sin\left(\frac{m\pi y'}{L}\right)\cos\left(\frac{n\pi z'}{L}\right)}{(l^2+m^2+n^2)}\right]. \tag{53}$$

Two sines on the upper face have been changed to cosines to satisfy the Neumann boundary conditions.

Then, the Neumann potential inside the cube is

$$\begin{aligned}\phi_N(\mathbf{r}) &= \int \mathbf{dS}' \cdot G_N(\mathbf{r},\mathbf{r}')[\nabla'\phi(\mathbf{r}')] \\ &= 4\pi \int dS' G_N(x,y,z;x',y',0)\sigma(x',y')) \\ &= \frac{128Q_0}{L^3}\int_0^L dx' \int_0^L dy' \sum_{lmn}\left[\frac{(-1)^n \sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)\cos\left(\frac{n\pi z}{L}\right)\sin\left(\frac{l\pi x'}{L}\right)\sin\left(\frac{m\pi y'}{L}\right)}{(l^2+m^2+n^2)}\right].\end{aligned} \tag{54}$$

The main difference between the two results is the factor n in the Dirichlet potential, and $\cos(n\pi z)$ in the Neumann potential, which make the interior potentials quite different.

6. Summary

To summarize, we have found six essential differences between the Neumann Green's function treated here and the Dirichlet Green's function which is generally treated in Math Physics or Electromagnetism texts for the solution of a partial differential equation of the form

$$\mathcal{L}_{\mathbf{r}} f(\mathbf{r}) = \nabla \cdot [p(\mathbf{r})\boldsymbol{\nabla} f(\mathbf{r}) = \rho(\mathbf{r}). \tag{55}$$

We list the different Neumann properties below:

(1) The Neumann boundary condition for the solution $f(\mathbf{r})$ is

$$\hat{\mathbf{n}} \cdot \nabla f(\mathbf{r}) = f(\mathbf{r}_S), \tag{56}$$

(2) with the constraint

$$\int \mathbf{dS} \cdot p(\mathbf{r})\nabla f(\mathbf{r}) = \oint \rho(\mathbf{r})]d^3r \tag{57}$$

required for any solution to exist.

(3) The Neumann Green's function satisfies the differential equation

$$\mathcal{L}_{\mathbf{r}'} G_N(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') - 1/V, \tag{58}$$

(4) with the homogeneous surface boundary condition

$$\hat{\mathbf{n}}' \cdot \nabla G_N(\mathbf{r},\mathbf{r}') = 0. \tag{59}$$

(5) The solution to Equation (55) is given by

$$f(\mathbf{r}) = < f > + \int G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')d^3r' - \int \mathbf{dS}' \cdot G(\mathbf{r},\mathbf{r}')p(x')\nabla' f(\mathbf{r}'), \tag{60}$$

where $< f >$ is an arbitrary constant that equals the average value of $f(\mathbf{r})$ in the volume.

(6) The Neumann Green's function is not necessarily symmetric, but can be made symmetric by adding a function of $\mathbf{r}$ to $G_N(\mathbf{r},\mathbf{r}')$.

Of these six differences, only numbers (1) and (4) are generally mentioned in Math, Physics, or Electromagnetism textbooks. Some books give the homogeneous boundary condition in Equation (59), but do not mention that it is inconsistent with omitting the $1/V$ term in Equation (58). Other books [2] recognize this by making the Green's function boundary condition inhomogeneous, but this is not a satisfactory remedy. A Green's function with inhomogeneous boundary conditions can be just as hard to find as the original solution. Constructing a Green's function by an expansion in eigenfunctions is difficult without homogeneous boundary conditions.

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