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$\mathcal{A}$-Joint Numerical Radius Bounds for Operator Pairs in Semi-Hilbert Spaces with Applications

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Abstract

Consider a complex Hilbert space $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ equipped with a positive (semidefinite) bounded linear operator $\mathcal{A}$ on $\mathbb{X}$. The $\mathcal{A}$-joint numerical radius for two $\mathcal{A}$-bounded operators $\mathcal{T}$ and $\mathcal{S}$ is defined as $\omega_{\mathcal{A},e}(\mathcal{T},\mathcal{S}) = \sup_{\|x\|_{\mathcal{A}}=1} \sqrt{|\langle \mathcal{T}x, x\rangle_{\mathcal{A}}|^2 + |\langle \mathcal{S}x, x\rangle_{\mathcal{A}}|^2}$. Among other results, in this study we demonstrate that $\omega^2_{e,\mathcal{A}}(\mathcal{T},\mathcal{S}) \leq 2^{1-\frac{1}{r}} \left[\max\left\{\|\mathcal{T}\|^{2r}_{\mathcal{A}}, \|\mathcal{S}\|^{2r}_{\mathcal{A}}\right\} + \omega^r_{\mathcal{A}}\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right]^{\frac{1}{r}}$ for $r \geq 1$ and that $\omega^2_{e,\mathcal{A}}(\mathcal{T},\mathcal{S}) \geq \frac{1}{2}\omega_{\mathcal{A}}(\mathcal{T}^2 + \mathcal{S}^2) + \frac{1}{2}\max\{\omega_{\mathcal{A}}(\mathcal{S}+\mathcal{T}), \omega_{\mathcal{A}}(\mathcal{S}-\mathcal{T})\}|\omega_{\mathcal{A}}(\mathcal{S}) - \omega_{\mathcal{A}}(\mathcal{T})|$. Additionally, we provide several inequalities related to the $\mathcal{A}$-numerical radius and $\mathcal{A}$-seminorm.

Keywords: joint $\mathcal{A}$-numerical radius; Euclidean $\mathcal{A}$-seminorm; semi-inner product; positive operator

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1. Introduction

In recent years, the study of operators on semi-Hilbert spaces has experienced a significant resurgence of interest driven by their ability to generalize classical Hilbert spaces and provide a versatile framework for operator theory, as evidenced by works such as [1–13]. Semi-Hilbert spaces are equipped with a semi-inner product induced by a positive operator $\mathcal{A}$, offering a robust and promising branch of functional analysis with applications in quantum theory and beyond. This framework enables the study of operators under modified inner products, facilitating the analysis of physical observables in quantum systems where traditional self-adjointness constraints are restrictive [8,10–12]. Specifically, physical states in a quantum system are represented on a Hilbert space $\mathbb{X}$ with an inner product $\langle \cdot, \cdot \rangle$. Non-Hermitian quantum theory introduces a new inner product, defined as $\langle \xi_1, \xi_2 \rangle_{\mathcal{A}} = \langle \mathcal{A}\xi_1, \xi_2 \rangle$ for all $\xi_1, \xi_2 \in \mathbb{X}$, relative to which operators are self-adjoint, thereby broadening the scope of quantum theory [13]. In quasi-Hermitian and pseudo-Hermitian quantum theory [8,10,11], $\mathcal{A}$ is invertible, self-adjoint, and positive, while in indefinite metric quantum theory [12] $\mathcal{A}$ is unitary and self-adjoint but not necessarily positive. This flexibility makes semi-Hilbert spaces critical for advancing operator theory and its applications in functional analysis, spectral theory, and quantum mechanics.

Mathematical inequalities, particularly norm and numerical radius inequalities, have received considerable attention in this context [9,14–19]. These inequalities provide insights into the geometric and spectral properties of operators, with significant implications for functional analysis. For instance, ref. [1] established metric properties of projections in semi-Hilbertian spaces, while [2] developed a framework for partial isometries that enhanced the understanding of operator behavior. The foundational theory of $\mathcal{A}$-normal operators was introduced in [20], while [3,4] respectively explored joint normality and closed operators in semi-Hilbertian spaces, offering results tied to the $\mathcal{A}$-joint numerical radius. Inequalities for the Euclidean operator radius of operator pairs in Hilbert spaces were derived in [17], inspiring generalizations to semi-Hilbert spaces in [16], which extended seminorm and numerical radius inequalities. Further advancements include bounds for the $\mathcal{A}$-joint numerical radius and $\mathcal{A}$-numerical radius of operator matrices [6,7], refinements of $\mathcal{A}$-numerical radius inequalities [21–23], and studies on 2×2 operator matrices [9,24]. Orthogonality and norm-parallelism in the $\mathcal{A}$-numerical radius were examined in [25], while [15,26–29] provided additional bounds and generalizations, including $\mathcal{A}$-translatable radii. Structured operator matrices were addressed in [18,19,30], complementing these results. The works in [31–33] further solidify the importance of semi-Hilbert spaces in studying operator inequalities.

In this paper, we establish novel inequalities for the $\mathcal{A}$-joint numerical radius and $\mathcal{A}$-numerical radius of operators in a semi-Hilbert space, building on frameworks from [1,3,7,17]. We also investigate connections between these quantities and the $\mathcal{A}$-seminorm, providing refined bounds and applications to operator theory. Our results leverage the generalized structure of semi-Hilbert spaces to extend classical operator theory, offering new tools for spectral analysis and quantum theory applications.

Prior to presenting our findings, we clarify notations and concepts. Let $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space with norm $\| \cdot \|$ induced by $\langle \cdot, \cdot \rangle$. The C^*-algebra of bounded linear operators on $\mathbb{X}$ with identity I is denoted by $\mathbf{L}(\mathbb{X})$. The sets $\mathbb{N}$ and $\mathbb{N}^*$ represent the non-negative and positive integers, respectively. Throughout this paper, an operator refers to an element of $\mathbf{L}(\mathbb{X})$.

An operator $\mathcal{T} \in \mathbf{L}(\mathbb{X})$ is called positive, denoted $\mathcal{T} \geq 0$, if $\langle \mathcal{T}x, x \rangle \geq 0$ for all $x \in \mathbb{X}$ and $n \in \mathbb{N}^*$. For a positive operator $\mathcal{T}$, its square root is denoted by $\mathcal{T}^{\frac{1}{2}}$. Henceforth, we assume that $\mathcal{A}$ is a nonzero positive operator inducing the semi-inner product

$$\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{C}, \ (\mathcal{T}, \mathcal{S}) \longmapsto \langle \mathcal{T}, \mathcal{S} \rangle_{\mathcal{A}} := \langle \mathcal{A}\mathcal{T}, \mathcal{S} \rangle.$$

Observe that $(\mathbb{X}, \| \cdot \|_{\mathcal{A}})$ is called a semi-Hilbert space. Here, $\| \cdot \|_{\mathcal{A}}$ denotes the seminorm induced by the semi-inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$, defined by $\|\xi\|_{\mathcal{A}} = \sqrt{\langle \xi, \xi \rangle_{\mathcal{A}}}$ for all $\xi \in \mathbb{X}$. It should be noted that $(\mathbb{X}, \| \cdot \|_{\mathcal{A}})$ is, in general, neither a normed space nor a complete one. However, it becomes a Hilbert space if and only if $\mathcal{A}$ is injective and $\overline{\mathrm{Ran}(\mathcal{A})} = \mathrm{Ran}(\mathcal{A})$. Here, $\overline{\mathrm{Ran}(\mathcal{A})}$ denotes the closure of $\mathrm{Ran}(\mathcal{A})$ with respect to the topology induced by the norm $\| \cdot \|$.

To illustrate, consider the complex field $\mathbb{C}$ and let ℓ^2 be the set of all sequences $\xi = \{\xi_i\}$ ($\xi_i \in \mathbb{C}$ for all $i \in \mathbb{N}^*$) satisfying $\sum_{m \in \mathbb{N}} |\xi_m|^2 < \infty$. The inner product on ℓ^2 is defined by

$$\langle \xi, \nu \rangle = \sum_{m \in \mathbb{N}^*} \xi_m \, \overline{\nu_m} \quad \text{for all } \xi = \{\xi_i\}, \ \nu = \{\nu_i\} \in \ell^2.$$

It is well known that $(\ell^2, \langle \cdot, \cdot \rangle)$ is a Hilbert space. We define the diagonal operator $\mathcal{A}$ on ℓ^2 by

$$\mathcal{A}e_1 = 0, \quad \text{and} \quad \mathcal{A}e_i = \frac{e_i}{i}, \ \text{for all } i \geq 2,$$

where $\{e_i\}$ is the canonical basis of ℓ^2. It can be easily verified that $\mathcal{A}$ is not injective and that $\overline{\mathrm{Ran}(\mathcal{A})} \neq \mathrm{Ran}(\mathcal{A})$. Therefore, the semi-Hilbert space $(\ell^2, \langle \cdot, \cdot \rangle_{\mathcal{A}})$ is neither normed nor complete.

An operator $\mathcal{M} \in \mathbf{L}(\mathbb{X})$ is called an $\mathcal{A}$-*adjoint* of $\mathcal{T}$ if

$$\langle \mathcal{T}\xi, \nu \rangle_{\mathcal{A}} = \langle \xi, \mathcal{M}\nu \rangle_{\mathcal{A}} \quad \text{for all } \xi, \nu \in \mathbb{X},$$

or equivalently, if $\mathcal{A}\mathcal{M} = \mathcal{T}^*\mathcal{A}$. It is important to note that not every operator $\mathcal{T} \in \mathbf{L}(\mathbb{X})$ admits an $\mathcal{A}$-adjoint. Moreover, even when such an operator $\mathcal{M}$ exists, it is not necessarily unique. This non-uniqueness introduces several difficulties in the study of operators on semi-Hilbert spaces.

In this framework, the celebrated *Douglas range inclusion theorem* [34] plays a crucial role. In essence, the Douglas theorem asserts that the operator equation $\mathcal{T}X = \mathcal{S}$ admits a solution $\mathcal{Y} \in \mathbf{L}(\mathbb{X})$ if and only if $\mathrm{Ran}(\mathcal{S}) \subseteq \mathrm{Ran}(\mathcal{T})$. Equivalently, there exists a constant $\kappa > 0$ such that

$$\|\mathcal{S}^*\xi\| \leq \kappa \|\mathcal{T}^*\xi\| \quad \text{for all } \xi \in \mathbb{X}.$$

By applying the Douglas theorem, it is possible to identify from among the operators $\mathcal{T}$ that possess $\mathcal{A}$-adjoints a distinguished operator denoted by $\mathcal{T}^{\sharp_{\mathcal{A}}}$. If $\mathcal{A}^\dagger$ stands for the Moore–Penrose pseudo-inverse of $\mathcal{A}$, then (see [2,32])

$$\mathcal{T}^{\sharp_{\mathcal{A}}} = \mathcal{A}^\dagger \mathcal{T}^* \mathcal{A}.$$

The operator $\mathcal{T}^{\sharp_{\mathcal{A}}}$ shares certain properties with $\mathcal{T}^*$, but not all. For instance, the equality $(\mathcal{T}^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}} = \mathcal{T}$ does not generally hold. However, it is valid precisely when $\mathrm{Ran}(\mathcal{T}) \subseteq \overline{\mathrm{Ran}(\mathcal{A})}$. Furthermore, we have

$$\left((\mathcal{T}^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}}\right)^{\sharp_{\mathcal{A}}} = \mathcal{T}^{\sharp_{\mathcal{A}}}. \tag{1}$$

An operator $\mathcal{T} \in \mathbf{L}(\mathbb{X})$ is said to be $\mathcal{A}$-selfadjoint if $\mathcal{A}\mathcal{T}$ is selfadjoint. It was shown in [35], Lemma 1, that if $\mathcal{T} \in \mathbf{L}(\mathbb{X})$ is an $\mathcal{A}$-selfadjoint operator, then $\mathcal{T}^{\sharp_{\mathcal{A}}}$ is also $\mathcal{A}$-selfadjoint and

$$(\mathcal{T}^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}} = \mathcal{T}^{\sharp_{\mathcal{A}}}. \tag{2}$$

Let $\mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $\mathbf{L}_{\sqrt{\mathcal{A}}}(\mathbb{X})$ denote the sets of all operators that admit $\mathcal{A}$-adjoints and $\sqrt{\mathcal{A}}$-adjoints, respectively. Using the Douglas theorem, it follows immediately that

$$\mathbf{L}_{\mathcal{A}}(\mathbb{X}) = \left\{ \mathcal{T} \in \mathbf{L}(\mathbb{X}) ;\ \mathrm{Ran}(\mathcal{T}^*\mathcal{A}) \subseteq \mathrm{Ran}(\mathcal{A}) \right\}$$

and

$$\mathbf{L}_{\sqrt{\mathcal{A}}}(\mathbb{X}) = \left\{ \mathcal{T} \in \mathbf{L}(\mathbb{X}) ;\ \exists \kappa > 0 \text{ such that } \|\mathcal{T}x\|_{\mathcal{A}} \leq \kappa \|x\|_{\mathcal{A}},\ \forall x \in \mathbb{X} \right\}.$$

Operators in $\mathbf{L}_{\sqrt{\mathcal{A}}}(\mathbb{X})$ are called $\mathcal{A}$-*bounded*. It is also worth noting that both $\mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $\mathbf{L}_{\sqrt{\mathcal{A}}}(\mathbb{X})$ are subalgebras of $\mathbf{L}(\mathbb{X})$, which are in general neither closed nor dense in the C^*-algebra $\mathbf{L}(\mathbb{X})$. Moreover, the proper inclusions

$$\mathbf{L}_{\mathcal{A}}(\mathbb{X}) \subseteq \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathbb{X}) \subseteq \mathbf{L}(\mathbb{X})$$

hold.

The $\mathcal{A}$-joint numerical radius of two $\mathcal{A}$-bounded operators $\mathcal{T}$ and $\mathcal{S}$ is defined in [36] as

$$\omega_{\mathcal{A},e}(\mathcal{T}, \mathcal{S}) = \sup_{\|x\|_{\mathcal{A}}=1} \sqrt{|\langle \mathcal{T}x, x \rangle_{\mathcal{A}}|^2 + |\langle \mathcal{S}x, x \rangle_{\mathcal{A}}|^2}.$$

In the same work [36], the third author established the following upper bounds:

$$\omega_{\mathcal{A},e}(\mathcal{T},\mathcal{S}) \leq \sqrt{\|\mathcal{T}\|_{\mathcal{A}}^4 + \|\mathcal{S}\|_{\mathcal{A}}^4 + 2\omega_{\mathcal{A}}^2(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T})} \leq \|\mathcal{T}\|_{\mathcal{A}}^2 + \|\mathcal{S}\|_{\mathcal{A}}^2,$$

$$\omega_{\mathcal{A},e}(\mathcal{T},\mathcal{S}) \leq \left[\omega_{\mathcal{A}}\left((\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T})^2 + (\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S})^2\right) + 2\omega_{\mathcal{A}}^2\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right]^{\frac{1}{4}},$$

$$\begin{aligned}\omega_{\mathcal{A},e}(\mathcal{T},\mathcal{S}) &\leq \frac{\sqrt{2}}{2}\sqrt{\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\|_{\mathcal{A}} + \|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} - \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\|_{\mathcal{A}} + 2\omega_{\mathcal{A}}(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T})} \\ &\leq \sqrt{\|\mathcal{T}\|_{\mathcal{A}}^2 + \|\mathcal{S}\|_{\mathcal{A}}^2 + \omega_{\mathcal{A}}(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T})},\end{aligned}$$

and

$$\omega_{\mathcal{A},e}(\mathcal{T},\mathcal{S}) \leq \sqrt{\max(\|\mathcal{T}\|_{\mathcal{A}}^2, \|\mathcal{S}\|_{\mathcal{A}}^2) + \omega_{\mathcal{A}}(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T})}. \tag{3}$$

Furthermore, the inequality in (3) is sharp.

The following upper bound also holds:

$$\omega_{\mathcal{A},e}(\mathcal{T},\mathcal{S}) \leq \sqrt{\max\{\omega_{\mathcal{A}}(\mathcal{T}), \omega_{\mathcal{A}}(\mathcal{S})\}\sqrt{\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\|_{\mathcal{A}} + 2\omega_{\mathcal{A}}(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T})}}. \tag{4}$$

Additional lower and upper bounds were derived in [36]:

$$\begin{aligned}\frac{\sqrt{2}}{2}\max\{\omega_{\mathcal{A}}(\mathcal{T}+\mathcal{S}), \omega_{\mathcal{A}}(\mathcal{T}-\mathcal{S})\} &\leq \omega_{\mathcal{A},e}(\mathcal{T},\mathcal{S}) \\ &\leq \frac{\sqrt{2}}{2}\sqrt{\omega_{\mathcal{A}}^2(\mathcal{T}+\mathcal{S}) + \omega_{\mathcal{A}}^2(\mathcal{T}-\mathcal{S})}.\end{aligned}$$

The constant $\frac{\sqrt{2}}{2}$ is sharp in both inequalities.

Additionally, the following bounds were recalled:

$$\frac{\sqrt{2}}{2}\sqrt{\omega_{\mathcal{A}}(\mathcal{T}^2 + \mathcal{S}^2)} \leq \omega_{\mathcal{A},e}(\mathcal{T},\mathcal{S}) \leq \sqrt{\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\|_{\mathcal{A}}}, \tag{5}$$

with $\frac{\sqrt{2}}{2}$ being a sharp constant.

In [36], the author also derived several inequalities involving the $\mathcal{A}$-Davis–Wielandt radius and the $\mathcal{A}$-numerical radii of $\mathcal{A}$-bounded operators.

2. Upper Bounds for $\mathcal{A}$-Euclidean Numerical Radius

R.P. Boas in 1941 [37], along with R. Bellman independently in 1944 [38], established a generalization of Bessel's inequality; for vectors $x, y_1, \ldots, y_n$ in an inner product space $(\mathbb{X}; \langle\cdot,\cdot\rangle)$, the following inequality holds:

$$\sum_{i=1}^{n}|\langle x, y_i\rangle|^2 \leq \|x\|^2\left[\max_{1\leq i\leq n}\|y_i\|^2 + \left(\sum_{1\leq i\neq j\leq n}|\langle y_i, y_j\rangle|^2\right)^{\frac{1}{2}}\right], \tag{6}$$

When $n = 2$, $y_1 = y$, and $y_2 = z$ in (6), we obtain

$$|\langle x, y\rangle|^2 + |\langle x, z\rangle|^2 \leq \|x\|^2\left[\max\left\{\|y\|^2, \|z\|^2\right\} + \sqrt{2}|\langle y, z\rangle|\right] \tag{7}$$

for all $x, y, z \in \mathbb{X}$.

In 2006, the second author [17], Equation (2.26) derived a sharper inequality than (7):

$$\left|\langle x,y\rangle\right|^2+\left|\langle x,z\rangle\right|^2\leq\|x\|^2\left[\max\left\{\|y\|^2,\|z\|^2\right\}+\left|\langle y,z\rangle\right|\right] \tag{8}$$

for all $x,y,z\in\mathbb{X}$.

In 2004, the second author [39] also obtained a result related to (6):

$$\sum_{i=1}^{n}\left|\langle x,y_i\rangle\right|^2\leq\|x\|\max_{1\leq i\leq n}\left|\langle x,y_i\rangle\right|\left\{\sum_{i=1}^{n}\|y_i\|^2+\sum_{1\leq i\neq j\leq n}\left|\langle y_i,y_j\rangle\right|\right\}^{\frac{1}{2}} \tag{9}$$

for any $x,y_1,\ldots,y_n$ in the inner product space $\left(\mathbb{X};\langle\cdot,\cdot\rangle\right)$. Setting $n=2$, $y_1=y$, and $y_2=z$ in (9), we get

$$\begin{aligned}&\left|\langle x,y\rangle\right|^2+\left|\langle x,z\rangle\right|^2 \\ &\leq\sqrt{2}\|x\|\max\left\{\left|\langle x,y\rangle\right|,\left|\langle x,z\rangle\right|\right\}\left\{\frac{\|y\|^2+\|z\|^2}{2}+\left|\langle y,z\rangle\right|\right\}^{\frac{1}{2}}\end{aligned} \tag{10}$$

for all $x,y,z\in\mathbb{X}$.

We require the following result for the $\mathcal{A}$-inner product and norm.

Lemma 1. *For all $u,v,w\in\mathbb{X}$, we have*

$$\left|\langle u,v\rangle_{\mathcal{A}}\right|^2+\left|\langle u,w\rangle_{\mathcal{A}}\right|^2\leq\|u\|_{\mathcal{A}}^2\left[\max\left\{\|v\|_{\mathcal{A}}^2,\|w\|_{\mathcal{A}}^2\right\}+\left|\langle v,w\rangle_{\mathcal{A}}\right|\right]. \tag{11}$$

Proof. By setting $x=\mathcal{A}^{\frac{1}{2}}u$, $y=\mathcal{A}^{\frac{1}{2}}v$, and $z=\mathcal{A}^{\frac{1}{2}}w$ in (8), we obtain

$$\begin{aligned}&\left|\langle\mathcal{A}^{\frac{1}{2}}u,\mathcal{A}^{\frac{1}{2}}v\rangle\right|^2+\left|\langle\mathcal{A}^{\frac{1}{2}}u,\mathcal{A}^{\frac{1}{2}}w\rangle\right|^2\\ &\leq\left\|\mathcal{A}^{\frac{1}{2}}u\right\|^2\left[\max\left\{\left\|\mathcal{A}^{\frac{1}{2}}v\right\|^2,\left\|\mathcal{A}^{\frac{1}{2}}w\right\|^2\right\}+\left|\langle\mathcal{A}^{\frac{1}{2}}v,\mathcal{A}^{\frac{1}{2}}w\rangle\right|\right],\end{aligned}$$

which simplifies to

$$\left|\langle\mathcal{A}u,v\rangle\right|^2+\left|\langle\mathcal{A}u,w\rangle\right|^2\leq\langle\mathcal{A}u,u\rangle\left[\max\left\{\langle\mathcal{A}v,v\rangle,\langle\mathcal{A}w,w\rangle\right\}+\left|\langle\mathcal{A}v,w\rangle\right|\right]$$

for all $u,v,w\in\mathbb{X}$, which is equivalent to (11). □

Our first result is as follows:

Theorem 1. *For all $\mathcal{T},\mathcal{S}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $r\geq1$, the following inequalities hold:*

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})\leq2^{1-\frac{1}{r}}\left[\max\left\{\|\mathcal{T}\|_{\mathcal{A}}^{2r},\|\mathcal{S}\|_{\mathcal{A}}^{2r}\right\}+\omega_{\mathcal{A}}^r\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right]^{\frac{1}{r}} \tag{12}$$

and

$$\begin{aligned}&\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})\\ &\leq2^{1-\frac{1}{r}}\left[\frac{\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^r+\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}-\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^r}{2}+\omega_{\mathcal{A}}^r\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right]^{\frac{1}{r}}.\end{aligned} \tag{13}$$

Proof. From (11), we obtain

$$\left|\langle x,y\rangle_{\mathcal{A}}\right|^2+\left|\langle x,z\rangle_{\mathcal{A}}\right|^2\leq 2\|x\|_{\mathcal{A}}^2\left[\frac{\max\left\{\|y\|_{\mathcal{A}}^2,\|z\|_{\mathcal{A}}^2\right\}+\left|\langle y,z\rangle_{\mathcal{A}}\right|}{2}\right]$$

for all $x,y,z\in\mathbb{X}$.

Raising both sides to the power $r\geq 1$ and leveraging the convexity of the power function for $r\geq 1$, we get

$$\begin{aligned}&\left(\left|\langle x,y\rangle_{\mathcal{A}}\right|^2+\left|\langle x,z\rangle_{\mathcal{A}}\right|^2\right)^r\\&\leq 2^r\|x\|_{\mathcal{A}}^{2r}\left[\frac{\max\left\{\|y\|_{\mathcal{A}}^2,\|z\|_{\mathcal{A}}^2\right\}+\left|\langle y,z\rangle_{\mathcal{A}}\right|}{2}\right]^r\\&\leq 2^r\|x\|_{\mathcal{A}}^{2r}\frac{\left(\max\left\{\|y\|_{\mathcal{A}}^2,\|z\|_{\mathcal{A}}^2\right\}\right)^r+\left|\langle y,z\rangle_{\mathcal{A}}\right|^r}{2}\\&=2^{r-1}\|x\|_{\mathcal{A}}^{2r}\left[\max\left\{\|y\|_{\mathcal{A}}^{2r},\|z\|_{\mathcal{A}}^{2r}\right\}+\left|\langle y,z\rangle_{\mathcal{A}}\right|^r\right]\end{aligned}\tag{14}$$

for all $x,y,z\in\mathbb{X}$.

Substituting $y=\mathcal{T}x,z=\mathcal{S}x$ with $x\in\mathbb{X}$ and $\|x\|_{\mathcal{A}}=1$ into (14), we obtain

$$\begin{aligned}&\left(\left|\langle x,\mathcal{T}x\rangle_{\mathcal{A}}\right|^2+\left|\langle x,\mathcal{S}x\rangle_{\mathcal{A}}\right|^2\right)^r\\&\leq 2^{r-1}\left[\left(\max\left\{\|\mathcal{T}x\|_{\mathcal{A}}^2,\|\mathcal{S}x\|_{\mathcal{A}}^2\right\}\right)^r+\left|\langle\mathcal{T}x,\mathcal{S}x\rangle_{\mathcal{A}}\right|^r\right]\\&=2^{r-1}\left[\left(\max\left\{\|\mathcal{T}x\|_{\mathcal{A}}^2,\|\mathcal{S}x\|_{\mathcal{A}}^2\right\}\right)^r+\left|\langle\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}x,x\rangle_{\mathcal{A}}\right|^r\right].\end{aligned}\tag{15}$$

Taking the supremum over $\|x\|_{\mathcal{A}}=1$, we get

$$\begin{aligned}&\omega_{e,\mathcal{A}}^{2r}(\mathcal{T},\mathcal{S})\\&=\sup_{\|x\|_{\mathcal{A}}=1}\left(\left|\langle x,\mathcal{T}x\rangle_{\mathcal{A}}\right|^2+\left|\langle x,\mathcal{S}x\rangle_{\mathcal{A}}\right|^2\right)^r\\&\leq 2^{r-1}\sup_{\|x\|_{\mathcal{A}}=1}\left[\left(\max\left\{\|\mathcal{T}x\|_{\mathcal{A}}^2,\|\mathcal{S}x\|_{\mathcal{A}}^2\right\}\right)^r+\left|\langle\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}x,x\rangle_{\mathcal{A}}\right|^r\right]\\&\leq 2^{r-1}\left[\left(\sup_{\|x\|_{\mathcal{A}}=1}\max\left\{\|\mathcal{T}x\|_{\mathcal{A}}^2,\|\mathcal{S}x\|_{\mathcal{A}}^2\right\}\right)^r\right.\\&\left.+\sup_{\|x\|_{\mathcal{A}}=1}\left|\langle\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}x,x\rangle_{\mathcal{A}}\right|^r\right]\\&=2^{r-1}\left[\left(\max\left\{\sup_{\|x\|_{\mathcal{A}}=1}\|\mathcal{T}x\|_{\mathcal{A}}^2,\sup_{\|x\|_{\mathcal{A}}=1}\|\mathcal{S}x\|_{\mathcal{A}}^2\right\}\right)^r\right.\\&\left.+\sup_{\|x\|_{\mathcal{A}}=1}\left|\langle\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}x,x\rangle_{\mathcal{A}}\right|^r\right]\\&=2^{r-1}\left[\left(\max\left\{\|\mathcal{T}\|_{\mathcal{A}}^2,\|\mathcal{S}\|_{\mathcal{A}}^2\right\}\right)^r+\omega_{\mathcal{A}}^r\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right]\\&=2^{r-1}\left[\max\left\{\|\mathcal{T}\|_{\mathcal{A}}^{2r},\|\mathcal{S}\|_{\mathcal{A}}^{2r}\right\}+\omega_{\mathcal{A}}^r\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right],\end{aligned}$$

which establishes (12).

Additionally, observe that

$$\begin{aligned}
&\max\left\{\|\mathcal{T}x\|_{\mathcal{A}}^2, \|\mathcal{S}x\|_{\mathcal{A}}^2\right\} \\
&= \frac{1}{2}\left(\|\mathcal{T}x\|_{\mathcal{A}}^2 + \|\mathcal{S}x\|_{\mathcal{A}}^2\right) + \frac{1}{2}\left|\|\mathcal{T}x\|_{\mathcal{A}}^2 - \|\mathcal{S}x\|_{\mathcal{A}}^2\right| \\
&= \frac{1}{2}\left(\langle \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}x, x\rangle_{\mathcal{A}} + \langle \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}x, x\rangle_{\mathcal{A}}\right) + \frac{1}{2}\left|\langle \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}x, x\rangle_{\mathcal{A}} - \langle \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}x, x\rangle_{\mathcal{A}}\right| \\
&= \frac{1}{2}\left(\langle\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)x, x\rangle_{\mathcal{A}} + \left|\langle\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} - \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)x, x\rangle_{\mathcal{A}}\right|\right).
\end{aligned}$$

Using the convexity of the power function for exponents greater than 1, we have

$$\begin{aligned}
&\left(\max\left\{\|\mathcal{T}x\|_{\mathcal{A}}^2, \|\mathcal{S}x\|_{\mathcal{A}}^2\right\}\right)^r \\
&= \left(\frac{\langle\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)x, x\rangle_{\mathcal{A}} + \left|\langle\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} - \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)x, x\rangle_{\mathcal{A}}\right|}{2}\right)^r \\
&\leq \frac{\langle\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)x, x\rangle_{\mathcal{A}}^r + \left|\langle\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} - \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)x, x\rangle_{\mathcal{A}}\right|^r}{2}.
\end{aligned}$$

From (15), we obtain

$$\begin{aligned}
&\left(\left|\langle x, \mathcal{T}x\rangle_{\mathcal{A}}\right|^2 + \left|\langle x, \mathcal{S}x\rangle_{\mathcal{A}}\right|^2\right)^r \\
&\leq 2^{r-1}\left[\frac{\langle\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)x, x\rangle_{\mathcal{A}}^r + \left|\langle\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} - \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)x, x\rangle_{\mathcal{A}}\right|^r}{2}\right. \\
&\left.+\left|\langle \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}x, x\rangle_{\mathcal{A}}\right|^r\right]
\end{aligned} \tag{16}$$

for $x \in \mathbb{X}$ with $\|x\|_{\mathcal{A}} = 1$.

Taking the supremum in (16) over $\|x\|_{\mathcal{A}} = 1$, we get

$$\begin{aligned}
&\omega_{e,\mathcal{A}}^{2r}(\mathcal{T}, \mathcal{S}) \\
&= \sup_{\|x\|_{\mathcal{A}}=1}\left(\left|\langle x, \mathcal{T}x\rangle_{\mathcal{A}}\right|^2 + \left|\langle x, \mathcal{S}x\rangle_{\mathcal{A}}\right|^2\right)^r \\
&\leq 2^{r-1}\sup_{\|x\|_{\mathcal{A}}=1}\left[\frac{\langle\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)x, x\rangle_{\mathcal{A}}^r + \left|\langle\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} - \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)x, x\rangle_{\mathcal{A}}\right|^r}{2}\right. \\
&\left.+\left|\langle \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}x, x\rangle_{\mathcal{A}}\right|^r\right] \\
&\leq 2^{r-1}\left[\frac{1}{2}\left[\sup_{\|x\|_{\mathcal{A}}=1}\langle\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)x, x\rangle_{\mathcal{A}}^r\right.\right. \\
&\left.\left.+ \sup_{\|x\|_{\mathcal{A}}=1}\left|\langle\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} - \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)x, x\rangle_{\mathcal{A}}\right|^r\right] + \sup_{\|x\|_{\mathcal{A}}=1}\left|\langle \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}x, x\rangle_{\mathcal{A}}\right|^r\right] \\
&= 2^{r-1}\left[\frac{\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^r + \left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} - \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^r}{2} + \omega_{\mathcal{A}}^r\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right],
\end{aligned}$$

which establishes (13). □

Corollary 1. *For all $\mathcal{T}, \mathcal{S} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $r \geq 1$, the following inequalities hold:*

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \leq \left[\frac{1}{2}\max\left\{\|\mathcal{T}+\mathcal{S}\|_{\mathcal{A}}^{2r}, \|\mathcal{T}-\mathcal{S}\|_{\mathcal{A}}^{2r}\right\} + \frac{1}{2}\omega_{\mathcal{A}}^r\left(\left(\mathcal{T}^{\sharp_{\mathcal{A}}}-\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}+\mathcal{S})\right)\right]^{\frac{1}{r}} \tag{17}$$

and

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \leq \left[2^{r-2}\left(\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^r + \left\|\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^r\right) + \frac{1}{2}\omega_{\mathcal{A}}^r\left(\left(\mathcal{T}^{\sharp_{\mathcal{A}}}-\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}+\mathcal{S})\right)\right]^{\frac{1}{r}}. \tag{18}$$

Proof. We first note that by the parallelogram identity for the $\mathcal{A}$-inner product, we have

$$\begin{aligned}
&\omega_{e,\mathcal{A}}^2(\mathcal{T}+\mathcal{S},\mathcal{T}-\mathcal{S}) \\
&= \sup_{\|x\|_{\mathcal{A}}=1}\left(\left|\langle x,(\mathcal{T}+\mathcal{S})x\rangle_{\mathcal{A}}\right|^2 + \left|\langle x,(\mathcal{T}-\mathcal{S})x\rangle_{\mathcal{A}}\right|^2\right) \\
&= 2\sup_{\|x\|_{\mathcal{A}}=1}\left(\left|\langle x,\mathcal{T}x\rangle_{\mathcal{A}}\right|^2 + \left|\langle x,\mathcal{S}x\rangle_{\mathcal{A}}\right|^2\right) = 2\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})
\end{aligned} \tag{19}$$

for $\mathcal{T}, \mathcal{S} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$.

Applying (12) to the pair $(\mathcal{T}+\mathcal{S}, \mathcal{T}-\mathcal{S})$, we obtain

$$\begin{aligned}
&\omega_{e,\mathcal{A}}^2(\mathcal{T}+\mathcal{S},\mathcal{T}-\mathcal{S}) \\
&\leq 2^{1-\frac{1}{r}}\left[\max\left\{\|\mathcal{T}+\mathcal{S}\|_{\mathcal{A}}^{2r}, \|\mathcal{T}-\mathcal{S}\|_{\mathcal{A}}^{2r}\right\} + \omega_{\mathcal{A}}^r\left(\left(\mathcal{T}^{\sharp_{\mathcal{A}}}-\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}+\mathcal{S})\right)\right]^{\frac{1}{r}}.
\end{aligned}$$

Using (19), this implies (17).

Next, we observe that

$$\begin{aligned}
&(\mathcal{T}+\mathcal{S})^{\sharp_{\mathcal{A}}}(\mathcal{T}+\mathcal{S}) + (\mathcal{T}-\mathcal{S})^{\sharp_{\mathcal{A}}}(\mathcal{T}-\mathcal{S}) \\
&= \left(\mathcal{T}^{\sharp_{\mathcal{A}}}+\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}+\mathcal{S}) + \left(\mathcal{T}^{\sharp_{\mathcal{A}}}-\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}-\mathcal{S}) \\
&= \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{S} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S} + \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} - \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{S} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S} \\
&= 2\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)
\end{aligned}$$

and

$$\begin{aligned}
&(\mathcal{T}+\mathcal{S})^{\sharp_{\mathcal{A}}}(\mathcal{T}+\mathcal{S}) - (\mathcal{T}-\mathcal{S})^{\sharp_{\mathcal{A}}}(\mathcal{T}-\mathcal{S}) \\
&= \left(\mathcal{T}^{\sharp_{\mathcal{A}}}+\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}+\mathcal{S}) - \left(\mathcal{T}^{\sharp_{\mathcal{A}}}-\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}-\mathcal{S}) \\
&= \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{S} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S} - \left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} - \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{S} + \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right) \\
&= 2\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{S}\right).
\end{aligned}$$

Applying (13), we obtain

$$\begin{aligned}
&\omega_{e,\mathcal{A}}^2(\mathcal{T}+\mathcal{S},\mathcal{T}-\mathcal{S})\\
&\leq 2^{1-\frac{1}{r}}\left[\frac{\left\|2\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)\right\|_{\mathcal{A}}^r+\left\|2\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{S}\right)\right\|_{\mathcal{A}}^r}{2}\right.\\
&\left.+\omega_{\mathcal{A}}^r\left(\left(\mathcal{T}^{\sharp_{\mathcal{A}}}-\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}+\mathcal{S})\right)\right]^{\frac{1}{r}}\\
&=2^{1-\frac{1}{r}}\left[2^{r-1}\left(\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^r+\left\|\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^r\right)\right.\\
&\left.+\omega_{\mathcal{A}}^r\left(\left(\mathcal{T}^{\sharp_{\mathcal{A}}}-\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}+\mathcal{S})\right)\right]^{\frac{1}{r}}.
\end{aligned}$$

Using (19), this implies (18). □

Remark 1. *Setting $r=1$ in Theorem 1, we obtain the inequalities*

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})\leq\max\left\{\|\mathcal{T}\|_{\mathcal{A}}^2,\|\mathcal{S}\|_{\mathcal{A}}^2\right\}+\omega_{\mathcal{A}}\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right), \tag{20}$$

which correspond to inequality (3) from the Introduction, along with

$$\begin{aligned}
\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})\leq&\frac{\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}+\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}-\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}}{2}\\
&+\omega_{\mathcal{A}}\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right).
\end{aligned}$$

For $r=2$, we derive

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})\leq\sqrt{2}\left[\max\left\{\|\mathcal{T}\|_{\mathcal{A}}^4,\|\mathcal{S}\|_{\mathcal{A}}^4\right\}+\omega_{\mathcal{A}}^2\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right]^{\frac{1}{2}}$$

and

$$\begin{aligned}
&\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})\\
&\leq\sqrt{2}\left[\frac{\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^2+\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}-\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^2}{2}+\omega_{\mathcal{A}}^2\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right]^{\frac{1}{2}}.
\end{aligned}$$

Similarly, setting $r=1$ in Corollary 1, we obtain

$$\begin{aligned}
&\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})\\
&\leq\frac{1}{2}\max\left\{\|\mathcal{T}+\mathcal{S}\|_{\mathcal{A}}^2,\|\mathcal{T}-\mathcal{S}\|_{\mathcal{A}}^2\right\}+\frac{1}{2}\omega_{\mathcal{A}}\left(\left(\mathcal{T}^{\sharp_{\mathcal{A}}}-\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}+\mathcal{S})\right)
\end{aligned}$$

and

$$\begin{aligned}
\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})\leq&\frac{1}{2}\left(\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}+\left\|\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}\right)\\
&+\frac{1}{2}\omega_{\mathcal{A}}\left(\left(\mathcal{T}^{\sharp_{\mathcal{A}}}-\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}+\mathcal{S})\right).
\end{aligned}$$

For $r = 2$, we get

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \leq \left[\frac{1}{2}\max\left\{\|\mathcal{T}+\mathcal{S}\|_{\mathcal{A}}^4, \|\mathcal{T}-\mathcal{S}\|_{\mathcal{A}}^4\right\} + \frac{1}{2}\omega_{\mathcal{A}}^2\left(\left(\mathcal{T}^{\sharp_{\mathcal{A}}}-\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}+\mathcal{S})\right)\right]^{\frac{1}{2}}$$

and

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \leq \left[\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^2 + \left\|\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^2 + \frac{1}{2}\omega_{\mathcal{A}}^2\left(\left(\mathcal{T}^{\sharp_{\mathcal{A}}}-\mathcal{S}^{\sharp_{\mathcal{A}}}\right)(\mathcal{T}+\mathcal{S})\right)\right]^{\frac{1}{2}}.$$

We now require the following vector inequality for the $\mathcal{A}$-inner product and $\mathcal{A}$-seminorm.

Lemma 2. *For all $u, v, w \in \mathbb{X}$, we have*

$$\left|\langle u,v\rangle_{\mathcal{A}}\right|^2 + \left|\langle u,w\rangle_{\mathcal{A}}\right|^2 \leq \sqrt{2}\|u\|_{\mathcal{A}}\max\{\left|\langle u,v\rangle_{\mathcal{A}}\right|, \left|\langle u,w\rangle_{\mathcal{A}}\right|\} \times \left\{\frac{\|v\|_{\mathcal{A}}^2+\|w\|_{\mathcal{A}}^2}{2} + \left|\langle v,w\rangle_{\mathcal{A}}\right|\right\}^{\frac{1}{2}}. \tag{21}$$

The proof follows by applying (10) with $x = \mathcal{A}^{\frac{1}{2}}u$, $y = \mathcal{A}^{\frac{1}{2}}v$, and $z = \mathcal{A}^{\frac{1}{2}}w$.
Furthermore, we have the following theorem.

Theorem 2. *For all $\mathcal{T}, \mathcal{S} \in \mathbf{L}(\mathbb{X})$ and $r \geq 2$, the following inequalities hold:*

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \leq 2^{1-\frac{1}{r}}\max\{\omega_{\mathcal{A}}(\mathcal{T}),\omega_{\mathcal{A}}(\mathcal{S})\} \times \left\{\left\|\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}}{2}\right\|_{\mathcal{A}}^{\frac{r}{2}} + \omega_{\mathcal{A}}^{\frac{r}{2}}\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right\}^{\frac{1}{r}}. \tag{22}$$

Additionally, we have

$$\begin{aligned}&\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \\ &\leq \max\{\omega_{\mathcal{A}}(\mathcal{T}+\mathcal{S}),\omega_{\mathcal{A}}(\mathcal{T}-\mathcal{S})\} \\ &\times\left\{\frac{1}{2}\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}^{\frac{r}{2}} + \frac{1}{2}\omega_{\mathcal{A}}^{\frac{r}{2}}\left((\mathcal{T}-\mathcal{S})^{\sharp_{\mathcal{A}}}(\mathcal{T}+\mathcal{S})\right)\right\}^{\frac{1}{r}}.\end{aligned} \tag{23}$$

Proof. By setting $v = \mathcal{T}x$, $w = \mathcal{S}x$ with $x \in \mathbb{X}$ and $\|x\|_{\mathcal{A}} = 1$ in (21), we obtain

$$\begin{aligned}&\left|\langle x,\mathcal{T}x\rangle_{\mathcal{A}}\right|^2 + \left|\langle x,\mathcal{S}x\rangle_{\mathcal{A}}\right|^2 \\ &\leq \sqrt{2}\max\{\left|\langle x,\mathcal{T}x\rangle_{\mathcal{A}}\right|, \left|\langle x,\mathcal{S}x\rangle_{\mathcal{A}}\right|\}\left\{\frac{\|\mathcal{T}x\|_{\mathcal{A}}^2+\|\mathcal{S}x\|_{\mathcal{A}}^2}{2} + \left|\langle \mathcal{T}x,\mathcal{S}x\rangle_{\mathcal{A}}\right|\right\}^{\frac{1}{2}} \\ &= \sqrt{2}\max\{\left|\langle x,\mathcal{T}x\rangle_{\mathcal{A}}\right|, \left|\langle x,\mathcal{S}x\rangle_{\mathcal{A}}\right|\}\left\{\left\langle\left(\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}}{2}\right)x,x\right\rangle_{\mathcal{A}} + \left|\left\langle\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}x,x\right\rangle_{\mathcal{A}}\right|\right\}^{\frac{1}{2}}.\end{aligned}$$

Raising both sides to the power $r \geq 2$ and applying the convexity of the power function, we get

$$\begin{aligned}
&\left(\left|\langle x,\mathcal{T}x\rangle_{\mathcal{A}}\right|^2+\left|\langle x,\mathcal{S}x\rangle_{\mathcal{A}}\right|^2\right)^r \qquad (24)\\
&\leq 2^{\frac{r}{2}}\max\left\{\left|\langle x,\mathcal{T}x\rangle_{\mathcal{A}}\right|^r,\left|\langle x,\mathcal{S}x\rangle_{\mathcal{A}}\right|^r\right\}\left\{\left\langle\left(\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}}{2}\right)x,x\right\rangle_{\mathcal{A}}+\left|\left\langle\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}x,x\right\rangle_{\mathcal{A}}\right|\right\}^{\frac{r}{2}}\\
&=2^{\frac{r}{2}}\max\left\{\left|\langle x,\mathcal{T}x\rangle_{\mathcal{A}}\right|^r,\left|\langle x,\mathcal{S}x\rangle_{\mathcal{A}}\right|^r\right\}2^{\frac{r}{2}}\left\{\frac{\left\langle\left(\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}}{2}\right)x,x\right\rangle_{\mathcal{A}}+\left|\left\langle\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}x,x\right\rangle_{\mathcal{A}}\right|}{2}\right\}^{\frac{r}{2}}\\
&\leq 2^r\max\left\{\left|\langle x,\mathcal{T}x\rangle_{\mathcal{A}}\right|^r,\left|\langle x,\mathcal{S}x\rangle_{\mathcal{A}}\right|^r\right\}\frac{\left\langle\left(\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}}{2}\right)x,x\right\rangle_{\mathcal{A}}^{\frac{r}{2}}+\left|\left\langle\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}x,x\right\rangle_{\mathcal{A}}\right|^{\frac{r}{2}}}{2}\\
&=2^{r-1}\max\left\{\left|\langle x,\mathcal{T}x\rangle_{\mathcal{A}}\right|^r,\left|\langle x,\mathcal{S}x\rangle_{\mathcal{A}}\right|^r\right\}\left[\left\langle\left(\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}}{2}\right)x,x\right\rangle_{\mathcal{A}}^{\frac{r}{2}}+\left|\left\langle\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}x,x\right\rangle_{\mathcal{A}}\right|^{\frac{r}{2}}\right]
\end{aligned}$$

for all $x\in\mathbb{X}$ with $\|x\|_{\mathcal{A}}=1$.

Taking the supremum in (24) over $\|x\|_{\mathcal{A}}=1$, we obtain (22).

Now, applying (22) to the pair $(\mathcal{T}+\mathcal{S},\mathcal{T}-\mathcal{S})$, we have

$$\begin{aligned}
&\omega^2_{e,\mathcal{A}}(\mathcal{T}+\mathcal{S},\mathcal{T}-\mathcal{S})\\
&\leq 2^{1-\frac{1}{r}}\max\{\omega_{\mathcal{A}}(\mathcal{T}+\mathcal{S}),\omega_{\mathcal{A}}(\mathcal{T}-\mathcal{S})\}\\
&\times\left[\left\|\frac{(\mathcal{T}+\mathcal{S})^{\sharp_{\mathcal{A}}}(\mathcal{T}+\mathcal{S})+(\mathcal{T}-\mathcal{S})^{\sharp_{\mathcal{A}}}(\mathcal{T}-\mathcal{S})}{2}\right\|_{\mathcal{A}}^{\frac{r}{2}}\right.\\
&\left.+\omega^{\frac{r}{2}}_{\mathcal{A}}\Big((\mathcal{T}-\mathcal{S})^{\sharp_{\mathcal{A}}}(\mathcal{T}+\mathcal{S})\Big)\right]^{\frac{1}{r}},
\end{aligned}$$

which by the equality in (19) yields (23). □

Remark 2. *If we take $r=2$ in (22), then we get*

$$\omega^2_{e,\mathcal{A}}(\mathcal{T},\mathcal{S})\leq\sqrt{2}\max\{\omega_{\mathcal{A}}(\mathcal{T}),\omega_{\mathcal{A}}(\mathcal{S})\}\left\{\left\|\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}}{2}\right\|_{\mathcal{A}}+\omega_{\mathcal{A}}(\mathcal{S}^*\mathcal{T})\right\}^{\frac{1}{2}}.$$

This is the result obtained in (4) from the Introduction.

For $r=4$, we obtain

$$\omega^2_{e,\mathcal{A}}(\mathcal{T},\mathcal{S})\leq 2\sqrt{2}\max\{\omega_{\mathcal{A}}(\mathcal{T}),\omega_{\mathcal{A}}(\mathcal{S})\}\left\{\left\|\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}}{2}\right\|^2_{\mathcal{A}}+\omega^2(\mathcal{S}^*\mathcal{T})\right\}^{\frac{1}{4}}.$$

If we take $r=2$ in (23), then we get

$$\begin{aligned}
\omega^2_{e,\mathcal{A}}(\mathcal{T},\mathcal{S})&\leq\max\{\omega_{\mathcal{A}}(\mathcal{T}+\mathcal{S}),\omega_{\mathcal{A}}(\mathcal{T}-\mathcal{S})\}\\
&\times\left\{\frac{1}{2}\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|_{\mathcal{A}}+\frac{1}{2}\omega\Big((\mathcal{T}-\mathcal{S})^{\sharp_{\mathcal{A}}}(\mathcal{T}+\mathcal{S})\Big)\right\}^{\frac{1}{2}},
\end{aligned}$$

while for $r=4$ we obtain

$$\begin{aligned}
\omega^2_{e,\mathcal{A}}(\mathcal{T},\mathcal{S})&\leq\max\{\omega_{\mathcal{A}}(\mathcal{T}+\mathcal{S}),\omega_{\mathcal{A}}(\mathcal{T}-\mathcal{S})\}\\
&\times\left\{\frac{1}{2}\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S}\right\|^2_{\mathcal{A}}+\frac{1}{2}\omega^2\Big((\mathcal{T}-\mathcal{S})^{\sharp_{\mathcal{A}}}(\mathcal{T}+\mathcal{S})\Big)\right\}^{\frac{1}{4}}.
\end{aligned}$$

Theorem 3. *For all $\mathcal{T}, \mathcal{S} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $|\alpha + \beta| = 1$ and $|\gamma + \delta| = 1$, we have*

$$\begin{aligned} &\omega_{e,\mathcal{A}}(\mathcal{T}, \mathcal{S}) \\ &\leq \omega_{e,\mathcal{A}}(\alpha\mathcal{T}, \gamma\mathcal{S}) + \omega_{e,\mathcal{A}}(\beta\mathcal{T}, \delta\mathcal{S}) \\ &\leq \left\| |\alpha|^2 \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + |\gamma|^2 \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S} \right\|_{\mathcal{A}}^{\frac{1}{2}} + \left\| |\beta|^2 \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + |\delta|^2 \mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S} \right\|_{\mathcal{A}}^{\frac{1}{2}} \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\omega_{e,\mathcal{A}}(\mathcal{T}, \mathcal{S}) \\ &\leq \omega_{e,\mathcal{A}}(\alpha\mathcal{T}, \gamma\mathcal{S}) + \omega_{e,\mathcal{A}}(\beta\mathcal{T}, \delta\mathcal{S}) \\ &\leq \left[\max\left\{ |\alpha|^2 \|\mathcal{T}\|_{\mathcal{A}}^2, |\gamma|^2 \|\mathcal{S}\|_{\mathcal{A}}^2 \right\} + |\alpha||\gamma| \omega_{\mathcal{A}}\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T} \right) \right]^{\frac{1}{2}} \\ &+ \left[\max\left\{ |\beta|^2 \|\mathcal{T}\|_{\mathcal{A}}^2, |\delta|^2 \|\mathcal{S}\|_{\mathcal{A}}^2 \right\} + |\beta||\delta| \omega_{\mathcal{A}}\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T} \right) \right]^{\frac{1}{2}}. \end{aligned} \tag{26}$$

Proof. We use the elementary inequality of Minkowski type:

$$\sqrt{|a+b|^2 + |c+d|^2} \leq \sqrt{|a|^2 + |c|^2} + \sqrt{|b|^2 + |d|^2}$$

which holds for the complex numbers a, b, c, d.

Becaude $|\overline{\alpha} + \overline{\beta}| = 1$ and $|\overline{\gamma} + \overline{\delta}| = 1$, we have

$$\begin{aligned} &\left(|\langle x, \mathcal{T}x \rangle_{\mathcal{A}}|^2 + |\langle x, \mathcal{S}x \rangle_{\mathcal{A}}|^2 \right)^{\frac{1}{2}} \\ &= \left(|\overline{\alpha}\langle x, \mathcal{T}x \rangle_{\mathcal{A}} + \overline{\beta}\langle x, \mathcal{T}x \rangle_{\mathcal{A}}|^2 + |\overline{\gamma}\langle x, \mathcal{S}x \rangle_{\mathcal{A}} + \overline{\delta}\langle x, \mathcal{S}x \rangle_{\mathcal{A}}|^2 \right)^{\frac{1}{2}} \\ &= \left(|\langle x, \alpha\mathcal{T}x \rangle_{\mathcal{A}} + \langle x, \beta\mathcal{T}x \rangle_{\mathcal{A}}|^2 + |\langle x, \gamma\mathcal{S}x \rangle_{\mathcal{A}} + \langle x, \delta\mathcal{S}x \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq \left(|\langle x, \alpha\mathcal{T}x \rangle_{\mathcal{A}}|^2 + |\langle x, \gamma\mathcal{S}x \rangle_{\mathcal{A}}|^2 \right)^{\frac{1}{2}} + \left(|\langle x, \beta\mathcal{T}x \rangle_{\mathcal{A}}|^2 + |\langle x, \delta\mathcal{S}x \rangle_{\mathcal{A}}|^2 \right)^{\frac{1}{2}} \end{aligned}$$

for all $x \in \mathbb{X}$, $\|x\|_{\mathcal{A}} = 1$.

If we take the supremum over $x \in \mathbb{X}$, $\|x\|_{\mathcal{A}} = 1$, then we get

$$\begin{aligned} \omega_{e,\mathcal{A}}(\mathcal{T}, \mathcal{S}) &= \sup_{\|x\|_{\mathcal{A}}=1} \left(|\langle x, \mathcal{T}x \rangle_{\mathcal{A}}|^2 + |\langle x, \mathcal{S}x \rangle_{\mathcal{A}}|^2 \right)^{\frac{1}{2}} \\ &\leq \sup_{\|x\|_{\mathcal{A}}=1} \left[\left(|\langle x, \alpha\mathcal{T}x \rangle_{\mathcal{A}}|^2 + |\langle x, \gamma\mathcal{S}x \rangle_{\mathcal{A}}|^2 \right)^{\frac{1}{2}} \right. \\ &\left. + \left(|\langle x, \beta\mathcal{T}x \rangle_{\mathcal{A}}|^2 + |\langle x, \delta\mathcal{S}x \rangle_{\mathcal{A}}|^2 \right)^{\frac{1}{2}} \right] \\ &\leq \sup_{\|x\|_{\mathcal{A}}=1} \left(|\langle x, \alpha\mathcal{T}x \rangle_{\mathcal{A}}|^2 + |\langle x, \gamma\mathcal{S}x \rangle_{\mathcal{A}}|^2 \right)^{\frac{1}{2}} \\ &+ \sup_{\|x\|_{\mathcal{A}}=1} \left(|\langle x, \beta\mathcal{T}x \rangle_{\mathcal{A}}|^2 + |\langle x, \delta\mathcal{S}x \rangle_{\mathcal{A}}|^2 \right)^{\frac{1}{2}} \\ &= \omega_{e,\mathcal{A}}(\alpha\mathcal{T}, \gamma\mathcal{S}) + \omega_{e,\mathcal{A}}(\beta\mathcal{T}, \delta\mathcal{S}), \end{aligned}$$

which proves the first part of (25).

We know that the following inequality is true for two operators $\mathcal{X}, \mathcal{Y} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$; see [36], Theorem 2.27:

$$\omega_{e,\mathcal{A}}(\mathcal{X}, \mathcal{Y}) \leq \left\| \mathcal{X}^{\sharp_{\mathcal{A}}}\mathcal{X} + \mathcal{Y}^{\sharp_{\mathcal{A}}}\mathcal{Y} \right\|_{\mathcal{A}}^{\frac{1}{2}}.$$

Using this inequality, we have

$$\omega_{e,\mathcal{A}}(\alpha\mathcal{T},\gamma\mathcal{S}) \le \left\| |\alpha|^2\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + |\gamma|^2\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S} \right\|_{\mathcal{A}}^{\frac{1}{2}}$$

and

$$\omega_{e,\mathcal{A}}(\beta\mathcal{T},\delta\mathcal{S}) \le \left\| |\beta|^2\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + |\delta|^2\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S} \right\|_{\mathcal{A}}^{\frac{1}{2}},$$

which proves (25).

By (20), we also have

$$\omega_{e,\mathcal{A}}(\alpha\mathcal{T},\gamma\mathcal{S}) \le \left[\max\left\{|\alpha|^2\|\mathcal{T}\|_{\mathcal{A}}^2, |\gamma|^2\|\mathcal{S}\|_{\mathcal{A}}^2\right\} + |\alpha||\gamma|\omega_{\mathcal{A}}\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right]^{\frac{1}{2}}$$

and

$$\omega_{e,\mathcal{A}}(\beta\mathcal{T},\delta\mathcal{S}) \le \left[\max\left\{|\beta|^2\|\mathcal{T}\|_{\mathcal{A}}^2, |\delta|^2\|\mathcal{S}\|_{\mathcal{A}}^2\right\} + |\beta||\delta|\omega_{\mathcal{A}}\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right]^{\frac{1}{2}},$$

which proves (26). □

Remark 3. *If we take $\alpha,\beta \in \mathbb{C}$ with $|\alpha+\beta| = 1$ and choose $\gamma = \beta$, $\delta = \alpha$ in Theorem 3, then we get*

$$\begin{aligned}\omega_{e,\mathcal{A}}(\mathcal{T},\mathcal{S}) &\le \omega_{e,\mathcal{A}}(\alpha\mathcal{T},\beta\mathcal{S}) + \omega_{e,\mathcal{A}}(\beta\mathcal{T},\alpha\mathcal{S}) \\ &\le \left\| |\alpha|^2\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + |\beta|^2\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S} \right\|_{\mathcal{A}}^{\frac{1}{2}} + \left\| |\beta|^2\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + |\alpha|^2\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{S} \right\|_{\mathcal{A}}^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}\omega_{e,\mathcal{A}}(\mathcal{T},\mathcal{S}) &\le \omega_{e,\mathcal{A}}(\alpha\mathcal{T},\beta\mathcal{S}) + \omega_{e,\mathcal{A}}(\beta\mathcal{T},\alpha\mathcal{S}) \\ &\le \left[\max\left\{|\alpha|^2\|\mathcal{T}\|_{\mathcal{A}}^2, |\beta|^2\|\mathcal{S}\|_{\mathcal{A}}^2\right\} + |\alpha||\beta|\omega_{\mathcal{A}}\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right]^{\frac{1}{2}} \\ &+ \left[\max\left\{|\beta|^2\|\mathcal{T}\|_{\mathcal{A}}^2, |\alpha|^2\|\mathcal{S}\|_{\mathcal{A}}^2\right\} + |\alpha||\beta|\omega_{\mathcal{A}}\left(\mathcal{S}^{\sharp_{\mathcal{A}}}\mathcal{T}\right)\right]^{\frac{1}{2}}.\end{aligned}$$

3. Lower Bounds

We start with the following main result.

Theorem 4. *For $\mathcal{T},\mathcal{S} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $\zeta,\eta,\lambda,\mu \in \mathbb{C}$ with $|\zeta|^2 + |\eta|^2 = |\lambda|^2 + |\mu|^2 = 1$, we have*

$$\begin{aligned}&\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \\ &\ge \frac{1}{2}\max\left\{\omega_{\mathcal{A}}\left[\left(\zeta^2+\lambda^2\right)\mathcal{T}^2 + \left(\eta^2+\mu^2\right)\mathcal{S}^2 + (\zeta\eta+\lambda\mu)(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})\right],\right. \\ &\left.\omega_{\mathcal{A}}\left[\left(\zeta^2+\lambda^2\right)\mathcal{T}^2 + \left(\eta^2+\mu^2\right)\mathcal{S}^2 - (\zeta\eta+\lambda\mu)(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})\right]\right\} \\ &+ \frac{1}{2}\max\{\omega_{\mathcal{A}}[(\zeta+\lambda)\mathcal{T} + (\eta+\mu)\mathcal{S}], \omega_{\mathcal{A}}[(\zeta-\lambda)\mathcal{T} + (\eta-\mu)\mathcal{S}]\} \\ &\times |\omega_{\mathcal{A}}(\zeta\mathcal{T}+\eta\mathcal{S}) - \omega_{\mathcal{A}}(\lambda\mathcal{T}+\mu\mathcal{S})|\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \\
&\geq \frac{1}{4}\max\Big\{\omega_{\mathcal{A}}\Big[\Big[(\zeta+\eta)^2+(\lambda+\mu)^2\Big]\mathcal{T}^2+\Big[(\zeta-\eta)^2+(\lambda-\mu)^2\Big]\mathcal{S}^2 \\
&+\Big(\zeta^2+\lambda^2-\eta^2-\mu^2\Big)(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T}), \\
&\omega_{\mathcal{A}}\Big[\Big[(\zeta-\eta)^2+(\lambda-\mu)^2\Big]\mathcal{T}^2+\Big[(\zeta+\eta)^2+(\lambda+\mu)^2\Big]\mathcal{S}^2 \\
&+\Big(\zeta^2+\lambda^2-\eta^2-\mu^2\Big)(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})\Big\} \\
&+\frac{1}{4}\max\{\omega_{\mathcal{A}}[(\zeta+\lambda+\eta+\mu)\mathcal{T}+(\zeta+\lambda-\eta-\mu)\mathcal{S}], \\
&\omega_{\mathcal{A}}[(\zeta+\eta-\lambda-\mu)\mathcal{T}+(\zeta-\eta-\lambda+\mu)\mathcal{S}]\} \\
&\times|\omega_{\mathcal{A}}((\zeta+\eta)\mathcal{T}+(\zeta-\eta)\mathcal{S})-\omega_{\mathcal{A}}((\lambda+\mu)\mathcal{T}+(\lambda-\mu)\mathcal{S})|.
\end{aligned} \tag{28}$$

Proof. By using the Cauchy–Schwarz inequality for $|\zeta|^2+|\eta|^2=1$, we have

$$\begin{aligned}
&\left|\langle\mathcal{T}x,x\rangle_{\mathcal{A}}\right|^2+\left|\langle\mathcal{S}x,x\rangle_{\mathcal{A}}\right|^2 \\
&=\Big(|\zeta|^2+|\eta|^2\Big)\Big(\left|\langle\mathcal{T}x,x\rangle_{\mathcal{A}}\right|^2+\left|\langle\mathcal{S}x,x\rangle_{\mathcal{A}}\right|^2\Big) \\
&\geq\left(|\zeta|\left|\langle\mathcal{T}x,x\rangle_{\mathcal{A}}\right|+|\eta|\left|\langle\mathcal{S}x,x\rangle_{\mathcal{A}}\right|\right)^2 \\
&=\left(\left|\langle\zeta\mathcal{T}x,x\rangle_{\mathcal{A}}\right|+\left|\langle\eta\mathcal{S}x,x\rangle_{\mathcal{A}}\right|\right)^2 \\
&\geq\left|\langle\zeta\mathcal{T}x,x\rangle_{\mathcal{A}}+\langle\eta\mathcal{S}x,x\rangle_{\mathcal{A}}\right|^2=\left|\langle(\zeta\mathcal{T}+\eta\mathcal{S})x,x\rangle_{\mathcal{A}}\right|^2
\end{aligned}$$

for all $x\in\mathbb{X}$.

If we take the supremum over $x\in\mathbb{X}$, $\|x\|_{\mathcal{A}}=1$, then we get

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})\geq\omega_{\mathcal{A}}^2(\zeta\mathcal{T}+\eta\mathcal{S}).$$

If $|\lambda|^2+|\mu|^2=1$, then we also have

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})\geq\omega_{\mathcal{A}}^2(\lambda\mathcal{T}+\mu\mathcal{S}).$$

Therefore,

$$\begin{aligned}
\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) &\geq \max\Big\{\omega_{\mathcal{A}}^2(\zeta\mathcal{T}+\eta\mathcal{S}),\omega_{\mathcal{A}}^2(\lambda\mathcal{T}+\mu\mathcal{S})\Big\} \\
&=\frac{1}{2}\Big[\omega_{\mathcal{A}}^2(\zeta\mathcal{T}+\eta\mathcal{S})+\omega_{\mathcal{A}}^2(\lambda\mathcal{T}+\mu\mathcal{S})\Big] \\
&+\frac{1}{2}\Big|\omega_{\mathcal{A}}^2(\zeta\mathcal{T}+\eta\mathcal{S})-\omega_{\mathcal{A}}^2(\lambda\mathcal{T}+\mu\mathcal{S})\Big| \\
&=\frac{1}{2}\Big[\omega_{\mathcal{A}}^2(\zeta\mathcal{T}+\eta\mathcal{S})+\omega_{\mathcal{A}}^2(\lambda\mathcal{T}+\mu\mathcal{S})\Big] \\
&+\frac{1}{2}[\omega_{\mathcal{A}}(\zeta\mathcal{T}+\eta\mathcal{S})+\omega_{\mathcal{A}}(\lambda\mathcal{T}+\mu\mathcal{S})] \\
&\times|\omega_{\mathcal{A}}(\zeta\mathcal{T}+\eta\mathcal{S})-\omega_{\mathcal{A}}(\lambda\mathcal{T}+\mu\mathcal{S})|.
\end{aligned} \tag{29}$$

Now, observing that

$$\begin{aligned}
&\omega_{\mathcal{A}}^2(\zeta\mathcal{T}+\eta\mathcal{S})+\omega_{\mathcal{A}}^2(\lambda\mathcal{T}+\mu\mathcal{S})\\
&\geq \omega_{\mathcal{A}}\Big[(\zeta\mathcal{T}+\eta\mathcal{S})^2\Big]+\omega_{\mathcal{A}}\Big[(\lambda\mathcal{T}+\mu\mathcal{S})^2\Big]\\
&\geq \max\Big\{\omega_{\mathcal{A}}\Big[(\zeta\mathcal{T}+\eta\mathcal{S})^2+(\lambda\mathcal{T}+\mu\mathcal{S})^2\Big],\\
&\omega_{\mathcal{A}}\Big[(\zeta\mathcal{T}+\eta\mathcal{S})^2-(\lambda\mathcal{T}+\mu\mathcal{S})^2\Big]\Big\}\\
&=\max\Big\{\omega_{\mathcal{A}}\Big[\Big(\zeta^2+\lambda^2\Big)\mathcal{T}^2+(\zeta\eta+\lambda\mu)(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})+\Big(\eta^2+\mu^2\Big)\mathcal{S}^2\Big],\\
&\omega_{\mathcal{A}}\Big[\Big(\zeta^2+\lambda^2\Big)\mathcal{T}^2-(\zeta\eta+\lambda\mu)(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})+\Big(\eta^2+\mu^2\Big)\mathcal{S}^2\Big]\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&\omega_{\mathcal{A}}(\zeta\mathcal{T}+\eta\mathcal{S})+\omega_{\mathcal{A}}(\lambda\mathcal{T}+\mu\mathcal{S})\\
&\geq \max\{\omega_{\mathcal{A}}[(\zeta+\lambda)\mathcal{T}+(\eta+\mu)\mathcal{S}],\omega_{\mathcal{A}}[(\zeta-\lambda)\mathcal{T}+(\eta-\mu)\mathcal{S}]\},
\end{aligned}$$

by (29) we derive (27).

Now, if we replace $\mathcal{T}$ with $\mathcal{S}+\mathcal{D}$ and $\mathcal{S}$ with $\mathcal{S}-\mathcal{D}$, we get

$$\begin{aligned}
&\omega_{e,\mathcal{A}}^2(\mathcal{S}+\mathcal{D},\mathcal{S}-\mathcal{D})\\
&\geq \frac{1}{2}\max\Big\{\omega_{\mathcal{A}}\Big[\Big(\zeta^2+\lambda^2\Big)(\mathcal{S}+\mathcal{D})^2+\Big(\eta^2+\mu^2\Big)(\mathcal{S}-\mathcal{D})^2\\
&+(\zeta\eta+\lambda\mu)((\mathcal{S}+\mathcal{D})(\mathcal{S}-\mathcal{D})+(\mathcal{S}-\mathcal{D})(\mathcal{S}+\mathcal{D})),\\
&\omega_{\mathcal{A}}\Big[\Big(\zeta^2+\lambda^2\Big)(\mathcal{S}+\mathcal{D})^2+\Big(\eta^2+\mu^2\Big)(\mathcal{S}-\mathcal{D})^2\\
&-(\zeta\eta+\lambda\mu)((\mathcal{S}+\mathcal{D})(\mathcal{S}-\mathcal{D})+(\mathcal{S}-\mathcal{D})(\mathcal{S}+\mathcal{D}))]\}\\
&+\frac{1}{2}\max\{\omega_{\mathcal{A}}[(\zeta+\lambda)(\mathcal{S}+\mathcal{D})+(\eta+\mu)(\mathcal{S}-\mathcal{D})],\\
&\omega_{\mathcal{A}}[(\zeta-\lambda)(\mathcal{S}+\mathcal{D})+(\eta-\mu)(\mathcal{S}-\mathcal{D})]\}\\
&\times|\omega_{\mathcal{A}}(\zeta(\mathcal{S}+\mathcal{D})+\eta(\mathcal{S}-\mathcal{D}))-\omega(\lambda(\mathcal{S}+\mathcal{D})+\mu(\mathcal{S}-\mathcal{D}))|.
\end{aligned}\tag{30}$$

Observe that

$$\begin{aligned}
&\Big(\zeta^2+\lambda^2\Big)(\mathcal{S}+\mathcal{D})^2+\Big(\eta^2+\mu^2\Big)(\mathcal{S}-\mathcal{D})^2\\
&=\Big(\zeta^2+\lambda^2\Big)\Big(\mathcal{S}^2+\mathcal{D}^2+\mathcal{S}\mathcal{D}+\mathcal{D}\mathcal{S}\Big)+\Big(\eta^2+\mu^2\Big)\Big(\mathcal{S}^2+\mathcal{D}^2-\mathcal{S}\mathcal{D}-\mathcal{D}\mathcal{S}\Big)\\
&=\Big(\zeta^2+\lambda^2+\eta^2+\mu^2\Big)\Big(\mathcal{S}^2+\mathcal{D}^2\Big)+\Big(\zeta^2+\lambda^2-\eta^2-\mu^2\Big)(\mathcal{S}\mathcal{D}+\mathcal{D}\mathcal{S})
\end{aligned}$$

$$\begin{aligned}
&(\mathcal{S}+\mathcal{D})(\mathcal{S}-\mathcal{D})+(\mathcal{S}-\mathcal{D})(\mathcal{S}+\mathcal{D})\\
&=\mathcal{S}^2-\mathcal{D}^2+\mathcal{D}\mathcal{S}-\mathcal{S}\mathcal{D}+\mathcal{S}^2-\mathcal{D}^2=\mathcal{D}\mathcal{S}+\mathcal{S}\mathcal{D}\\
&=2\Big(\mathcal{S}^2-\mathcal{D}^2\Big);
\end{aligned}$$

then,

$$\begin{aligned}
&\left(\zeta^2+\lambda^2\right)(\mathcal{S}+\mathcal{D})^2+\left(\eta^2+\mu^2\right)(\mathcal{S}-\mathcal{D})^2\\
&+(\zeta\eta+\lambda\mu)((\mathcal{S}+\mathcal{D})(\mathcal{S}-\mathcal{D})+(\mathcal{S}-\mathcal{D})(\mathcal{S}+\mathcal{D}))\\
&=\left(\zeta^2+\lambda^2+\eta^2+\mu^2\right)\left(\mathcal{S}^2+\mathcal{D}^2\right)+\left(\zeta^2+\lambda^2-\eta^2-\mu^2\right)(\mathcal{S}\mathcal{D}+\mathcal{D}\mathcal{S})\\
&+2(\zeta\eta+\lambda\mu)\left(\mathcal{S}^2-\mathcal{D}^2\right)\\
&=\left(\zeta^2+\lambda^2+\eta^2+\mu^2+2(\zeta\eta+\lambda\mu)\right)\mathcal{S}^2\\
&+\left(\zeta^2+\lambda^2+\eta^2+\mu^2-2(\zeta\eta+\lambda\mu)\right)\mathcal{D}^2\\
&+\left(\zeta^2+\lambda^2-\eta^2-\mu^2\right)(\mathcal{S}\mathcal{D}+\mathcal{D}\mathcal{S})\\
&=\left[(\zeta+\eta)^2+(\lambda+\mu)^2\right]\mathcal{S}^2+\left[(\zeta-\eta)^2+(\lambda-\mu)^2\right]\mathcal{D}^2\\
&+\left(\zeta^2+\lambda^2-\eta^2-\mu^2\right)(\mathcal{S}\mathcal{D}+\mathcal{D}\mathcal{S})
\end{aligned}$$

and

$$\begin{aligned}
&\left(\zeta^2+\lambda^2\right)(\mathcal{S}+\mathcal{D})^2+\left(\eta^2+\mu^2\right)(\mathcal{S}-\mathcal{D})^2\\
&-(\zeta\eta+\lambda\mu)((\mathcal{S}+\mathcal{D})(\mathcal{S}-\mathcal{D})+(\mathcal{S}-\mathcal{D})(\mathcal{S}+\mathcal{D}))\\
&=\left(\zeta^2+\lambda^2+\eta^2+\mu^2\right)\left(\mathcal{S}^2+\mathcal{D}^2\right)+\left(\zeta^2+\lambda^2-\eta^2-\mu^2\right)(\mathcal{S}\mathcal{D}+\mathcal{D}\mathcal{S})\\
&-2(\zeta\eta+\lambda\mu)\left(\mathcal{S}^2-\mathcal{D}^2\right)\\
&=\left(\zeta^2+\lambda^2+\eta^2+\mu^2-2(\zeta\eta+\lambda\mu)\right)\mathcal{S}^2\\
&+\left(\zeta^2+\lambda^2+\eta^2+\mu^2+2(\zeta\eta+\lambda\mu)\right)\mathcal{D}^2\\
&+\left(\zeta^2+\lambda^2-\eta^2-\mu^2\right)(\mathcal{S}\mathcal{D}+\mathcal{D}\mathcal{S})\\
&=\left[(\zeta-\eta)^2+(\lambda-\mu)^2\right]\mathcal{S}^2+\left[(\zeta+\eta)^2+(\lambda+\mu)^2\right]\mathcal{D}^2\\
&+\left(\zeta^2+\lambda^2-\eta^2-\mu^2\right)(\mathcal{S}\mathcal{D}+\mathcal{D}\mathcal{S}).
\end{aligned}$$

Additionally,

$$(\zeta+\lambda)(\mathcal{S}+\mathcal{D})+(\eta+\mu)(\mathcal{S}-\mathcal{D})=(\zeta+\lambda+\eta+\mu)\mathcal{S}+(\zeta+\lambda-\eta-\mu)\mathcal{D}$$

and

$$(\zeta-\lambda)(\mathcal{S}+\mathcal{D})+(\eta-\mu)(\mathcal{S}-\mathcal{D})=(\zeta+\eta-\lambda-\mu)\mathcal{S}+(\zeta-\lambda-\eta+\mu)\mathcal{D}.$$

Moreover,

$$\omega^2_{e,\mathcal{A}}(\mathcal{S}+\mathcal{D},\mathcal{S}-\mathcal{D})=2\omega^2_{e,\mathcal{A}}(\mathcal{S},\mathcal{D}),$$

and by (30) we get

$$
\begin{aligned}
&2\omega_{e,\mathcal{A}}^2(\mathcal{S},\mathcal{D})\\
&\geq \frac{1}{2}\max\Big\{\omega_{\mathcal{A}}\Big[\Big[(\zeta+\eta)^2+(\lambda+\mu)^2\Big]\mathcal{S}^2+\Big[(\zeta-\eta)^2+(\lambda-\mu)^2\Big]\mathcal{D}^2\\
&+\Big(\zeta^2+\lambda^2-\eta^2-\mu^2\Big)(\mathcal{S}\mathcal{D}+\mathcal{D}\mathcal{S}),\\
&\omega_{\mathcal{A}}\Big[\Big[(\zeta-\eta)^2+(\lambda-\mu)^2\Big]\mathcal{S}^2+\Big[(\zeta+\eta)^2+(\lambda+\mu)^2\Big]\mathcal{D}^2\\
&+\Big(\zeta^2+\lambda^2-\eta^2-\mu^2\Big)(\mathcal{S}\mathcal{D}+\mathcal{D}\mathcal{S})\Big\}\\
&+\frac{1}{2}\max\{\omega_{\mathcal{A}}[(\zeta+\lambda+\eta+\mu)\mathcal{S}+(\zeta+\lambda-\eta-\mu)\mathcal{D}],\\
&\omega_{\mathcal{A}}[(\zeta+\eta-\lambda-\mu)\mathcal{S}+(\zeta-\lambda-\eta+\mu)\mathcal{D}]\}\\
&\times|\omega_{\mathcal{A}}((\zeta+\eta)\mathcal{S}+(\zeta-\eta)\mathcal{D})-\omega((\lambda+\mu)\mathcal{S}+(\lambda-\mu)\mathcal{D})|.
\end{aligned}
$$

By replacing $\mathcal{S}$ with $\mathcal{T}$ and $\mathcal{D}$ with $\mathcal{S}$, the inequality in (28) is obtained. □

Remark 4. *By setting $\zeta=0$, $\eta=1$, $\lambda=1$, and $\mu=0$ in (27), we obtain*

$$
\begin{aligned}
\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) &\geq \frac{1}{2}\omega_{\mathcal{A}}\Big(\mathcal{T}^2+\mathcal{S}^2\Big)\\
&+\frac{1}{2}\max\{\omega_{\mathcal{A}}(\mathcal{S}+\mathcal{T}),\omega_{\mathcal{A}}(\mathcal{S}-\mathcal{T})\}|\omega_{\mathcal{A}}(\mathcal{S})-\omega_{\mathcal{A}}(\mathcal{T})|,
\end{aligned}
\tag{31}
$$

while from (28) we get

$$
\begin{aligned}
\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) &\geq \frac{1}{2}\omega_{\mathcal{A}}\Big(\mathcal{T}^2+\mathcal{S}^2\Big)+\frac{1}{2}\max\{\omega_{\mathcal{A}}(\mathcal{T}),\omega_{\mathcal{A}}(\mathcal{S})\}\\
&\times|\omega_{\mathcal{A}}(\mathcal{T}-\mathcal{S})-\omega_{\mathcal{A}}(\mathcal{T}+\mathcal{S})|.
\end{aligned}
\tag{32}
$$

Both inequalities (31) and (32) refine the inequality

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \geq \frac{1}{2}\omega_{\mathcal{A}}\Big(\mathcal{T}^2+\mathcal{S}^2\Big)$$

established in (5) from the Introduction.

Remark 5. *By setting $\lambda=\zeta$ and $\mu=\eta$ in (27), we obtain*

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \geq \max\Big\{\omega_{\mathcal{A}}\Big[(\zeta\mathcal{T}+\eta\mathcal{S})^2\Big],\omega_{\mathcal{A}}\Big[(\zeta\mathcal{T}-\eta\mathcal{S})^2\Big]\Big\}$$

for $\mathcal{T},\mathcal{S}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $\zeta,\eta\in\mathbb{C}$ with $|\zeta|^2+|\eta|^2=1$. For $\zeta=\eta=\frac{\sqrt{2}}{2}$, this yields

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \geq \frac{1}{2}\max\Big\{\omega_{\mathcal{A}}\Big[(\mathcal{T}+\mathcal{S})^2\Big],\omega_{\mathcal{A}}\Big[(\mathcal{T}-\mathcal{S})^2\Big]\Big\}. \tag{33}$$

Setting $\lambda=\eta$ and $\mu=\zeta$ in (27), we get

$$\begin{aligned}
&\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \\
&\geq \frac{1}{2}\max\Big\{\omega_{\mathcal{A}}\Big[\Big(\zeta^2+\eta^2\Big)\Big(\mathcal{T}^2+\mathcal{S}^2\Big)+2\zeta\eta(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})\Big], \\
&\omega_{\mathcal{A}}\Big[\Big(\zeta^2+\eta^2\Big)\Big(\mathcal{T}^2+\mathcal{S}^2\Big)-2\zeta\eta(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})\Big]\Big\} \\
&+\frac{1}{2}\max\{|\zeta+\eta|\omega_{\mathcal{A}}(\mathcal{T}+\mathcal{S}),|\zeta-\eta|\omega_{\mathcal{A}}(\mathcal{T}-\mathcal{S})\} \\
&\times|\omega_{\mathcal{A}}(\zeta\mathcal{T}+\eta\mathcal{S})-\omega_{\mathcal{A}}(\eta\mathcal{T}+\zeta\mathcal{S})|
\end{aligned} \tag{34}$$

for $\mathcal{T},\mathcal{S}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$ *and* $\zeta,\eta\in\mathbb{C}$ *with* $|\zeta|^2+|\eta|^2=1$. *For* $\zeta=\eta=\frac{\sqrt{2}}{2}$, *this again yields (33).*

Setting $\lambda=i\zeta$ *and* $\mu=-i\eta$ *in (27), we obtain*

$$\begin{aligned}
\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) &\geq |\zeta\eta|\omega_{\mathcal{A}}(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T}) \\
&+\frac{1}{2}\max\{\omega_{\mathcal{A}}[\zeta(1+i)\mathcal{T}+\eta(1-i)\mathcal{S}], \\
&\omega_{\mathcal{A}}[\zeta(1-i)\mathcal{T}+\eta(1+i)\mathcal{S}]\} \\
&\times|\omega_{\mathcal{A}}(\zeta\mathcal{T}+\eta\mathcal{S})-\omega_{\mathcal{A}}(\zeta\mathcal{T}-\eta\mathcal{S})|
\end{aligned} \tag{35}$$

for $\mathcal{T},\mathcal{S}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$ *and* $\zeta,\eta\in\mathbb{C}$ *with* $|\zeta|^2+|\eta|^2=1$. *For* $\zeta=\eta=\frac{\sqrt{2}}{2}$, *we get*

$$\begin{aligned}
\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) &\geq \frac{1}{2}\omega_{\mathcal{A}}(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T}) \\
&+\frac{1}{4}\max\{\omega_{\mathcal{A}}[(1+i)\mathcal{T}+(1-i)\mathcal{S}], \\
&\omega_{\mathcal{A}}[(1-i)\mathcal{T}+(1+i)\mathcal{S}]\} \\
&\times|\omega_{\mathcal{A}}(\mathcal{T}+\mathcal{S})-\omega_{\mathcal{A}}(\mathcal{T}-\mathcal{S})|.
\end{aligned} \tag{36}$$

Setting $\lambda=i\eta$ *and* $\mu=-i\zeta$ *in (27), we obtain*

$$\begin{aligned}
&\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \\
&\geq \frac{1}{2}\max\Big\{\omega_{\mathcal{A}}\Big[\Big(\eta^2-\zeta^2\Big)\Big(\mathcal{S}^2-\mathcal{T}^2\Big)+2\zeta\eta(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})\Big], \\
&\omega_{\mathcal{A}}\Big[\Big(\eta^2-\zeta^2\Big)\Big(\mathcal{S}^2-\mathcal{T}^2\Big)-2\zeta\eta(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})\Big]\Big\} \\
&+\frac{1}{2}\max\{\omega_{\mathcal{A}}[(\zeta+i\eta)\mathcal{T}+(\eta-i\zeta)\mathcal{S}],\omega_{\mathcal{A}}[(\zeta-i\eta)\mathcal{T}+(\eta+i\zeta)\mathcal{S}]\} \\
&\times|\omega_{\mathcal{A}}(\zeta\mathcal{T}+\eta\mathcal{S})-\omega_{\mathcal{A}}(\eta\mathcal{T}-\zeta\mathcal{S})|
\end{aligned}$$

for $\mathcal{T},\mathcal{S}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$ *and* $\zeta,\eta\in\mathbb{C}$ *with* $|\zeta|^2+|\eta|^2=1$. *For* $\zeta=\frac{\sqrt{2}}{2}i$ *and* $\eta=\frac{\sqrt{2}}{2}$, *we get*

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})\geq\frac{1}{2}\max\Big\{\omega_{\mathcal{A}}\Big[(\mathcal{S}+i\mathcal{T})^2\Big],\omega_{\mathcal{A}}\Big[(\mathcal{S}-i\mathcal{T})^2\Big]\Big\}.$$

By choosing $\zeta=\sin\theta$, $\eta=\cos\theta$, $\theta\in\mathbb{R}$, *we can also state the trigonometric inequalities*

$$\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S})\geq\max\Big\{\omega_{\mathcal{A}}\Big[(\sin\theta\mathcal{T}+\cos\theta\mathcal{S})^2\Big],\omega_{\mathcal{A}}\Big[(\sin\theta\mathcal{T}-\cos\theta\mathcal{S})^2\Big]\Big\},$$

$$\begin{aligned}\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \geq \frac{1}{2}\max\Big\{&\omega_{\mathcal{A}}\Big[\mathcal{T}^2+\mathcal{S}^2+\sin(2\theta)(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})\Big],\\ &\omega_{\mathcal{A}}\Big[\mathcal{T}^2+\mathcal{S}^2-\sin(2\theta)(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})\Big]\Big\}\\ &+\frac{1}{2}\max\{|\sin\theta+\cos\theta|\omega_{\mathcal{A}}(\mathcal{T}+\mathcal{S}),|\sin\theta-\cos\theta|\omega_{\mathcal{A}}(\mathcal{T}-\mathcal{S})\}\\ &\times|\omega_{\mathcal{A}}(\sin\theta\mathcal{T}+\cos\theta\mathcal{S})-\omega_{\mathcal{A}}(\cos\theta\mathcal{T}+\sin\theta\mathcal{S})|,\end{aligned}$$

$$\begin{aligned}\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \geq{}&\frac{1}{2}|\sin(2\theta)|\omega_{\mathcal{A}}(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})\\ &+\frac{1}{2}\max\{\omega_{\mathcal{A}}[\sin\theta(1+i)\mathcal{T}+\cos\theta(1-i)\mathcal{S}],\\ &\omega_{\mathcal{A}}[\sin\theta(1-i)\mathcal{T}+\cos\theta(1+i)\mathcal{S}]\}\\ &\times|\omega_{\mathcal{A}}(\sin\theta\mathcal{T}+\cos\theta\mathcal{S})-\omega_{\mathcal{A}}(\sin\theta\mathcal{T}-\cos\theta\mathcal{S})|,\end{aligned}$$

and

$$\begin{aligned}\omega_{e,\mathcal{A}}^2(\mathcal{T},\mathcal{S}) \geq \frac{1}{2}\max\Big\{&\omega_{\mathcal{A}}\Big[\cos(2\theta)\Big(\mathcal{S}^2-\mathcal{T}^2\Big)+\sin(2\theta)(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})\Big],\\ &\omega_{\mathcal{A}}\Big[\cos(2\theta)\Big(\mathcal{S}^2-\mathcal{T}^2\Big)-\sin(2\theta)(\mathcal{T}\mathcal{S}+\mathcal{S}\mathcal{T})\Big]\Big\}\\ &+\frac{1}{2}\max\{\omega_{\mathcal{A}}[(\sin\theta+i\cos\theta)\mathcal{T}+(\cos\theta-i\sin\theta)\mathcal{S}],\\ &\omega_{\mathcal{A}}[(\sin\theta-i\cos\theta)\mathcal{T}+(\cos\theta+i\sin\theta)\mathcal{S}]\}\\ &\times|\omega_{\mathcal{A}}(\sin\theta\mathcal{T}+\cos\theta\mathcal{S})-\omega_{\mathcal{A}}(\cos\theta\mathcal{T}-\sin\theta\mathcal{S})|\end{aligned}$$

for $\mathcal{T},\mathcal{S}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$ *and* $\theta\in\mathbb{R}$.

4. Applications to a Single Operator

For $\mathcal{T}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$, setting $\mathcal{T}=\mathcal{T}^{\sharp_{\mathcal{A}}}$ and $\mathcal{S}=\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\right)^{\sharp_{\mathcal{A}}}$, we have

$$\omega_{e,\mathcal{A}}^2\left(\mathcal{T},\mathcal{T}^{\sharp_{\mathcal{A}}}\right)=2\omega_{\mathcal{A}}^2\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\right)=2\omega_{\mathcal{A}}^2(\mathcal{T}).$$

By applying Theorem 1 combined with (1) and the properties that $\omega_{\mathcal{A}}(\mathcal{X}^{\sharp_{\mathcal{A}}})=\omega_{\mathcal{A}}(\mathcal{X})$ and $\|\mathcal{X}^{\sharp_{\mathcal{A}}}\|_{\mathcal{A}}=\|\mathcal{X}\|_{\mathcal{A}}$ for all $\mathcal{X}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$, we obtain for $r\geq 1$ that

$$\omega_{\mathcal{A}}^2(\mathcal{T})\leq\frac{1}{2^{\frac{1}{r}}}\left[\|\mathcal{T}\|_{\mathcal{A}}^{2r}+\omega_{\mathcal{A}}^r\left(\mathcal{T}^2\right)\right]^{\frac{1}{r}}$$

and

$$\begin{aligned}&\omega_{\mathcal{A}}^2(\mathcal{T})\\ &\leq\frac{1}{2^{\frac{1}{r}}}\left[\frac{\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}^r+\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}-\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}^r}{2}+\omega_{\mathcal{A}}^r\left(\mathcal{T}^2\right)\right]^{\frac{1}{r}}.\end{aligned}$$

For $r=1$, we derive

$$\omega_{\mathcal{A}}^2(\mathcal{T})\leq\frac{1}{2}\left[\|\mathcal{T}\|_{\mathcal{A}}^2+\omega_{\mathcal{A}}\left(\mathcal{T}^2\right)\right],\tag{37}$$

which was established in [36], Corollary 2.16 and

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{1}{2}\left[\frac{\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}+\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}-\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}}{2}+\omega_{\mathcal{A}}\left(\mathcal{T}^2\right)\right].$$

For $r=2$, we obtain

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{\sqrt{2}}{2}\left[\|\mathcal{T}\|_{\mathcal{A}}^4+\omega_{\mathcal{A}}^2\left(\mathcal{T}^2\right)\right]^{\frac{1}{2}}$$

and

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{\sqrt{2}}{2}\left[\frac{\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}^2+\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}-\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}^2}{2}+\omega_{\mathcal{A}}^2\left(\mathcal{T}^2\right)\right]^{\frac{1}{2}}.$$

Similarly, by setting $\mathcal{T}=\mathcal{T}^{\sharp_{\mathcal{A}}}$ and $\mathcal{S}=(\mathcal{T}^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}}$ in Theorem 2, we obtain for $r\geq 2$ that

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{1}{4^{\frac{1}{r}}}\left\{\left\|\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}}{2}\right\|_{\mathcal{A}}^{\frac{r}{2}}+\omega_{\mathcal{A}}^{\frac{r}{2}}\left(\mathcal{T}^2\right)\right\}^{2/r}.$$

For $r=2$, we obtain

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{1}{2}\left\{\left\|\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}}{2}\right\|_{\mathcal{A}}+\omega_{\mathcal{A}}\left(\mathcal{T}^2\right)\right\}, \tag{38}$$

while for $r=4$ we get

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{\sqrt{2}}{2}\left\{\left\|\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}}{2}\right\|_{\mathcal{A}}^2+\omega_{\mathcal{A}}^2\left(\mathcal{T}^2\right)\right\}^{\frac{1}{2}}.$$

Because

$$\left\|\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}}{2}\right\|_{\mathcal{A}} \leq \frac{1}{2}\left(\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}\right\|_{\mathcal{A}}+\left\|\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}\right)=\|\mathcal{T}\|_{\mathcal{A}}^2,$$

it follows that (38) refines (37).

Now, let $\mathcal{T}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and consider its $\mathcal{A}$-Cartesian decomposition $\mathcal{T}=\mathcal{B}+i\mathcal{C}$, where $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{A}$-selfadjoint operators defined by

$$\mathcal{B}=\Re_{\mathcal{A}}(\mathcal{T}):=\frac{1}{2}\left(\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\right) \quad \text{and} \quad \mathcal{C}=\Im_{\mathcal{A}}(\mathcal{T}):=\frac{1}{2i}\left(\mathcal{T}-\mathcal{T}^{\sharp_{\mathcal{A}}}\right).$$

It can be seen that

$$\omega_{e,\mathcal{A}}^2\left(\mathcal{B}^{\sharp_{\mathcal{A}}},\mathcal{C}^{\sharp_{\mathcal{A}}}\right)=\omega_{\mathcal{A}}^2\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\right)=\omega_{\mathcal{A}}^2(\mathcal{T}), \tag{39}$$

$$\left(\mathcal{B}^{\sharp_{\mathcal{A}}}\right)^2+\left(\mathcal{C}^{\sharp_{\mathcal{A}}}\right)^2=\left(\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}}{2}\right)^{\sharp_{\mathcal{A}}}, \tag{40}$$

and

$$\left(\mathcal{B}^{\sharp_{\mathcal{A}}}\right)^2-\left(\mathcal{C}^{\sharp_{\mathcal{A}}}\right)^2=\left(\frac{\mathcal{T}^2+\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\right)^2}{2}\right)^{\sharp_{\mathcal{A}}}. \tag{41}$$

By applying Theorem 1 to the $\mathcal{A}$-Cartesian decomposition of $\mathcal{T}^{\sharp_{\mathcal{A}}}$, combined with (2), (39), and the properties that $\omega_{\mathcal{A}}(\mathcal{X}^{\sharp_{\mathcal{A}}}) = \omega_{\mathcal{A}}(\mathcal{X})$ and $\|\mathcal{X}^{\sharp_{\mathcal{A}}}\|_{\mathcal{A}} = \|\mathcal{X}\|_{\mathcal{A}}$ for all $\mathcal{X} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$, we obtain for $r \geq 1$ that

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{1}{2^{1+\frac{1}{r}}} \left[\max\left\{ \left\| \mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}} \right\|_{\mathcal{A}}^{2r}, \left\| \mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}} \right\|_{\mathcal{A}}^{2r} \right\} + \omega_{\mathcal{A}}^r\left(\left(\mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \left(\mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \right) \right]^{\frac{1}{r}}. \tag{42}$$

Further, by taking into account (40) and (41), we get

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{1}{2^{\frac{1}{r}}} \left[\frac{\left\| \mathcal{T}^{\sharp_{\mathcal{A}}} \mathcal{T} + \mathcal{T} \mathcal{T}^{\sharp_{\mathcal{A}}} \right\|_{\mathcal{A}}^{r} + \left\| \mathcal{T}^2 + \left(\mathcal{T}^{\sharp_{\mathcal{A}}} \right)^2 \right\|_{\mathcal{A}}^{r}}{2} + \frac{1}{2^r} \omega_{\mathcal{A}}^r\left(\left(\mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \left(\mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \right) \right]^{\frac{1}{r}}. \tag{43}$$

By setting $r = 1$ in (42) and (43), we obtain results analogous to those in ([17], Equation (2.29)) and ([36], Corollary 2.18), i.e., we have

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{1}{4} \max\left\{ \left\| \mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}} \right\|_{\mathcal{A}}^{2}, \left\| \mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}} \right\|_{\mathcal{A}}^{2} \right\} + \frac{1}{4} \omega_{\mathcal{A}}\left(\left(\mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \left(\mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \right)$$

and

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{\left\| \mathcal{T}^{\sharp_{\mathcal{A}}} \mathcal{T} + \mathcal{T} \mathcal{T}^{\sharp_{\mathcal{A}}} \right\|_{\mathcal{A}} + \left\| \mathcal{T}^2 + \left(\mathcal{T}^{\sharp_{\mathcal{A}}} \right)^2 \right\|_{\mathcal{A}}}{4} + \frac{1}{4} \omega_{\mathcal{A}}\left(\left(\mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \left(\mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \right).$$

For $r = 2$ in (42) and (43), we obtain

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{\sqrt{2}}{4} \left[\max\left\{ \left\| \mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}} \right\|_{\mathcal{A}}^{4}, \left\| \mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}} \right\|_{\mathcal{A}}^{4} \right\} + \omega_{\mathcal{A}}^2\left(\left(\mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \left(\mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \right) \right]^{\frac{1}{2}}$$

and

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{\sqrt{2}}{2} \left[\frac{\left\| \mathcal{T}^{\sharp_{\mathcal{A}}} \mathcal{T} + \mathcal{T} \mathcal{T}^{\sharp_{\mathcal{A}}} \right\|_{\mathcal{A}}^{2} + \left\| \mathcal{T}^2 + \left(\mathcal{T}^{\sharp_{\mathcal{A}}} \right)^2 \right\|_{\mathcal{A}}^{2}}{2} + \frac{1}{4} \omega_{\mathcal{A}}^2\left(\left(\mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \left(\mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \right) \right]^{\frac{1}{2}}.$$

Similarly, applying Theorem 2 to the $\mathcal{A}$-Cartesian decomposition of $\mathcal{T}$, for $r \geq 2$ we obtain

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{1}{2^{1+\frac{1}{r}}} \max\left\{ \left\| \mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}} \right\|_{\mathcal{A}}, \left\| \mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}} \right\|_{\mathcal{A}} \right\} \times \left\{ \left\| \mathcal{T}^{\sharp_{\mathcal{A}}} \mathcal{T} + \mathcal{T} \mathcal{T}^{\sharp_{\mathcal{A}}} \right\|_{\mathcal{A}}^{\frac{r}{2}} + \omega_{\mathcal{A}}^{\frac{r}{2}}\left(\left(\mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \left(\mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}} \right) \right) \right\}^{\frac{1}{r}}.$$

For $r = 2$, we obtain

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{\sqrt{2}}{4}\max\left\{\left\|\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}, \left\|\mathcal{T}-\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}\right\} \times \left\{\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}} + \omega_{\mathcal{A}}\left(\left(\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\right)\left(\mathcal{T}-\mathcal{T}^{\sharp_{\mathcal{A}}}\right)\right)\right\}^{\frac{1}{2}},$$

while for $r = 4$ we get

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \leq \frac{\sqrt[4]{8}}{4}\max\left\{\left\|\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}, \left\|\mathcal{T}-\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}\right\} \times \left\{\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}^2 + \omega_{\mathcal{A}}^2\left(\left(\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\right)\left(\mathcal{T}-\mathcal{T}^{\sharp_{\mathcal{A}}}\right)\right)\right\}^{\frac{1}{4}}.$$

For $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$, by proceeding as above and setting $\mathcal{T} = \mathcal{T}^{\sharp_{\mathcal{A}}}$ and $\mathcal{S} = (\mathcal{T}^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}}$ in Theorem 3, we obtain for $t \in [0,1]$ that

$$\omega_{\mathcal{A}}(\mathcal{T}) \leq \left\|\frac{(1-t)^2\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+t^2\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}}{2}\right\|_{\mathcal{A}}^{\frac{1}{2}} + \left\|\frac{t^2\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+(1-t)^2\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}}{2}\right\|_{\mathcal{A}}^{\frac{1}{2}}$$

and

$$\omega_{\mathcal{A}}(\mathcal{T}) \leq \sqrt{2}\left[\left(\frac{1}{2}+\left|t-\frac{1}{2}\right|\right)^2\|\mathcal{T}\|_{\mathcal{A}}^2 + (1-t)t\omega_{\mathcal{A}}\left(\mathcal{T}^2\right)\right]^{\frac{1}{2}} \tag{44}$$

for all $t \in [0,1]$.

Setting $t = \frac{1}{2}$ in (44), we recover (37).

Now, by setting $\mathcal{T} = \mathcal{T}$ and $\mathcal{S} = \mathcal{T}^{\sharp_{\mathcal{A}}}$ in Theorem 4, we obtain

$$\begin{aligned}
&\omega_{\mathcal{A}}^2(\mathcal{T}) \\
&\geq \frac{1}{4}\max\left\{\omega_{\mathcal{A}}\left[\left(\zeta^2+\lambda^2\right)\mathcal{T}^2+\left(\eta^2+\mu^2\right)\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\right)^2+(\zeta\eta+\lambda\mu)\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right)\right],\right. \\
&\left.\omega_{\mathcal{A}}\left[\left(\zeta^2+\lambda^2\right)\mathcal{T}^2+\left(\eta^2+\mu^2\right)\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\right)^2-(\zeta\eta+\lambda\mu)\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right)\right]\right\} \\
&+\frac{1}{4}\max\left\{\omega_{\mathcal{A}}\left[(\zeta+\lambda)\mathcal{T}+(\eta+\mu)\mathcal{T}^{\sharp_{\mathcal{A}}}\right], \omega_{\mathcal{A}}\left[(\zeta-\lambda)\mathcal{T}+(\eta-\mu)\mathcal{T}^{\sharp_{\mathcal{A}}}\right]\right\} \\
&\times\left|\omega_{\mathcal{A}}\left(\zeta\mathcal{T}+\eta\mathcal{T}^{\sharp_{\mathcal{A}}}\right)-\omega_{\mathcal{A}}\left(\lambda\mathcal{T}+\mu\mathcal{T}^{\sharp_{\mathcal{A}}}\right)\right|
\end{aligned}$$

and for $\zeta, \eta, \lambda, \mu \in \mathbb{C}$ with $|\zeta|^2+|\eta|^2 = |\lambda|^2+|\mu|^2 = 1$, we have

$$\begin{aligned}
\omega_{\mathcal{A}}^2(\mathcal{T}) \geq \frac{1}{8}\max&\left\{\omega_{\mathcal{A}}\left[a\mathcal{T}^2+b(\mathcal{T}^{\sharp_{\mathcal{A}}})^2+c(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}})\right],\right. \\
&\left.\omega_{\mathcal{A}}\left[b\mathcal{T}^2+a(\mathcal{T}^{\sharp_{\mathcal{A}}})^2+c(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}})\right]\right\} \\
+\frac{1}{8}\max&\left\{\omega_{\mathcal{A}}\left[d\mathcal{T}+e\mathcal{T}^{\sharp_{\mathcal{A}}}\right], \omega_{\mathcal{A}}\left[f\mathcal{T}+g\mathcal{T}^{\sharp_{\mathcal{A}}}\right]\right\} \\
&\times\left|\omega_{\mathcal{A}}\left(h\mathcal{T}+i\mathcal{T}^{\sharp_{\mathcal{A}}}\right)-\omega_{\mathcal{A}}\left(j\mathcal{T}+k\mathcal{T}^{\sharp_{\mathcal{A}}}\right)\right|,
\end{aligned}$$

where $a = (\zeta+\eta)^2+(\lambda+\mu)^2$, $b = (\zeta-\eta)^2+(\lambda-\mu)^2$, $c = \zeta^2+\lambda^2-\eta^2-\mu^2$, $d = \zeta+\lambda+\eta+\mu$, $e = \zeta+\lambda-\eta-\mu$, $f = \zeta+\eta-\lambda-\mu$, $g = \zeta-\eta-\lambda+\mu$, $h = \zeta+\eta$, $i = \zeta-\eta$, $j = \lambda+\mu$, and $k = \lambda-\mu$.

Setting $(\mathcal{T},\mathcal{S}) = (\mathcal{T},\mathcal{T}^{\sharp_{\mathcal{A}}})$ and $\lambda = \zeta$, $\mu = \eta$ in (27), we obtain

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \geq \frac{1}{2}\max\left\{\omega_{\mathcal{A}}\left[\left(\zeta\mathcal{T}+\eta\mathcal{T}^{\sharp_{\mathcal{A}}}\right)^2\right], \omega_{\mathcal{A}}\left[\left(\zeta\mathcal{T}-\eta\mathcal{T}^{\sharp_{\mathcal{A}}}\right)^2\right]\right\}$$

for $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $\zeta,\eta \in \mathbb{C}$ with $|\zeta|^2+|\eta|^2 = 1$. For $\zeta = \eta = \frac{\sqrt{2}}{2}$, we obtain

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \geq \frac{1}{4}\max\left\{\left\|\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}^2, \left\|\mathcal{T}-\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}^2\right\}.$$

Setting $(\mathcal{T},\mathcal{S}) = (\mathcal{T},\mathcal{T}^{\sharp_{\mathcal{A}}})$ and $\lambda = \eta$, $\mu = \zeta$ in (27), we obtain

$$\begin{aligned}
&\omega_{\mathcal{A}}^2(\mathcal{T})\\
&\geq \frac{1}{4}\max\left\{\omega_{\mathcal{A}}\left[\left(\zeta^2+\eta^2\right)\left(\mathcal{T}^2+\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\right)^2\right)+2\zeta\eta\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right)\right],\right.\\
&\left.\omega_{\mathcal{A}}\left[\left(\zeta^2+\eta^2\right)\left(\mathcal{T}^2+\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\right)^2\right)-2\zeta\eta\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right)\right]\right\}\\
&+\frac{1}{4}\max\left\{|\zeta+\eta|\left\|\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}, |\zeta-\eta|\left\|\mathcal{T}-\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}\right\}\\
&\times\left|\omega_{\mathcal{A}}\left(\zeta\mathcal{T}+\eta\mathcal{T}^{\sharp_{\mathcal{A}}}\right)-\omega_{\mathcal{A}}\left(\eta\mathcal{T}+\zeta\mathcal{T}^{\sharp_{\mathcal{A}}}\right)\right|
\end{aligned}$$

for $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $\zeta,\eta \in \mathbb{C}$ with $|\zeta|^2+|\eta|^2 = 1$.

For $\zeta = 0$ and $\eta = 1$, we get

$$\omega_{\mathcal{A}}^2(\mathcal{T}) \geq \frac{1}{4}\omega_{\mathcal{A}}\left[\mathcal{T}^2+\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\right)^2\right] = \frac{1}{4}\left\|\mathcal{T}^2+\left(\mathcal{T}^{\sharp_{\mathcal{A}}}\right)^2\right\|_{\mathcal{A}}$$

for $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$.

Further, from (35) with $(\mathcal{T},\mathcal{S}) = (\mathcal{T},\mathcal{T}^{\sharp_{\mathcal{A}}})$, we obtain

$$\begin{aligned}
\omega_{\mathcal{A}}^2(\mathcal{T}) &\geq \frac{1}{2}|\zeta\eta|\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}\\
&+\frac{1}{4}\max\left\{\omega_{\mathcal{A}}\left[\zeta(1+i)\mathcal{T}+\eta(1-i)\mathcal{T}^{\sharp_{\mathcal{A}}}\right],\right.\\
&\left.\omega_{\mathcal{A}}\left[\zeta(1-i)\mathcal{T}+\eta(1+i)\mathcal{T}^{\sharp_{\mathcal{A}}}\right]\right\}\\
&\times\left|\omega_{\mathcal{A}}\left(\zeta\mathcal{T}+\eta\mathcal{T}^{\sharp_{\mathcal{A}}}\right)-\omega_{\mathcal{A}}\left(\zeta\mathcal{T}-\eta\mathcal{T}^{\sharp_{\mathcal{A}}}\right)\right|
\end{aligned}$$

for $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $\zeta,\eta \in \mathbb{C}$ with $|\zeta|^2+|\eta|^2 = 1$. For $\zeta = \eta = \frac{\sqrt{2}}{2}$, we get

$$\begin{aligned}
\omega_{\mathcal{A}}^2(\mathcal{T}) &\geq \frac{1}{4}\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}\\
&+\frac{1}{8}\max\left\{\omega_{\mathcal{A}}\left[(1+i)\mathcal{T}+(1-i)\mathcal{T}^{\sharp_{\mathcal{A}}}\right],\right.\\
&\left.\omega_{\mathcal{A}}\left[(1-i)\mathcal{T}+(1+i)\mathcal{T}^{\sharp_{\mathcal{A}}}\right]\right\}\\
&\times\left|\left\|\mathcal{T}+\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}-\left\|\mathcal{T}-\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}\right|
\end{aligned}$$

for $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$.

Moreover, setting $(\mathcal{T},\mathcal{S}) = (\mathcal{T},\mathcal{T}^{\sharp_{\mathcal{A}}})$ in (36), we obtain

$$
\begin{aligned}
&\omega_{\mathcal{A}}^2(\mathcal{T}) \\
&\geq \frac{1}{4}\max\Big\{\omega_{\mathcal{A}}\Big[\Big(\eta^2-\zeta^2\Big)\Big(\Big(\mathcal{T}^{\sharp_{\mathcal{A}}}\Big)^2-\mathcal{T}^2\Big)+2\zeta\eta\Big(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\Big)\Big], \\
&\omega_{\mathcal{A}}\Big[\Big(\eta^2-\zeta^2\Big)\Big(\Big(\mathcal{T}^{\sharp_{\mathcal{A}}}\Big)^2-\mathcal{T}^2\Big)-2\zeta\eta\Big(\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}+\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\Big)\Big]\Big\} \\
&+\frac{1}{4}\max\Big\{\omega_{\mathcal{A}}\Big[(\zeta+i\eta)\mathcal{T}+(\eta-i\zeta)\mathcal{T}^{\sharp_{\mathcal{A}}}\Big], \\
&\omega_{\mathcal{A}}\Big[(\zeta-i\eta)\mathcal{T}+(\eta+i\zeta)\mathcal{T}^{\sharp_{\mathcal{A}}}\Big]\Big\} \\
&\times\Big|\omega_{\mathcal{A}}\Big(\zeta\mathcal{T}+\eta\mathcal{T}^{\sharp_{\mathcal{A}}}\Big)-\omega_{\mathcal{A}}\Big(\eta\mathcal{T}-\zeta\mathcal{T}^{\sharp_{\mathcal{A}}}\Big)\Big|
\end{aligned} \tag{45}
$$

for $\mathcal{T}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $\zeta,\eta\in\mathbb{C}$ with $|\zeta|^2+|\eta|^2=1$. For $\zeta=\frac{\sqrt{2}}{2}i$ and $\eta=\frac{\sqrt{2}}{2}$, we get

$$
\omega_{\mathcal{A}}^2(\mathcal{T})\geq\frac{1}{4}\max\Big\{\omega_{\mathcal{A}}\Big[\Big(\mathcal{T}^{\sharp_{\mathcal{A}}}+i\mathcal{T}\Big)^2\Big],\omega_{\mathcal{A}}\Big[\Big(\mathcal{T}^{\sharp_{\mathcal{A}}}-i\mathcal{T}\Big)^2\Big]\Big\}.
$$

Additionally, setting $\zeta=0$ and $\eta=1$ in (45), we obtain

$$
\omega_{\mathcal{A}}^2(\mathcal{T})\geq\frac{1}{4}\omega_{\mathcal{A}}\Big[\Big(\mathcal{T}^{\sharp_{\mathcal{A}}}\Big)^2-\mathcal{T}^2\Big].
$$

If we apply the inequality in (33) with $\mathcal{T}=\Re_{\mathcal{A}}(\mathcal{T})$ and $\mathcal{S}=\Im_{\mathcal{A}}(\mathcal{T})$, we get

$$
\begin{aligned}
&\omega_{\mathcal{A}}^2(\mathcal{T}) \\
&\geq\max\Big\{\omega_{\mathcal{A}}\Big[(\zeta\Re_{\mathcal{A}}(\mathcal{T})+\eta\Im_{\mathcal{A}}(\mathcal{T}))^2\Big],\omega_{\mathcal{A}}\Big[(\zeta\Re_{\mathcal{A}}(\mathcal{T})-\eta\Im_{\mathcal{A}}(\mathcal{T}))^2\Big]\Big\}
\end{aligned}
$$

for $\mathcal{T}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $\zeta,\eta\in\mathbb{C}$ with $|\zeta|^2+|\eta|^2=1$. For $\zeta=\eta=\frac{\sqrt{2}}{2}$, we obtain

$$
\omega_{\mathcal{A}}^2(\mathcal{T})\geq\frac{1}{2}\max\Big\{\|\Re_{\mathcal{A}}(\mathcal{T})+\Im_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}}^2,\|\Re_{\mathcal{A}}(\mathcal{T})-\Im_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}}^2\Big\}.
$$

From (34) with $\mathcal{T}=\Re_{\mathcal{A}}(\mathcal{T})$ and $\mathcal{S}=\Im_{\mathcal{A}}(\mathcal{T})$, we also have

$$
\begin{aligned}
&\omega_{\mathcal{A}}^2(\mathcal{T}) \\
&\geq\frac{1}{2}\max\Big\{\omega_{\mathcal{A}}\Big[\Big(\zeta^2+\eta^2\Big)\Big(\Re_{\mathcal{A}}^2(\mathcal{T})+\Im_{\mathcal{A}}^2(\mathcal{T})\Big)+2\zeta\eta(\Re_{\mathcal{A}}(\mathcal{T})\Im_{\mathcal{A}}(\mathcal{T})+\Im_{\mathcal{A}}(\mathcal{T})\Re_{\mathcal{A}}(\mathcal{T}))\Big], \\
&\omega_{\mathcal{A}}\Big[\Big(\zeta^2+\eta^2\Big)\Big(\Re_{\mathcal{A}}^2(\mathcal{T})+\Im_{\mathcal{A}}^2(\mathcal{T})\Big)-2\zeta\eta(\Re_{\mathcal{A}}(\mathcal{T})\Im_{\mathcal{A}}(\mathcal{T})+\Im_{\mathcal{A}}(\mathcal{T})\Re_{\mathcal{A}}(\mathcal{T}))\Big]\Big\} \\
&+\frac{1}{2}\max\{|\zeta+\eta|\|\Re_{\mathcal{A}}(\mathcal{T})+\Im_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}},|\zeta-\eta|\|\Re_{\mathcal{A}}(\mathcal{T})-\Im_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}}\} \\
&\times|\omega_{\mathcal{A}}(\zeta\Re_{\mathcal{A}}(\mathcal{T})+\eta\Im_{\mathcal{A}}(\mathcal{T}))-\omega_{\mathcal{A}}(\eta\Re_{\mathcal{A}}(\mathcal{T})+\zeta\Im_{\mathcal{A}}(\mathcal{T}))|
\end{aligned}
$$

for $\mathcal{T}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $\zeta,\eta\in\mathbb{C}$ with $|\zeta|^2+|\eta|^2=1$.

Setting $\zeta=0$ and $\eta=1$ in this inequality, we obtain

$$
\begin{aligned}
\omega_{\mathcal{A}}^2(\mathcal{T})\geq{}&\frac{1}{2}\Big\|\Re_{\mathcal{A}}^2(\mathcal{T})+\Im_{\mathcal{A}}^2(\mathcal{T})\Big\|_{\mathcal{A}} \\
&+\frac{1}{2}\max\{\|\Re_{\mathcal{A}}(\mathcal{T})+\Im_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}},\|\Re_{\mathcal{A}}(\mathcal{T})-\Im_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}}\} \\
&\times|\|\Im_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}}-\|\Re_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}}|
\end{aligned}
$$

for $\mathcal{T}\in\mathbf{L}_{\mathcal{A}}(\mathbb{X})$.

Because $\Re^2_{\mathcal{A}}(\mathcal{T}) + \Im^2_{\mathcal{A}}(\mathcal{T}) = \frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}}{2}$, we have

$$\begin{aligned}\omega^2_{\mathcal{A}}(\mathcal{T}) \geq \frac{1}{4}\left\|\frac{\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T} + \mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}}{2}\right\|_{\mathcal{A}} \\ + \frac{1}{2}\max\{\|\Re_{\mathcal{A}}(\mathcal{T}) + \Im_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}}, \|\Re_{\mathcal{A}}(\mathcal{T}) - \Im_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}}\} \\ \times |\|\Im_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}} - \|\Re_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}}|.\end{aligned}$$

Further, setting $\mathcal{T} = \Re_{\mathcal{A}}(\mathcal{T})$ and $\mathcal{S} = \Im_{\mathcal{A}}(\mathcal{T})$ in (35), we obtain

$$\begin{aligned}\omega^2_{\mathcal{A}}(\mathcal{T}) \geq |\zeta\eta|\|\Re_{\mathcal{A}}(\mathcal{T})\Im_{\mathcal{A}}(\mathcal{T}) + \Im_{\mathcal{A}}(\mathcal{T})\Re_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}} \\ + \frac{1}{2}\max\{\omega_{\mathcal{A}}[\zeta(1+i)\Re_{\mathcal{A}}(\mathcal{T}) + \eta(1-i)\Im_{\mathcal{A}}(\mathcal{T})], \\ \omega_{\mathcal{A}}[\zeta(1-i)\Re_{\mathcal{A}}(\mathcal{T}) + \eta(1+i)\Im_{\mathcal{A}}(\mathcal{T})]\} \\ \times |\omega_{\mathcal{A}}(\zeta\Re_{\mathcal{A}}(\mathcal{T}) + \eta\Im_{\mathcal{A}}(\mathcal{T})) - \omega_{\mathcal{A}}(\zeta\Re_{\mathcal{A}}(\mathcal{T}) - \eta\Im_{\mathcal{A}}(\mathcal{T}))|\end{aligned}$$

for $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$ and $\zeta, \eta \in \mathbb{C}$ with $|\zeta|^2 + |\eta|^2 = 1$. For $\zeta = \eta = \frac{\sqrt{2}}{2}$, we obtain

$$\begin{aligned}\omega^2_{\mathcal{A}}(\mathcal{T}) \geq \frac{1}{2}\|\Re_{\mathcal{A}}(\mathcal{T})\Im_{\mathcal{A}}(\mathcal{T}) + \Im_{\mathcal{A}}(\mathcal{T})\Re_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}} \\ + \frac{1}{4}\max\{\omega_{\mathcal{A}}[(1+i)\Re_{\mathcal{A}}(\mathcal{T}) + (1-i)\Im_{\mathcal{A}}(\mathcal{T})], \\ \omega_{\mathcal{A}}[(1-i)\Re_{\mathcal{A}}(\mathcal{T}) + (1+i)\Im_{\mathcal{A}}(\mathcal{T})]\} \\ \times |\|\Re_{\mathcal{A}}(\mathcal{T}) + \Im_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}} - \|\Re_{\mathcal{A}}(\mathcal{T}) - \Im_{\mathcal{A}}(\mathcal{T})\|_{\mathcal{A}}|\end{aligned}$$

for $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathbb{X})$.

5. Conclusions

In this paper, we have established a series of new inequalities for the $\mathcal{A}$-joint numerical radius of operators in the generalized setting of semi-Hilbert spaces. Our primary contributions include the derivation of novel upper and lower bounds for $\omega_{e,\mathcal{A}}(\mathcal{T}, \mathcal{S})$, which refine and extend several known results in operator theory. Key findings include power-dependent inequalities that offer tighter estimations by connecting the $\mathcal{A}$-joint numerical radius with the $\mathcal{A}$-seminorms and individual $\mathcal{A}$-numerical radii of related operators. Furthermore, the application of our main theorems to a single operator and its $\mathcal{A}$-adjoint has yielded several new bounds for the standard $\mathcal{A}$-numerical radius, providing deeper insights into the properties of operators.

The results presented herein not only advance the study of operator inequalities but also open avenues for further investigation. This work may serve as a foundational starting point for future projects. A natural and significant extension would be the generalization of the $\mathcal{A}$-joint numerical radius to higher powers. Investigating and proving novel inequalities for the generalized radius, potentially of the form

$$\omega_{\mathcal{A},e,\rho}(\mathcal{T}, \mathcal{S}) = \sup_{\|\xi\|_{\mathcal{A}}=1} \left(|\langle \mathcal{T}\xi, \xi\rangle_{\mathcal{A}}|^{\rho} + |\langle \mathcal{S}\xi, \xi\rangle_{\mathcal{A}}|^{\rho}\right)^{\frac{1}{\rho}}$$

for $\rho \geq 1$, could lead to a rich new class of inequalities and a more comprehensive understanding of operator behavior. We believe that these potential directions will continue to enrich the theory of operators on semi-Hilbert spaces.

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