

Article

Analysis of Quantum Multiplicative Calculus and Related Inequalities

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Abstract

This article investigates the exact meaning of a quantum derivative result and the corresponding definition of a quantum definite integral in multiplicative calculus from a geometrical viewpoint. After this critical analysis, we give an accurate definition of the q-multiplicative definite integral and the corresponding derivative result. Additionally, an example pertaining to q-multiplicative definite integrals is presented, and rigorous analysis to prove several fundamental results is provided. In addition, two other concepts are defined: the left q-multiplicative derivative and definite integral and the right q-multiplicative derivative and definite integral. Finally, several q-multiplicative Hermite–Hadamard-type inequalities are constructed, and related examples are shown to support our recent findings.

Keywords: q-multiplicative calculus; multiplicative calculus; geometric calculus; Hermite–Hadamard inequality

MSC: 05A30; 26A51; 26D10; 26D15; 39A13



Academic Editor: Vasily E. Tarasov

Received: 12 September 2025

Revised: 20 October 2025

Accepted: 20 October 2025

Published: 23 October 2025

Citation: Aftab, M.N.; Butt, S.I.; Alammr, M.; Seol, Y. Analysis of Quantum Multiplicative Calculus and Related Inequalities. *Mathematics* **2025**, *13*, 3381. <https://doi.org/10.3390/math13213381>

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1. Introduction

Between 1967 and 1970, Michael Grossman and Robert Katz defined an alternative form of derivatives and integrals, which altered the addition and subtraction operations to division and multiplication. This provided the origin of the entirely new calculus known as multiplicative calculus, which is also frequently called non-Newtonian calculus. In multiplicative calculus, two well-known operators, the derivative and integral, are defined below.

Definition 1 ([1]). If we consider $\mathcal{J} : \mathcal{I} = [\alpha, \beta] \rightarrow \mathbb{R}$ to be a positive function with $\mathcal{J}(s) \neq 0$, then

$$\mathfrak{D}^* \mathcal{J}(s) = \mathcal{J}^*(s) = \lim_{h \rightarrow 0} \left(\frac{\mathcal{J}(s+h)}{\mathcal{J}(s)} \right)^{\frac{1}{h}}$$

is said to be a multiplicative derivative or $*$ derivative of $\mathcal{J}$ at $s \in \mathbb{R}$. Also,

$$\mathcal{J}^*(s) = e^{\left(\frac{\mathcal{J}'(s)}{\mathcal{J}(s)}\right)} = e^{((\ln \mathcal{J}(s))')}.$$

Here, $(\)'$ shows a classical derivative.

Definition 2 ([1]). If we assume that $\mathcal{J}$ is a Riemann integral and positive function, then for $s \in \mathcal{I}$,

$$\int_{\alpha}^{\beta} \mathcal{J}(s) \, ds = e^{\left(\int_{\alpha}^{\beta} \ln(\mathcal{J}(s)) \, ds\right)}$$

is said to be a $*$ integral or multiplicative integral of $\mathcal{J}$ over $\mathcal{I}$.

Some characteristics of a multiplicative integral for an arbitrary function are described below.

Proposition 1 ([1]). If we assume that $\mathcal{J}$ and $\mathcal{K}$ are two multiplicative integrals and positive functions, with $\alpha \leq c \leq \beta$ and $n \in \mathbb{R}$, then

- $\int_{\alpha}^{\alpha} \mathcal{J}(s) \, ds = 1$.
- $\int_{\alpha}^{\beta} \mathcal{J}(s) \, ds = \left(\int_{\beta}^{\alpha} \mathcal{J}(s) \, ds\right)^{-1}$.
- $\int_{\alpha}^{\beta} \mathcal{J}(s) \, ds = \ln\left(\int_{\alpha}^{\beta} (e^{\mathcal{J}(s)}) \, ds\right)$.
- $\int_{\alpha}^{\beta} (\mathcal{J}(s)^n) \, ds = \left(\int_{\alpha}^{\beta} (\mathcal{J}(s)) \, ds\right)^n$.
- $\int_{\alpha}^{\beta} \mathcal{J}(s) \, ds = \int_{\alpha}^c \mathcal{J}(s) \, ds \cdot \int_c^{\beta} \mathcal{J}(s) \, ds$.
- $\int_{\alpha}^{\beta} (\mathcal{J}(s)\mathcal{K}(s)) \, ds = \int_{\alpha}^{\beta} \mathcal{J}(s) \, ds \cdot \int_{\alpha}^{\beta} \mathcal{K}(s) \, ds$.
- $\int_{\alpha}^{\beta} \left(\frac{\mathcal{J}(s)}{\mathcal{K}(s)}\right) \, ds = \frac{\int_{\alpha}^{\beta} \mathcal{J}(s) \, ds}{\int_{\alpha}^{\beta} \mathcal{K}(s) \, ds}$.

Researchers in different mathematical fields have recently made contributions pertaining to this calculus. The literature of non-Newtonian calculus includes, for example, the fundamental theorem of multiplicative calculus [1], multiplicative stochastic integrals [2], complex multiplicative calculus [3], multiplicative differential equations [4], and multiplicative Hermite–Hadamard inequalities for different kinds of convex functions [5–7].

Definition 3 ([8]). If we assume that $\mathcal{J}$ is a positive function on $\mathcal{I}$, if for all $s, h \in \mathcal{I}$ and $0 \leq \alpha_1 \leq 1$,

$$\mathcal{J}(s\alpha_1 + h(1 - \alpha_1)) \leq (\mathcal{J}(s))^{\alpha_1} (\mathcal{J}(h))^{1-\alpha_1}$$

exists, then $\mathcal{J}$ is a multiplicative convex function.

The Hermite–Hadamard inequalities for multiplicative calculus are stated below.

Theorem 1 ([5]). If we assume that $\mathcal{J}$ is a multiplicative convex and positive function on $\mathcal{I}$, then

$$\mathcal{J}\left(\frac{\alpha + \beta}{2}\right) \leq \left(\int_{\alpha}^{\beta} (\mathcal{J}(s)) \, ds\right)^{\frac{1}{\beta-\alpha}} \leq \sqrt{\mathcal{J}(\alpha)\mathcal{J}(\beta)} \quad (1)$$

holds.

Theorem 2 ([5]). If we assume that $\mathcal{J}$ and $\mathcal{K}$ are two multiplicative convex and positive functions on $\mathcal{I}$, then

$$\mathcal{J}\left(\frac{\alpha+\beta}{2}\right)\mathcal{K}\left(\frac{\alpha+\beta}{2}\right) \leq \left(\int_{\alpha}^{\beta}(\mathcal{J}(s))^{\mathrm{d}s} \cdot \int_{\alpha}^{\beta}(\mathcal{K}(s))^{\mathrm{d}s}\right)^{\frac{1}{\beta-\alpha}} \leq \sqrt{\mathcal{J}(\alpha)\mathcal{J}(\beta)\mathcal{K}(\alpha)\mathcal{K}(\beta)} \quad (2)$$

holds.

Theorem 3 ([5]). *If we assume that $\mathcal{J}$ and $\mathcal{K}$ are two multiplicative convex and positive functions on $\mathcal{I}$, then*

$$\mathcal{J}\left(\frac{\alpha+\beta}{2}\right): \mathcal{K}\left(\frac{\alpha+\beta}{2}\right) \leq \left(\int_{\alpha}^{\beta}(\mathcal{J}(s))^{\mathrm{d}s} : \int_{\alpha}^{\beta}(\mathcal{K}(s))^{\mathrm{d}s}\right)^{\frac{1}{\beta-\alpha}} \leq \sqrt{\mathcal{J}(\alpha)\mathcal{J}(\beta)} : \sqrt{\mathcal{K}(\alpha)\mathcal{K}(\beta)} \quad (3)$$

or, equivalently,

$$\frac{\mathcal{J}\left(\frac{\alpha+\beta}{2}\right)}{\mathcal{K}\left(\frac{\alpha+\beta}{2}\right)} \leq \left(\frac{\int_{\alpha}^{\beta}(\mathcal{J}(s))^{\mathrm{d}s}}{\int_{\alpha}^{\beta}(\mathcal{K}(s))^{\mathrm{d}s}}\right)^{\frac{1}{\beta-\alpha}} \leq \sqrt{\frac{\mathcal{J}(\alpha)\mathcal{J}(\beta)}{\mathcal{K}(\alpha)\mathcal{K}(\beta)}}$$

holds.

Recently, some authors have extended the research on integral inequalities using the generalized multiplicative convex functions. For instance, in [9], the authors introduced a new type of function called a multiplicatively (P, m)-convex function. They studied their basic properties and developed Hermite–Hadamard-type inequalities of these functions. Multiplicative k-Riemann–Liouville fractional integrals are also defined in this paper; the integrability, continuity, and commutativity of these integrals are discussed; and the new fractional Hermite–Hadamard- and Newton-type inequalities are obtained using multiplicative k-Riemann–Liouville integrals. Moreover, in [10], the authors constructed a series of three-point Newton–Cotes-type inequalities based on multiplicative convexity conditions and used multiplicative Riemann–Liouville fractional integrals for double multiplicative differentiable functions to generate a fractional integral identity. Our primary motivation includes inequalities (1), (2), and (3), Definition 3, and the works published in [9,10], which may also be developed in a q-multiplicative calculus by obtaining some new definitions of the definite integral in this calculus. In Section 3 of this article, we investigated the definition of definite integrals and the derivative results in q-multiplicative calculus, which was developed in 2015 [11], using the graphical point of view, and gave the correct definition of definite integrals and the derivative’s results. In Section 4, we introduced two more definitions for the definite integral and derivative as well. In addition, we derived some fundamental results of the definite integral based on newly obtained results. Moreover, we established some Hermite–Hadamard-type inequalities and constructed related examples that support our results.

2. Materials and Methods

The fundamental definitions of q-calculus that are necessary for comprehension will be demonstrated in this article. The q-calculus is a limitless version of classical calculus. Its analysis can be seen in difference equations, number theory, knot theory, general relativity, particle and chemical physics, molecular and nuclear spectroscopy, the hydrogen atom, string theory, and hypergeometric functions [12]. For further comprehension, its general and symmetric version can be read in these referred manuscripts [13,14]. The definitions of derivatives and definite integrals of q-calculus are mentioned below.

Definition 4 ([15]). If we consider a continuous function $\mathcal{J}$ defined on $\mathcal{I}$, then for $0 < q < 1$

$$\mathfrak{D}_q \mathcal{J}(s) = \frac{\mathcal{J}(qs) - \mathcal{J}(s)}{s(q-1)}$$

is called a quantum or q -derivative of $\mathcal{J}$ at $s \in \mathbb{R}$.

Definition 5 ([15]). If we assume a continuous function $\mathcal{J}$, then for $s_1 \in \mathcal{I}$,

$$\int_{\alpha}^{s_1} \mathcal{J}(s) d_q s = (1-q)(s_1 - \alpha) \sum_{n=0}^{\infty} q^n \mathcal{J}(q^n s_1)$$

is called a q - or quantum integral of $\mathcal{J}$.

In 2013 [16], Tariboon and his co-author defined a new definition of definite integrals in q -calculus using the left endpoint α of $\mathcal{I}$ as written below.

Definition 6 ([16]). If we assume a continuous function $\mathcal{J}$, then for $s_1 \in \mathcal{I}$,

$$\int_{\alpha}^{s_1} \mathcal{J}(s)_{\alpha} d_q s = (s_1 - \alpha)(1-q) \sum_{n=0}^{\infty} q^n \mathcal{J}((1-q^n)\alpha + q^n s_1)$$

is called a ${}_{\alpha}q$ -integral of $\mathcal{J}$.

The definite integral in q -calculus using the right endpoint β of $\mathcal{I}$ can also be seen in the literature and is written below.

Definition 7 ([17]). If we assume a continuous function $\mathcal{J}$, then for $s_1 \in \mathcal{I}$,

$$\int_{s_1}^{\beta} \mathcal{J}(s)^{\beta} d_q s = (\beta - s_1)(1-q) \sum_{n=0}^{\infty} q^n \mathcal{J}((1-q^n)\beta + q^n s_1)$$

is called a ${}^{\beta}q$ -integral of $\mathcal{J}$.

In 2015 [11], Yener et al. derived a q -analogue of multiplicative calculus and defined basic definitions and results of this calculus that are written below.

Definition 8 ([11]). If we consider a positive function $\mathcal{J}$ defined on $\mathcal{I}$ with $\mathcal{J}(s) \neq 0$, then the q -multiplicative derivative of $\mathcal{J}$ at $s \in \mathbb{R}$ and is written as

$$\mathfrak{D}_q^* \mathcal{J}(s) = \left(\frac{\mathcal{J}(qs)}{\mathcal{J}(s)} \right)^{\frac{1}{(q-1)s}}.$$

Definition 9 ([11]). If we assume that $\mathcal{J}$ is a positive and bounded function on $0 < \alpha < \beta$, then for $s \in \mathcal{I}$

$$\int_{\alpha}^{\beta} \mathcal{J}(s) d_q s = e_q^{\int_{\alpha}^{\beta} \ln(\mathcal{J}(s)) d_q s} = e_q^{\left(\int_0^{\beta} \ln(\mathcal{J}(s)) d_q s - \int_0^{\alpha} \ln(\mathcal{J}(s)) d_q s \right)} \quad (4)$$

is called a q^* - or q -multiplicative integral of $\mathcal{J}$ over $\mathcal{I}$.

Some properties regarding q -multiplicative derivatives and integrals of a function are written below.

Theorem 4 ([11]). If we assume a positive and q -differentiable function $\mathcal{J}$, then the

$$\mathfrak{D}_q^* \mathcal{J}(s) = e_q^{\mathfrak{D}_q \ln(\mathcal{J}(s))} \quad (5)$$

equality holds.

Theorem 5 ([11]). If we assume two positive q -multiplicative integrable functions $\mathcal{J}$ and $\mathcal{K}$ with $c \in \mathcal{I}$ and $n \in \mathbb{R}$, then

- $\int_{\alpha}^{\beta} (\mathcal{J}(s)^n) d_{qs} = \left(\int_{\alpha}^{\beta} (\mathcal{J}(s)) d_{qs} \right)^n$;
- $\int_{\alpha}^{\beta} \mathcal{J}(s) d_{qs} = \int_{\alpha}^c \mathcal{J}(s) d_{qs} \cdot \int_c^{\beta} \mathcal{J}(s) d_{qs}$;
- $\int_{\alpha}^{\beta} (\mathcal{J}(s) \cdot \mathcal{K}(s)) d_{qs} = \int_{\alpha}^{\beta} \mathcal{J}(s) d_{qs} \cdot \int_{\alpha}^{\beta} \mathcal{K}(s) d_{qs}$;
- $\int_{\alpha}^{\beta} \left(\frac{\mathcal{J}(s)}{\mathcal{K}(s)} \right) d_{qs} = \frac{\int_{\alpha}^{\beta} \mathcal{J}(s) d_{qs}}{\int_{\alpha}^{\beta} \mathcal{K}(s) d_{qs}}$.

3. Corrections

First, we critically examine (4) and (5). Both equations are incorrect because $\ln$ and e_q are not inverse to each other. In fact, e_q (natural q -exponential function) is the inverse of $\ln_q$ (natural q -log function), and $\ln$ (natural log function) is the inverse of e (natural exponential function). Moreover, e_q and $\ln_q$ are the generalization of e and $\ln$, respectively (see [18]). Another generalization of these functions is introduced by Tsallis [19] and written as follows:

$$e_q(s) = [1 + (1 - q)s]^{\frac{1}{(1-q)}}$$

where $q \neq 1$ and $1 + (1 - q)s > 0$. And for $s > 0$,

$$\ln_q(s) = \frac{s^{1-q} - 1}{1 - q}.$$

Now, we demonstrate our claim graphically in Figures 1 and 2.

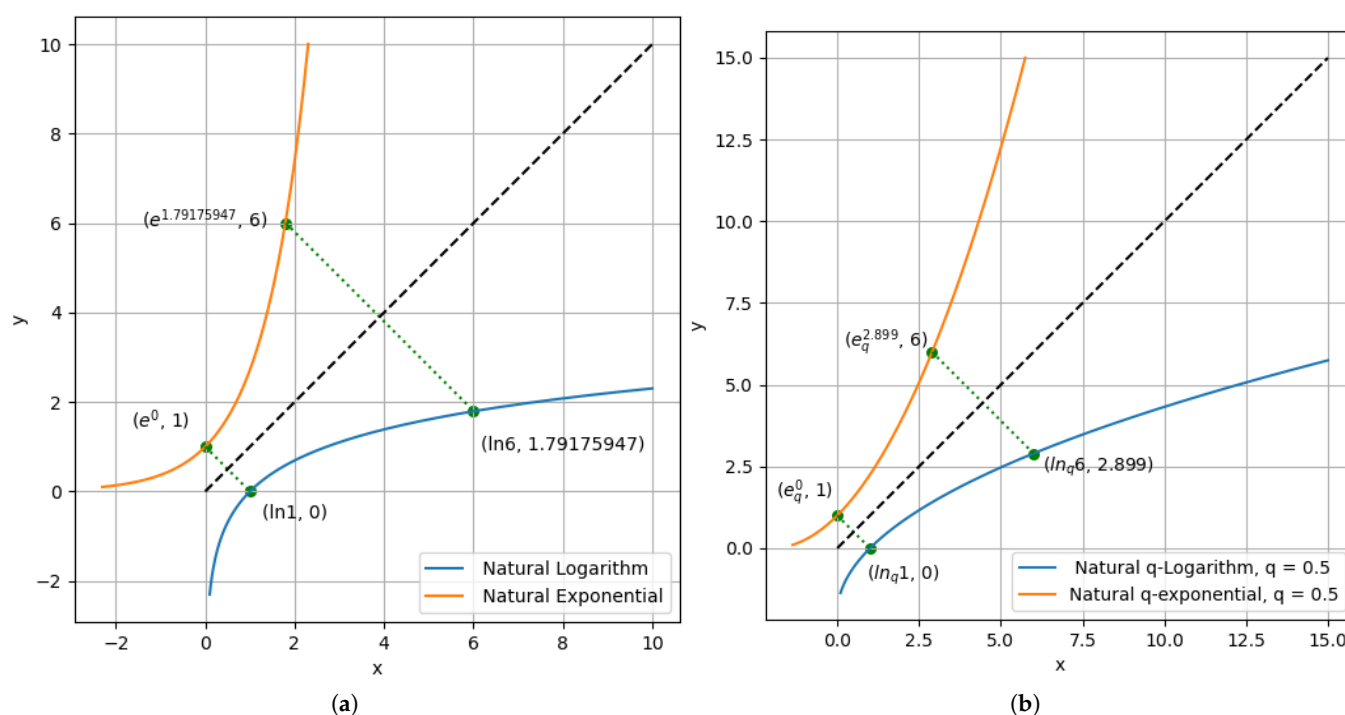


Figure 1. (a) Graph of natural log and its inverse. (b) Graph of natural q -log and its inverse.

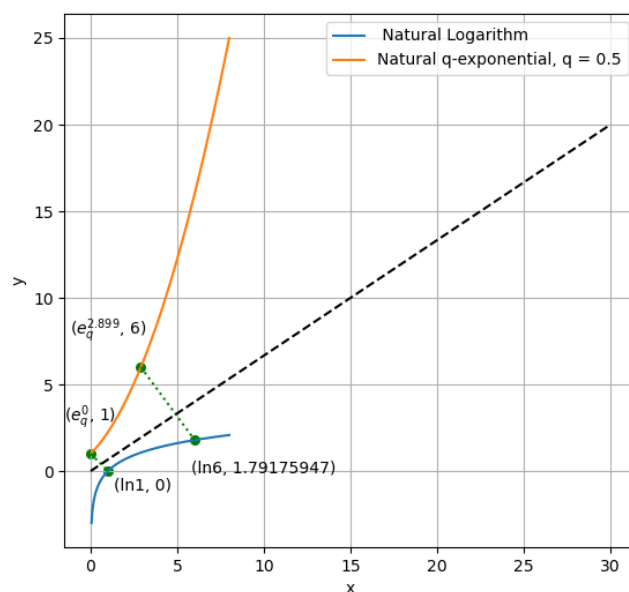


Figure 2. Graph of natural log and q-exponential functions.

From Figure 1, it can be seen that the graph of functions and their inverses form symmetry about the line (which passes through the origin and bisects the first and third quadrant); however, Figure 2 shows that $\ln$ and e_q are not inverse to each other. Therefore, the correct form of Definition 9 for the definite integral in q-multiplicative calculus should be written as below.

Definition 10. If we assume that $\mathcal{J}$ is a positive and bounded function on $0 < \alpha < \beta$, then for $s \in \mathcal{I}$

$$\int_{\alpha}^{\beta} \mathcal{J}(s) \, d_q s = e^{\int_{\alpha}^{\beta} \ln(\mathcal{J}(s)) \, d_q s} = e^{\left(\int_0^{\beta} \ln(\mathcal{J}(s)) \, d_q s - \int_0^{\alpha} \ln(\mathcal{J}(s)) \, d_q s \right)} \quad (6)$$

is called a q^* - or q-multiplicative integral of $\mathcal{J}$ over $\mathcal{I}$.

Remark 1. Since equality (5) of Theorem 4 is also wrong due to the following reasons,

1. The quantum natural exponential function (e_q) is not the inverse of the natural logarithmic ($\ln$) function for $q \neq 1$;
2. The Chain rule in q-calculus does not hold generally (see [20]).

The first point is handled in our manuscript, but there is still a second reason to refute this theorem. However, this theorem can be reconstructed for a particular case; for instance, if we let the inner function $\mathcal{J}$ be a monomial having any positive integer power and the outer function be a natural logarithmic for the Chain rule, then the second reason can also be fixed. i.e., if we assume $\mathcal{J}(s) = s^n$, then

$$\begin{aligned} \mathfrak{D}_q \ln(\mathcal{J}(s)) &= \mathfrak{D}_q \ln(s^n) = \frac{\ln(q^n s^n) - \ln(s^n)}{qs - s} \\ &= \frac{\ln(q^n s^n) - \ln(s^n)}{q^n s^n - s^n} \frac{q^n s^n - s^n}{qs - s} \\ &= \frac{\ln(q^n \mathcal{J}(s)) - \ln(\mathcal{J}(s))}{q^n \mathcal{J}(s) - \mathcal{J}(s)} \frac{\mathcal{J}(qs) - \mathcal{J}(s)}{qs - s} \\ &= \mathfrak{D}_{q^n} \ln(\mathcal{J}(s)) \mathfrak{D}_q \mathcal{J}(s). \end{aligned}$$

Therefore, after some refinements, Theorem 4 can be reconsidered as the proposition that is mentioned below.

Proposition 2. If we assume a q -differentiable and positive function $\mathcal{J}$, if $\mathcal{J}$ is a monomial having any positive integer power, then

$$\mathfrak{D}_q^* \mathcal{J}(s) = e^{\mathfrak{D}_q \ln(\mathcal{J}(s))}$$

holds.

4. Novel Definitions and Results in q -Multiplicative Calculus

First, we define new definitions of derivatives and definite integrals in q -multiplicative calculus using the left endpoint α of $\mathcal{I}$.

Definition 11. If we consider a positive function $\mathcal{J}$ defined on $\mathcal{I}$ with $\mathcal{J}(s) \neq 0$, then the ${}_{\alpha}q$ -multiplicative or left q -multiplicative derivative of $\mathcal{J}$ at $s \in \mathbb{R}$ and is written as

$${}_{\alpha}\mathfrak{D}_q^* \mathcal{J}(s) = \left(\frac{\mathcal{J}(\alpha(1-q) + qs)}{\mathcal{J}(s)} \right)^{\frac{1}{(s-\alpha)(q-1)}}.$$

Example 1. If we let $\mathcal{J}(s) = s^2 + 1$ be a positive function on $[2, 5]$ and fix $q = 0.5$, then its ${}_{\alpha}q$ -multiplicative derivative can be calculated as

$$\begin{aligned} {}_{\alpha}\mathfrak{D}_q^* \mathcal{J}(s) &= \left(\frac{\mathcal{J}(\alpha(1-q) + qs)}{\mathcal{J}(s)} \right)^{\frac{1}{(s-\alpha)(q-1)}} = \left(\frac{\alpha(1-q) + q(s^2 + 1)}{s^2 + 1} \right)^{\frac{1}{(s-\alpha)(q-1)}} \\ {}_2\mathfrak{D}_{0.5}^* \mathcal{J}(s) &= \left(\frac{1 + \frac{s^2 + 1}{2}}{s^2 + 1} \right)^{\frac{1}{(s-2)(0.5-1)}} = \left(\frac{s^2 + 3}{2s^2 + 2} \right)^{\frac{2}{2-s}}. \end{aligned}$$

Definition 12. If we assume a bounded function on $0 < \alpha < \beta$ and a positive function $\mathcal{J}$, then for $s \in \mathcal{I}$

$$\int_{\alpha}^{\beta} \mathcal{J}(s) {}_{\alpha}d_q s = e^{\int_{\alpha}^{\beta} \ln(\mathcal{J}(s)) {}_{\alpha}d_q s} = e^{\left(\int_{\alpha}^{\beta} \ln(\mathcal{J}(s)) {}_{\alpha}d_q s - \int_{\alpha}^{\alpha} \ln(\mathcal{J}(s)) {}_{\alpha}d_q s \right)} \quad (7)$$

is called a left q - or ${}_{\alpha}q$ -multiplicative integral of $\mathcal{J}$ over $\mathcal{I}$.

Example 2. If we let $\mathcal{J}(s) = e^{s^2}$, then using (7) and Definition 6,

$$\begin{aligned} \int_{\alpha}^{\beta} e^{s^2} {}_{\alpha}d_q s &= e^{\int_{\alpha}^{\beta} \ln(e^{s^2}) {}_{\alpha}d_q s} \\ &= e^{\int_{\alpha}^{\beta} s^2 {}_{\alpha}d_q s} \\ &= e^{(1-q)(\beta-\alpha) \sum_{n=0}^{\infty} q^n ((1-q^n)^2 \alpha^2 + q^{2n} \beta^2 + 2q^n (1-q^n) \alpha \beta)} \\ &= e^{(\beta-\alpha) \left[\frac{(\beta-\alpha)^2}{q^2+q+1} + \frac{2\alpha(\beta-\alpha)}{q+1} + \alpha^2 \right]}. \end{aligned} \quad (8)$$

Some basic properties of left q -multiplicative integrals of function are written below.

Theorem 6. If we assume two positive ${}_{\alpha}q$ -multiplicative integrable functions $\mathcal{J}$ and $\mathcal{K}$ with $c \in \mathcal{I}$ and $n \in \mathbb{R}$, then

1. $\int_{\alpha}^{\beta} (\mathcal{J}(s)^n) {}_{\alpha}d_q s = \left(\int_{\alpha}^{\beta} \mathcal{J}(s) {}_{\alpha}d_q s \right)^n$,
2. $\int_{\alpha}^{\beta} \mathcal{J}(s) {}_{\alpha}d_q s = \int_{\alpha}^c \mathcal{J}(s) {}_{\alpha}d_q s + \int_c^{\beta} \mathcal{J}(s) {}_{\alpha}d_q s$,
3. $\int_{\alpha}^{\beta} (\mathcal{J}(s) \cdot \mathcal{K}(s)) {}_{\alpha}d_q s = \int_{\alpha}^{\beta} \mathcal{J}(s) {}_{\alpha}d_q s \cdot \int_{\alpha}^{\beta} \mathcal{K}(s) {}_{\alpha}d_q s$,
4. $\int_{\alpha}^{\beta} \left(\frac{\mathcal{J}(s)}{\mathcal{K}(s)} \right) {}_{\alpha}d_q s = \frac{\int_{\alpha}^{\beta} \mathcal{J}(s) {}_{\alpha}d_q s}{\int_{\alpha}^{\beta} \mathcal{K}(s) {}_{\alpha}d_q s}$.

Proof. When (1) and (3) are obvious, the following is carried out.
Equation (7) is used to prove (2).

$$\begin{aligned}\int_{\alpha}^{\beta} \mathcal{J}(s) {}_{\alpha}d_{\mathbf{q}}s &= e^{\int_{\alpha}^{\beta} \ln(\mathcal{J}(s)) {}_{\alpha}d_{\mathbf{q}}s} \\ &= e^{\int_{\alpha}^c \ln(\mathcal{J}(s)) {}_{\alpha}d_{\mathbf{q}}s + \int_c^{\beta} \ln(\mathcal{J}(s)) {}_{\alpha}d_{\mathbf{q}}s} \\ &= \int_{\alpha}^c \mathcal{J}(s) {}_{\alpha}d_{\mathbf{q}}s \cdot \int_c^{\beta} \mathcal{J}(s) {}_{\alpha}d_{\mathbf{q}}s.\end{aligned}$$

In a similar fashion, (4) can be proven:

$$\begin{aligned}\int_{\alpha}^{\beta} \left(\frac{\mathcal{J}(s)}{\mathcal{K}(s)} \right) {}_{\alpha}d_{\mathbf{q}}s &= e^{\int_{\alpha}^{\beta} \ln \left(\frac{\mathcal{J}(s)}{\mathcal{K}(s)} \right) {}_{\alpha}d_{\mathbf{q}}s} \\ &= e^{\int_{\alpha}^{\beta} \ln(\mathcal{J}(s)) {}_{\alpha}d_{\mathbf{q}}s - \int_{\alpha}^{\beta} \ln(\mathcal{K}(s)) {}_{\alpha}d_{\mathbf{q}}s} \\ &= e^{\int_{\alpha}^{\beta} \ln(\mathcal{J}(s)) {}_{\alpha}d_{\mathbf{q}}s} \cdot e^{-\int_{\alpha}^{\beta} \ln(\mathcal{K}(s)) {}_{\alpha}d_{\mathbf{q}}s} \\ &= \frac{\int_{\alpha}^{\beta} \mathcal{J}(s) {}_{\alpha}d_{\mathbf{q}}s}{\int_{\alpha}^{\beta} \mathcal{K}(s) {}_{\alpha}d_{\mathbf{q}}s}.\end{aligned}$$

□

Now, we construct some ${}_{\alpha}\mathbf{q}$ -multiplicative Hermite–Hadamard inequalities.

Theorem 7. If we assume a positive ${}_{\alpha}\mathbf{q}$ -multiplicative convex differentiable function $\mathcal{J}$ on $\mathcal{I}$, then

$$\mathcal{J}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \leq \left(\int_{\alpha}^{\beta} \mathcal{J}(s) {}_{\alpha}d_{\mathbf{q}}s\right)^{\frac{1}{\beta - \alpha}} \leq [\mathcal{J}(\alpha)^{\mathbf{q}} \cdot \mathcal{J}(\beta)]^{\frac{1}{\mathbf{q} + 1}}. \quad (9)$$

holds.

Proof. Due to the ${}_{\alpha}\mathbf{q}$ -multiplicative differentiability of $\mathcal{J}$ on (α, β) , the tangential line for $\mathcal{J}$ at point $\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1} \in (\alpha, \beta)$ can be expressed as

$$h(s) = \mathcal{J}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \mathcal{J}'\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right)^{\left(s - \frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right)}.$$

By the convexity of $\mathcal{J}$,

$$h(s) = \mathcal{J}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \mathcal{J}'\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right)^{\left(s - \frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right)} \leq \mathcal{J}(s).$$

For all $s \in [\alpha, \beta]$,

$$\ln h(s) = \ln \mathcal{J}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) + \left(s - \frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \ln \mathcal{J}'\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right).$$

By ${}_a q$ -integrating, we obtain

$$\begin{aligned} \int_a^\beta \ln \mathcal{J}(s) {}_a d_q s &\geq \int_a^\beta \ln \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) {}_a d_q s + \int_a^\beta \left(s - \frac{q\alpha + \beta}{q+1}\right) {}_a d_q s \ln \mathcal{J}'\left(\frac{q\alpha + \beta}{q+1}\right) \\ &= \ln \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) (\beta - \alpha) + (0) \ln \mathcal{J}'\left(\frac{q\alpha + \beta}{q+1}\right) \\ &= \ln \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) (\beta - \alpha) \\ \frac{\int_a^\beta \ln \mathcal{J}(s) {}_a d_q s}{\beta - \alpha} &\geq \ln \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right). \end{aligned}$$

Taking the exponential on both sides, we get

$$e^{\frac{1}{\beta-\alpha} \int_a^\beta \ln \mathcal{J}(s) {}_a d_q s} \geq e^{\ln \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right)}.$$

Hence,

$$\mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) \leq \left(\int_a^\beta (\mathcal{J}(s)) {}_a d_q s\right)^{\frac{1}{\beta-\alpha}}. \quad (10)$$

Additionally, the line segment that engages $(\alpha, \mathcal{J}(\alpha))$ and $(\beta, \mathcal{J}(\beta))$ is a function that can be generated:

$$h_1(s) = \mathcal{J}(\alpha) \left[\frac{\mathcal{J}(\beta)}{\mathcal{J}(\alpha)} \right]^{\left(\frac{s-\alpha}{\beta-\alpha}\right)}.$$

By the convexity of $\mathcal{J}$,

$$\mathcal{J}(s) \leq h_1(s) = \mathcal{J}(\alpha) \left[\frac{\mathcal{J}(\beta)}{\mathcal{J}(\alpha)} \right]^{\left(\frac{s-\alpha}{\beta-\alpha}\right)}.$$

For all $s \in [\alpha, \beta]$, we have

$$\ln \mathcal{J}(s) \leq \ln h_1(s) = [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)] \left(\frac{s-\alpha}{\beta-\alpha}\right) + \ln \mathcal{J}(\alpha).$$

By ${}_a q$ -integrating, we obtain

$$\begin{aligned} \int_a^\beta \ln \mathcal{J}(s) {}_a d_q s &\leq \int_a^\beta (s - \alpha) {}_a d_q s \left[\frac{\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)}{\beta - \alpha} \right] + \int_a^\beta \ln \mathcal{J}(\alpha) {}_a d_q s \\ &= \left(\frac{\beta - \alpha}{q+1}\right) [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)] + \ln(\mathcal{J}(\alpha))(\beta - \alpha) \\ &= (q+1) \left(\frac{\beta - \alpha}{q+1}\right) \ln \mathcal{J}(\alpha) + \left(\frac{\beta - \alpha}{q+1}\right) [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)] \\ &= \left(\frac{\beta - \alpha}{q+1}\right) [q \ln \mathcal{J}(\alpha) + \ln \mathcal{J}(\beta)] \\ \frac{1}{\beta - \alpha} \int_a^\beta \ln \mathcal{J}(s) {}_a d_q s &\leq \frac{1}{q+1} \ln [\mathcal{J}(\alpha)^q \mathcal{J}(\beta)]. \end{aligned}$$

Taking the exponential, we obtain

$$\begin{aligned} e^{\frac{1}{\beta-\alpha} \int_a^\beta \ln \mathcal{J}(s) {}_a d_q s} &\leq e^{\ln [\mathcal{J}(\alpha)^q \mathcal{J}(\beta)]^{\frac{1}{q+1}}} \\ \left(\int_a^\beta \mathcal{J}(s) {}_a d_q s\right)^{\frac{1}{\beta-\alpha}} &\leq [\mathcal{J}(\alpha)^q \mathcal{J}(\beta)]^{\frac{1}{q+1}}. \end{aligned} \quad (11)$$

From (10) and (11), the desired result has been achieved. $\square$

Remark 2. As $q \rightarrow 1$ in (9), it is reduced into (1), which is the Hermite–Hadamard inequality via multiplicative calculus.

Example 3. If we put $\mathcal{J}(s) = e^{s^2}$ in Theorem 7 and use (8), then we have following cases:

Case 1 If we fix $\alpha = 1$, $\beta = 2$, and $0 < q < 1$, then inequality (9) becomes

$$\begin{aligned} \mathcal{J}\left(\frac{q+2}{q+1}\right) &\leq \left(\int_{\alpha}^{\beta} e^{s^2} d_q s\right)^{\frac{1}{\beta-\alpha}} \leq [\mathcal{J}(1)^q \cdot \mathcal{J}(2)]^{\frac{1}{q+1}} \\ e^{\left(\frac{q+2}{q+1}\right)^2} &\leq \left[e^{(\beta-\alpha)\left(\frac{(\beta-\alpha)^2}{q^2+q+1} + \frac{2\alpha(\beta-\alpha)}{q+1} + \alpha^2\right)}\right]^{\frac{1}{\beta-\alpha}} \leq [e^q \cdot e^4]^{\frac{1}{q+1}} \\ e^{\left(\frac{q+2}{q+1}\right)^2} &\leq e^{\left[\frac{1}{q^2+q+1} + \frac{2}{q+1} + 1\right]} \leq [e^{q+4}]^{\frac{1}{q+1}} \\ e^{\left(\frac{q+2}{q+1}\right)^2} &\leq e^{\left(\frac{4+5q+4q^2+q^3}{1+2q+2q^2+q^3}\right)} \leq e^{\left(\frac{q+4}{q+1}\right)}. \end{aligned} \quad (12)$$

Case 2 If we fix $q = 0.5$, $1 \leq \alpha \leq 1.1$, and $3 \leq \beta \leq 5$ in (9), then we have

$$\begin{aligned} \mathcal{J}\left(\frac{\alpha+2\beta}{3}\right) &\leq \left(\int_{\alpha}^{\beta} e^{s^2} d_q s\right)^{\frac{1}{\beta-\alpha}} \leq [\mathcal{J}(\alpha)^{\frac{1}{2}} \cdot \mathcal{J}(\beta)]^{\frac{2}{3}} \\ e^{\left(\frac{\alpha+2\beta}{3}\right)^2} &\leq \left[e^{(\beta-\alpha)\left(\frac{(\beta-\alpha)^2}{q^2+q+1} + \frac{2\alpha(\beta-\alpha)}{q+1} + \alpha^2\right)}\right]^{\frac{1}{\beta-\alpha}} \leq [e^{\frac{\alpha^2}{2}} \cdot e^{\beta^2}]^{\frac{2}{3}} \\ e^{\left(\frac{\alpha+2\beta}{3}\right)^2} &\leq e^{\left(\frac{4(\beta-\alpha)^2}{7} + \frac{4\alpha(\beta-\alpha)}{3} + \alpha^2\right)} \leq \left(e^{\frac{\alpha^2+2\beta^2}{2}}\right)^{\frac{2}{3}} \\ e^{\left(\frac{\alpha+2\beta}{3}\right)^2} &\leq e^{\left(\frac{5\alpha^2+12\beta^2+4\alpha\beta}{21}\right)} \leq (e^{\alpha^2+2\beta^2})^{\frac{1}{3}}. \end{aligned} \quad (13)$$

The pictorial visualization of (12) and (13) can be seen in Figure 3.

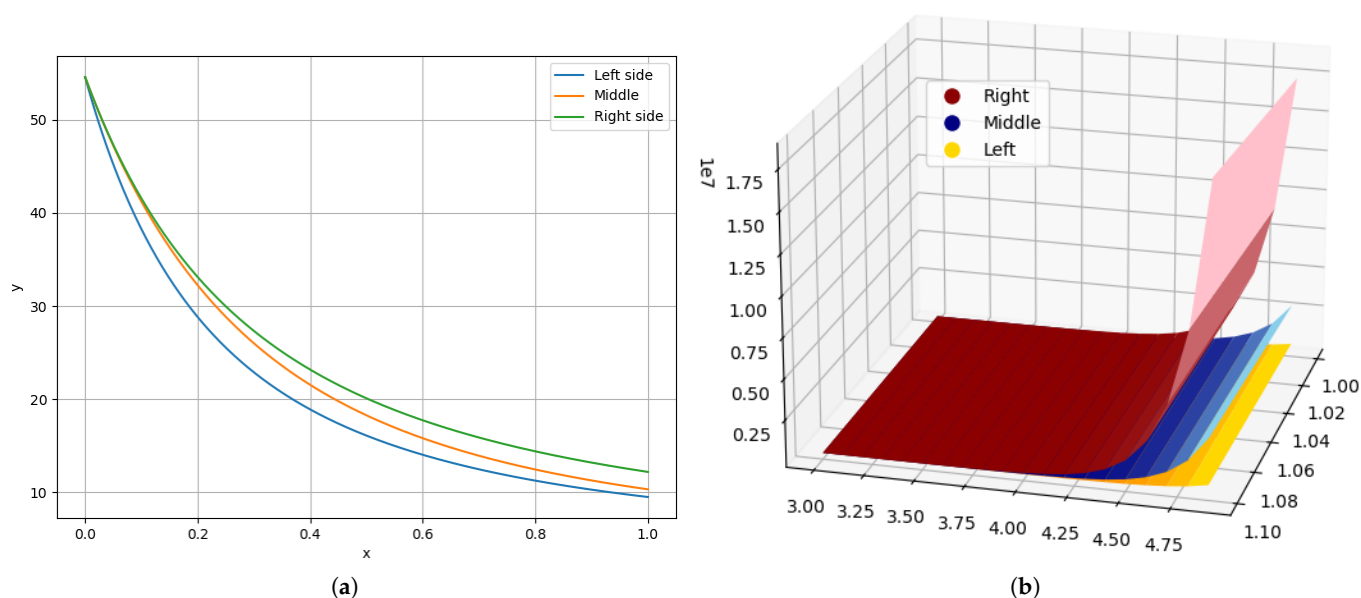


Figure 3. (a) Pictorial form of (12). (b) Pictorial form of (13).

Theorem 8. If we assume two positive ${}_a q$ -multiplicative convex differentiable functions $\mathcal{J}$ and $\mathcal{K}$ on $\mathcal{I}$, then

$$\begin{aligned} \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) \cdot \mathcal{K}\left(\frac{q\alpha + \beta}{q+1}\right) &\leq \left(\int_{\alpha}^{\beta} \mathcal{J}(s) {}_a d_q s \cdot \int_{\alpha}^{\beta} \mathcal{K}(s) {}_a d_q s\right)^{\frac{1}{\beta-\alpha}} \\ &\leq [\mathcal{J}(\alpha)^q \cdot \mathcal{J}(\beta) \cdot \mathcal{K}(\alpha)^q \cdot \mathcal{K}(\beta)]^{\frac{1}{q+1}} \end{aligned} \quad (14)$$

holds.

Proof. Since $\mathcal{J}$ and $\mathcal{K}$ are ${}_a q$ -multiplicative differentiable functions on (α, β) , their tangents' equations at the point $\frac{q\alpha + \beta}{q+1} \in (\alpha, \beta)$ can therefore be expressed as

$$h(s) = \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) \mathcal{J}'\left(\frac{q\alpha + \beta}{q+1}\right)^{\left(s - \frac{q\alpha + \beta}{q+1}\right)}$$

and

$$h_2(s) = \mathcal{K}\left(\frac{q\alpha + \beta}{q+1}\right) \mathcal{K}'\left(\frac{q\alpha + \beta}{q+1}\right)^{\left(s - \frac{q\alpha + \beta}{q+1}\right)}.$$

Due to convexity of $\mathcal{J}$ and $\mathcal{K}$, we can write

$$h(s) = \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) \mathcal{J}'\left(\frac{q\alpha + \beta}{q+1}\right)^{\left(s - \frac{q\alpha + \beta}{q+1}\right)} \leq \mathcal{J}(s) \quad (15)$$

and

$$h_2(s) = \mathcal{K}\left(\frac{q\alpha + \beta}{q+1}\right) \mathcal{K}'\left(\frac{q\alpha + \beta}{q+1}\right)^{\left(s - \frac{q\alpha + \beta}{q+1}\right)} \leq \mathcal{K}(s). \quad (16)$$

By multiplying (15) and (16) and then taking $\ln$, we get

$$\begin{aligned} \ln h(s) + \ln h_2(s) &= \ln \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) + \left(s - \frac{q\alpha + \beta}{q+1}\right) \ln \mathcal{J}'\left(\frac{q\alpha + \beta}{q+1}\right) + \ln \mathcal{K}\left(\frac{q\alpha + \beta}{q+1}\right) \\ &\quad + \left(s - \frac{q\alpha + \beta}{q+1}\right) \ln \mathcal{K}'\left(\frac{q\alpha + \beta}{q+1}\right) \leq \ln \mathcal{J}(s) + \ln \mathcal{K}(s). \end{aligned}$$

By ${}_a q$ -integrating, we obtain

$$\begin{aligned} \int_{\alpha}^{\beta} \ln \mathcal{J}(s) {}_a d_q s + \int_{\alpha}^{\beta} \ln \mathcal{K}(s) {}_a d_q s &\geq \int_{\alpha}^{\beta} \ln \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) {}_a d_q s + \ln \mathcal{J}'\left(\frac{q\alpha + \beta}{q+1}\right) \int_{\alpha}^{\beta} \left(s - \frac{q\alpha + \beta}{q+1}\right) {}_a d_q s \\ &\quad + \int_{\alpha}^{\beta} \ln \mathcal{K}\left(\frac{q\alpha + \beta}{q+1}\right) {}_a d_q s + \ln \mathcal{K}'\left(\frac{q\alpha + \beta}{q+1}\right) \int_{\alpha}^{\beta} \left(s - \frac{q\alpha + \beta}{q+1}\right) {}_a d_q s \\ &= (\beta - \alpha) \ln \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) + \ln \mathcal{J}'\left(\frac{q\alpha + \beta}{q+1}\right)(0) + (\beta - \alpha) \ln \mathcal{K}\left(\frac{q\alpha + \beta}{q+1}\right) \\ &\quad + \ln \mathcal{K}'\left(\frac{q\alpha + \beta}{q+1}\right)(0) \\ &= (\beta - \alpha) \left\{ \ln \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) + \ln \mathcal{K}\left(\frac{q\alpha + \beta}{q+1}\right) \right\} \\ \frac{1}{\beta - \alpha} \left(\int_{\alpha}^{\beta} \ln \mathcal{J}(s) {}_a d_q s + \int_{\alpha}^{\beta} \ln \mathcal{K}(s) {}_a d_q s \right) &\geq \ln \mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) + \ln \mathcal{K}\left(\frac{q\alpha + \beta}{q+1}\right). \end{aligned}$$

Taking the exponential, we obtain

$$e^{\frac{1}{\beta-\alpha} \left(\int_{\alpha}^{\beta} \ln \mathcal{J}(s) {}_a d_q s + \int_{\alpha}^{\beta} \ln \mathcal{K}(s) {}_a d_q s \right)} \geq e^{\ln \left(\mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) \cdot \mathcal{K}\left(\frac{q\alpha + \beta}{q+1}\right) \right)}.$$

Hence,

$$\mathcal{J}\left(\frac{q\alpha + \beta}{q+1}\right) \cdot \mathcal{K}\left(\frac{q\alpha + \beta}{q+1}\right) \leq \left(\int_{\alpha}^{\beta} \mathcal{J}(s) {}_a d_q s \cdot \int_{\alpha}^{\beta} \mathcal{K}(s) {}_a d_q s\right)^{\frac{1}{\beta-\alpha}}. \quad (17)$$

Furthermore, for the lines connecting points $(\alpha, \mathcal{J}(\alpha))$, $(\beta, \mathcal{J}(\beta))$, and $(\alpha, \mathcal{K}(\alpha))$, $(\beta, \mathcal{K}(\beta))$ can be expressed as functions

$$h_1(s) = \mathcal{J}(\alpha) \left[\frac{\mathcal{J}(\beta)}{\mathcal{J}(\alpha)} \right]^{\left(\frac{s-\alpha}{\beta-\alpha}\right)}$$

and

$$h_3(s) = \mathcal{K}(\alpha) \left[\frac{\mathcal{K}(\beta)}{\mathcal{K}(\alpha)} \right]^{\left(\frac{s-\alpha}{\beta-\alpha}\right)}$$

respectively. Due to the convexity of $\mathcal{J}$ and $\mathcal{K}$, we have

$$\mathcal{J}(s) \leq h_1(s) = \mathcal{J}(\alpha) \left[\frac{\mathcal{J}(\beta)}{\mathcal{J}(\alpha)} \right]^{\left(\frac{s-\alpha}{\beta-\alpha}\right)} \quad (18)$$

and

$$\mathcal{K}(s) \leq h_3(s) = \mathcal{K}(\alpha) \left[\frac{\mathcal{K}(\beta)}{\mathcal{K}(\alpha)} \right]^{\left(\frac{s-\alpha}{\beta-\alpha}\right)}. \quad (19)$$

By multiplying (18) and (19) and then taking $\ln$, we have

$$\begin{aligned} \ln \mathcal{J}(s) + \ln \mathcal{K}(s) &\leq \ln \mathcal{J}(\alpha) + \left(\frac{s}{\beta-\alpha} - \frac{\alpha}{\beta-\alpha} \right) [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)] + \ln \mathcal{K}(\alpha) \\ &\quad + \left(\frac{s}{\beta-\alpha} - \frac{\alpha}{\beta-\alpha} \right) [\ln \mathcal{K}(\beta) - \ln \mathcal{K}(\alpha)]. \end{aligned}$$

Taking ${}_a q$ -integral, we obtain

$$\begin{aligned} \int_{\alpha}^{\beta} \ln \mathcal{J}(s) {}_a d_q s + \int_{\alpha}^{\beta} \ln \mathcal{K}(s) {}_a d_q s &\leq \int_{\alpha}^{\beta} \ln \mathcal{J}(\alpha) {}_a d_q s + \int_{\alpha}^{\beta} \left(\frac{s}{\beta-\alpha} - \frac{\alpha}{\beta-\alpha} \right) {}_a d_q s [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)] \\ &\quad + \int_{\alpha}^{\beta} \ln \mathcal{K}(\alpha) {}_a d_q s + \int_{\alpha}^{\beta} \left(\frac{s}{\beta-\alpha} - \frac{\alpha}{\beta-\alpha} \right) {}_a d_q s [\ln \mathcal{K}(\beta) - \ln \mathcal{K}(\alpha)] \\ &= (\beta - \alpha) \ln \mathcal{J}(\alpha) + \left(\frac{\beta - \alpha}{q+1} \right) [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)] + (\beta - \alpha) \ln \mathcal{K}(\alpha) \\ &\quad + \left(\frac{\beta - \alpha}{q+1} \right) [\ln \mathcal{K}(\beta) - \ln \mathcal{K}(\alpha)] \\ &= \left(\frac{\beta - \alpha}{q+1} \right) [q \ln \mathcal{J}(\alpha) + \ln \mathcal{J}(\beta)] + \left(\frac{\beta - \alpha}{q+1} \right) [q \ln \mathcal{K}(\alpha) + \ln \mathcal{K}(\beta)] \\ \frac{1}{\beta - \alpha} \left(\int_{\alpha}^{\beta} \ln \mathcal{J}(s) {}_a d_q s + \int_{\alpha}^{\beta} \ln \mathcal{K}(s) {}_a d_q s \right) &\leq \frac{1}{q+1} \ln [\mathcal{J}(\alpha)^q \mathcal{J}(\beta)] + \frac{1}{q+1} \ln [\mathcal{K}(\alpha)^q \mathcal{K}(\beta)]. \end{aligned}$$

By taking the exponential, we obtain

$$\begin{aligned} e^{\frac{1}{\beta-\alpha} \left(\int_{\alpha}^{\beta} \ln \mathcal{J}(s) {}_a d_q s + \int_{\alpha}^{\beta} \ln \mathcal{K}(s) {}_a d_q s \right)} &\leq e^{\frac{1}{q+1} \ln [\mathcal{J}(\alpha)^q \mathcal{J}(\beta)] + \frac{1}{q+1} \ln [\mathcal{K}(\alpha)^q \mathcal{K}(\beta)]} \\ \left(\int_{\alpha}^{\beta} \mathcal{J}(s) {}_a d_q s \cdot \int_{\alpha}^{\beta} \mathcal{K}(s) {}_a d_q s \right)^{\frac{1}{\beta-\alpha}} &\leq [\mathcal{J}(\alpha)^q \cdot \mathcal{J}(\beta) \cdot \mathcal{K}(\alpha)^q \cdot \mathcal{K}(\beta)]^{\frac{1}{q+1}}. \quad (20) \end{aligned}$$

By combining (17) and (20), we get (14). $\square$

Remark 3. As $q \rightarrow 1$ in (14), it is reduced into (2), which is another Hermite–Hadamard inequality via multiplicative calculus.

Example 4. If we put $\mathcal{J}(s) = e^{s^2}$ and $\mathcal{K}(s) = e^s$ in Theorem 8, and also use (8) and Definition 6, then we have following cases:

Case 1 If we fix $\alpha = 1$, $\beta = 2$, and $0 < q < 1$, then inequality (14) becomes

$$\begin{aligned} \mathcal{J}\left(\frac{q+2}{q+1}\right) \cdot \mathcal{K}\left(\frac{q+2}{q+1}\right) &\leq \left(\int_{\alpha}^{\beta} e^{s^2} d_q s \cdot \int_{\alpha}^{\beta} e^s d_q s\right)^{\frac{1}{\beta-\alpha}} \leq [\mathcal{J}(1)^q \cdot \mathcal{J}(2) \cdot \mathcal{K}(1)^q \cdot \mathcal{K}(2)]^{\frac{1}{q+1}} \\ e^{\left(\frac{q+2}{q+1}\right)^2} \cdot e^{\left(\frac{q+2}{q+1}\right)} &\leq \left[e^{(\beta-\alpha)\left(\frac{(\beta-\alpha)^2}{q^2+q+1} + \frac{2\alpha(\beta-\alpha)}{q+1} + \alpha^2\right)} \cdot e^{(\beta-\alpha)\left(\frac{\beta+q\alpha}{q+1}\right)}\right]^{\frac{1}{\beta-\alpha}} \leq [e^q \cdot e^4 \cdot e^q \cdot e^2]^{\frac{1}{q+1}} \\ e^{\left(\frac{2q^2+7q+6}{q^2+2q+1}\right)} &\leq e^{\left[\frac{1}{q^2+q+1} + \frac{2}{q+1} + 1 + \frac{2+q}{q+1}\right]} \leq [e^{2q+6}]^{\frac{1}{q+1}} \\ e^{\left(\frac{2q^2+7q+6}{q^2+2q+1}\right)} &\leq e^{\left(\frac{6+8q+7q^2+2q^3}{1+2q+2q^2+q^3}\right)} \leq e^{\left(\frac{2q+6}{q+1}\right)}. \end{aligned} \quad (21)$$

Case 2 If we fix $q = 0.5$, $1 \leq \alpha \leq 1.1$, and $3 \leq \beta \leq 5$ in (14), then we have

$$\begin{aligned} \mathcal{J}\left(\frac{\alpha+2\beta}{3}\right) \cdot \mathcal{K}\left(\frac{\alpha+2\beta}{3}\right) &\leq \left(\int_{\alpha}^{\beta} e^{s^2} d_q s \cdot \int_{\alpha}^{\beta} e^s d_q s\right)^{\frac{1}{\beta-\alpha}} \leq [\mathcal{J}(\alpha)^{\frac{1}{2}} \cdot \mathcal{J}(\beta) \cdot \mathcal{K}(\alpha)^{\frac{1}{2}} \cdot \mathcal{K}(\beta)]^{\frac{2}{3}} \\ e^{\left(\frac{\alpha+2\beta}{3}\right)^2} \cdot e^{\left(\frac{\alpha+2\beta}{3}\right)} &\leq \left[e^{(\beta-\alpha)\left(\frac{(\beta-\alpha)^2}{q^2+q+1} + \frac{2\alpha(\beta-\alpha)}{q+1} + \alpha^2\right)} \cdot e^{(\beta-\alpha)\left(\frac{\beta+q\alpha}{q+1}\right)}\right]^{\frac{1}{\beta-\alpha}} \leq [e^{\frac{\alpha^2}{2}} \cdot e^{\beta^2} \cdot e^{\frac{\alpha}{2}} \cdot e^{\beta}]^{\frac{2}{3}} \\ e^{\left(\frac{\alpha^2+4\beta^2+3\alpha+6\beta+4\alpha\beta}{9}\right)} &\leq e^{\left(\frac{4(\beta-\alpha)^2}{7} + \frac{4\alpha(\beta-\alpha)}{3} + \alpha^2 + \frac{2\beta+\alpha}{3}\right)} \leq \left(e^{\frac{\alpha^2+2\beta^2+\alpha+2\beta}{2}}\right)^{\frac{2}{3}} \\ e^{\left(\frac{\alpha^2+4\beta^2+3\alpha+6\beta+4\alpha\beta}{9}\right)} &\leq e^{\left(\frac{5\alpha^2+12\beta^2+4\alpha\beta+7\alpha+14\beta}{21}\right)} \leq \left(e^{\frac{\alpha^2+2\beta^2+\alpha+2\beta}{3}}\right). \end{aligned} \quad (22)$$

The pictorial visualization of (21) and (22) can be seen in Figure 4.

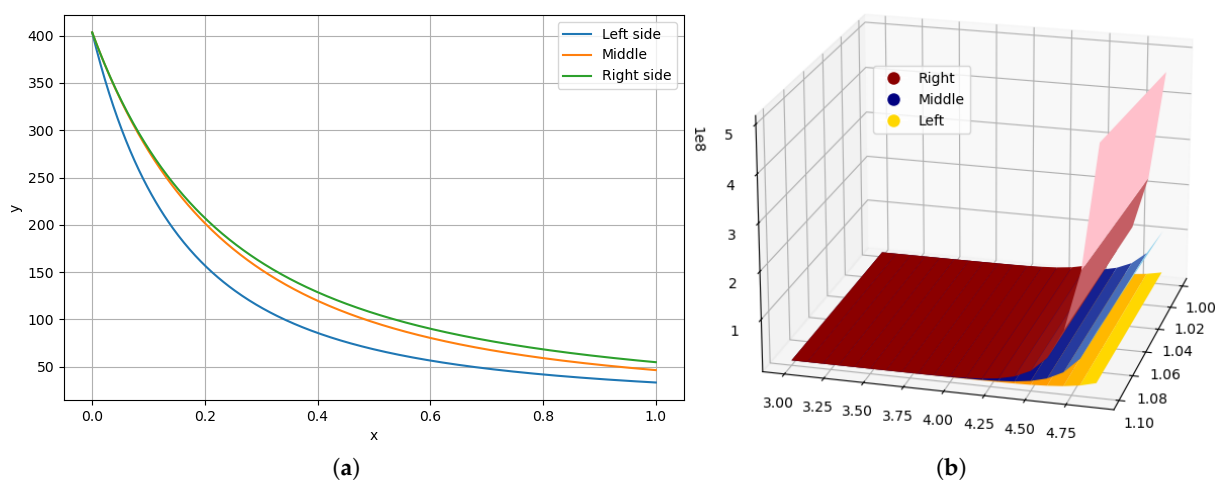


Figure 4. (a) Graphical representation of (21). (b) Graphical representation of (22).

Theorem 9. If we assume two positive ${}_a\mathbf{q}$ -multiplicative convex differentiable functions $\mathcal{J}$ and $\mathcal{K}$ on $\mathcal{I}$, then

$$\begin{aligned} \mathcal{J}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) : \mathcal{K}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) &\leq \left(\int_{\alpha}^{\beta} \mathcal{J}(s) {}_a\mathbf{d}_{\mathbf{q}}s : \int_{\alpha}^{\beta} \mathcal{K}(s) {}_a\mathbf{d}_{\mathbf{q}}s\right)^{\frac{1}{\beta-\alpha}} \\ &\leq [\mathcal{J}(\alpha)^{\mathbf{q}} \cdot \mathcal{J}(\beta) : \mathcal{K}(\alpha)^{\mathbf{q}} \cdot \mathcal{K}(\beta)]^{\frac{1}{\mathbf{q}+1}} \end{aligned} \quad (23)$$

holds, where “:” represents division.

Proof. By dividing (15) from (16) and taking $\ln$, then

$$\begin{aligned} \ln h(s) - \ln h_2(s) &= \ln \mathcal{J}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) + \left(s - \frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \ln \mathcal{J}'\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) - \ln \mathcal{K}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \\ &\quad - \left(s - \frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \ln \mathcal{K}'\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \leq \ln \mathcal{J}(s) - \ln \mathcal{K}(s). \end{aligned}$$

By ${}_a\mathbf{q}$ -integrating, we obtain

$$\begin{aligned} \int_{\alpha}^{\beta} \ln \mathcal{J}(s) {}_a\mathbf{d}_{\mathbf{q}}s - \int_{\alpha}^{\beta} \ln \mathcal{K}(s) {}_a\mathbf{d}_{\mathbf{q}}s &\geq \int_{\alpha}^{\beta} \ln \mathcal{J}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) {}_a\mathbf{d}_{\mathbf{q}}s + \ln \mathcal{J}'\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \int_{\alpha}^{\beta} \left(s - \frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) {}_a\mathbf{d}_{\mathbf{q}}s \\ &\quad - \int_{\alpha}^{\beta} \ln \mathcal{K}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) {}_a\mathbf{d}_{\mathbf{q}}s - \ln \mathcal{K}'\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \int_{\alpha}^{\beta} \left(s - \frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) {}_a\mathbf{d}_{\mathbf{q}}s \\ &= (\beta - \alpha) \ln \mathcal{J}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) + \ln \mathcal{J}'\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) (0) - (\beta - \alpha) \ln \mathcal{K}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \\ &\quad - \ln \mathcal{K}'\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) (0) \\ &= (\beta - \alpha) \ln \mathcal{J}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) - (\beta - \alpha) \ln \mathcal{K}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \\ \frac{1}{\beta - \alpha} \left(\int_{\alpha}^{\beta} \ln \mathcal{J}(s) {}_a\mathbf{d}_{\mathbf{q}}s - \int_{\alpha}^{\beta} \ln \mathcal{K}(s) {}_a\mathbf{d}_{\mathbf{q}}s \right) &\geq \ln \mathcal{J}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) - \ln \mathcal{K}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right). \end{aligned}$$

Taking the exponential, we obtain

$$e^{\frac{1}{\beta-\alpha} \left(\int_{\alpha}^{\beta} \ln \mathcal{J}(s) {}_a\mathbf{d}_{\mathbf{q}}s - \int_{\alpha}^{\beta} \ln \mathcal{K}(s) {}_a\mathbf{d}_{\mathbf{q}}s \right)} \geq e^{\ln \left(\mathcal{J}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) : \mathcal{K}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \right)}.$$

Hence,

$$\mathcal{J}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) : \mathcal{K}\left(\frac{\mathbf{q}\alpha + \beta}{\mathbf{q} + 1}\right) \leq \left(\int_{\alpha}^{\beta} \mathcal{J}(s) {}_a\mathbf{d}_{\mathbf{q}}s : \int_{\alpha}^{\beta} \mathcal{K}(s) {}_a\mathbf{d}_{\mathbf{q}}s \right)^{\frac{1}{\beta-\alpha}}. \quad (24)$$

Similarly, by dividing (18) from (19) and taking $\ln$, we have

$$\begin{aligned} \ln \mathcal{J}(s) - \ln \mathcal{K}(s) &\leq \ln \mathcal{J}(\alpha) + \left(\frac{s}{\beta - \alpha} - \frac{\alpha}{\beta - \alpha} \right) [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)] - \ln \mathcal{K}(\alpha) \\ &\quad - \left(\frac{s}{\beta - \alpha} - \frac{\alpha}{\beta - \alpha} \right) [\ln \mathcal{K}(\beta) - \ln \mathcal{K}(\alpha)]. \end{aligned}$$

Taking ${}_a\mathbf{q}$ -integral, we obtain

$$\begin{aligned}
\int_{\alpha}^{\beta} \ln \mathcal{J}(s)_{\alpha} d_{\mathbf{q}} s - \int_{\alpha}^{\beta} \ln \mathcal{K}(s)_{\alpha} d_{\mathbf{q}} s &\leq \int_{\alpha}^{\beta} \ln \mathcal{J}(\alpha)_{\alpha} d_{\mathbf{q}} s + \int_{\alpha}^{\beta} \left(\frac{s-\alpha}{\beta-\alpha} \right)_{\alpha} d_{\mathbf{q}} s [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)] \\
&\quad - \int_{\alpha}^{\beta} \ln \mathcal{K}(\alpha)_{\alpha} d_{\mathbf{q}} s - \int_{\alpha}^{\beta} \left(\frac{s-\alpha}{\beta-\alpha} \right)_{\alpha} d_{\mathbf{q}} s [\ln \mathcal{K}(\beta) - \ln \mathcal{K}(\alpha)] \\
&= (\beta - \alpha) \ln \mathcal{J}(\alpha) + \left(\frac{\beta - \alpha}{\mathbf{q} + 1} \right) [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)] - (\beta - \alpha) \ln \mathcal{K}(\alpha) \\
&\quad - \left(\frac{\beta - \alpha}{\mathbf{q} + 1} \right) [\ln \mathcal{K}(\beta) - \ln \mathcal{K}(\alpha)] \\
&= \left(\frac{\beta - \alpha}{\mathbf{q} + 1} \right) [\mathbf{q} \ln \mathcal{J}(\alpha) + \ln \mathcal{J}(\beta)] - \left(\frac{\beta - \alpha}{\mathbf{q} + 1} \right) [\mathbf{q} \ln \mathcal{K}(\alpha) + \ln \mathcal{K}(\beta)] \\
\frac{1}{\beta - \alpha} \left(\int_{\alpha}^{\beta} \ln \mathcal{J}(s)_{\alpha} d_{\mathbf{q}} s - \int_{\alpha}^{\beta} \ln \mathcal{K}(s)_{\alpha} d_{\mathbf{q}} s \right) &\leq \frac{1}{\mathbf{q} + 1} \ln [\mathcal{J}(\alpha)^{\mathbf{q}} \mathcal{J}(\beta)] - \frac{1}{\mathbf{q} + 1} \ln [\mathcal{K}(\alpha)^{\mathbf{q}} \mathcal{K}(\beta)].
\end{aligned}$$

By taking the exponential, we obtain

$$\begin{aligned}
e^{\frac{1}{\beta-\alpha} \left(\int_{\alpha}^{\beta} \ln \mathcal{J}(s)_{\alpha} d_{\mathbf{q}} s - \int_{\alpha}^{\beta} \ln \mathcal{K}(s)_{\alpha} d_{\mathbf{q}} s \right)} &\leq e^{\frac{1}{\mathbf{q}+1} \ln [\mathcal{J}(\alpha)^{\mathbf{q}} \mathcal{J}(\beta)] - \frac{1}{\mathbf{q}+1} \ln [\mathcal{K}(\alpha)^{\mathbf{q}} \mathcal{K}(\beta)]} \\
\left(\int_{\alpha}^{\beta} \mathcal{J}(s)_{\alpha} d_{\mathbf{q}} s : \int_{\alpha}^{\beta} \mathcal{K}(s)_{\alpha} d_{\mathbf{q}} s \right)^{\frac{1}{\beta-\alpha}} &\leq [\mathcal{J}(\alpha)^{\mathbf{q}} \cdot \mathcal{J}(\beta) : \mathcal{K}(\alpha)^{\mathbf{q}} \cdot \mathcal{K}(\beta)]^{\frac{1}{\mathbf{q}+1}}. \quad (25)
\end{aligned}$$

Combining (24) and (25), we get (23). $\square$

Remark 4. As $\mathbf{q} \rightarrow 1$ in (23), it is reduced to (3), which is another Hermite–Hadamard-type inequality via multiplicative calculus.

Example 5. If we put $\mathcal{J}(s) = e^{s^2}$ and $\mathcal{K}(s) = e^s$ in Theorem 9, and also use (8) and Definition 6, then we have the following cases:

Case 1 If we fix $\alpha = 1$, $\beta = 2$, and $0 < \mathbf{q} < 1$, then inequality (23) becomes

$$\begin{aligned}
\mathcal{J}\left(\frac{\mathbf{q}+2}{\mathbf{q}+1}\right) : \mathcal{K}\left(\frac{\mathbf{q}+2}{\mathbf{q}+1}\right) &\leq \left(\int_{\alpha}^{\beta} e^{s^2}_{\alpha} d_{\mathbf{q}} s : \int_{\alpha}^{\beta} e^s_{\alpha} d_{\mathbf{q}} s \right)^{\frac{1}{\beta-\alpha}} \leq [\mathcal{J}(1)^{\mathbf{q}} \cdot \mathcal{J}(2) : \mathcal{K}(1)^{\mathbf{q}} \cdot \mathcal{K}(2)]^{\frac{1}{\mathbf{q}+1}} \\
e^{\left(\frac{\mathbf{q}+2}{\mathbf{q}+1}\right)^2} \cdot e^{-\left(\frac{\mathbf{q}+2}{\mathbf{q}+1}\right)} &\leq \left[e^{(\beta-\alpha)\left(\frac{(\beta-\alpha)^2}{\mathbf{q}^2+\mathbf{q}+1} + \frac{2\alpha(\beta-\alpha)}{\mathbf{q}+1} + \alpha^2\right)} \cdot e^{-(\beta-\alpha)\left(\frac{\beta+\mathbf{q}\alpha}{\mathbf{q}+1}\right)} \right]^{\frac{1}{\beta-\alpha}} \leq [e^{\mathbf{q}} \cdot e^4 \cdot e^{-\mathbf{q}} \cdot e^{-2}]^{\frac{1}{\mathbf{q}+1}} \\
e^{\left(\frac{\mathbf{q}+2}{\mathbf{q}^2+2\mathbf{q}+1}\right)} &\leq e^{\left[\frac{1}{\mathbf{q}^2+\mathbf{q}+1} + \frac{2}{\mathbf{q}+1} + 1 - \frac{2+\mathbf{q}}{\mathbf{q}+1}\right]} \leq [e^2]^{\frac{1}{\mathbf{q}+1}} \\
e^{\left(\frac{\mathbf{q}+2}{\mathbf{q}^2+2\mathbf{q}+1}\right)} &\leq e^{\left(\frac{2+2\mathbf{q}+\mathbf{q}^2}{1+2\mathbf{q}+2\mathbf{q}^2+\mathbf{q}^3}\right)} \leq e^{\left(\frac{2}{\mathbf{q}+1}\right)}. \quad (26)
\end{aligned}$$

Case 2 If we fix $\mathbf{q} = 0.5$, $1 \leq \alpha \leq 1.1$, and $3 \leq \beta \leq 5$ in (23), then we have

$$\begin{aligned}
\mathcal{J}\left(\frac{\alpha+2\beta}{3}\right) : \mathcal{K}\left(\frac{\alpha+2\beta}{3}\right) &\leq \left(\int_{\alpha}^{\beta} e^{s^2_{\alpha} d_q s} : \int_{\alpha}^{\beta} e^{s^2_{\alpha} d_q s} \right)^{\frac{1}{\beta-\alpha}} \leq \left[\mathcal{J}(\alpha)^{\frac{1}{2}} \cdot \mathcal{J}(\beta) : \mathcal{K}(\alpha)^{\frac{1}{2}} \cdot \mathcal{K}(\beta) \right]^{\frac{2}{3}} \\
e^{\left(\frac{\alpha+2\beta}{3}\right)^2} \cdot e^{-\left(\frac{\alpha+2\beta}{3}\right)} &\leq \left[e^{(\beta-\alpha)\left(\frac{(\beta-\alpha)^2}{q^2+q+1} + \frac{2\alpha(\beta-\alpha)}{q+1} + \alpha^2\right)} \cdot e^{-(\beta-\alpha)\left(\frac{\beta+q\alpha}{q+1}\right)} \right]^{\frac{1}{\beta-\alpha}} \leq \left[e^{\frac{\alpha^2}{2}} \cdot e^{\beta^2} \cdot e^{-\frac{\alpha}{2}} \cdot e^{-\beta} \right]^{\frac{2}{3}} \\
e^{\left(\frac{\alpha^2+4\beta^2-3\alpha-6\beta+4\alpha\beta}{9}\right)} &\leq e^{\left(\frac{4(\beta-\alpha)^2}{7} + \frac{4\alpha(\beta-\alpha)}{3} + \alpha^2 - \frac{2\beta+\alpha}{3}\right)} \leq \left(e^{\frac{\alpha^2+2\beta^2-\alpha-2\beta}{2}} \right)^{\frac{2}{3}} \\
e^{\left(\frac{\alpha^2+4\beta^2-3\alpha-6\beta+4\alpha\beta}{9}\right)} &\leq e^{\left(\frac{5\alpha^2+12\beta^2+4\alpha\beta-7\alpha-14\beta}{21}\right)} \leq \left(e^{\frac{\alpha^2+2\beta^2-\alpha-2\beta}{3}} \right). \tag{27}
\end{aligned}$$

The pictorial visualization of (26) and (27) can be seen in Figure 5.

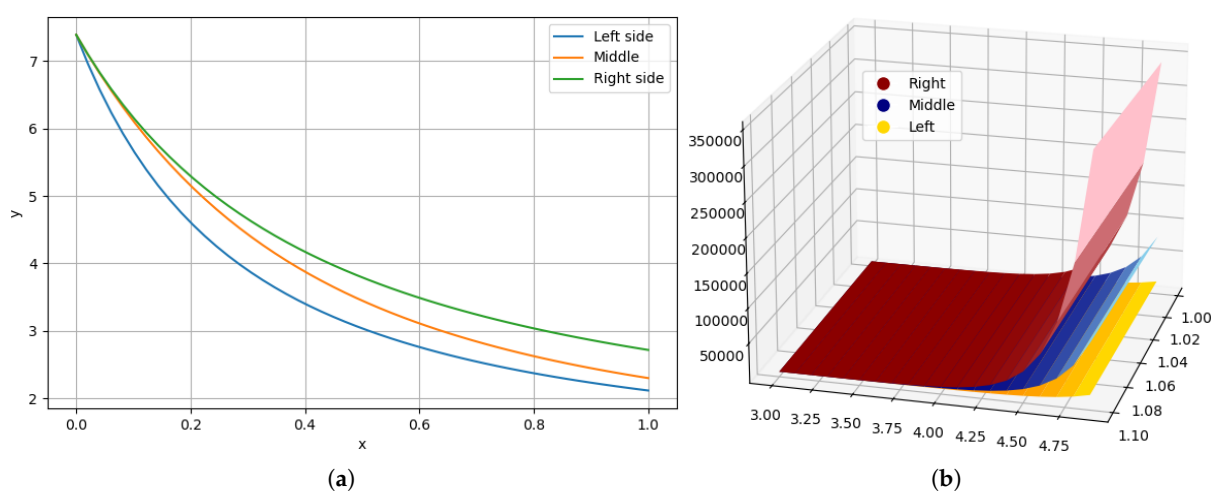


Figure 5. (a) Graphical visualization of (26). (b) Graphical visualization of (27).

Now, we define new definitions of derivatives and definite integrals in the q -multiplicative calculus using the right endpoint β of $\mathcal{I}$.

Definition 13. If we consider a positive function $\mathcal{J}$ defined on $\mathcal{I}$ with $\mathcal{J}(s) \neq 0$, then the ${}^{\beta}q$ -multiplicative or right q -multiplicative derivative of $\mathcal{J}$ at $s \in \mathbb{R}$ and is written as

$${}^{\beta}\mathfrak{D}_q^* \mathcal{J}(s) = \left(\frac{\mathcal{J}(\beta(1-q) + qs)}{\mathcal{J}(s)} \right)^{\frac{1}{(\beta-s)(1-q)}}.$$

Example 6. If we let $\mathcal{J}(s) = s^2 + 1$ be a positive function on $[2, 5]$ and fix $q = 0.5$, then its ${}^{\beta}q$ -multiplicative derivative can be calculated as

$$\begin{aligned}
{}^{\beta}\mathfrak{D}_q^* \mathcal{J}(s) &= \left(\frac{\mathcal{J}(\beta(1-q) + qs)}{\mathcal{J}(s)} \right)^{\frac{1}{(s-\beta)(q-1)}} = \left(\frac{\beta(1-q) + q(s^2 + 1)}{s^2 + 1} \right)^{\frac{1}{(s-\beta)(q-1)}} \\
{}^5\mathfrak{D}_{0.5}^* \mathcal{J}(s) &= \left(\frac{\frac{5}{2} + \frac{s^2 + 1}{2}}{s^2 + 1} \right)^{\frac{1}{(s-5)(0.5-1)}} = \left(\frac{s^2 + 6}{2s^2 + 2} \right)^{\frac{2}{5-s}}.
\end{aligned}$$

Definition 14. If we assume a bounded function on $0 < \alpha < \beta$ and a positive function $\mathcal{J}$, then for $s \in \mathcal{I}$

$$\int_{\alpha}^{\beta} \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s = e^{\int_{\alpha}^{\beta} \ln(\mathcal{J}(s))^{\beta} d_{\mathbf{q}} s} = e^{\left(\int_{\alpha}^{\beta} \ln(\mathcal{J}(s))^{\beta} d_{\mathbf{q}} s - \int_{\alpha}^{\alpha} \ln(\mathcal{J}(s))^{\beta} d_{\mathbf{q}} s\right)} \quad (28)$$

is called a right $\mathbf{q}$ - or ${}^{\beta}\mathbf{q}$ -multiplicative integral of $\mathcal{J}$ over $\mathcal{I}$.

Some basic properties of the right ${}^{\beta}\mathbf{q}$ -multiplicative integral of a function are written below.

Theorem 10. If we assume two positive ${}^{\beta}\mathbf{q}$ -multiplicative integrable functions $\mathcal{J}$ and $\mathcal{K}$ with $c \in \mathcal{I}$ and $n \in \mathbb{R}$, then

1. $\int_{\alpha}^{\beta} (\mathcal{J}(s)^n)^{\beta} d_{\mathbf{q}} s = \left(\int_{\alpha}^{\beta} (\mathcal{J}(s))^{\beta} d_{\mathbf{q}} s\right)^n$;
2. $\int_{\alpha}^{\beta} \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s = \int_{\alpha}^c \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s \cdot \int_c^{\beta} \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s$;
3. $\int_{\alpha}^{\beta} (\mathcal{J}(s) \cdot \mathcal{K}(s))^{\beta} d_{\mathbf{q}} s = \int_{\alpha}^{\beta} \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s \cdot \int_{\alpha}^{\beta} \mathcal{K}(s)^{\beta} d_{\mathbf{q}} s$;
4. $\int_{\alpha}^{\beta} \left(\frac{\mathcal{J}(s)}{\mathcal{K}(s)}\right)^{\beta} d_{\mathbf{q}} s = \frac{\int_{\alpha}^{\beta} \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s}{\int_{\alpha}^{\beta} \mathcal{K}(s)^{\beta} d_{\mathbf{q}} s}$.

Proof. The proof is equivalent to Theorem 6. $\square$

Now, we construct some ${}^{\beta}\mathbf{q}$ -multiplicative Hermite–Hadamard inequalities.

Theorem 11. If we assume a positive ${}^{\beta}\mathbf{q}$ -multiplicative convex differentiable function $\mathcal{J}$ on $\mathcal{I}$, then

$$\mathcal{J}\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right) \leq \left(\int_{\alpha}^{\beta} \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s\right)^{\frac{1}{\beta - \alpha}} \leq [\mathcal{J}(\alpha) \cdot \mathcal{J}(\beta)^{\mathbf{q}}]^{\frac{1}{\mathbf{q} + 1}} \quad (29)$$

holds.

Proof. Using the same argument in Theorem 7 for the point $\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1} \in (\alpha, \beta)$, we have

$$h_4(s) = \mathcal{J}\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right) \mathcal{J}'\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right)^{\left(s - \frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right)}.$$

By the convexity of $\mathcal{J}$, we obtain

$$h_4(s) = \mathcal{J}\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right) \mathcal{J}'\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right)^{\left(s - \frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right)} \leq \mathcal{J}(s).$$

For all $s \in [\alpha, \beta]$, we have

$$\ln h_4(s) = \ln \mathcal{J}\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right) + \left(s - \frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right) \ln \mathcal{J}'\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right).$$

By ${}^{\beta}\mathbf{q}$ -integrating, we have

$$\begin{aligned} \int_{\alpha}^{\beta} \ln \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s &\geq \int_{\alpha}^{\beta} \ln \mathcal{J}\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right) + \int_{\alpha}^{\beta} \left(s - \frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right)^{\beta} d_{\mathbf{q}} s \cdot \ln \mathcal{J}'\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right) \\ &= (\beta - \alpha) \ln \mathcal{J}\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right) + (0) \ln \mathcal{J}'\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right) \\ &= (\beta - \alpha) \ln \mathcal{J}\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right) \\ \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \ln \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s &\geq \ln \mathcal{J}\left(\frac{\alpha + \mathbf{q}\beta}{\mathbf{q} + 1}\right). \end{aligned}$$

By taking the exponential on both sides, we get

$$e^{\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \ln \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s} \geq e^{\ln \mathcal{J}\left(\frac{\alpha+\mathbf{q}\beta}{\mathbf{q}+1}\right)}.$$

Hence,

$$\mathcal{J}\left(\frac{\alpha+\mathbf{q}\beta}{\mathbf{q}+1}\right) \leq \left(\int_{\alpha}^{\beta} (\mathcal{J}(s))^{\beta} d_{\mathbf{q}} s\right)^{\frac{1}{\beta-\alpha}}. \quad (30)$$

Again, using the same argument in Theorem 7, we have

$$h_1(s) = \mathcal{J}(\alpha) \left[\frac{\mathcal{J}(\beta)}{\mathcal{J}(\alpha)} \right]^{\left(\frac{s-\alpha}{\beta-\alpha}\right)}.$$

By the convexity of $\mathcal{J}$, we have

$$\mathcal{J}(s) \leq h_1(s) = \mathcal{J}(\alpha) \left[\frac{\mathcal{J}(\beta)}{\mathcal{J}(\alpha)} \right]^{\left(\frac{s-\alpha}{\beta-\alpha}\right)}.$$

For all $s \in [\alpha, \beta]$, we have

$$\ln \mathcal{J}(s) \leq \ln h_1(s) = \ln \mathcal{J}(\alpha) + \left(\frac{s}{\beta-\alpha} - \frac{\alpha}{\beta-\alpha} \right) [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)].$$

By ${}^{\beta}\mathbf{q}$ -integrating, we have

$$\begin{aligned} \int_{\alpha}^{\beta} \ln \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s &\leq \int_{\alpha}^{\beta} \ln \mathcal{J}(\alpha)^{\beta} d_{\mathbf{q}} s + \int_{\alpha}^{\beta} \left(\frac{s}{\beta-\alpha} - \frac{\alpha}{\beta-\alpha} \right)^{\beta} d_{\mathbf{q}} s [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)] \\ &= (\beta-\alpha) \ln \mathcal{J}(\alpha) + \frac{\mathbf{q}(\beta-\alpha)}{\mathbf{q}+1} [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)] \\ &= (\mathbf{q}+1) \left(\frac{\beta-\alpha}{\mathbf{q}+1} \right) \ln \mathcal{J}(\alpha) + \frac{\mathbf{q}(\beta-\alpha)}{\mathbf{q}+1} [\ln \mathcal{J}(\beta) - \ln \mathcal{J}(\alpha)] \\ &= \left(\frac{\beta-\alpha}{\mathbf{q}+1} \right) [\ln \mathcal{J}(\alpha) + \mathbf{q} \ln \mathcal{J}(\beta)] \\ \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \ln \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s &\leq \frac{1}{\mathbf{q}+1} \ln [\mathcal{J}(\alpha) \cdot \mathcal{J}(\beta)^{\mathbf{q}}]. \end{aligned}$$

By taking the exponential, we obtain

$$\begin{aligned} e^{\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \ln \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s} &\leq e^{\ln [\mathcal{J}(\alpha) \cdot \mathcal{J}(\beta)^{\mathbf{q}}]^{\frac{1}{\mathbf{q}+1}}} \\ \left(\int_{\alpha}^{\beta} \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s \right)^{\frac{1}{\beta-\alpha}} &\leq [\mathcal{J}(\alpha) \cdot \mathcal{J}(\beta)^{\mathbf{q}}]^{\frac{1}{\mathbf{q}+1}}. \end{aligned} \quad (31)$$

From (30) and (31), the desired result has been achieved. $\square$

Theorem 12. If we assume two positive ${}^{\beta}\mathbf{q}$ -multiplicative convex differentiable functions $\mathcal{J}$ and $\mathcal{K}$ on $\mathcal{I}$, then

$$\begin{aligned} \mathcal{J}\left(\frac{\alpha+\mathbf{q}\beta}{\mathbf{q}+1}\right) \cdot \mathcal{K}\left(\frac{\alpha+\mathbf{q}\beta}{\mathbf{q}+1}\right) &\leq \left(\int_{\alpha}^{\beta} \mathcal{J}(s)^{\beta} d_{\mathbf{q}} s \cdot \int_{\alpha}^{\beta} \mathcal{K}(s)^{\beta} d_{\mathbf{q}} s \right)^{\frac{1}{\beta-\alpha}} \\ &\leq [\mathcal{J}(\alpha) \cdot \mathcal{J}(\beta)^{\mathbf{q}} \cdot \mathcal{K}(\alpha) \cdot \mathcal{K}(\beta)^{\mathbf{q}}]^{\frac{1}{\mathbf{q}+1}} \end{aligned} \quad (32)$$

holds.

Proof. It can be proven in the same manner as Theorem 8. $\square$

Theorem 13. If we assume two positive ${}^{\beta}q$ -multiplicative convex differentiable functions $\mathcal{J}$ and $\mathcal{K}$ on $\mathcal{I}$, then

$$\begin{aligned} \mathcal{J}\left(\frac{\alpha + q\beta}{q+1}\right) : \mathcal{K}\left(\frac{\alpha + q\beta}{q+1}\right) &\leq \left(\int_{\alpha}^{\beta} \mathcal{J}(s) {}^{\beta}d_{qs} : \int_{\alpha}^{\beta} \mathcal{K}(s) {}^{\beta}d_{qs}\right)^{\frac{1}{\beta-\alpha}} \\ &\leq [\mathcal{J}(\alpha) \cdot \mathcal{J}(\beta)^q : \mathcal{K}(\alpha) \cdot \mathcal{K}(\beta)^q]^{\frac{1}{q+1}} \end{aligned} \quad (33)$$

holds, where “:” represents division.

Proof. It can be proven in the same manner as Theorem 9. $\square$

5. Concluding Remarks

In this manuscript, we investigated the q -multiplicative definite integral and the results of the q -multiplicative derivative. After this, we introduced the correct definition of q -multiplicative definite integrals and the relevant results. Moreover, we derived some basic results and an example regarding this definition. Furthermore, we defined novel definitions in q -multiplicative calculus regarding derivatives and integration at the left endpoint $\alpha \in \mathcal{I}$ and derived some ${}_{\alpha}q$ -multiplicative Hermite–Hadamard-type inequalities. All of these inequalities are reduced into multiplicative Hermite–Hadamard inequalities when $q \rightarrow -1$, and corresponding examples also justified our newly obtained inequalities. Finally, we defined new definitions in q -multiplicative calculus regarding derivatives and integration at the right endpoint $\beta \in \mathcal{I}$ and derived some ${}^{\beta}q$ -multiplicative Hermite–Hadamard-type inequalities. This article might be the cornerstone for upcoming research in q -multiplicative calculus. For instance, this paper might be the origin for further works on integral inequalities in q -multiplicative calculus and its generalization, like Hahn multiplicative calculus.

Author Contributions: Conceptualization, S.I.B. and M.N.A.; methodology, S.I.B. and M.N.A.; software, M.A. and Y.S.; validation, S.I.B. and M.N.A.; formal analysis, S.I.B.; investigation, M.A.; resources, M.A. and Y.S.; data curation, Y.S. and M.N.A.; writing—original draft preparation, M.N.A.; writing—review and editing, S.I.B. and M.N.A.; visualization, S.I.B. and M.A.; supervision, S.I.B.; project administration, S.I.B.; funding acquisition, Y.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: This work was supported by the Dong-A University research fund. This research was supported by Global—Learning & Academic research institution for Master’s-PhD students, and Postdocs (LAMP) Program of the National Research Foundation of Korea (NRF) grant funded by the Ministry of Education (RS-2025-25440216).

Conflicts of Interest: The authors declare no conflicts of interest.

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