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Counting Rainbow Solutions of a Linear Equation over $\mathbb{F}_p$ via Fourier-Analytic Methods

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Abstract

We study rainbow solutions to linear equations modulo a prime p, where the residue classes are partitioned into n color classes. Using the Fourier method, we derive a universal lower bound that depends only on the class densities and a single spectral parameter: the Fourier bias (the largest nontrivial Fourier coefficient) of each class. When the biases are at the square-root cancellation scale $p^{-1/2}$ (for random colorings, up to a $\sqrt{\log p}$ factor), the bound recovers the optimal growth p^{n-1} with an explicit leading constant and negligible error. Our results complement recent work: in low-bias regimes (pseudorandom or random) they yield sharper quantitative bounds with transparent constants, and the bound requires no extra hypotheses such as coefficient separability.

Keywords: rainbow solutions; finite fields; additive combinatorics; Fourier analysis; Fourier bias; Gauss sums; random colorings

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1. Introduction

Let $\mathbb{F}_p$ denote the finite field of order p (with p prime), and write $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$. Fix $n \in \mathbb{N}$, set $[n] = \{1, \dots, n\}$ (the set of colors), and call an n-coloring of $\mathbb{F}_p$ a map $\chi : \mathbb{F}_p \to [n]$. For $i = 1, \dots, n$, let $C_i = \chi^{-1}(i)$; then the chromatic classes $C_1, \dots, C_n$ form a partition $\mathbb{F}_p = C_1 \sqcup \cdots \sqcup C_n$.

Given nonzero coefficients $a_1, \dots, a_n \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_p$, consider the linear equation

$$a_1 x_1 + \cdots + a_n x_n = b. \tag{1}$$

A solution is *monochromatic* if all coordinates lie in the same chromatic class, and *rainbow (ordered)* if each color appears exactly once (i.e., the tuple intersects every C_i). Throughout, following the notation in previous work [1], we denote by $R(\chi, L)$ the set of rainbow solutions of L under the coloring χ, and by $|R(\chi, L)|$ its cardinality.

While lower bounds for monochromatic solutions are well established (see, e.g., [2–5]), the rainbow case is comparatively less explored. Early work focused on structural and existence aspects; see, for instance, [6–11]. Quantitative bounds for specific equations have also appeared. For $n = 3$, bounds for the equation $L' : x_1 + x_2 - 2x_3 = 0$ under certain colorings were proved ([12] Proposition 1); see also ([13] Proposition 11), and later extended to more general equations and colorings [14]. A quantitative bound for the general case

$|R(\chi, L)|$ was obtained only recently ([1] Theorem 1.2). That result ensures that, given $n \geq 3$ and $w \in (0,1)$ with

$$w \;\geq\; \left(\frac{10^{6401}}{p}\right)^{1/4},$$

if the coefficients satisfy the separability condition $a_i \notin \{\pm a_j\}$ for some distinct $i, j \in [n]$, then for every n-coloring satisfying $\min_{1\leq i\leq n} |C_i| \geq wp + 1$ one has

$$|R(\chi, L)| \;\geq\; c\, p^{n-1}, \tag{2}$$

where one may take $c = \left(\frac{w}{10^{1600}}\right)^{36(n-1)}$. The constant c can nevertheless be improved, leaving room for sharper quantitative bounds. Our contribution is to refine this lower bound, especially in regimes where the coloring behaves (pseudo-)randomly. This constitutes the main result of the paper.

A second contribution is methodological: we employ a Fourier-analytic approach to counting rainbow solutions. For background and intuition we refer to [15] and references therein. In brief, linear equations are naturally expressed and counted in frequency space: the Fourier expansion turns L into an average of additive characters, and the resulting count decomposes into a main term depending only on the sizes of the chromatic classes and an error term that measures their correlation with additive characters (Fourier bias). More broadly, spectral techniques via Gowers uniformity norms and higher-order Fourier analysis underpin many quantitative results in additive combinatorics; see [16–18]. Our bounds use only classical (U^2-level) Fourier information.

The outline is as follows. In Section 2 we introduce the basic Fourier-analytic notions over $\mathbb{F}_p$ used in the paper. In Section 3 we use additive orthogonality to derive a lower bound on the relevant counts via the triangle inequality. Section 4 states the main result and discusses (pseudo-)random regimes in which the bound becomes tight. Section 5 analyzes the optimality of our bounds in these cases and compares them with the state of the art, and Section 6 summarizes our contributions.

2. Fourier Basics in $\mathbb{F}_p$

We begin by recalling standard Fourier-analytic notions over finite fields; for details see ([15] Ch. 4), ([19] Ch. 6), and ([20] Ch. 11).

Fix a prime p and work additively in $\mathbb{F}_p$. Set $e_p(t) = e^{2\pi i t/p}$ and define the *Fourier transform* as follows:

$$\widehat{f}(\xi) \;=\; \frac{1}{p} \sum_{x\in\mathbb{F}_p} f(x)\, e_p(\xi x), \qquad \xi \in \mathbb{F}_p.$$

Set $t \in \mathbb{F}_p$. The well-known *additive orthogonality* (see, e.g., [15] Lemma 4.5) is

$$\frac{1}{p} \sum_{\xi\in\mathbb{F}_p} e_p(\xi t) \;=\; \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases} \tag{3}$$

For $A \subseteq \mathbb{F}_p$, the *Fourier bias* ([15] Definition 4.12) of the set A is the quantity:

$$\|A\|_u \;=\; \max_{\xi\neq 0} \left|\widehat{\mathbf{1}_A}(\xi)\right|,$$

where $\mathbf{1}_A$ denotes the indicator of A.

Note that for every $A \subseteq \mathbb{F}_p$ one has

$$0 \;\leq\; \|A\|_u \;\leq\; \frac{|A|}{p} \;\leq\; 1,$$

because by the definition of $\widehat{\mathbf{1}_A}$ and the triangle inequality, we have

$$\left|\widehat{\mathbf{1}_A}(\xi)\right| \;=\; \left|\frac{1}{p}\sum_{x\in A} e_p(\xi x)\right| \;\le\; \frac{1}{p}\sum_{x\in A}\left|e_p(\xi x)\right| \;=\; \frac{|A|}{p}$$

for every $\xi \in \mathbb{F}_p$. Taking the maximum over $\xi \neq 0$ gives $\|A\|_u \le |A|/p$. In addition, $\widehat{\mathbf{1}_A}(0) = |A|/p$, while the bias only involves nonzero frequencies.

As a general rule, the Fourier bias tracks how uniform a set looks. If the bias is small, no single frequency stands out and repeated sumsets tend to smooth the distribution, much like random data. If the bias is large, a few frequencies dominate; this points to visible additive structure, for example long arithmetic progressions or other near-periodic patterns, and the set stops looking random.

3. A Triangular Lower Bound

Let L be a linear equation such as in Equation (1). For subsets $A_1, \dots, A_n \subseteq \mathbb{F}_p$ we denote by $N(A_1, \dots, A_n)$ the number of tuples $(x_1, \dots, x_n) \in A_1 \times \cdots \times A_n$ solving L.

Lemma 1 (Orthogonality identity)**.** *For any $A_1, \dots, A_n \subseteq \mathbb{F}_p$, it holds*

$$N(A_1, \dots, A_n) \;=\; p^{n-1}\sum_{\xi\in\mathbb{F}_p} e_p(-\xi b)\,\prod_{i=1}^{n}\widehat{\mathbf{1}_{A_i}}(a_i\xi).$$

Proof. By additive orthogonality in Equation (3), we have

$$\mathbf{1}_{\{\sum_{i=1}^n a_i x_i = b\}} = \frac{1}{p}\sum_{\xi\in\mathbb{F}_p} e_p\left(\xi\left(\sum_{i=1}^{n} a_i x_i - b\right)\right).$$

Summing over $(x_1, \dots, x_n) \in A_1 \times \cdots \times A_n$ and exchanging sums gives

$$N(A_1, ..., A_n) = \frac{1}{p}\sum_{\xi\in\mathbb{F}_p} e_p(-\xi b)\,\prod_{i=1}^{n}\sum_{x_i\in A_i} e_p(\xi a_i x_i) = \frac{1}{p}\sum_{\xi\in\mathbb{F}_p} e_p(-\xi b)\,\prod_{i=1}^{n}\left(p\,\widehat{\mathbf{1}_{A_i}}(a_i\xi)\right).$$

Thus, we obtain $N(A_1, ..., A_n) = p^{n-1}\sum_{\xi\in\mathbb{F}_p} e_p(-\xi b)\,\prod_{i=1}^{n}\widehat{\mathbf{1}_{A_i}}(a_i\xi)$ as we claimed. □

Lemma 2. *For any $A_1, \dots, A_n \subseteq \mathbb{F}_p$, it holds*

$$N(A_1, \dots, A_n) \;\ge\; p^{n-1}\left(\prod_{i=1}^{n}\frac{|A_i|}{p} \;-\; (p-1)\prod_{i=1}^{n}\|A_i\|_u\right).$$

Proof. Apply Lemma 1 and split the sum into $\xi = 0$ and $\xi \neq 0$. The $\xi = 0$ term equals

$$p^{n-1}\prod_{i=1}^{n}\widehat{\mathbf{1}_{A_i}}(0) = p^{n-1}\prod_{i=1}^{n}\frac{|A_i|}{p}.$$

For the nonzero modes, by the triangle inequality and $|\prod z_i| \le \prod |z_i|$ we have

$$-\left|p^{n-1}\sum_{\xi\neq 0} e_p(-\xi b)\,\prod_{i=1}^{n}\widehat{\mathbf{1}_{A_i}}(a_i\xi)\right| \;\ge\; -\,p^{n-1}\sum_{\xi\neq 0}\prod_{i=1}^{n}\left|\widehat{\mathbf{1}_{A_i}}(a_i\xi)\right|.$$

Now, since we assume $a_i \in \mathbb{F}_p^*$, then for every $\xi \neq 0$ we have $a_i\xi \neq 0$, hence by the definition of $\|\cdot\|_u$,

$$\left|\widehat{\mathbf{1}_{A_i}}(a_i\xi)\right| \le \|A_i\|_u.$$

Therefore,

$$-\,p^{n-1}\sum_{\xi\neq 0}\prod_{i=1}^{n}|\widehat{\mathbf{1}_{A_i}}(a_i\xi)| \;\geq\; -\,p^{n-1}\,(p-1)\prod_{i=1}^{n}\|A_i\|_u.$$

Combining the $\xi = 0$ contribution with this lower bound yields the claim. □

4. Main Result

Recall that a rainbow (ordered) solution uses each color exactly once in some order. Let S_n be the permutation group of n elements. For the partition $\mathbb{F}_p = C_1 \sqcup \cdots \sqcup C_n$ we have

$$|R(\chi, L)| = \sum_{\sigma\in S_n} N\big(C_{\sigma(1)}, \ldots, C_{\sigma(n)}\big).$$

Theorem 1. *Let L be a linear equation as in Equation (1), and let $\chi : \mathbb{F}_p \to [n]$ be an n-coloring with chromatic classes $C_1, \ldots, C_n$ and densities $w_i = |C_i|/p$. Then*

$$|R(\chi, L)| \;\geq\; \underbrace{n!\,p^{n-1}\prod_{i=1}^{n} w_i}_{\text{main term}} \;-\; \underbrace{n!\,p^{n-1}(p-1)\prod_{i=1}^{n}\|C_i\|_u}_{\text{spectral error}}. \tag{4}$$

Proof. Apply Lemma 2 with $A_i = C_{\sigma(i)}$ and sum over $\sigma \in S_n$. □

Note that when $n = 1$ there is only one color, so rainbow and monochromatic coincide and the permutation sum disappears. The equation $a_1x_1 = b$ has exactly one solution, so the case $n = 1$ is trivial. The bound above is aimed at the genuinely multicolor case $n \geq 2$.

In what follows in this section we show an application of the previous result for n-colorings with (pseudo-)random behavior, obtaining tight bounds.

4.1. Pseudorandom Colorings: Bias at the $p^{-1/2}$ Scale

As a canonical example, take $\mathbb{F}_p$ with p odd. The Gauss sum estimate ([15] Lemma 4.14) gives that the 2–coloring defined by the quadratic residues and their complement

$$C_1 = \{x^2 :\; x \in \mathbb{F}_p\}, \qquad C_2 = \mathbb{F}_p \setminus C_1 \tag{5}$$

satisfies (see also [15] (Exercise 4.3.2) for the complement)

$$\|C_1\|_u = \|C_2\|_u \;\leq\; \tfrac{1}{2}\,p^{-1} + \tfrac{1}{2}\,p^{-1/2} \;\leq\; p^{-1/2}.$$

The same Gauss–sum mechanism extends to algebraic colorings, such as those built from multiplicative characters on $\mathbb{F}_p^*$. For instance, partitioning $\mathbb{F}_p^*$ according to the value of a fixed multiplicative character (and placing 0 in any class) yields color classes whose indicators are short linear combinations of that character, and hence their nontrivial additive Fourier coefficients are linear combinations of Gauss sums. By the classical Gauss bound (see, e.g., [20], Proposition 11.5), each such Gauss sum has size $\ll \sqrt{p}$. Consequently, there is a constant $K > 0$ (independent of p) such that

$$\|C_i\|_u \;\leq\; K\,p^{-1/2}.$$

More generally, if one forms an n-coloring by taking joint level sets of a fixed finite family of multiplicative characters (and merging cells if needed to obtain exactly n classes), the same argument shows that $\|C_i\|_u \leq K_n\,p^{-1/2}$ for all i, with K_n depending on that finite family.

Plugging $\|C_i\|_u \le K_n\, p^{-1/2}$ into Equation (4) gives

$$|R(\chi, L)| \;\ge\; n!\, p^{n-1}\left(\prod_{i=1}^{n} w_i \;-\; K_n'\, p^{1-\frac{n}{2}}\right), \tag{6}$$

for some constant $K_n' > 0$ depending only on n. Thus, for every fixed $n \ge 3$, the main term dominates as $p \to \infty$.

4.2. Random Colorings: Bias at the $p^{-1/2}\sqrt{\log p}$ Scale (with High Probability)

Assign each $x \in \mathbb{F}_p$ to exactly one color $i \in [n]$ independently over x, with the law $\mathbb{P}[\chi(x) = i] = w_i$ where $\sum_{i=1}^{n} w_i = 1$. By ([15] Lemma 4.16) (applied with $A = \mathbb{F}_p$ and with the lemma's density parameter τ set to w_i), there exists an absolute constant $K > 0$ such that, for each i,

$$\mathbb{P}\left(\;\|C_i\|_u \;\le\; K\sqrt{\frac{w_i(1-w_i)\,\log p}{p}}\;\right) \;\ge\; 1 - O(p^{-100}).$$

Now, by a union bound over i, with probability $1 - O(np^{-100}) = 1 - O(p^{-100})$ we have simultaneously for all $i \in [n]$ that

$$\|C_i\|_u \;\le\; K\sqrt{\frac{w_i(1-w_i)\,\log p}{p}}$$

Consequently, with the same probability,

$$|R(\chi, L)| \;\ge\; n!\, p^{n-1}\left(\;\prod_{i=1}^{n} w_i \;-\; (p-1)\prod_{i=1}^{n} K\sqrt{\frac{w_i(1-w_i)\,\log p}{p}}\;\right).$$

Since $w_i(1-w_i) \le \frac{1}{4}$, we bound the product and absorb constants to obtain

$$|R(\chi, L)| \;\ge\; n!\, p^{n-1}\Big(\;\prod_{i=1}^{n} w_i \;-\; K_n\, p^{1-\frac{n}{2}}\,(\log p)^{n/2}\;\Big), \tag{7}$$

for some constant $K_n > 0$ depending only on n. In particular, for every fixed $n \ge 3$ the main term dominates as $p \to \infty$.

5. Discussion

To orient the reader, Table 1 summarizes assumptions, constants, and asymptotic regimes side by side.

Table 1. Comparison between [1] and the present Fourier-analytic bound.

	[1]	**This Paper**
Coefficients	Separability: $a_i \notin \{\pm a_j\}$ for some distinct $i \ne j$	Only $a_i \ne 0$
Class sizes	$\min_i \lvert C_i\rvert \ge wp + 1$ with $w \in (0,1)$	Arbitrary densities $w_i = \lvert C_i\rvert / p$
Lower bound	$c\, p^{n-1}$ with $c = \left(\frac{w}{10^{1600}}\right)^{36(n-1)}$	$n!\, p^{n-1}\left(\prod_i w_i - (p-1)\prod_i \|C_i\|_u\right)$
Asymptotic tightness	Uniform over admissible colorings; constant not optimized	Low-bias colorings: main term $n!\, p^{n-1}\prod_i w_i$ dominates

Throughout this section we assume $|C_1| \le |C_2| \le \cdots \le |C_n|$ and set

$$w \;=\; \min_{1\le i\le n} w_i \;=\; \min_{1\le i\le n} \frac{|C_i|}{p}.$$

In the state of art ([1] Theorem 1.2) the minimal class size is parametrised by a number $w \in (0,1)$ via $\min_i |C_i| \ge wp + 1$. Up to the additive $+1$ (an inessential rounding when passing to densities), this matches our convention $w = \min_{1\le i\le n} w_i$. We will use the same letter w for the minimal density throughout.

The result below shows an upper bound (cf. [1] Equation (1)) for $|R(\chi, L)|$ showing the natural order p^{n-1} for this quantity.

Proposition 1. *Let L be a linear equation as in Equation (1), and let $\chi : \mathbb{F}_p \to [n]$ be an n-coloring with chromatic classes $C_1, \ldots, C_n$. Then*

$$|R(\chi, L)| \;\le\; n!\, p^{\,n-1}\, \frac{\prod_{i=1}^{n} w_i}{w}. \tag{8}$$

In particular, $|R(\chi, L)| = O\big(p^{\,n-1}\big)$ with an implied constant depending only on the w_i and n.

Proof. Fix a permutation $\sigma \in S_n$ and let $j = \sigma(n)$ be the color used in the last coordinate. If we choose $(x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}) \in C_{\sigma(1)} \times \cdots \times C_{\sigma(n-1)}$ arbitrarily, the equation L determines at most one value of $x_{\sigma(n)}$ because $a_{\sigma(n)} \neq 0$. Thus

$$N\big(C_{\sigma(1)}, \ldots, C_{\sigma(n)}\big) \;\le\; \prod_{i\neq j} |C_i|.$$

Summing over all σ and grouping by the last color j (there are $(n-1)!$ permutations with $\sigma(n) = j$) gives

$$|R(\chi, L)| = \sum_{\sigma\in S_n} N\big(C_{\sigma(1)}, \ldots, C_{\sigma(n)}\big) \;\le\; (n-1)! \sum_{j=1}^{n} \prod_{i\neq j} |C_i|.$$

Finally, for any j, $\prod_{i\neq j} |C_i| \le \prod_{i=2}^{n} |C_i|$ (because the product over any $n-1$ classes is maximized by omitting the smallest one, C_1). Hence

$$|R(\chi, L)| \;\le\; (n-1)!\, n \prod_{i=2}^{n} |C_i| \;=\; n! \prod_{i=2}^{n} |C_i| \;=\; n!\, p^{\,n-1}\, \frac{\prod_{i=1}^{n} w_i}{w}$$

as we claimed. □

Note that for the (pseudo-)random cases, when p is large enough so that the spectral error is negligible, the lower bounds in Equations (6) and (7) are far from the upper bound in Equation (8) by a term $1/w$. For example, take the 2–coloring partition given by the quadratic residues in Equation (5). Then $|C_1| = (p+1)/2$, hence

$$w = \min\{w_1, w_2\} = \frac{\min\{|C_1|,\, p - |C_1|\}}{p} = \frac{1}{2} - \frac{1}{2p}.$$

Thus, $1/w = 2 + O\Big(\frac{1}{p}\Big)$. Also, for instance, for n-coloring given by the balanced random model, assign each $x \in \mathbb{F}_p$ independently with $\mathbb{P}[\chi(x) = i] = \frac{1}{n}$. By Chernoff bounds, $1/w = n\,(1 + o(1))$ as $p \to \infty$, with high probability. Thus, in the (pseudo-)random settings, once p is large enough for the error term to be negligible, our lower bounds match the

upper bound up to a constant factor governed by $1/w$ bounded (asymptotically) by the number of colors n.

Now, we consider the comparison with the prior bound in Equation (2) in low-bias regimes. Whenever the hypotheses of ([1] Theorem 1.2) hold ($n \geq 3$, $a_i \notin \{\pm a_j\}$ for some $i \neq j$, and $\min_{1\leq i\leq n} |C_i| \geq wp + 1$ with $w \geq (10^{6401}/p)^{1/4}$), one has

$$|R(\chi, L)| \;\geq\; c\, p^{\,n-1}, \qquad c = \Big(\tfrac{w}{10^{1600}}\Big)^{36(n-1)}.$$

In the low-bias regimes treated above, our Fourier bound is essentially sharp while the constant c is extremely small.

Consider the pseudorandom scale with $\|C_i\|_u \leq K_n\, p^{-1/2}$ for all i. For p large enough (depending only on n and the w_i) the spectral error in Equation (6) is at most $\frac{1}{2}\prod_i w_i$, hence

$$|R(\chi, L)| \;\geq\; \tfrac{n!}{2}\Big(\prod_{i=1}^{n} w_i\Big)\, p^{\,n-1} \;\geq\; \tfrac{n!}{2}\, w^n\, p^{\,n-1}.$$

Comparing the leading constants gives

$$\frac{\frac{n!}{2}\, w^n}{\big(\frac{w}{10^{1600}}\big)^{36(n-1)}} \;=\; \frac{n!}{2}\; w^{-35n+36}\; 10^{\,57600(n-1)},$$

which is astronomically larger for any fixed $n \geq 3$ and $w \in (0, 1)$. Thus, in this case, the prior lower bound is far from the true constant-level scale, while the Fourier bound attains the correct p^{n-1} order with the correct constant up to a fixed factor. Equivalently, the same conclusion as above holds with high probability in the random regime.

Furthermore, we note that our Fourier bound does not require coefficient separability ($a_i \notin \{\pm a_j\}$); it applies verbatim for arbitrary nonzero coefficients. Therefore, it also covers non-separable instances that lie outside the scope of the previous related result.

We also note that our result should be read as complementary to previous work. When the color classes have strong additive structure, the spectral term can dominate and make our inequality vacuous, while the bound in Equation (2) remains positive.

As a concrete example, partition $\mathbb{F}_p$ into n long, disjoint arithmetic progressions (for instance, n consecutive cyclic intervals of length $m \simeq p/n$). Then $w_i = |C_i|/p \simeq 1/n$. For a single cyclic interval of the form $C_i = \{x_i, x_i + 1, \ldots, x_i + m - 1\}$ one has, for $\xi \neq 0$,

$$\widehat{\mathbf{1}_{C_i}}(\xi) = \frac{1}{p}\sum_{k=0}^{m-1} e_p\big(\xi(x_i + k)\big) = \frac{e_p(\xi x_i)}{p}\sum_{k=0}^{m-1} e_p(\xi)^k = \frac{e_p(\xi x_i)}{p}\,\frac{1 - e_p(\xi m)}{1 - e_p(\xi)},$$

hence taking module values and using $|1 - e^{2\pi i\theta}| = 2|\sin(\pi\theta)|$ gives

$$\big|\widehat{\mathbf{1}_{C_i}}(\xi)\big| = \frac{1}{p}\,\frac{\big|1 - e^{2\pi i\xi m/p}\big|}{\big|1 - e^{2\pi i\xi/p}\big|} = \frac{1}{p}\,\frac{2|\sin(\pi\xi m/p)|}{2|\sin(\pi\xi/p)|} = \frac{1}{p}\,\frac{|\sin(\pi\xi m/p)|}{|\sin(\pi\xi/p)|}.$$

Taking $\xi = 1$ and using $\sin(\pi/p) \leq \pi/p$ gives

$$\|C_i\|_u \;\geq\; \big|\widehat{\mathbf{1}_{C_i}}(1)\big| \;\geq\; \frac{1}{\pi}\,\sin\Big(\frac{\pi m}{p}\Big).$$

If $m/p \leq \frac{1}{2}$ (which holds when $n \geq 2$ and $m \simeq p/n$), the elementary bound $\sin x \geq \frac{2}{\pi}x$ for $x \in [0, \pi/2]$ yields

$$\|C_i\|_u \;\geq\; \frac{2}{\pi}\,\frac{m}{p} \;\simeq\; \frac{K}{n} \qquad \text{with } K = \frac{2}{\pi}.$$

The same estimate applies to complements (see [15], Exercise 4.3.2). Thus, for the coloring by n consecutive cyclic intervals we have

$$w_i \simeq \frac{1}{n} \quad \text{and} \quad \|C_i\|_u \geq \frac{K}{n}.$$

From our general bound Equation (4) and using $\prod_{i=1}^{n} \|C_i\|_u \geq (K/n)^n$, we obtain the upper estimate

$$\prod_{i=1}^{n} w_i - (p-1) \prod_{i=1}^{n} \|C_i\|_u \leq (1/n)^n - (p-1)(K/n)^n.$$

In particular, if

$$(p-1)(K/n)^n > (1/n)^n \iff p > K^{-n} + 1,$$

then the right-hand side above is negative, so our Fourier lower bound becomes trivial in this highly structured regime.

By contrast, the hypotheses of ([1] Theorem 1.2) are satisfied here. For large p we have $w \simeq 1/n \geq (10^{6401}/p)^{1/4}$, and one can choose L with at least two distinct non-opposite coefficients (i.e., not equal up to sign). Therefore Equation (2) still yields a strictly positive lower bound (with a constant that seems improvable). It remains interesting to optimize that constant in these cases.

6. Conclusions

This work provides new lower bounds for counting rainbow solutions to linear equations over a finite field of prime order p partitioned into n chromatic classes. We employ Fourier-analytic methods that leverage the linear structure to obtain quantitative bounds in terms of class densities and a single spectral parameter (the Fourier bias). Furthermore, our result does not require additional assumptions such as coefficient separability.

To sum up, for colorings with strong additive structure, the spectral error can dominate and our Fourier bound may become vacuous, while the previous combinatorial approach remains nontrivial. Conversely, in low-bias regimes (pseudorandom and random), our bound recovers the optimal p^{n-1} scaling with the correct leading constant up to a fixed factor; to our knowledge, this quantitative form is new. It remains an open problem to obtain finer bounds in strongly additive-structured settings.

Finally, the same Fourier-analytic template extends to systems of linear equations over finite abelian groups and to higher-dimensional colorings via multidimensional characters. For certain nonlinear patterns, partial extensions are available through linearization or completion of sums, whereas a broader nonlinear theory would likely call for higher-order uniformity tools.

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