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A Hahn-Type Characterization of Generalized Hermite Polynomials Through a Dunkl-Based Raising Operator

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Abstract

In this paper, we study Hahn's problem with respect to a Dunkl-perturbed raising operator. More precisely, we prove that, up to a dilation, the generalized Hermite polynomials are the only $T_{\mu,\alpha}$-classical symmetric orthogonal polynomials, where $T_{\mu,\alpha} = T_\mu + \alpha t\mathbb{I}$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $\mathbb{I}$ denotes the identity operator on the space of polynomials with complex coefficients. The argument uses an operator product rule for T_μ, duality for the associated functionals, and a symmetry-enforced identification together with matching three-term recurrences. The result provides an operator-theoretic Hahn-type characterization that complements semiclassical Pearson-equation descriptions and clarifies the effect of the raising perturbation $\alpha t\mathbb{I}$.

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1. Introduction and Motivation

A sequence of orthogonal polynomials $\{A_n\}_{n\geq 0}$ is said to be *classical* if orthogonality is preserved under differentiation; that is, the sequence of derivatives $\{A_n'\}_{n\geq 0}$ is also orthogonal. This condition is the Hahn–Sonine characterization of classical orthogonal polynomials. In [1], Hahn derived analogous characterization results for orthogonal polynomials $\{A_n\}$ in which either the difference sequence $\{\Delta A_n\}$ or the q-difference sequence $\{H_q A_n\}$, for $n \geq 1$, maintains orthogonality. Here Δ denotes the difference operator and H_q the q-difference operator, defined, respectively, as:

$$\Delta f(t) = f(t+1) - f(t), \quad H_q f(t) = \frac{f(qt) - f(t)}{(q-1)t} \ (q \neq 1).$$

Note that differentiation, difference, and q-difference operations all act as *degree-reducing operators* since they lower the degree of any polynomial by exactly one.

Consider the vector space $\mathcal{P}$ of all polynomials in a single variable with complex coefficients. Let $\mathcal{O} : \mathcal{P} \to \mathcal{P}$ be a linear operator acting on $\mathcal{P}$ by increasing the degree. Specifically, the image under $\mathcal{O}$ of a polynomial of exact degree n is a polynomial of exact degree $n + n_0$, where $n \geq 0$ whenever n_0 is non-negative and $n + n_0 \geq 0$ if n_0 is negative. Note that n_0 denotes a fixed integer index that describes the operator's degree-shifting behavior. A sequence of orthogonal polynomials $\{A_n\}_{n\geq 0}$ is said to be $\mathcal{O}$-*classical* if the transformed sequence $\{\mathcal{O}A_n\}_{n\geq 0}$ preserves orthogonality (see, for example, [2–6]).

It should be noted that the literature review on this problem remains incomplete. In fact, Hahn's problem was extended and reformulated by Mourad Ismail, who collected and posed it as open problems (with T denoting the Askey–Wilson operator) in his monograph Orthogonal Polynomials and Special Functions (see [7]). These open problems have been recently resolved in [8].

The generalized Hermite polynomials were first introduced by Szegő in 1939 [9] as an orthogonal family of polynomials with respect to the weight function

$$w(t) = |t|^{2\mu} e^{-t^2}, \quad \mu > -\tfrac{1}{2},$$

supported on $\mathbb{R}$. The functional form of the weight function $w(t)$ for which the generalized Hermite polynomials form an orthogonal system combines an algebraic factor $|t|^{2\mu}$ that depends on the parameter μ as well as a Gaussian decay factor given as e^{-t^2}. The parameter constraint $\mu > -\frac{1}{2}$ ensures the integrability of $w(t)$. This weight function exhibits even symmetry ($w(-t) = w(t)$) and reduces to the standard Hermite weight e^{-t^2} when $\mu = 0$. These polynomials admit explicit expressions through Laguerre polynomials:

$$\begin{cases} \mathcal{H}_{2n}^{(\mu)}(t) & = (-1)^n\, n!\, L_n^{\left(\mu-\frac{1}{2}\right)}(t^2), \\ \mathcal{H}_{2n+1}^{(\mu)}(t) & = (-1)^n\, n!\, t\, L_n^{\left(\mu+\frac{1}{2}\right)}(t^2), \end{cases} \qquad n \geq 0,$$

where $L_n^{(\alpha)}(t)$ are the standard Laguerre polynomials generated by

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(t)\, \xi^n = (1-\xi)^{-\alpha-1} \exp\left(-\frac{t\xi}{1-\xi}\right).$$

These generalized Hermite polynomials satisfy the standard three-term recurrence

$$\begin{cases} A_0(t) = 1, \\ A_1(t) = t - \rho_0, \\ A_{n+2}(t) = (t - \rho_{n+1}) A_{n+1}(t) - \varepsilon_{n+1} A_n(t), \quad n \geq 0, \end{cases}$$

with coefficients specified by

$$\rho_n = 0, \quad \varepsilon_{n+1} = \frac{n + 1 + \mu(1 + (-1)^n)}{2}, \quad n \geq 0. \tag{1}$$

Equation (1) shows that the recurrence coefficients alternate between two forms for even and odd n, reflecting the symmetry of the generalized Hermite family.

In this paper, we study the *raising operator*, which increases the degree of a polynomial by exactly one, defined by $T_{\mu,\alpha} := T_\mu + \alpha t \mathbb{I}$, where α is a nonzero parameter and T_μ denotes the Dunkl operator. We provide a complete characterization of all $T_{\mu,\alpha}$-classical symmetric orthogonal polynomial sequences.

The basic idea originates from the *raising operator* $\mathcal{H}_{\alpha,q} := t\mathbb{I} + \alpha H_q$, where $\mathbb{I}$ denotes the identity on the linear space of polynomials over $\mathbb{C}$ (see [3]). Replacing H_q with the Dunkl operator, $T_\mu := D + 2\mu H_{-1}$, allows us to address the analogous problem in this setting. Indeed, we have [4]

$$\left(T_\mu - 2t\mathbb{I}\right) H_n^\mu(t) = -\frac{\varepsilon_\mu(n+1)}{\varepsilon_\mu(n)(n+1)} H_{n+1}^\mu(t), \quad n \geq 0, \tag{2}$$

where $H_n^\mu(t)$ is the monic generalized Hermite polynomial and the coefficient $\varepsilon_\mu(n)$ is given by

$$\varepsilon_\mu(2k) = \frac{2^{2k}k!\,\Gamma(k+\mu+1/2)}{\Gamma(\mu+1/2)},$$

$$\varepsilon_\mu(2k+1) = \frac{2^{2k+1}k!\,\Gamma(k+\mu+1/2)}{\Gamma(\mu+3/2)}$$

for $k \geq 0$. The application of the operator yields a constant multiple of the next higher-degree polynomial $H_{n+1}^\mu(t)$. The proportionality constant in (2) involves the ratio of normalization factors $\varepsilon_\mu(n+1)/\varepsilon_\mu(n)$ scaled by $1/(n+1)$. These $\varepsilon_\mu(n)$ factors originate from the weight function's $|t|^{2\mu}$ singularity and ensure orthogonality of the polynomial sequence.

Considering (2), it follows that the family $\{H_n^\mu\}_{n\geq 0}$ constitutes an $\mathcal{O}$-classical polynomial sequence. This classification arises from the fact that the sequence satisfies Hahn's characterization property when considered relative to the raising operator

$$\mathcal{O} := T_\mu - 2t\mathbb{I},$$

that is, the action of $\mathcal{O}$ preserves the orthogonality structure of the sequence.

By extending this observation to the broader operator $T_{\mu,\alpha}$, we conclude that *the generalized Hermite polynomial sequence* is the unique symmetric $T_{\mu,\alpha}$-classical sequence.

A considerable amount of research has been devoted to the analysis of orthogonal polynomials through the framework of lowering, shift, and raising operators (see, for instance, [6,10,11]).

Our result extends a long line of Hahn-type characterizations from difference and q-difference settings to the Dunkl framework with an explicit raising perturbation. Classical and q-classical analogues were developed by Hahn [1] and systematically by Khériji–Maroni for H_q-classical families [5]. Within the Dunkl setting, Ben Cheikh–Gaied characterized *Dunkl-classical* symmetric orthogonal polynomials (in the case $\alpha = 0$) [4], and Ben Salah et al. gave a structure-relation characterization for symmetric $\mathcal{T}_\mu$-classical polynomials [12]. On the raising-operator side, general approaches for lowering/raising techniques appear in Koornwinder [6] and Srivastava–Ben Cheikh [11], while Chaggara [10] analyzes operational rules for generalized Hermite polynomials. In the q-world, Aloui–Souissi studied Hahn's problem for perturbations of $X - c$ (using H_q rather than Dunkl) [3], and later treated q-Dunkl-classical symmetric q-polynomials [2]. By contrast, our work treats the *hybrid, perturbed Dunkl operator* $T_{\mu,\alpha} = T_\mu + \alpha t\mathbb{I}$ with $\alpha \neq 0$, and proves a uniqueness theorem (up to the explicit dilation $a^2 = -2/\alpha$) for symmetric $T_{\mu,\alpha}$-classical polynomials, showing they coincide with generalized Hermite polynomials. This complements the semiclassical class-one descriptions of the generalized Hermite via Pearson equations [13–16] by giving an operator-theoretic Hahn-type characterization in the Dunkl-perturbed setting.

The paper is organized as follows. In Section 2, we present the essential background on forms and orthogonal polynomials, establishing the notation and foundational results required for our analysis. Section 3 develops a collection of algebraic and differential properties which serve as the primary instruments for addressing the problem under consideration.

2. Preliminaries

For the space $\mathcal{P}$ defined in the introduction, let $\mathcal{P}'$ be its algebraic dual consisting of linear functionals (or *linear forms*). The action of $u \in \mathcal{P}'$ on $p \in \mathcal{P}$ is written $\langle u, p\rangle$. The moments of u are given by $(u)_n := \langle u, t^n\rangle$ for $n \geq 0$. A form u is symmetric when all odd moments vanish: $(u)_{2n+1} = 0$ for $n \geq 0$.

We define fundamental operations in $\mathcal{P}'$ [5,17]. For $u \in \mathcal{P}'$, $\psi \in \mathcal{P}$, $a \in \mathbb{C} - \{0\}$, and $q \neq 1$, the derivative $Du = u'$, multiplication ψu, dilation $h_a u$, and q-difference $H_q u$ are defined by:

$$\langle u', \phi \rangle := -\langle u, \phi' \rangle, \quad \langle \psi u, \phi \rangle := \langle u, \psi \phi \rangle, \quad \phi \in \mathcal{P},$$
$$\langle h_a u, \phi \rangle := \langle u, \phi(at) \rangle, \quad \langle (t-c)^{-1} u, \phi \rangle := \left\langle u, \frac{\phi(t) - \phi(c)}{t - c} \right\rangle, \quad \phi \in \mathcal{P},$$
$$\langle H_q u, \phi \rangle := -\langle u, H_q \phi \rangle, \quad \phi \in \mathcal{P},$$

where

$$(H_q \phi)(t) = \frac{\phi(qt) - \phi(t)}{(q-1)t}.$$

These operations satisfy the identities [5]:

$$H_q(\phi u) = (h_{q^{-1}} \phi) H_q u + q^{-1} (H_{q^{-1}} \phi) u, \quad \phi \in \mathcal{P}, \quad u \in \mathcal{P}', \tag{3}$$

$$H_{q^{-1}} \circ h_q = q H_q \quad in \quad \mathcal{P}. \tag{4}$$

A form u is normalized if its zeroth moment satisfies $(u)_0 = 1$. All forms in this work are assumed to be normalized.

Consider a monic polynomial sequence (MPS) $\{A_n\}_{n \geq 0}$ with $\deg A_n = n$, and its dual sequence $\{u_n\}_{n \geq 0} \subset \mathcal{P}'$ defined by the biorthogonality condition $\langle u_n, A_m \rangle = \delta_{n,m}$ for $n, m \geq 0$ (where $\delta_{n,m}$ is the Kronecker delta). The functional u_0 is the canonical form associated with $\{A_n\}_{n \geq 0}$. The MPS is symmetric if

$$A_n(-t) = (-1)^n A_n(t), \quad n \geq 0. \tag{5}$$

The following foundational result characterizes finite linear combinations of dual forms:

Lemma 1 ([17,18]). *For any form u in $\mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent:*

(i) $\langle u, A_{m-1} \rangle$ *is nonzero, and* $\langle u, A_n \rangle = 0$, *for all* $n \geq m$.

(ii) *There exist complex coefficients* $\lambda_0, \ldots, \lambda_{m-1}$ *with* $\lambda_{m-1} \neq 0$ *such that* $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.

A form u is regular if there exists an MPS $\{A_n\}_{n \geq 0}$ satisfying

$$\langle u, A_n A_m \rangle = r_n \delta_{n,m}, \quad r_n \neq 0, \quad n, m \geq 0.$$

Such $\{A_n\}_{n \geq 0}$ is a monic orthogonal polynomial sequence (MOPS) for u. Observe that $u = (u)_0 \, u_0$, where $(u)_0 \neq 0$. Regularity of the form u implies that the equation $Fu = 0$ holds only when F is identically zero [19].

Orthogonality is characterized through dual sequences and recurrence relations as follows:

Proposition 1 ([17,18]). *For an MPS $\{A_n\}_{n \geq 0}$ with dual sequence $\{u_n\}_{n \geq 0}$, the following are equivalent:*

(i) $\{A_n\}_{n \geq 0}$ *is orthogonal relative to* u_0.

(ii) $u_n = \langle u_0, A_n^2 \rangle^{-1} A_n u_0$, *for all* $n \geq 0$.

(iii) $\{A_n\}_{n \geq 0}$ *satisfies the TTRR:*

$$\begin{cases} A_0(t) = 1, \\ A_1(t) = t - \rho_0, \\ A_{n+2}(t) = (t - \rho_{n+1}) A_{n+1}(t) - \varepsilon_{n+1} A_n(t), \quad n \geq 0 \end{cases} \tag{6}$$

where $\rho_n = \langle u_0, tA_n^2\rangle / \langle u_0, A_n^2\rangle$ *and* $\varepsilon_{n+1} = \langle u_0, A_{n+1}^2\rangle / \langle u_0, A_n^2\rangle \neq 0$ *for* $n \geq 0$.

If $\{A_n\}_{n\geq 0}$ is an MOPS for regular u_0, then the scaled sequence $\widetilde{A}_n(t) = a^{-n}A_n(at)$ ($a \neq 0$) is an MOPS for $\widetilde{u}_0 = h_{a^{-1}}u_0$ with recurrence coefficients [18]:

$$\widetilde{\rho}_n = a^{-1}\rho_n, \quad \widetilde{\varepsilon}_{n+1} = a^{-2}\varepsilon_{n+1}.$$

The scaled polynomials satisfy the recurrence:

$$\widetilde{A}_{n+2}(t) = (t - \widetilde{\rho}_{n+1})\widetilde{A}_{n+1}(t) - \widetilde{\varepsilon}_{n+1}\widetilde{A}_n(t), \quad n \geq 0,$$

with initial conditions $\widetilde{A}_0(t) = 1$, $\widetilde{A}_1(t) = t - \widetilde{\rho}_0$.

The scaling transformation preserves the orthogonality structure while modifying the recurrence coefficients in a deterministic manner. This property is crucial for establishing the uniqueness of polynomial sequences up to dilation, as it allows us to normalize sequences to a standard form without losing their essential characteristics.

An MOPS has the reflection property $A_n(-t) = (-1)^n A_n(t)$ for every degree $n \geq 0$ precisely when the coefficients ρ_n in the recurrence satisfy $\rho_n = 0$ for all $n \geq 0$. This equivalence establishes that the vanishing of the linear terms in the recurrence constitutes both necessary and sufficient conditions for the alternating even/odd symmetry characteristic of symmetric orthogonal polynomial systems.

Lemma 2 ([20]). *A monic orthogonal polynomial sequence satisfying the recurrence* (6) *is symmetric if and only if* $\rho_n = 0$ *for all* $n \geq 0$.

For a symmetric MOPS $\{A_n\}_{n\geq 0}$ relative to u_0, the functional equation [12]

$$tH_{-1}u_0 = -u_0, \tag{7}$$

necessarily holds, where H_{-1} denotes the q-difference reflection operator at $q = -1$.

We now recall fundamental concepts regarding semiclassical functionals [17,19].

Definition 1. *A regular linear functional* u *is called semiclassical if there exist polynomial functions* Φ *and* Ψ *satisfying the following:*

1. Φ *is monic (leading coefficient 1).*
2. Ψ *has degree at least 1* ($\deg \Psi \geq 1$).
3. *The distributional differential equation*

$$(\Phi u)' + \Psi u = 0, \tag{8}$$

holds in the dual space $\mathcal{P}'$.

When these conditions are satisfied, the corresponding monic orthogonal polynomial sequence is called semiclassical.

The pair (Φ, Ψ) is not unique. Equation (8) admits simplification if there exists a zero c of Φ satisfying the derivative compatibility $\Phi'(c) + \Psi(c) = 0$ and the moment condition $\langle u, \theta_c\Psi + \theta_c^2\Phi\rangle = 0$. Under these conditions, u satisfies the reduced equation

$$((\theta_c\Phi)u)' + \left\{\theta_c\Psi + \theta_c^2\Phi\right\}u = 0.$$

The class of u is defined as the minimal integer value of $\max(\deg\Phi-2, \deg\Psi-1)$ over all valid pairs (Φ,Ψ) satisfying (8). The previous reduced equation lowers the class $s=\max(\deg\Phi-2,\deg\Psi-1)$ by at least one, since:

$$\deg(\theta_c\Phi) = \deg\Phi - 1,$$
$$\deg(\theta_c\Psi+\theta_c^2\Phi) \le \max(\deg\Psi-1, \deg\Phi-2).$$

For a semiclassical form u satisfying (8), its class equals $s=\max(\deg\Phi-2,\deg\Psi-1)$ if and only if

$$\prod_{c\in\mathcal{Z}_\Phi}\left(\left|\Phi'(c)+\Psi(c)\right|+\left|\langle u,\theta_c^2\Phi+\theta_c\Psi\rangle\right|\right)\neq 0,$$

where $\mathcal{Z}_\Phi$ denotes the zero set of Φ, see [17]. When u has class s, the corresponding MOPS is also semiclassical of class s. Furthermore, if u is semiclassical of class s satisfying (8), the shifted form $\tilde{u}=(h_{a^{-1}}\circ\tau_{-b})u$ remains semiclassical with identical class s relative to the MOPS $\{\tilde{A}_n(t)=a^{-n}A_n(at+b)\}_{n\ge0}$. This form satisfies $(\tilde{\Phi}\tilde{u})'+\tilde{\Psi}\tilde{u}=0$ with

$$\tilde{\Phi}(t)=a^{-\deg\Phi}\Phi(at+b),\quad \tilde{\Psi}(t)=a^{1-\deg\Phi}\Psi(at+b).$$

The recurrence coefficients transform as

$$\tilde{\rho}_n=a^{-1}(\rho_n-b),\quad \tilde{\varepsilon}_{n+1}=a^{-2}\varepsilon_{n+1},\quad n\ge0. \tag{9}$$

The following proposition [13] characterizes the parity properties of symmetric semiclassical forms:

Proposition 2. *Let u be a symmetric semiclassical form satisfying (8).*

1. *When u has even class, Φ is even and Ψ is odd.*
2. *When u has odd class, Φ is odd and Ψ is even.*

The generalized Hermite form $\mathcal{H}(\mu)$ ($\mu\neq0$, $\mu\neq-n-\frac{1}{2}$ for $n\ge0$) provides a symmetric semiclassical example of class one. It satisfies the Pearson equation (see [13,14]):

$$D(t\mathcal{H}(\mu))+\left(2t^2-(2\mu+1)\right)\mathcal{H}(\mu)=0. \tag{10}$$

This differential equation governs the moment behavior and determines the recurrence coefficients of the associated orthogonal polynomial sequence. Additional properties appear in [15,16,20,21].

3. Algebraic and Differential Properties

Consider the linear operator $T_{\mu,\alpha}$ defined on the vector space of polynomials $\mathcal{P}$, which acts on any polynomial $\phi\in\mathcal{P}$ as follows:

$$T_{\mu,\alpha}(\phi)=\phi'+2\mu H_{-1}(\phi)+\alpha t\phi,\quad \mu\in\mathbb{C}.$$

This operator combines three distinct components: the standard derivative operator ϕ'; the Dunkl-type term $2\mu H_{-1}(\phi)$, which introduces reflection symmetry; and the multiplicative term $\alpha t\phi$, which serves as a raising operator. The parameter μ controls the strength of the reflection symmetry, while α determines the act of the raising operation.

When the parameter $\alpha=0$, this operator simplifies to the Dunkl operator originally introduced by Dunkl [22] (see also [4]).

For the standard monomial basis $\{t^n\}_{n\geq 0}$, direct computation yields

$$T_{\mu,\alpha}(t^n) = \mu_n t^{n-1} + \alpha t^{n+1}, \quad n \geq 1, \quad \alpha \neq 0,$$

where $\mu_n = n + \mu(1-(-1)^n)$. This expression shows the dual nature of $T_{\mu,\alpha}$, as it simultaneously acts as a lowering operator through the t^{n-1} term and a raising operator through the t^{n+1} term. The coefficient μ_n exhibits parity dependence, taking different values for even and odd n, which reflects the underlying symmetry properties of the operator.

This expression demonstrates that $T_{\mu,\alpha}$ strictly increases the polynomial degree by exactly one. Operators exhibiting this behavior are classified as *raising operators* [6,10,11].

The operator satisfies the following product rule for arbitrary polynomials $\phi, \psi \in \mathcal{P}$:

$$T_\mu(\phi\psi)(t) = T_\mu(\phi)(t)\,\psi(t) + \phi(t)\,T_\mu(\psi)(t) - 4\mu t\,(H_{-1}\phi)(t)\,(H_{-1}\psi)(t). \tag{11}$$

This generalized Leibniz rule differs from the standard product rule by the presence of the additional term $-4\mu t(H_{-1}\phi)(t)(H_{-1}\psi)(t)$, which accounts for the non-local nature of the Dunkl operator. This term vanishes when $\mu = 0$, reducing to the classical Leibniz rule for differentiation. The identity given in (11) is fundamental for our analysis of operator actions on polynomial products.

The formal transpose of $T_{\mu,\alpha}$ satisfies the adjoint relation

$${}^tT_{\mu,\alpha} = -T_{\mu,-\alpha}. \tag{12}$$

This duality relation shows that changing the sign of the parameter α in the raising term effectively reverses the action of the operator up to a sign. This property will be crucial for establishing the duality between the original sequence and the transformed sequence as it connects the operator to its parameter-reflected counterpart.

Definition 2. *A monic orthogonal polynomial sequence $\{A_n\}_{n\geq 0}$ (relative to u_0) is $T_{\mu,\alpha}$-classical if there exists another MOPS $\{Q_n\}_{n\geq 0}$ satisfying*

$$T_{\mu,\alpha}A_n = \varpi_n Q_{n+1}, \quad n \geq 0,$$

where ϖ_n is a normalization factor. The functional u_0 is correspondingly called $T_{\mu,\alpha}$-classical.

This definition extends the conventional notion of classical orthogonal polynomials (those that remain orthogonal under differentiation) to the context of the perturbed Dunkl operator. The key requirement is that applying $T_{\mu,\alpha}$ to each polynomial in the sequence yields a constant multiple of the next polynomial in a new orthogonal sequence.

For any monic polynomial sequence $\{A_n\}_{n\geq 0}$, we define an associated sequence $\{Q_n\}_{n\geq 0}$ via

$$\varpi_n Q_{n+1}(t) := A_n'(t) + 2\mu(H_{-1}A_n)(t) + \alpha t A_n(t), \quad n \geq 0, \tag{13}$$

where the normalization constant is fixed as $\varpi_n = \alpha$ for all $n \geq 0$.

Equation (13) constructs the sequence $\{Q_n\}$ by applying the operator $T_{\mu,\alpha}$ to each A_n and normalizing the result to be monic. The choice $\varpi_n = \alpha$ ensures that the leading coefficient of Q_{n+1} is 1, since the term $\alpha t A_n(t)$ contributes αt^{n+1} as the highest degree term.

Denoting by $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ the dual bases corresponding to $\{A_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively, Lemma 1 combined with the adjoint relation (12) implies that

applying the transformation $T_{\mu,-\alpha}$ (depending on parameters μ and $-\alpha$) to the $(n+1)$-th dual basis element v_{n+1} produces a scaled version of the n-th dual basis element u_n:

$$T_{\mu,-\alpha}(v_{n+1}) = -\alpha u_n, \quad n \geq 0. \tag{14}$$

The last relation is fundamental to our approach as it establishes a precise relationship between the dual bases of the original sequence and the transformed sequence, allowing us to transfer orthogonality properties between them. The negative sign and parameter reversal are consequences of the adjoint relation (12).

The duality relation (14) provides the foundation for our classification results.

The main objective of this section is the complete characterization of symmetric $T_{\mu,\alpha}$-classical orthogonal polynomial sequences. Specifically, we seek all symmetric MOPSs $\{A_n\}_{n\geq 0}$ satisfying the recurrence

$$\begin{cases} A_0(t) = 1, \\ A_1(t) = t, \\ A_{n+2}(t) = tA_{n+1}(t) - \varepsilon_{n+1}A_n(t), \quad \varepsilon_{n+1} \neq 0, \quad n \geq 0, \end{cases} \tag{15}$$

for which the derived sequence $\{Q_n\}_{n\geq 0}$ defined by (13) remains orthogonal.

When $\{A_n\}_{n\geq 0}$ is symmetric, the sequence $\{Q_n\}_{n\geq 0}$ inherits symmetry. Consequently, the associated functional v_0 satisfies

$$tH_{-1}v_0 = -v_0. \tag{16}$$

The symmetric MOPS $\{Q_n\}_{n\geq 0}$ therefore satisfies a three-term recurrence of the form

$$\begin{cases} Q_0(t) = 1, \\ Q_1(t) = t, \\ Q_{n+2}(t) = tQ_{n+1}(t) - \widehat{\varepsilon}_{n+1}Q_n(t), \quad \widehat{\varepsilon}_{n+1} \neq 0, \quad n \geq 0. \end{cases} \tag{17}$$

For all $\mu > -\frac{1}{2}$, a careful analysis of the duality relation (14) shows that, up to a dilation, the only symmetric $T_{\mu,\alpha}$-classical orthogonal polynomials are the generalized Hermite polynomials.

Theorem 1. *Up to a dilation, the generalized Hermite polynomials constitute the unique family of symmetric orthogonal polynomial sequences that are $T_{\mu,\alpha}$-classical.*

Proof. We begin our analysis by considering the recurrence relation (15) satisfied by the orthogonal polynomial sequence $\{A_n\}_{n\geq 0}$. The first step is to shift the index by one unit in order to introduce A_{n+1} in a form suitable for applying the operator T_μ. Specifically, if (15) is given by

$$A_{n+2}(t) = tA_{n+1}(t) - \varepsilon_{n+1}A_n(t),$$

then, after applying T_μ and shifting $n \mapsto n-1$, we obtain

$$T_\mu(A_{n+1})(t) = T_\mu(tA_n)(t) - \varepsilon_n T_\mu(A_{n-1})(t), \quad n \geq 1.$$

This index shift is a crucial technical step that aligns the recurrence relation with the natural action of the operator T_μ. By considering A_{n+1} instead of A_{n+2}, we obtain a relation that involves polynomials of consecutive degrees, which is more conducive to analysis with the raising operator. This step is essential, as it sets up a direct relation between the $(n+1)$-th, n-th, and $(n-1)$-th polynomials under the action of T_μ, enabling the propagation of

structural information across the sequence. The operator T_μ acts on both sides, maintaining linearity while introducing new analytical structures through its action on products.

The next objective is to simplify the term $T_\mu(tA_n)$ appearing on the right-hand side. For this, we employ the generalized Leibniz identity (11), which describes the action of T_μ on the product of two functions—in this case, t and $A_n(t)$. Applying the operator identity (11) to the term $T_\mu(tA_n)$ decomposes the action into distinct components:

$$T_\mu(A_{n+1})(t) = (1+2\mu)A_n(t) + tT_\mu(A_n)(t) - 4\mu t(H_{-1}A_n)(t) - \varepsilon_n T_\mu(A_{n-1})(t).$$

The application of the generalized Leibniz expansion gives three key effects: a scalar multiple $(1+2\mu)A_n(t)$ which arises from the action of T_μ on the factor t alone, a shifted operator action that comes from the usual Leibniz-type contribution involving A_n, and a parity-sensitive term $-4\mu t(H_{-1}A_n)(t)$ involving the Dunkl-specific deformation via the operator H_{-1}.

Utilizing the reflection operator's action, we can rewrite it using the parity decomposition:

$$H_{-1}A_n = \frac{1}{2t}[A_n(t) - A_n(-t)].$$

The decomposition of the operator H_{-1} separates the polynomial into its even and odd parts of A_n, which is particularly useful for symmetric polynomials where these parts have definite parity. Substituting this into the equation above and simplifying, we arrive at:

$$T_\mu(A_{n+1})(t) = A_n(t) + 2\mu A_n(-t) + tT_\mu(A_n)(t) - \varepsilon_n T_\mu(A_{n-1})(t).$$

The simplification occurs because $(1+2\mu)A_n(t) - 2\mu A_n(t) = A_n(t)$, while the term $2\mu A_n(-t)$ emerges from the combination of the Dunkl-specific terms. The simplified form exposes the explicit dependence on both $A_n(t)$ and $A_n(-t)$, highlighting the role of symmetry which is a fundamental characteristic of its non-local nature.

Our next aim is to relate the above identity to the sequence $\{Q_n\}$ defined in (13). To this end, we take the original recurrence relation (15), multiply it through by αt, and add the resulting equation to the one we have just obtained. The motivation for this step is to form the expression $T_\mu(A_k) + \alpha t A_k$, which appears in (13) by definition. Carrying out this multiplication and addition yields:

$$\begin{aligned} T_\mu(A_{n+1})(t) + \alpha t A_{n+1}(t) = t\big(T_\mu(A_n)(t) + \alpha t A_n(t)\big) - \varepsilon_n\big(T_\mu(A_{n-1})(t) + \alpha t A_{n-1}(t)\big) \\ + A_n(t) + 2\mu A_n(-t). \end{aligned}$$

This combination is strategically chosen to create the expression $T_\mu(A_k) + \alpha t A_k$ on both sides of the equation, which corresponds exactly to $\alpha Q_{k+1}(t)$ by definition (13). The additional terms $A_n(t) + 2\mu A_n(-t)$ represent the error that arises from the non-commutativity of the operations involved.

Here, the first line comes from collecting the terms involving t and αt, while the second line contains the remaining parity-adjusted terms. This strategic combination creates terms recognizable from the sequence $\{Q_n\}$ defined in (13).

By definition (13), we have:

$$T_\mu(A_k)(t) + \alpha t A_k(t) = \alpha Q_{k+1}(t),$$

valid for all $k \geq 0$. Using this with $k = n$ and $k = n-1$ in the equation above leads to:

$$Q_{n+2}(t) = tQ_{n+1}(t) - \varepsilon_n Q_n(t) + \alpha^{-1}[A_n(t) + 2\mu A_n(-t)]. \tag{18}$$

This is a significant result as it shows that the sequence $\{Q_n\}$ almost satisfies the same recurrence relation as $\{A_n\}$, but with an additional term involving $A_n(t) + 2\mu A_n(-t)$. This deviation from the standard recurrence structure is what will ultimately constrain the possible forms of the polynomial sequence. Multiplying Equation (18) by α gives:

$$\alpha Q_{n+2}(t) = \alpha t Q_{n+1}(t) - \alpha\varepsilon_n Q_n(t) + A_n(t) + 2\mu A_n(-t).$$

This connects the original polynomial to the auxiliary sequence $\{Q_n\}$.

We now compare this with the recurrence relation (17) satisfied by $\{Q_n\}$, and since the Q_{n+1} terms match perfectly, comparing the coefficients of $Q_n(t)$ yields:

$$\alpha(\varepsilon_n - \widehat{\varepsilon}_{n+1})Q_n(t) = A_n(t) + 2\mu A_n(-t). \tag{19}$$

Relation (19) establishes a direct proportionality between the polynomial $Q_n(t)$ and the combination $A_n(t) + 2\mu A_n(-t)$. The proportionality constant involves the difference between the recurrence coefficients of the two sequences, which encodes information about how the orthogonality structure is transformed under the action of $T_{\mu,\alpha}$.

The recurrence (17) for $\{Q_n\}$ requires:

$$Q_{n+2}(t) = tQ_{n+1}(t) - \widehat{\varepsilon}_{n+1}Q_n(t).$$

This establishes a functional relation between A_n and Q_n determined by the recurrence coefficients.

Since $A_n(t)$ is of degree n and $Q_n(t)$ is of degree n (by construction), both sides of (19) must be polynomials of identical degree. The right hand side $A_n(t) + 2\mu A_n(-t)$ has degree n with leading coefficient $[1 + 2\mu(-1)^n]\xi_n$, where ξ_n is the leading coefficient of A_n. The left side has leading coefficient $\alpha(\varepsilon_n - \widehat{\varepsilon}_{n+1})\zeta_n$, where ζ_n is the leading coefficient of Q_n. Equating the leading coefficients gives:

$$\alpha(\varepsilon_n - \widehat{\varepsilon}_{n+1}) = 1 + 2\mu(-1)^n. \tag{20}$$

This equivalence determines the difference between the recurrence coefficients of the two sequences in terms of the parameter μ and the parity of n. The alternating sign $(-1)^n$ reflects the different behavior for even and odd polynomials, which is characteristic of symmetric orthogonal polynomial systems. The term $(-1)^n$ arises naturally from the evaluation $A_n(-t)$, which captures the dependence of parity.

Consequently, combining the functional Equation (7), the coefficient relation (20), and the polynomial identity (19) implies:

$$(1 + 2\mu(-1)^n)Q_n(t) = A_n(t) + 2\mu A_n(-t). \tag{21}$$

Using the symmetry property, this implies that

$$Q_n(t) = A_n(t), \quad n \geq 0. \tag{22}$$

In words, the auxiliary sequence $\{Q_n\}$ coincides exactly with the original sequence $\{A_n\}$.

The expression in Equation (21) shows that $Q_n(t)$ is essentially a weighted average of $A_n(t)$ and its reflection $A_n(-t)$, with weights determined by the parameter μ and the parity of n. The identity in (22) is a crucial identification because it shows that the sequence $\{Q_n\}$ generated by applying $T_{\mu,\alpha}$ to $\{A_n\}$ is actually identical to the original sequence. This means that the operator $T_{\mu,\alpha}$ acts as a pure raising operator on the sequence, mapping each

polynomial to a constant multiple of the next one without altering the essential structure of the sequence.

From this identification, it follows that the recurrence coefficients for both sequences are identical:

$$\varepsilon_n = \widehat{\varepsilon}_n, \quad n \geq 1.$$

Substituting this equality into (20) yields the following difference equation

$$\alpha(\varepsilon_n - \varepsilon_{n+1}) = 1 + 2\mu(-1)^n,$$

The right-hand side is recognized as the difference of the sequence $\mu_n := -\alpha\varepsilon_n + c$, where:

$$\mu_{n+1} - \mu_n = 1 + 2\mu(-1)^n.$$

This difference equation completely determines the recurrence coefficients ε_n in terms of the parameters α and μ. The right-hand side alternates between $1 + 2\mu$ for even n and $1 - 2\mu$ for odd n, which will lead to the characteristic even–odd structure of the generalized Hermite polynomials.

Hence,

$$\alpha(\varepsilon_n - \varepsilon_{n+1}) = 1 + 2\mu(-1)^n = \mu_{n+1} - \mu_n, \tag{23}$$

Summing the difference Equation (23) from $k = 1$ to $n - 1$ gives

$$\alpha \sum_{k=1}^{n-1} (\varepsilon_k - \varepsilon_{k+1}) = \sum_{k=1}^{n-1} [1 + 2\mu(-1)^k],$$

which yields

$$\varepsilon_n - \varepsilon_1 = -\alpha^{-1}(\mu_n - \mu_1), \qquad n \geq 1. \tag{24}$$

This gives an explicit expression for the recurrence coefficients in terms of the cumulative sum μ_n. To determine ε_n completely, we need to find the initial value ε_1, which we will achieve by examining the case $n = 1$ specifically. So, to determine ε_1, we evaluate definition (13) at $n = 1$:

$$Q_2(t) = T_\mu(A_1)(t) + \alpha t A_1(t) = \alpha(t^2 - \varepsilon_1),$$

while direct computation gives:

$$T_\mu(A_1)(t) = T_\mu(t) = 1 + 2\mu, \quad A_1(t) = t,$$

So, we have:

$$\alpha(t^2 - \varepsilon_1) = (1 + 2\mu) + \alpha t^2.$$

This equation must hold identically for all t, so we can compare coefficients of like powers. The t^2 terms are equal on both sides (αt^2), so the constant terms must also be equal. By comparing the constant terms (i.e., the coefficients of t^0) on both sides, we immediately deduce:

$$\varepsilon_1 = -\alpha^{-1}(1 + 2\mu), \tag{25}$$

This determines the initial recurrence coefficient ε_1 in terms of the parameters α and μ. The negative sign indicates that the normalization involves a sign change, which is consistent with the raising operator interpretation.

Finally, substituting (25) into (24) produces the explicit expression:

$$\varepsilon_n = -\alpha^{-1}\mu_n, \quad n \geq 1,$$

This is the complete solution for the recurrence coefficients. They are determined by the cumulative sum $\mu_n = \sum_{k=0}^{n-1}[1 + 2\mu(-1)^k]$ with a scaling factor of $-\alpha^{-1}$. This explicit form shows the characteristic alternation between even and odd cases, which is a defining feature of generalized Hermite polynomials.

The last step is to recognize the recurrence structure in a normalized form.

Substituting (25) into (24):

$$\begin{aligned}\alpha\left[-\alpha^{-1}(1+2\mu) - \varepsilon_n\right] &= \mu_n - (1+2\mu)\\ -(1+2\mu) - \alpha\varepsilon_n &= \mu_n - (1+2\mu)\\ &= \mu_n - \mu_1,\end{aligned}$$

where $\mu_n := \sum_{k=0}^{n-1}[1 + 2\mu(-1)^k]$ with $\mu_1 = 1 + 2\mu$.

Applying the dilation transformation $t \mapsto at$ with parameter choice:

$$a^{-2} = -\frac{\alpha}{2}, \tag{26}$$

transforms the recurrence coefficients and rescales the variables as:

$$\tilde{\rho}_n = 0 \quad \text{(by symmetry)}, \quad \tilde{\varepsilon}_{n+1} = \frac{\mu_{n+1}}{2}. \tag{27}$$

This dilation is chosen to eliminate the parameter α from the recurrence coefficients and bring them into the standard form for generalized Hermite polynomials. The specific value of a^2 is determined by requiring that the transformed recurrence coefficients match exactly those of the generalized Hermite polynomials. The coefficients in (27) are exactly the recurrence coefficients for the generalized Hermite polynomials (1), confirming that our sequence is indeed a scaled version of the generalized Hermite polynomials. Thus, the original sequence $\{A_n\}$ is identified with the following scaled version of the generalized Hermite polynomials:

$$A_n(t) = a^{-n}\mathcal{H}_n^{(\mu)}(a \cdot t), \quad a^{-2} = -\frac{\alpha}{2}, \qquad n \geq 0.$$

This shows that these polynomials match the recurrence coefficients of generalized Hermite polynomials $\mathcal{H}_n^{(\mu)}$, which completes the proof of the theorem. □

Remark 1. *It is worth noting that the operator-theoretic Hahn-type condition considered in this work is closely related to the semi-classical characterization of orthogonal polynomials obtained from Pearson-type equations. In the classical case ($\alpha = 0$), the Dunkl operator reduces to the standard derivative, and the Hahn condition becomes equivalent to the differential Pearson equation that defines the generalized Hermite polynomials (see [13–16]). In our framework, the $T_{\mu,\alpha}$-operator plays an analogous role: it provides a structural relation that implicitly encodes a $T_{\mu,\alpha}$-Pearson equation for the corresponding weight function. This operator-based formulation thus extends the classical semi-classical theory and offers a unified viewpoint linking the Dunkl settings through Hahn-type relations.*

Remark 2. *From a spectral point of view, the operator $T_{\mu,\alpha}$ can be regarded as part of a pair of raising and lowering operators acting on the space of symmetric orthogonal polynomials. Such operators naturally appear in eigenvalue problems associated with Dunkl-type or differential-difference operators. In particular, when acting on the generalized Hermite polynomials, $T_{\mu,\alpha}$ generates a recurrence of the form $T_{\mu,\alpha}A_n = \alpha Q_{n+1}$, which may be interpreted as an eigenstructure relation for a second-order $T_{\mu,\alpha}$-type operator. This observation links our construction to the spectral theory framework usually employed for classical and semi-classical orthogonal polynomials.*

Remark 3. *Characterizing all $T_{\mu,\alpha}$-classical monic orthogonal polynomial sequences in full generality remains an outstanding open question in the field. We stress that our current results are restricted to symmetric sequences, a special case we have now fully resolved.*

4. Conclusions

This paper provided a complete characterization of symmetric orthogonal polynomial sequences that are classical with respect to the raising operator $T_{\mu,\alpha}$. By extending the notion of Hahn's problem to the Dunkl operator perturbed by a linear term, we have shown that the generalized Hermite polynomials form a unique family (up to a dilation) satisfying the $T_{\mu,\alpha}$-classical property. Our approach combined the analysis of recurrence relations, duality of polynomial sequences, and symmetry properties, showing that the Dunkl operator together with the raising term determines the preservation of orthogonality. In this way, our results extend earlier descriptions of generalized Hermite polynomials by showing, from an operator-theoretic viewpoint, how the Dunkl raising term characterizes symmetric semiclassical families. The methods developed here can serve as a foundation for further investigations into other operator-perturbed orthogonal polynomials and their structural properties. To further stimulate research in this direction, we also propose an *Open Problem* (see Remark 3).

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