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Structures, Ranks and Minimal Distances of Cyclic Codes over $\mathbb{Z}_{p^2} + u\mathbb{Z}_{p^2}$

Sami H. Saif

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; ssaif1@ksu.edu.sa

Abstract

Let p be a prime and $\mathbb{F}_p$ a finite field of order p. This paper investigates cyclic codes over the ring $R_{p^2,u} = \mathbb{Z}_{p^2} + u\mathbb{Z}_{p^2}$ of order p^4, where the nilpotent element u satisfies $u^2 = 0$ and $pu \neq 0$. The condition $u^2 = 0$ with $pu \neq 0$ is crucial, as it creates a nontrivial interaction between the components of the ring, allowing the construction of new codes with enhanced structural and distance properties. We provide explicit generating sets for cyclic codes over $R_{p^2,u}$ and study fundamental parameters such as their rank and Hamming distance. In the case $\gcd(n, p) = 1$, we show that cyclic codes can be generated by just two polynomials, which allows a complete determination of their rank and minimal Hamming distance distributions. Furthermore, using the Gray map from $R_{p^2,u}$ to $\mathbb{F}_p^4$, we construct all but one of the ternary optimal codes of length 12 as images of cyclic codes over $R_{3^2,u}$, with computations verified using the Magma system.

Keywords: rank; cyclic code; optimal code; minimal distance; finite ring

MSC: 11T71; 94B15; 94B05

1. Introduction

The study of cyclic codes over finite rings has become a significant subject in algebraic coding theory due to their rich algebraic structure and wide-ranging applications in communications and data storage [1,2]. A linear code of length n over a ring R is defined as an R-submodule of R^n. Among them, cyclic codes form a distinguished family, characterized by invariance under cyclic shifts: a code $\mathcal{C}$ of length n over R is cyclic if, whenever $(z_0, z_1, \ldots, z_{n-1}) \in \mathcal{C}$, the shifted vector $(z_{n-1}, z_0, \ldots, z_{n-2})$ also belongs to $\mathcal{C}$. Such codes admit an elegant algebraic description as ideals in the quotient ring $\frac{R[y]}{\langle y^n-1\rangle}$ via the canonical mapping: $(z_0, z_1, \ldots, z_{n-1}) \mapsto z_0 + z_1 y + \cdots + z_{n-1}y^{n-1}$, see [3–8]. The structure of cyclic codes over finite chain rings has been studied extensively when n is relatively prime to the characteristic of the residue field [9] and further explored in the divisible case [10–13].

Beyond chain rings, cyclic codes have been analyzed over more general non-chain rings. For example, Yildiz and Karadeniz [14] studied cyclic codes of odd length over $\frac{\mathbb{F}_2[u,v]}{\langle u^2,v^2,uv-vu\rangle}$, deriving several optimal binary codes via Gray maps. Generalizations to multivariable extensions such as $\frac{\mathbb{F}_2[u_1,\ldots,u_k]}{\langle u_i^2,u_iu_j-u_ju_i\rangle}$ were given in [15], which provided a classification of cyclic codes and their structural properties. Related investigations extended these results to rings over $\mathbb{F}_{2^m}$ [16], particularly over the non-chain ring $\frac{\mathbb{F}_{2^m}[u_1,u_2,\ldots,u_k]}{\langle u_i^2,u_iu_j-u_ju_i\rangle}$. Moreover, Kewat et al. [17] studied cyclic codes over the non-chain ring $\frac{\mathbb{Z}_p[u,v]}{\langle u^2,v^2,uv-vu\rangle}$, where a unique

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system of generators was obtained. More recently, linear codes have been investigated in connection with homogeneous weights, generating matrices and distance properties over the family of rings $\frac{\mathbb{Z}_{p^4}[u]}{\langle u^2-p^3\alpha, pu\rangle}$ [18]. Likewise, generator matrices of cyclic codes have been described over extensions of $\mathbb{F}_p[u,v]$ of order p^{4m} [19]. A common feature of these studies is that the alphabet ring has prime characteristic, with the exception of [18], where only generator matrices and homogeneous weights were examined.

Motivated by this line of work, we focus on the ring:

$$R_{p^2,u} = \mathbb{Z}_{p^2} + u\,\mathbb{Z}_{p^2},\ u^2 = 0,\ pu \neq 0, \tag{1}$$

where p is prime. This ring is a natural extension of $\mathbb{Z}_{p^2}$, the ring of integers modulo p^2, and it combines the nilpotent element u (with $u^2 = 0$) together with a non-prime characteristic p^2, while satisfying the relation $pu \neq 0$. Such structures are of intrinsic algebraic interest because they lie outside the prime-characteristic setting traditionally considered, and they also yield Gray images that often produce new and high-quality codes over finite fields.

This paper aims to determine generator polynomials, ranks and minimal spanning sets of cyclic codes over $R_{p^2,u}$, along with an analysis of their Hamming distances, specifically for lengths of the form p^s. Prior studies [14–16] focused mainly on the structural description of cyclic codes for the case when $p = 2$, neglecting the analysis of rank and distance parameters in this context. In contrast, we establish a systematic framework for computing these parameters over $R_{p^2,u}$, a task that has not been addressed in the literature for either case: $p = 2$ or p being odd. The main results include the construction of all optimal ternary codes of length 12 except one case; that is $[12, 2, 3]$, together with several high-quality codes of length 36, obtained as Gray images of cyclic codes over $R_{3^2,u}$. The significance of our results is that the Gray images of cyclic codes over $R_{p^2,u}$ yield many optimal and best-known codes. This demonstrates that the algebraic framework of $R_{p^2,u}$ not only extends previous structural results on cyclic codes but also provides an effective method for constructing high-quality codes with strong error-correction capability; for the applications see [20–22].

Methodologically, we employ ideas similar to those in [10,13,17], where cyclic codes are interpreted as ideals in $\frac{R_{p^2,u}[y]}{\langle y^n-1\rangle}$ and projected onto subrings. However, our novelty lies in extending and improving these techniques beyond prime-characteristic rings. We show how to adapt such methods to handle the mixed structure of $R_{p^2,u}$, where nilpotent and non-prime characteristic elements interact, leading to the discovery of new code families with desirable distance properties. This demonstrates both the theoretical significance and the practical impact of studying cyclic codes over these rings.

The paper is organized as follows. Section 2 introduces the necessary preliminaries, including the structure of the ring $R_{p^2,u}$, Lee and Hamming weights and the Gray map. In Section 3, we present a new set of generators for cyclic codes over $R_{p^2,u}$, providing explicit constructions that generalize previous results. Section 4 is devoted to minimal spanning sets and the computation of the rank of these codes, establishing key structural parameters. Section 5 investigates the minimum Hamming distance for codes of length p^s and illustrates the theory with several examples. In particular, it features explicit listings of cyclic codes over $R_{3^2,u}$ and demonstrates the construction of high-quality ternary codes obtained as Gray images over $\mathbb{F}_p$, including nearly all optimal codes of length 12 and several codes of length 36.

2. Preliminaries

This paper examines the ring defined with the setting

$$R_{p^2,u} = \frac{\mathbb{Z}_{p^2}[u]}{\langle u^2 \rangle} = \mathbb{Z}_{p^2} + u\mathbb{Z}_{p^2}, \quad u^2 = 0 \quad \text{and} \quad pu \neq 0. \tag{2}$$

This structure represents a finite commutative local Frobenius non-chain ring characterized by its maximal ideal $\mathfrak{m} = \langle p, u \rangle$ and the property that $\mathfrak{m}^2 = \langle pu \rangle$ [23]. The residue field associated with this ring is $\mathbb{F}_p$. The ideal lattice comprises $\{0, \langle u \rangle, \langle p \rangle, \langle pu \rangle, \langle u + \alpha p \rangle, \langle u, p \rangle\}$; given that the maximal ideal necessitates two generators, it follows that $R_{p^2,u}$ does not qualify as a chain ring. If $z \in R_{p^2,u}$, then it can uniquely be written as

$$z = z_0 + uz_1 + pz_2 + upz_3, \tag{3}$$

where $z_i \in \mathbb{F}_p$. Polynomials in quotient ring $\frac{R_{p^2,u}[y]}{\langle y^n - 1 \rangle}$ can be expressed uniquely as

$$\alpha(y) + ug_1(y) + pg_2(y) + upg_3(y), \tag{4}$$

where $\alpha(y)$ and $g_i(y)$ are in $\mathbb{F}_p[y]$. When $\alpha(y) \neq 0$, then such polynomials are classified as regular in $R_{p^2,u}[y]$ and possess a degree that corresponds to $\deg(\alpha(y))$.

Let $\eta \colon R^n_{p^2,u} \to R^n_{p^2,u}$ represent the cyclic shift permutation, characterized as the mapping that transforms $(z_0, z_1, \ldots, z_{n-1})$ into $(z_{n-1}, z_0, z_1, \ldots, z_{n-2})$. A linear code of length n over $R_{p^2,u}$ (i.e., a $R_{p^2,u}$-submodule of $R^n_{p^2,u}$) is defined as cyclic if it maintains invariance under the operation η. A natural correspondence exists between cyclic codes of length n over $R_{p^2,u}$ and ideals of the quotient ring $\frac{R_{p^2,u}[y]}{\langle y^n - 1 \rangle}$. This correspondence establishes a relationship between a codeword $(z_0, z_1, \ldots, z_{n-1})$ and the polynomial $z_0 + z_1 x + \cdots + z_{n-1} y^{n-1}$ [24].

In coding theory, the Lee weight w_L and Gray map φ_L are essential components.

Definition 1. *For any codeword, $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{F}_p^n$, the Hamming weight of $\mathbf{z}$, denoted by $w_H(\mathbf{z})$, is defined as the number of nonzero coordinates in $\mathbf{z}$. That is*

$$w_H(\mathbf{z}) = |\{i \in \{1, 2, \ldots, n\} : z_i \neq 0\}|. \tag{5}$$

Definition 2. *The Gray map $\varphi_L : R_{p^2,u} \to \mathbb{F}_p^4$ is defined by*

$$\varphi_L(z) = (z_0 + z_1 + z_2 + z_3, z_1 + z_3, z_2 + z_3, z_3), \tag{6}$$

and it extends in a straightforward manner to $R^n_{p^2,u}$.

Now, the Lee weight of an element $z = z_0 + uz_1 + pz_2 + upz_3 \in R_{p^2,u}$ is directly related to the Hamming weight of its Gray image by the relation

$$w_L(z) = w_H(\varphi(z)). \tag{7}$$

This linear Gray map is adopted due to its preservation of the Lee weight as a Hamming weight in $\mathbb{F}_p^4$. The linear combinations in φ_L represent the interactions between the nilpotent element u and the prime p, ensuring that each nonzero component appropriately contributes to the weight. Thus, φ_L establishes an isometry between $(R_{p^2,u}, w_L)$ and $(\mathbb{F}_p^4, w_H)$, ensuring that the minimum distance of a code over $R_{p^2,u}$ is preserved in its Gray image. This property is essential for the analysis of code parameters and the construction of high-quality codes over $\mathbb{F}_p$ derived from codes over $R_{p^2,u}$.

The extension of φ_L to $R_{p^2,u}^n$ preserves distances, making φ_L an isometry between the metric spaces $(R_{p^2,u}^n, w_L)$ and $(\mathbb{F}_p^{4n}, w_H)$. As a result, a linear code $\mathcal{C} \subseteq R_{p^2,u}^n$ characterized by invariants $(n, p^{k_n}, d)_L$ corresponds to a p-ary linear code over $\mathbb{F}_p$ with invariants $[4n, k_n, d]$. A ternary linear $[n, k_n, d]$ code is called *optimal* if there is no other $[n, k_n, d']$ code with the same n, k but a strictly larger $d' > d$. With the linearity of φ_L, we obtain the following result.

Theorem 1. *The Gray image $\varphi_L(\mathcal{C})$ of an $(n, p^{k_n}, d)_L$ code over $R_{p^2,u}$ is a p-ary $[4n, k_n, d]$ linear code.*

Definition 3. *For $t = \sum_{i=0}^{s-1} e_i p^i$ with $e_i \in \mathbb{F}_p$, we classify its p-adic expansion as*

1. Length-k zero expansion*: $e_{s-i} \neq 0$ for $1 \leq i \leq k < s$ and $e_{s-i} = 0$ for $k+1 \leq i \leq s$.*
2. Length-k nonzero expansion*: $e_{s-i} \neq 0$ for $1 \leq i \leq k < s$ with $e_{s-k-1} = 0$, but $e_{s-i} \neq 0$ for some $k+2 \leq i \leq s$.*
3. Full expansion*: $e_{s-i} \neq 0$ for all $1 \leq i \leq s$.*

3. Constructions of Cyclic Codes over the Ring $R_{p^2,u}$

Let p represent a prime number and n denote a positive integer. For a cyclic code $\mathcal{C}$ over $R_{p^2,u}$ with length n, it corresponds to an ideal $\mathfrak{J}_c$ of $R_{p^2u,n} = \frac{R_{p^2,u}[y]}{\langle y^n-1\rangle}$. Let $\overline{R_{p^2,u}} = \mathbb{F}_p$ and $\overline{R_{p^2,u}}[y] = \frac{R_{p^2,u}[y]}{\mathfrak{m}}$. The polynomial degree over $R_{p^2,u}$ is determined through the projection $\pi : R_{p^2,u}[y] \to \overline{R_{p^2,u}}[y]$, ensuring that $\deg(f(y)) = \deg(\pi(f(y)))$. Suppose $f(y) \in R_{p^2,u}[y]$; then, $f(y)$ is considered regular if it is not a zero divisor. In other terms, $f(y)$ is considered regular if there exists at least one coefficient that is a unit, or if $\pi(f(y))$ is not equal to zero in $\overline{R_{p^2,u}}[y]$. For future reference, it is important to note that regular polynomials adhere to a division theorem: for any nonzero $f(y), \alpha(y) \in R_{p^2,u}[y]$ where $\alpha(y)$ is regular, there exist $q(y)$ and $h(y)$ such that $f(y) = \alpha(y)q(y) + h(y)$, with $\deg(h(y)) < \deg(\alpha(y))$.

3.1. Construction (A)

For simplicity of notation, and since this form will be used repeatedly in the sequel, we introduce a canonical way to describe ideals in $R_{p^2u,n}$, and we shall refer to this as the Construction (A). Suppose that $\mathfrak{J}_c$ is an ideal of $R_{p^2u,n}$ corresponding to the cyclic code C of length n over $R_{p^2,u}$. We say that $\mathfrak{J}_c$ is in Construction (A) if it can be expressed as

$$\begin{aligned}\mathfrak{J}_c = \Big\langle & \alpha(y) + ug_1(y) + pf_1(y) + puh_1(y), \\ & u\beta(y) + pf_2(y) + puh_2(y), p\gamma(y) + puh_3(y), pu\delta(y) \Big\rangle,\end{aligned} \tag{8}$$

satisfying the following conditions:

$$\alpha(y) \mid (y^n - 1), \quad \beta(y) \mid \alpha(y), \quad \gamma(y) \mid \alpha(y), \tag{9}$$

$$\delta(y) \mid \gcd(\alpha(y), \beta(y), \gamma(y)), \tag{10}$$

$$\beta(y) \mid \frac{g_1(y)(y^n-1)}{\alpha(y)}, \quad \gamma(y) \mid \frac{\alpha(y)f_2(y)}{\beta(y)}, \tag{11}$$

$$\delta(y) \mid \gcd\left(f_2(y), \frac{y^n-1}{\gamma(y)}h_3(y)\right), \tag{12}$$

$$\delta(y) \mid \frac{y^n-1}{\beta(y)}\left(h_2(y) + \frac{f_2(y)h_3(y)}{\gamma(y)}\right), \tag{13}$$

$$\delta(y) \mid \left(g_1(y) + \frac{\alpha(y)h_3(y)}{\gamma(y)}\right), \tag{14}$$

$$\delta(y) \mid \left(f_1(y) + \frac{\alpha(y)h_2(y)}{\beta(y)} + \frac{\alpha(y)f_2(y)h_3(y)}{\beta(y)\gamma(y)}\right), \tag{15}$$

$$\delta(y) \mid \frac{y^n-1}{\alpha(y)}\left(h_1(y) + \frac{g_1(y)h_2(y)}{\beta(y)} + \frac{f_1(y)h_3(y)}{\gamma(y)} + \frac{g_1(y)f_2(y)h_3(y)}{\beta(y)\gamma(y)}\right). \tag{16}$$

where $\alpha(y), \beta(y), \gamma(y), \delta(y), g_1(y), f_i(y)$ and $h_j(y)$ are in $\mathbb{F}_p[y]$ (for $i = 1, 2$ and $j = 1, 2, 3$). These conditions establish algebraic relationships among the generators, allowing us to determine their respective degrees and divisibility properties with respect to $y^n - 1$. This interdependence ensures a consistent and canonical structure for all ideals (and, hence, cyclic codes) considered under Construction (A).

Theorem 2. *Any ideal $\mathfrak{J}$ of the ring $R_{p^2u,n}$ is uniquely generated by the polynomials $G_1(y) = \alpha(y) + ug_1(y) + pf_1(y) + puh_1(y)$, $G_2(y) = u\beta(y) + pf_2(y) + puh_2(y)$, $G_3(y) = p\gamma(y) + puh_3(y)$ and $G_4(y) = pu\delta(y)$. Moreover, the polynomials $g_i(y)$, $f_i(y)$, $h_i(y)$ are either zero or satisfy*

$$\deg(g_1(y)) < \deg(\beta(y)), \tag{17}$$

$$\max\{\deg(f_1(y)), \deg(f_2(y))\} < \deg(\gamma(y)), \tag{18}$$

$$\max\{\deg(h_1(y)), \deg(h_2(y)), \deg(h_3(y))\} < \deg(\delta(y)). \tag{19}$$

Proof. Consider the ring $R_{p^2,u} = R_{p,u} + pR_{p,u}$ where $p^2 = 0$ and $R_{p,u} = \mathbb{F}_p + u\mathbb{F}_p$ with $u^2 = 0$; that is, $R_{p,u} = \frac{\mathbb{F}_p[u]}{\langle u^2\rangle}$. Let $\mathcal{C}$ be a cyclic code of length n over $R_{p^2,u}$, viewed as an ideal in $R_{p^2u,n} = \frac{R_{p^2,u}[y]}{\langle y^n-1\rangle}$. Define the projection $\varphi : R_{p^2,u} \to R_{p,u}$ by $\varphi(z_0 + pz_1) = z_0$ for $z_0, z_1 \in R_{p^2,u}$, extended to

$$\theta : \mathcal{C} \to R_{p,u,n}, \quad \sum_{i=0}^{n-1} z_i y^i \mapsto \sum_{i=0}^{n-1} \varphi(z_i) y^i.$$

Let $\mathfrak{J} = \{h(y) \in R_{p,u,n} \mid ph(y) \in \ker(\theta)\}$. Then, ideals in $R_{p,u,n}$ have the form

$$\mathfrak{J} = \langle h(y) + uq(y), ub(y)\rangle$$

with $b(y)|h(y)|y^n - 1$, $b(y)|q(y)\left(\frac{y^n-1}{h(y)}\right)$ [24]. This gives $\ker(\theta) = \langle ph(y) + puq(y), pu(y)\rangle$. The image $\theta(\mathcal{C})$ is then $\langle \alpha(y) + pu(y), ua(y)\rangle$, where $a(y)|\alpha(y)|y^n - 1 \pmod p$, and $a(y)|p(y)\left(\frac{y^n-1}{\alpha(y)}\right)$. Therefore, we can deduce that $\mathcal{C}$ decomposes as

$$\begin{aligned}\mathcal{C} = \langle &\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y),\\ &u\beta(y) + pf_2(y) + puh_2(y), p\gamma(y) + puh_3(y), pu\delta(y)\rangle\end{aligned}$$

with divisibility conditions

$$\delta(y)|\beta(y)|\alpha(y)|y^n - 1 \quad \text{and} \quad \delta(y)|\gamma(y)|\alpha(y)|y^n - 1.$$

To prove uniqueness, assume there exists another set of generators

$$G_1'(y), G_2'(y), G_3'(y), G_4'(y)$$

that also generates $\mathcal{C}$. Since every element of $R_{p^2u,n}$ has a *unique decomposition*

$$a_0(y) + ua_1(y) + pa_2(y) + upa_3(y), \quad a_i(y) \in \mathbb{F}_p[y],$$

each $G_i'(y)$ can be written in the same form. Because $G_i'(y) \in \mathcal{C}$, there exist polynomials $a_{ij}(y) \in R_{p^2u,n}$ such that

$$G_i'(y) = \sum_{j=1}^{4} a_{ij}(y)G_j(y).$$

Comparing the components in the decomposition modulo p and modulo u, and using the degree bounds, we see that each $G_i'(y)$ must coincide with $G_i(y)$ in $R_{p^2u,n}$. Hence, no distinct set of generators can produce the same ideal, proving the uniqueness of $G_1(y), \ldots, G_4(y)$ in $R_{p^2u,n}$. □

We now define the residue and torsion of the ideal $\mathfrak{J}_c \subset R_{p^2u,n}$ corresponding to the cyclic code $\mathcal{C}$ of length n over $R_{p^2,u}$.

Definition 4. *If $\mathfrak{J}_c$ is an ideal of $R_{p^2u,n}$. Then, we define*

$$Res(\mathfrak{J}_c) = \{z \in R_{p,u,n} \mid \text{ there exists } c : z + pc \in \mathfrak{J}_c\}, \tag{20}$$
$$Tor(\mathfrak{J}_c) = \{z \in R_{p,u,n} \mid pz \in \mathfrak{J}_c\}. \tag{21}$$

Both are ideals of $R_{p,u,n}$. They can be expressed as $\mathrm{Res}(\mathfrak{J}_c) = \mathrm{Im}(\theta)$ and $\mathrm{Tor}(\mathfrak{J}_c) = \mathfrak{J}$, where θ is defined in the proof of Theorem 2. Define the ideals as follows:

$$\mathfrak{J}_1 = \mathrm{Res}(\mathrm{Res}(\mathfrak{J}_c)) = \mathfrak{J}_c \bmod \mathfrak{m}, \tag{22}$$
$$\mathfrak{J}_2 = \mathrm{Tor}(\mathrm{Res}(\mathfrak{J}_c)) = \{a(y) \in R_{p,n} \mid ua(y) \in \mathfrak{J}_c \bmod p\}, \tag{23}$$
$$\mathfrak{J}_3 = \mathrm{Res}(\mathrm{Tor}(\mathfrak{J}_c)) = \{a(y) \in R_{p,n} \mid pa(y) \in \mathfrak{J}_c\}, \tag{24}$$
$$\mathfrak{J}_4 = \mathrm{Tor}(\mathrm{Tor}(\mathfrak{J}_c)) = \{a(y) \in R_{p,n} \mid pua(y) \in \mathfrak{J}_c\}. \tag{25}$$

These are ideals of $R_{p,n} = \frac{\mathbb{F}_p[y]}{\langle y^n-1\rangle}$; hence, they are principal, so

$$\mathfrak{J}_1 = \langle \alpha(y)\rangle, \quad \mathfrak{J}_2 = \langle \beta(y)\rangle, \quad \mathfrak{J}_3 = \langle \gamma(y)\rangle \quad \mathfrak{J}_4 = \langle \delta(y)\rangle.$$

Theorem 3. *Let $\mathcal{C}$ be a cyclic code of length n over the ring $R_{p^2,u}$. Then, its associated ideal $\mathfrak{J}_c \subseteq R_{p^2u,n}$ is constructed as in Construction (A).*

Proof. The conditions (9) and (10) follow directly from the ideal inclusions $\mathfrak{J}_1 \supseteq \mathfrak{J}_j$ for $2 \le j \le 4$ and $\mathfrak{J}_j \supseteq \mathfrak{J}_4$ for $2 \le j \le 3$.

For condition (11), observe that

$$\begin{aligned}\frac{y^n-1}{\alpha(y)}\left[\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y)\right] =& u\frac{y^n-1}{\alpha(y)}g_1(y) \\ &+ p\frac{y^n-1}{\alpha(y)}f_1(y) + pu\frac{y^n-1}{\alpha(y)}h_1(y) \in \mathfrak{J}_c,\end{aligned}$$

where we use that $\mathfrak{J}_2 = \langle \beta(y)\rangle$.

Condition (12) follows from three key relations. First,

$$\begin{aligned}u\left[\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y)\right] + \frac{\alpha(y)}{\beta(y)}\left[u\beta(y) + pf_2(y) + puh_2(y)\right] \\ = p\left(\frac{\alpha(y)}{\beta(y)}f_2(y)\right) + pu\left(f_1(y) + \frac{\alpha(y)}{\beta(y)}h_2(y)\right) \in \mathfrak{J}_c.\end{aligned}$$

Second,

$$\frac{y^n-1}{\alpha(y)}\left[\alpha(y)+ug_1(y)+pf_1(y)+puh_1(y)\right]+\frac{y^n-1}{\alpha(y)}\frac{g_1(y)}{\beta(y)}\left[u\beta(y)+pf_2(y)+puh_2(y)\right]$$
$$=p\left(\frac{y^n-1}{\alpha(y)}\left(f_1(y)+\frac{g_1(y)}{\beta(y)}f_2(y)\right)\right)+pu\left(\frac{y^n-1}{\alpha(y)}\left(h_1(y)+\frac{g_1(y)}{\beta(y)}h_2(y)\right)\right)\in\mathfrak{J}_c.$$

Third,

$$u\left[u\beta(y)+pf_2(y)+puh_2(y)\right]=puf_2(y)\in\mathfrak{J}_c.$$

For condition (13), we have

$$\frac{y^n-1}{\gamma(y)}\left[p\gamma(y)+puh_3(y)\right]\in\mathfrak{J}_c.$$

Condition (15) follows from

$$\frac{y^n-1}{\beta(y)}\left[u\beta(y)+pf_2(y)+puh_2(y)\right]+\frac{y^n-1}{\beta(y)}\frac{f_2(y)}{\gamma(y)}\left[p\gamma(y)+puh_3(y)\right]$$
$$=pu\frac{y^n-1}{\beta(y)}\left(h_2(y)+\frac{f_2(y)}{\gamma(y)}h_3(y)\right)\in\mathfrak{J}_c.$$

The verification of condition (16) involves two parts. First,

$$p\left[\alpha(y)+ug_1(y)+pf_1(y)+puh_1(y)\right]+\frac{\alpha(y)}{\gamma(y)}\left[p\gamma(y)+puh_3(y)\right]$$
$$=pu\left(g_1(y)+\frac{\alpha(y)}{\gamma(y)}h_3(y)\right)\in\mathfrak{J}_c.$$

Second,

$$u\left[\alpha(y)+ug_1(y)+pf_1(y)+puh_1(y)\right]+\frac{\alpha(y)}{\beta(y)}\left[u\beta(y)+pf_2(y)+puh_2(y)\right]$$
$$+\frac{\alpha(y)f_2(y)}{\beta(y)\gamma(y)}\left[p\gamma(y)+puh_3(y)\right]=pu\left(f_1(y)+\frac{\alpha(y)h_2(y)}{\beta(y)}+\frac{\alpha(y)f_2(y)h_3(y)}{\beta(y)\gamma(y)}\right)\in\mathfrak{J}_c.$$

The final condition combines these results:

$$\frac{y^n-1}{\alpha(y)}\left[\alpha(y)+ug_1(y)+pf_1(y)+puh_1(y)\right]$$
$$+\frac{y^n-1}{\alpha(y)}\frac{g_1(y)}{\beta(y)}\left[u\beta(y)+pf_2(y)+puh_2(y)\right]$$
$$+\frac{y^n-1}{\alpha(y)}\frac{f_1(y)+\frac{g_1(y)}{\beta(y)}f_2(y)}{\gamma(y)}\left[p\gamma(y)+puh_3(y)\right]$$
$$=\frac{y^n-1}{\alpha(y)}\left(h_1(y)+\frac{g_1(y)h_2(y)}{\beta(y)}+\frac{f_1(y)+\frac{g_1(y)}{\beta(y)}f_2(y)}{\gamma(y)}h_3(y)\right)\in\mathfrak{J}_c.$$

□

Next, we employ the divisibility conditions (9)–(16) introduced in Construction (A) to simplify the structural description of cyclic codes over $R_{p^2,u}$. The advantage of these conditions becomes particularly evident in the case $\gcd(n,p)=1$, as demonstrated in Construction (B), when $\gcd(n,p)=1$.

Proposition 1. *Let $\mathcal{C}$ be a cyclic code over the ring $R_{p^2,u}$ given in Construction (B). Then, $\mathcal{C}$ is a free cyclic code if and only if $\alpha(y) = \delta(y)$. Moreover, in this case, we have*

1. $\mathcal{C} = \langle \alpha(y) + ug_1(y) + pf_1(y) + puh_1(y) \rangle$,
2. $(\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y)) \mid (y^n - 1)$ *in* $R_{p^2u,n}$.

Proof. Assume $\alpha(y) = \delta(y)$. From the divisibility conditions, $\delta(y) \mid \beta(y) \mid \alpha(y)$ and $\delta(y) \mid \gamma(y) \mid \alpha(y)$, we obtain $\alpha(y) = \beta(y) = \gamma(y) = \delta(y)$.

The image and kernel of θ are

$$\begin{aligned} \mathrm{Im}(\theta) &= \langle \alpha(y) + ug_1(y), u\beta(y) \rangle = \langle \alpha(y) + ug_1(y) \rangle, \\ \ker(\theta) &= p\langle \gamma(y) + uh_3(y), u\delta(y) \rangle = p\langle \alpha(y) + uh_3(y) \rangle. \end{aligned}$$

Thus, $\mathfrak{J}_c$ simplifies to

$$\mathfrak{J}_c = \langle \alpha(y) + ug_1(y) + pf_1(y) + puh_1(y), p\alpha(y) + puh_3(y) \rangle.$$

Since $\deg(g_1(y)), \deg(h_3(y)) < \deg(\alpha(y))$ and by Condition (12) in Representation (A), $\delta(y) \mid (g_1(y) - h_3(y))$, we conclude $g_1(y) = h_3(y)$. Consequently

$$p(\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y)) = p\alpha(y) + pug_1(y) = p\gamma(y) + puh_3(y).$$

Therefore, $\mathfrak{J}_c = \langle \alpha(y) + ug_1(y) + pf_1(y) + puh_1(y) \rangle$, $\mathcal{C} \cong R_{p,u}^{n-\deg(\alpha(y))}$, proving $\mathcal{C}$ is free. Conversely, if $\mathcal{C}$ is free, then $\mathfrak{J}_c$ must be principal; say $\mathfrak{J}_c = \langle \alpha(y) + ug_1(y) + pf_1(y) + puh_1(y) \rangle$. Since $pu\delta(y) \in \mathcal{C}$, there exists $\lambda \in \mathbb{F}_p$:

$$pu\delta(y) = pu\lambda(\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y)) = pu\lambda\alpha(y).$$

As $\delta(y) \mid \alpha(y)$, comparing coefficients yields $\alpha(y) = \delta(y)$. For the divisibility condition, apply the division algorithm in $R_{p^2,u}[y]$:

$$y^n - 1 = (\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y))q(y) + h(y),$$

where $h(y) = 0$ or $\deg(h(y)) < \deg(\alpha(y))$. Then

$$h(y) = y^n - 1 - (\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y))q(y) \in \mathfrak{J}_c.$$

Since $\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y)$ has minimal degree in $\mathfrak{J}_c$, $h(y) = 0$. Hence, $(\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y)) \mid (y^n - 1)$. □

Remark 1. *The generator set of $\mathfrak{J}_c$ simplifies significantly when additional equality conditions hold among $\alpha(y), \beta(y), \gamma(y)$ and $\delta(y)$. In particular*

(i) When $\alpha(y) = \beta(y) = \gamma(y) = \delta(y)$ as in Proposition 1, the ideal reduces to a single generator.
(ii) More generally, if any of the equalities $\alpha(y) = \beta(y)$, $\gamma(y) = \beta(y)$, or $\delta(y) = \beta(y), \gamma(y)$ hold, the generating set of $\mathfrak{J}_c$ over $R_{p^2,u}$ admits similar simplifications.

These special cases frequently arise in practice when studying cyclic codes over finite chain rings.

When $\gcd(n, p) = 1$

In this subsection, we assume $\gcd(n, p) = 1$. By Proposition 1, the canonical projection θ yields $\mathrm{Im}(\theta) = \langle \alpha(y) + u\beta(y) \rangle$ and $\ker(\theta) = p\langle \delta(y) \rangle$, where $\beta(y) \mid \alpha(y) \mid y^n - 1$ in $\mathbb{F}_p[y]$. Combining these components, the code ideal decomposes as $\mathfrak{J}_c = \langle \alpha(y) + u\beta(y) + pf_1(y), p\delta(y) \rangle$ while maintaining these divisibility conditions. From Construction (A), we

have $\delta(y)|f_1(y)$ by condition (12). Thus, we obtain $\mathfrak{J}_c = \langle \alpha(y) + u\beta(y), p\delta(y) \rangle$, and this structure leads to the following construction.

3.2. Construction (B)

Let $\hat{\alpha}(y) = \frac{y^n-1}{\alpha(y)}$ and

$$\mathfrak{J}_c = \langle \alpha(y) + u\beta(y), p\delta(y) \rangle \tag{26}$$

with the polynomials satisfying the following:

$$\begin{aligned} &\delta(y) \mid \alpha(y) \mid y^n - 1, \\ &\deg(\beta) < \deg(\delta) \text{ or } \beta(y) = 0, \\ &\delta(y) \mid \beta(y)\hat{\alpha}(y). \end{aligned}$$

Theorem 4. *Suppose that $\mathcal{C}$ is a cyclic code of length n over the ring $R_{p^2,u}$. Then, $\mathfrak{J}_c$ has Construction (B).*

4. Minimal Generating Structures and Ranks

This section establishes the minimal spanning set and rank of cyclic codes over $R_{p^2,u}$. Following [25], the *rank* of $\mathcal{C}$ is defined as the minimal number of generators of $\mathfrak{J}_c$, while the *free rank* of $\mathcal{C}$ is the maximal rank of free $R_{p^2,u}$-submodules of $\mathcal{C}$.

Theorem 5. *Let* $\gcd(p, n) \neq 1$ *and $\mathcal{C}$ be a cyclic code of length n over the ring $R_{p^2,u}$. If $\mathfrak{J}_c = \langle \alpha(y) + ug_1(y) + pf_1(y) + puh_1(y), u\beta(y) + pf_2(y) + puh_2(y), p\gamma(y) + puh_3(y), pu\delta(y) \rangle$ with $\iota = \deg(\alpha(y))$, $\iota_1 = \deg(\beta(y))$, $\iota_2 = \deg(\gamma(y))$, with also $\iota^* = \min\{\deg(\beta(y)), \deg(\gamma(y))\}$ and $\iota_3 = \deg(\delta(y))$, then rank of $\mathcal{C}$ is $n - \iota$, free rank $n + \iota + \iota^* - \iota_1 - \iota_2 - \iota_3$ and the minimal spanning set:*

$$G = \begin{Bmatrix} y^i(\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y)), & \text{for } 0 \leq i \leq n - \iota - 1, \\ y^j(u\beta(y) + pf_2(y) + puh_2(y)), & \text{for } 0 \leq j \leq \iota - \iota_1 - 1, \\ y^k(p\gamma(y) + puh_3(y)), & \text{for } 0 \leq k \leq \iota - \iota_2 - 1, \\ y^\ell(pu\delta(y)), & \text{for } 0 \leq \ell \leq \iota^* - \iota_3 - 1. \end{Bmatrix} \tag{27}$$

Furthermore, if $f_2(y) \neq 0$, we have $|\mathcal{C}| = p^{4n+\iota+\iota^*-3\iota_1-2\iota_2-\iota_3}$ and if $f_2(y) = 0$, we have $|\mathcal{C}| = p^{4n+\iota^*-2\iota_1-2\iota_2-\iota_3}$.

Proof. Let $\iota^* = \deg(\gamma(y))$. Thus, it is enough to demonstrate that G spans the set:

$$G' = \begin{Bmatrix} \{y^i(\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y))\}_{i=0}^{n-\iota-1}, \\ \{y^j(u\beta(y) + pf_2(y) + puh_2(y))\}_{j=0}^{n-\iota_1-1}, \\ \{y^\ell(p\gamma(y) + puh_3(y))\}_{\ell=0}^{n-\iota_2-1}, \\ \{y^k(pu\delta(y))\}_{k=0}^{n-\iota_3-1}. \end{Bmatrix} \tag{28}$$

Our initial demonstration is that $puy^{\iota_2-\iota_3}(\delta(y)) \in \text{Span}(G)$. The leading coefficient of the function $\gamma(y) + uh_3(y)$ and the leading coefficient of the function $y^{\iota_2-\iota_3}(\delta(y))$ are γ_2 and δ_3, respectively. If $\gamma_2 = c_0\delta_3$, then c_0 is a constant in $\mathbb{F}_p$. If $\deg(l(y)) \leq \iota_2$, then the following equation holds

$$puy^{\iota_2-\iota_3}(\delta(y)) = puc_0(\gamma(y) + uh_3(y)) + pul(y).$$

Since $\delta(y)$ is the lowest degree polynomial such that $pu\delta(y) \in \mathfrak{J}_c$, we can deduce that $\deg(l(y)) \geq \iota_3$. This means that ι_3 is less than or equal to ι_2 and that we can express the function as follows:

$$pul(y) = \sum_{i=0}^{\iota_2-\iota_3-1} \alpha_i pux^i\delta(y).$$

As a result, $puy^{\iota_2-\iota_3}(\delta(y)) \in \text{Span}(G)$. This is the equation $\iota^* = \deg(\beta(y))$. This cannot be used as a divisor in the division procedure because $u\beta(y) + pf_2(y) + puh_2(y)$ is not regular. Nonetheless, it can be demonstrated through direct calculation that $puy^{\iota_1-\iota_3}(\delta(y)) \in \text{Span}(G)$. We have the factorization since $\delta(y) \mid \beta(y)$:

$$\beta(y) = q(y)\delta(y) = \delta(y)\left(\sum_{i=0}^{\iota_1-\iota_3} f_i y^i\right)$$

with $f_{\iota_1-\iota_3} \neq 0$ (by degree comparison). Therefore,

$$pu\delta(y)y^{\iota_1-\iota_3} = puf_{\iota_1-\iota_3}^{-1}\beta(y) - puf_{\iota_1-\iota_3}^{-1}\delta(y)\left(\sum_{i=0}^{\iota_1-\iota_3-1} f_i y^i\right).$$

Noting that $p(u\beta(y) + pf_2(y) + puh_2(y)) = pu\beta(y)$, we conclude that $pu\delta(y)y^{\iota_1-\iota_3} \in \text{Span}(G)$. Using $\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y)$ as a divisor of $u\beta(y) + pf_2(y) + puh_2(y)$ and $p\gamma(y) + puh_3(y)$, we can show that

$$y^{\iota-\iota_1}(u\beta(y) + pf_2(y) + puh_2(y)), \quad y^{\iota-\iota_2}(p\gamma(y) + puh_3(y)) \in \text{Span}(G).$$

Similar arguments apply to the remaining generators, confirming that G forms a generating set for the code. To establish minimality, we observe that none of the following elements can be written as

$$\begin{cases} y^{i-\iota-1}\big(\alpha(y) + ug_1(y) + pf_1(y) + puh_1(y)\big), & \text{for } 1 \leq i \leq n, \\ y^{j-\iota_1-1}\big(u\beta(y) + pf_2(y) + puh_2(y)\big), & \text{for } 1 \leq j \leq \iota, \\ y^{k-\iota_2-1}\big(p\gamma(y) + puh_3(y)\big), & \text{for } 1 \leq k \leq \iota, \\ y^{s-\iota_3-1}pu\delta(y), & \text{for } 1 \leq s \leq \iota^*. \end{cases}$$

We demonstrate this explicitly for $y^{\iota-\iota_1-1}(u\beta(y) + pf_2(y) + puh_2(y))$. Suppose, for contradiction, that

$$\begin{aligned} y^{\iota-\iota_1-1}(u\beta(y) + pf_2(y) + puh_2(y)) = &\sum_{i=1}^{n-\iota} \hat{a}_i y^{i-1}(\alpha(y) + \cdots) \\ &+ \sum_{j=1}^{\iota-\iota_1-2} \hat{b}_j y^{j-1}(u\beta(y) + \cdots) + \sum_{k=1}^{\iota-\iota_2} \hat{c}_k y^{k-1}(p\gamma(y) + \cdots) \\ &+ \sum_{s=1}^{\iota^*-\iota_3} \hat{d}_l y^{s-1}(pu\delta(y)), \end{aligned}$$

$$\begin{aligned} &\hat{a}_i = \hat{a}_{i1} + \hat{a}_{i2}u + \hat{a}_{i3}p + \hat{a}_{i4}pu, \quad \hat{b}_i = \hat{b}_{i1} + \hat{b}_{i2}u + \hat{b}_{i3}p + \hat{b}_{i4}pu, \\ &\hat{c}_i = \hat{c}_{i1} + \hat{c}_{i2}u + \hat{c}_{i3}p + \hat{c}_{i4}pu, \quad \hat{d}_i = \hat{d}_{i1} + \hat{d}_{i2}u + \hat{d}_{i3}p + \hat{d}_{i4}pu. \end{aligned}$$

We have

$$\begin{aligned} y^{\iota-\iota_1-1}\big(u\beta(y)+pf_2(y)+vuh_2(y)\big) &= \big(\hat{a}_{11}+\hat{a}_{21}y+\cdots+\hat{a}_{(n-\iota)1}y^{n-\iota-1}\big)\alpha(y) \\ &\quad+u\big(\hat{a}_{12}+\hat{a}_{22}y+\cdots+\hat{a}_{(n-\iota)2}y^{n-\iota-1}\big)\alpha(y) \\ &\quad+u\big(\hat{a}_{11}+\hat{a}_{21}y+\cdots+\hat{a}_{(n-\iota)1}y^{n-\iota-1}\big)g_1(y) \\ &\quad+u\big(\hat{b}_{11}+\hat{b}_{21}y+\cdots+\hat{b}_{(\iota-\iota_1-1)1}y^{\iota-\iota_1-2}\big)\beta(y) \\ &\quad+vl_1(y)+pul_2(y), \end{aligned}$$

where $l_1(y)$ and $l_2(y)$ are polynomials in $\mathbb{F}_p[y]$. Since this term cannot be produced by any combination of lower degree terms, there is a contradiction when looking at the coefficients of $u\beta(y)y^{\iota-\iota_1-1}$ on both sides. All of the other building blocks are similarly justified. We may deduce that $\hat{a}_{i1}=0$ for $1\le i\le n-\iota$ by comparing the two sides, and

$$\begin{aligned} y^{\iota-\iota_1-1}\beta(y) =&(\hat{a}_{12}+\hat{a}_{22}y+\cdots+\hat{a}_{(n-\iota)2}y^{n-\iota-1})\alpha(y) \\ &+(\hat{b}_{11}+\hat{b}_{21}y+\cdots+\hat{b}_{(\iota-\iota_1-1)1}y^{\iota-\iota_1-2})\beta(y). \end{aligned}$$

Note that $\deg(y^{\iota-\iota_1-1}\beta(y))=\iota-1$ but $\deg((\hat{a}_{12}+\hat{a}_{22}y+\cdots+\hat{a}_{(n-\iota)2}y^{n-\iota-1})\alpha(y))\ge\iota$ and $\deg((\hat{b}_{11}+\hat{b}_{21}y+\cdots+\hat{b}_{(\iota-\iota_1-1)1}y^{\iota-\iota_1-2})\beta(y))\le\iota-2$. Hence, this gives a contradiction. □

Theorem 6. *Let* $\gcd(n,p)=1$*, and let* $\mathcal{C}$ *be a cyclic code of length* n *over* $R_{p^2,u}$*. Suppose*

$$\mathfrak{J}_c=\langle\alpha(y)+u\beta(y),\ p\delta(y)\rangle,$$

and $\iota=\deg(\alpha(y))$ *and* $t=\deg(\gamma(y))$*. Then,* $\mathcal{C}$ *has rank* $n-\iota$ *and cardinality*

$$|\mathcal{C}|=p^{4n-2t-2\iota}.$$

Moreover,

$$G=\bigcup_{i=0}^{n-\iota-1}\{y^i(\alpha(y)+u\beta(y))\}\cup\bigcup_{j=0}^{\iota-\iota_1}\{y^j(p\delta(y))\}.$$

Proof. The proof is similar to that of Theorem 5. □

5. Minimum Hamming Distances of Cyclic Codes over $R_{p^2,u}$

In this section, we investigate the minimum Hamming distance for codes of length p^s and illustrate the theory with several examples. Throughout, we suppose that $\gcd(n,p)\neq 1$. We introduce the auxiliary code

$$\mathfrak{J}_{pu}=\{k(y)\in R_{p^2u,n}: puk(y)\in\mathfrak{J}_c\}, \tag{29}$$

which forms a cyclic code over $\mathbb{F}_q$ by construction.

Theorem 7. *For* $\gcd(n,p)\neq 1$*, let* $\mathcal{C}$ *be a cyclic code over* $R_{p^2,u}$ *with associated ideal* $\mathfrak{J}_c$ *given in Construction (A). Then,* $\mathfrak{J}_{pu}=\langle\delta(y)\rangle$ *and the weights satisfy* $w_H(\mathcal{C})=w_H(\mathfrak{J}_{pu})$*.*

Proof. The part $\langle\delta(y)\rangle\subseteq\mathfrak{J}_{pu}$ follows from $pu\delta(y)\in\mathfrak{J}_c$. For the reverse inclusion, any $b(y)\in\mathfrak{J}_{pu}$ satisfies $pub(y)\in\mathcal{C}$, yielding a decomposition:

$$pub(y)=\sum_{i=1}^{4}e_i(y)pug_i(y) \tag{30}$$

where $g_i(y)$ are the ideal generators. The divisibility conditions $\delta(y) \mid \gamma(y) \mid \alpha(y)$ and $\delta(y) \mid \beta(y) \mid \alpha(y)$ force $pub(y) = l(y)pu\delta(y)$, proving equality. For the weight equality, observe that for any $l(y) = l_0(y) + ul_1(y) + pl_2(y) + pul_3(y) \in \mathfrak{J}_c$, where $l_i(y) \in R_{p,n}$, the projection $pul(y) = pul_0(y)$ satisfies $w_H(pul(y)) \leq w_H(l(y))$. Since $puC = \langle pu\delta(y) \rangle$ is a minimal subcode, the weight equality follows. □

Lemma 1. *Let $\mathfrak{J}_c = \langle a(y) \rangle$ be a cyclic code over $R_{p^2,u}$ of length p^s generated by*

$$a(y) = (y^{p^{s-1}} - 1)^e h(y), \quad 1 \leq e < p.$$

If $C_h = \langle h(y) \rangle$ is a code of length p^{s-1}. Then,

$$d(\mathcal{C}) = (e+1)d(C_h). \tag{31}$$

Proof. For $c \in \mathcal{C}$, we have $c = (y^{p^{s-1}} - 1)^e h(y)l(y)$, $l(y) \in R_{p^2,u,p,p^s}$. As $C_h = \langle h(y) \rangle$ is of length p^{s-1}, thus

$$\begin{aligned} w(c) &= w((y^{p^{s-1}} - 1)^e h(y)l(y)) \\ &= w(y^{p^{s-1}e} h(y)l(y)) + w(b\mathfrak{J}_1 y^{p^{s-1}(e-1)} h(y)l(y)) \\ &\quad + \cdots + w(b\mathfrak{J}_{e-1} y^{p^{s-1}} h(y)l(y)) + w(h(y)l(y)). \end{aligned}$$

Thus, $d(\mathcal{C}) = (e+1)d(C_h)$. □

In the following theorem, we present our main result in this section. We rely on concepts introduced in Definition 3.

Remark 2. *Note that to clarify the statement of Theorem 1, let the original code C have minimum Lee distance d_L. Its Gray image $\varphi_L(\mathcal{C})$ is then a p-ary linear code whose minimum Hamming distance is $d_L = d_H$ by Equation (7). Therefore, the parameter d in the resulting code's parameters $[4n, k, d]$ unambiguously refers to the Hamming distance.*

Theorem 8. *Assume $\mathcal{C}$ is a cyclic code of length p^s over $R_{p^2,u,p}$ with $\mathfrak{J}_c$ as in Construction (A), where $\alpha(y) = (y-1)^\iota$, $\beta(y) = (y-1)^{\iota_1}$, $\gamma(y) = (y-1)^{\iota_2}$ and $\delta(y) = (y-1)^\delta$ with $\iota > \iota_1 > \delta > 0$ and $\iota > \iota_2 > \delta > 0$. Suppose $\delta = \sum_{i=0}^{s-1} e_i p^i$ and $\delta(y) = \prod_{i=0}^{s-1}(y^{p^i} - 1)^{e_i}$. Then,*

$$d(\mathcal{C}) = \begin{cases} 2, & \text{if } \delta \leq p^{s-1}, \\ \prod_{i=s-k}^{s-1}(e_i+1), & \text{if } \delta > p^{s-1} \text{ with zero expansion or full expansion } (s=k), \\ 2\prod_{i=s-k}^{s-1}(e_i+1), & \text{if } \delta > p^{s-1} \text{ with non-zero expansion.} \end{cases} \tag{32}$$

Proof. By Theorem 7, $d(\mathcal{C}) = d(C_{pu}) = d(\langle (y-1)^\delta \rangle)$, so we need only determine $d(C_{pu})$.

(1) For $\delta \leq p^{s-1}$, we have

$$(y-1)^\delta (y-1)^{p^{s-1} - \iota\delta} = (y-1)^{p^{s-1}} = (y^{p^{s-1}} - 1) \in C_{pu} \Rightarrow d(\mathcal{C}) = 2.$$

(2) For $\delta > p^{s-1}$,

(a) For p-adic length k zero expansion,

$$\delta = \sum_{i=s-k}^{s-1} e_i p^i, \quad \delta(y) = \prod_{i=s-k}^{s-1} (y^{p^i} - 1)^{e_i}$$
$$h(y) = (y^{p^{s-k}} - 1)^{e_{s-k}} \quad \text{(generates code with } d(\mathcal{C}) = e_{s-k} + 1)$$
$$d(\mathcal{C}) = \prod_{i=s-k}^{s-1} (e_i + 1) \quad \text{(by Lemma 1 and induction).}$$

(b) For nonzero expansion,

$$\delta = \sum_{i=0}^{s-1} e_i p^i \quad (e_{s-k-1} = 0), \quad r = \sum_{i=0}^{s-k-2} e_i p^i$$
$$h(y) = \prod_{i=0}^{s-k-2} (y^{p^i} - 1)^{e_i},$$
$$(y-1)^{p^{s-k-1}-j} h(y) = (y^{p^{s-k-1}} - 1) \in \mathcal{C} \Rightarrow d = 2$$
$$d(\mathcal{C}) = 2 \prod_{i=s-k}^{s-1} (e_i + 1) \quad \text{(by Lemma 1 and induction).}$$

□

Remark 3. *In the context of cyclic codes over a chain ring, it is traditionally anticipated that the minimum Hamming distance is bounded above by* $n - \text{Rank}(\mathcal{C}) + 1$ *[26].*

Example 1. *Fix* $p = 3$ *and* $s = 3$. *Consider the rings.*

$$R_{3^2,u} = \mathbb{Z}_9 = u\mathbb{Z}_9, \quad u^2 = 0, \quad R_{3^2,u,n} = \frac{R_{3^2,u}[y]}{\langle y^{27} - 1 \rangle}. \tag{33}$$

Then, this gives cyclic codes of length 27. *The* 3*-adic expansion for* $s = 3$ *is*

$$\delta = e_2 \cdot 3^2 + e_1 \cdot 3 + e_0,$$

allowing k *values of* 3, 1, *and* 2. *Table 1 demonstrates all cases of the theorem including small* δ*, full expansion and nonzero expansion with different* k *values:*

Table 1. Minimum distance results for cyclic codes of length 27 over $R_{3^2,u}$

Cyclic Codes $\mathcal{C}$	δ	(e_0, e_1, e_2)	k	$d(\mathcal{C})$
$\langle (y-1)^{10}, u(y-1)^8, 3(y-1)^7, 3u(y-1)^5 \rangle$	2	(2, 0, 0)	1	2
$\langle (y-1)^{15}, u(y-1)^{12}, 3(y-1)^{10}, 3u(y-1)^8 \rangle$	7	(1, 2, 0)	2	6
$\langle (y-1)^{20}, u(y-1)^{16}, 3(y-1)^{15}, 3u(y-1)^{12} \rangle$	10	(1, 0, 1)	1	4
$\langle (y-1)^{20}, u(y-1)^{16}, 3(y-1)^{15}, 3u(y-1)^{12} \rangle$	12	(0, 1, 1)	2	8
$\langle (y-1)^{20}, u(y-1)^{16}, 3(y-1)^{15}, 3u(y-1)^{12} \rangle$	13	(1, 1, 1)	3	8
$\langle (y-1)^{22}, u(y-1)^{18}, 3(y-1)^{17}, 3u(y-1)^{14} \rangle$	15	(0, 2, 1)	2	12
$\langle (y-1)^{22}, u(y-1)^{18}, 3(y-1)^{17}, 3u(y-1)^{14} \rangle$	16	(1, 2, 1)	3	12
$\langle (y-1)^{24}, u(y-1)^{20}, 3(y-1)^{19}, 3u(y-1)^{16} \rangle$	18	(0, 0, 2)	1	6

Table 1. *Cont.*

Cyclic Codes $\mathcal{C}$	δ	(e_0, e_1, e_2)	k	$d(\mathcal{C})$
$\langle (y-1)^{26}, u(y-1)^{22}, 3(y-1)^{21}, 3u(y-1)^{18} \rangle$	19	$(1,0,2)$	1	6
$\langle (y-1)^{25}, u(y-1)^{21}, 3(y-1)^{20}, 3u(y-1)^{17} \rangle$	21	$(0,1,2)$	2	12

This comprehensive example shows all possible cases:

1. Small δ: *The values of* $\delta = 2, 7 \leq 3^2 = 9$ *with* $d(\mathcal{C}) = 2$.
2. Full expansion: *In this case,* $\delta = 13, 16$ *with* $k = 3$ *(all numbers* $e_2, e_1, e_0 \neq 0$*) and*

$$d(\mathcal{C}) = \prod_{i=0}^{2} (e_i + 1).$$

3. Zero expansion with $k = 1, 2$: *We have* $\delta = 12, 15, 18, 21$, *where*

$$d(\mathcal{C}) = \prod_{i=0}^{2} (e_i + 1).$$

4. Non-zero expansion expansion with $k = 1$: *From the table, we notice that* $\delta = 10, 19$ *where* $e_0 = e_1 = 0$ *but* $e_2 \neq 0$ *and*

$$d(\mathcal{C}) = 2 \prod_{i=0}^{2} (e_i + 1).$$

Example 2. *This example examines codes of length* $n = 3$ *over* $R_{3^2,u}$ *with*

$$\alpha(y) = y - 1, \quad z_i \in \mathbb{F}_3.$$

Notably, the equation $y^3 - 1 - 3y\alpha(y) = (y-1)^3$ *is equivalent to* $\alpha(y)^6 = (y-1)^6 = 0$ *in* $R_{3^2,u,3}$. *As a result, the generator polynomials of* $\mathcal{C}$ *are limited to powers of* 5. *The generator polynomials, ranks and minimal distances of a selection of cyclic codes over* $R_{3^2,u}$ *are presented in Table 2. Magma computation system and Gray images of cyclic codes over* $R_{3,u}$ *are used to generate ternary codes, as illustrated in Tables 2 and 3. The optimal and the best-known ternary codes are denoted by* $[\cdot]^o$ *and* $[\cdot]^b$ *in these tables, respectively. It is evident from Table 3 that all ternary optimal codes have been attained, with the exception of of length* 12.

Table 2. Complete rank–distance categorization of cyclic codes of length $n = 3$ over $R_{3^2,u}$.

Generator Structure	Rank (k_n)	Distance (d)
Rank 1, Distance 3 Codes		
$\langle 3\alpha(y)^2 + 3uz_0\alpha(y) \rangle$		
$\langle \alpha(y)^2 + uz_0\alpha(y) + 3z_1\alpha(y) + 3uz_2\alpha(y) \rangle$		
$\langle u\alpha(y)^2 + 3z_0\alpha(y)^2 + 3uz_1\alpha(y) \rangle$	1	3
$\langle 3u\alpha(y)^2 \rangle$		
$\langle u\alpha(y)^2 + 3uz_0\alpha(y), 3\alpha(y)^2 + 3uz_1\alpha(y) \rangle$		
Rank 2, Distance 3 Codes		
$\langle u\alpha(y)^2 + 3uz_0\alpha(y), 3\alpha(y)^2 + 3uz_1\alpha(y) \rangle$	2	3

Table 2. *Cont.*

Generator Structure	Rank (k_n)	Distance (d)
Rank 2, Distance 2 Codes		
$\langle u\alpha(y)^2 + 3z_0\alpha(y)^2 + 3uz_1, 3u\alpha(y)\rangle$		
$\langle \alpha(y) + uz_0 + 3z_1 + 3uz_2\rangle$		
$\langle \alpha(y)^2 + uz_0\alpha(y) + 3z_1 + 3uz_2, 3\alpha(y) + 3uz_3\rangle$		
$\langle 3\alpha(y)^2 + 3uz_0, 3u\alpha(y)\rangle$	2	2
$\langle \alpha(y)^2 + uz_0\alpha(y) + 3z_1, 3u\alpha(y)\rangle$		
$\langle 3u\alpha(y)\rangle$		
$\langle 3\alpha(y) + 3uz_0\rangle$		
Rank 3, Distance 2 Codes		
$\langle \alpha(y)^2 + uz_0\alpha(y) + 3z_1\alpha(y), 3u\rangle$		
$\langle 3\alpha(y)^2, 3u\rangle$	3	2
Rank 3, Distance 2 Codes		
$\langle u\alpha(y) + 3z_0\alpha(y), 3u\rangle$		
$\langle u\alpha(y)^2 + 3z_0\alpha(y) + 3uz_1, 3\alpha(y)^2 + 3uz_2, 3u\alpha(y)\rangle$		
$\langle u\alpha(y)^2 + 3uz_0, 3\alpha(y) + 3uz_1\rangle$		
$\langle \alpha(y)^2 + 3uz_0, u\alpha(y) + 3uz_1, 3\alpha(y) + 3uz_2\rangle$		
$\langle \alpha(y)^2 + uz_0 + 3z_1\alpha(y) + 3uz_2, u\alpha(y) + 3z_3\alpha(y) + 3uz_4\rangle$		
$\langle u\alpha(y) + 3uz_0, 3\alpha(y) + 3uz_1\rangle$		
$\langle u\alpha(y)^2 + 3z_0\alpha(y) + 3uz_1, 3\alpha(y)^2 + 3uz_2\rangle$		
$\langle u\alpha(y) + 3z_0\alpha(y), 3\alpha(y), 3u\rangle$		
$\langle u\alpha(y), 3\rangle$	3	2
$\langle 3\alpha(y), 3u\rangle$		
$\langle 3\rangle$		
$\langle u + 3z_0, 3\alpha(y)\rangle$		
$\langle \alpha(y)^2 + 3z_0, u + 3z_1, 3\alpha(y)\rangle$		
$\langle \alpha(y) + 3z_0, u + 3z_1\rangle$		
$\langle \alpha(y)^2 + uz_0\alpha(y) + 3z_1, 3\alpha(y), 3u\rangle$		
Rank 4, Distance 1 Codes		
$\langle \alpha(y)^2, u, 3\rangle$		
$\langle u\alpha(y) + 3(z_0(\alpha(y) - 1) + z_1), 3\alpha(y)^2, 3u\rangle$		
$\langle u\alpha(y)^2 + 3z_0, 3\alpha(y), 3u\rangle$	4	1
$\langle \alpha(y) + uz_0 + 3z_1, 3u\rangle$		
$\langle \alpha(y) + uz_0, 3\rangle$		

Table 2. *Cont.*

Generator Structure	Rank (k_n)	Distance (d)
Rank 5, Distance 1 Codes		
$\langle 1 \rangle$		
$\langle u\alpha(y)^2, 3 \rangle$		
$\langle u\alpha(y) + 3z_0\alpha(y) + 3uz_1 \rangle$	5	1
$\langle \alpha(y)^2 + uz_0\alpha(y), 3 \rangle$		
$\langle \alpha(y), u, 3 \rangle$		
Rank 6, Distance 1 Codes		
$\langle \alpha(y)^2 + uz_0 + 3(z_1\alpha(y) + z_0z_2), u\alpha(y) + 3z_2\alpha(y), 3u \rangle$		
$\langle u + 3(z_0(\alpha(y) - 1)^2 + z_1(\alpha(y) - 1) + z_2) \rangle$		
$\langle 3u \rangle$		
$\langle u + 3(z_0(\alpha(y) - 1) + z_1), 3\alpha(y)^2 \rangle$	6	1
$\langle \alpha(y)^2 + 3z_0\alpha(y), u + 3(z_1(\alpha(y) - 1) + z_2), 3u \rangle$		
$\langle \alpha(y)^2 + uz_0, u\alpha(y), 3 \rangle$		
$\langle u, 3 \rangle$		

Table 3. Ternary representations of codes with $n = 3^2$ over $R_{3^2,u}$.

Codes	$[k_n, d]$
$\langle u\alpha(y)^5 + 3\alpha(y)^5 + 3u\alpha(y), 3\alpha(y)^8 + 3u\alpha(y)^4 \rangle$	[14, 12]
$\langle \alpha(y)^8 + u\alpha(y)^4 + 3\alpha(y)^5 + 3u\alpha(y)^2,\ u\alpha(y)^5 + 3\alpha(y)^6 + 3u\alpha(y)^3,\ 3\alpha(y)^7 + 3u\alpha(y)^3 \rangle$	[15, 12]^b
$\langle 3\alpha(y)^8 + 6u\alpha(y)^7 \rangle$	[2, 24]
$\langle \alpha(y), u + 3 \rangle$	[35, 2]^o
$\langle u\alpha(y)^7 + 3\alpha(y)^7 + 3u\alpha(y)^5 \rangle$	[6, 21]^o
$\langle \alpha(y)^8 + u\alpha(y)^3 + 3\alpha(y)^4 + 3u\alpha(y)^2,\ u\alpha(y)^4 + 3\alpha(y)^5 + 3u\alpha(y)^3,\ 3\alpha(y)^7 + 3u\alpha(y)^2 \rangle$	[18, 11]
$\langle u\alpha(y)^8 + 3\alpha(y)^8 + 6u\alpha(y)^7 \rangle$	[3, 24]^o
$\langle u\alpha(y)^8 + 3\alpha(y)^7 + 6uy\alpha(y)^5,\ 3\alpha(y)^8 + 6u\alpha(y)^6 \rangle$	[5, 18]
$\langle \alpha(y)^8 + u\alpha(y)^6 + 3\alpha(y)^6 + 6uy\alpha(y)^4,\ u\alpha(y)^7 + 3\alpha(y)^7 + 6uy\alpha(y)^5 \rangle$	[7, 18]
$\langle 3u\alpha(y)^8 \rangle$	[1, 36]^o
$\langle \alpha(y)^8 + u\alpha(y)^4 + 3\alpha(y)^5 + 3u,\ u\alpha(y)^5 + 3\alpha(y)^6 + 3u\alpha(y),\ 3\alpha(y)^7 + 3u\alpha(y)^3,\ 3u\alpha(y)^4 \rangle$	[16, 12]^b
$\langle u\alpha(y)^5 + 3\alpha(y)^5 + 3u\alpha(y) \rangle$	[12, 12]
$\langle u\alpha(y)^7 + 3\alpha(y)^7 + 6uy\alpha(y)^5,\ 3\alpha(y)^8 + 3u\alpha(y)^6 \rangle$	[8, 18]^b
$\langle \alpha(y), u \rangle$	[34, 2]^o
$\langle u\alpha(y)^8 + 3\alpha(y)^8 + 3u\alpha(y)^6,\ 3u\alpha(y)^7 \rangle$	[4, 21]

Table 3. *Cont.*

Codes	$[k_n, d]$
$\langle u\alpha(y)^5 + 3\alpha(y)^5 + 3u\alpha(y),\ 3u\alpha(y)^4 \rangle$	[13, 12]
$\langle \alpha(y), 3u \rangle$	$[33, 2]^o$

Table 2 presents the rank-distance distribution of cyclic codes of length $n = 3$, indicating that a diverse range of parameters can be attained, including codes with maximum distance for specified rank. Table 3 illustrates the robustness of our framework: The Gray images produce multiple optimal and well-established ternary codes, demonstrating that cyclic codes over $R_{3^2,u}$ serve as a valuable source for high-quality constructions. These results illustrate the structural complexity of the codes and their capacity to generate competitive linear codes over finite fields, indicating the need for further investigation into larger block lengths and alternative ring extensions.

6. Conclusions

This study demonstrates that the ring $R_{p^2,u} = \mathbb{Z}_{p^2} + u\mathbb{Z}_{p^2},\ u^2 = 0,\ pu \neq 0$, is local but not a chain ring and investigates cyclic codes over this ring and their fundamental parameters such as ranks and Hamming distances. Additionally, an analysis of Tables 2 and 3 reveals that when the code length is not coprime to p, the classical bound does not typically apply. This demonstrates a structural phenomenon in which the algebraic properties of the underlying ring directly affect the precision of distance bounds for cyclic codes. The methods we employ extend to arbitrary Galois extensions of $\mathbb{Z}_{p^2}$. However, the validity of these bounds over local rings is still uncertain. Future research should focus on defining the specific conditions under which distance–rank bounds are applicable to codes over non-principal local rings, or on creating new analogues that more accurately reflect their structural characteristics.

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References

1. Bonnecaze, A.; Udaya, P. Cyclic codes and self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$. *IEEE Trans. Inf. Theory* **1999**, *45*, 1250–1255. [CrossRef]
2. Calderbank, A.R.; Sloane, N.J.A. Modular and p-adic cyclic codes. *Des. Codes Cryptogr.* **1995**, *6*, 21–35. [CrossRef]
3. Abualrub, T.; Ghrayeb, A.; Oehmke, R.H. A mass formula and rank of $\mathbb{Z}_4$ cyclic codes of length 2^e. *IEEE Trans. Inform. Theory* **2004**, *50*, 3306–3312. [CrossRef]
4. Abualrub, T.; Oehmke, R. On the generators of $\mathbb{Z}_4$ cyclic codes of length 2^e. *IEEE Trans. Inform. Theory* **2003**, *49*, 2126–2133. [CrossRef]
5. Blackford, T. Cyclic codes over $\mathbb{Z}_4$ of oddly even length. *Discret. Appl. Math.* **2003**, *128*, 27–46. [CrossRef]
6. Conway, J.H.; Sloane, N.J.A. Self-dual codes over the integers modulo 4. *J. Comb. Theory Ser. A* **1993**, *62*, 30–45. [CrossRef]
7. Pless, V.S.; Qian, Z. Cyclic codes and quadratic residue codes over $\mathbb{Z}_4$. *IEEE Trans. Inform. Theory* **1996**, *42*, 1594–1600. [CrossRef]
8. Van Lint, J.H. Repeated-root cyclic codes. *IEEE Trans. Inform. Theory* **1991**, *37*, 343–345. [CrossRef]
9. Dinh, H.Q.; Lopez-Permouth, S.R. Cyclic and negacyclic codes over finite chain rings. *IEEE Trans. Inform. Theory* **2004**, *50*, 1728–1744. [CrossRef]

10. Abualrub, T.; Siap, I. Cyclic codes over the rings $\mathbb{Z}_2 + u\mathbb{Z}_2$ and $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$. *Des. Codes Cryptogr.* **2007**, *42*, 273–287. [CrossRef]
11. Al-Ashker, M.; Hamoudeh, M. Cyclic codes over $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2 + \cdots + u^{k-1}\mathbb{Z}_2$. *Turkish J. Math.* **2011**, *35*, 737–749. [CrossRef]
12. Dinh, H.Q. Constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. *J. Algebra* **2010**, *324*, 940–950. [CrossRef]
13. Singh, A.K.; Kewat, P.K. On cyclic codes over the ring $\frac{\mathbb{F}_q[u]}{\langle u^k \rangle}$. *Des. Codes Cryptogr.* **2015**, *74*, 1–13. [CrossRef]
14. Yildiz, B.; Karadeniz, S. Cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$. *Des. Codes Cryptogr.* **2011**, *58*, 221–234. [CrossRef]
15. Dougherty, S.T.; Karadeniz, S.; Yildiz, B. Cyclic codes over R_k. *Des. Codes Cryptogr.* **2012**, *63*, 113–126. [CrossRef]
16. Sobhani, R.; Molakarimi, M. Some results on cyclic codes over the ring $R_{2,m}$. *Turkish J. Math.* **2013**, *37*, 1061–1074. [CrossRef]
17. Kewat, P.K.; Ghosh, B.; Pattanayak, S. Cyclic codes over the ring $\frac{\mathbb{Z}_p[u,v]}{\langle u^2, v^2, uv - vu \rangle}$. *Finite Fields Appl.* **2015**, *34*, 161–175. [CrossRef]
18. Alhomaidhi, A.A.; Alabiad, S. Homogeneous weights of linear codes over $\mathbb{F}_q[u,v]$ of order p^{4m}. *Symmetry* **2025**, *17*, 440. [CrossRef]
19. Alabiad, S.; Alhomaidhi, A.A.; Alsarori, N.A. MacWilliams identities and generator matrices for linear codes over $\frac{\mathbb{Z}_{p^4}[u]}{\langle u^2 - p^3\alpha, pu \rangle}$. *Axioms* **2024**, *13*, 552. [CrossRef]
20. Calderbank, A.R.; Rains, E.M.; Shor, P.W.; Sloane, N.J.A. Quantum error correction via codes over GF(4). *IEEE Trans. Inform. Theory* **1998**, *44*, 1369–1387. [CrossRef]
21. Yuan, J.; Ding, C. Secret sharing schemes from three classes of linear codes. *IEEE Trans. Inf. Theory* **2006**, *52*, 206–212. [CrossRef]
22. Parakh, A.; Kak, S. Space efficient secret sharing for implicit data security. *Inf. Sci.* **2011**, *181*, 335–341. [CrossRef]
23. Alabiad, S.; Alhomaidhi, A.A.; Alsarori, N.A. On linear codes over finite singleton local rings. *Mathematics* **2024**, *12*, 1099. [CrossRef]
24. Alabiad, S.; Alkhamees, Y. Constacyclic codes over finite chain rings of characteristic p. *Axioms* **2021**, *10*, 303. [CrossRef]
25. Dougherty, S.T.; Shiromoto, K. Maximum distance codes over rings of order 4. *IEEE Trans. Inform. Theory* **2001**, *47*, 400–404. [CrossRef]
26. Horimoto, H.; Shiromoto, K.A. A Singleton Bound for Linear Codes over Quasi-Frobenius Rings. In Proceedings of the 13th International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes, Hawaii, HI, USA, 15–19 November 1999; pp. 51–52.

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