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Closed-Form Solutions of a Nonlinear Rational Second-Order Three-Dimensional System of Difference Equations

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Abstract

In this paper, we investigate the behavior of solutions to a nonlinear system of rational difference equations of order two, defined by $x_{n+1} = \frac{x_n y_{n-1}}{y_n(a+bx_n y_{n-1})}, y_{n+1} = \frac{y_n z_{n-1}}{z_n(c+dy_n z_{n-1})}$, $z_{n+1} = \frac{z_n x_{n-1}}{x_n(e+fz_n x_{n-1})}$, where n denotes a nonzero integer; the parameters a, b, c, d, e, f are real constants; and the initial conditions $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0$ are nonzero real numbers. By applying a suitable variable transformation, we reduce the original coupled system to three independent rational difference equations. This allows for separate analysis using established methods for second-order nonlinear recurrence relations. We derive explicit solutions and examine the qualitative behavior, including boundedness and periodicity, under different conditions. Our findings contribute to the theory of rational difference equations and offer insights for higher-order systems in applied sciences.

Keywords: system of difference equations; solution; nonlinear systems; rational difference equations

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1. Introduction

Difference equations play a fundamental role in modeling discrete-time dynamical systems where the current state depends on one or more past states. In particular, nonlinear systems of difference equations where the state at time $k+1$ is determined by nonlinear functions of the previous k states are essential for understanding the behavior of complex systems across various scientific disciplines. These include biology, ecology, physics, and economics, where such models are often used to describe population dynamics, economic cycles, and physical processes [1–4]. Difference equations are widely applied across diverse fields. In economics, they model the evolution of key indicators such as GDP and inflation rates, which are naturally observed at discrete time

intervals. In biology, they capture population dynamics, where nonlinear difference equations are frequently employed to describe growth processes, as highlighted by Elaydi [5]. Leinartas [6] introduced a systematic framework for understanding the fundamental solutions of multidimensional homogeneous linear difference equations with constant coefficients, employing multiple Laurent series as a powerful tool for solving the associated Cauchy problem. Linear recurrences with constant coefficients also form a cornerstone of discrete mathematics, particularly in the study of sequences and generating functions. Building on this foundation, Bousquet-Mélou and Petkovšek [7] extended the theory to the multivariate case, offering a unified framework for analyzing such recurrences with important implications in combinatorics, computer science, and mathematical education.

In recent years, there has been a growing interest in the theoretical analysis of nonlinear difference systems, with particular attention being paid to qualitative properties such as boundedness, periodicity, and global asymptotic stability [8–18]. These properties are fundamental for predicting long-term behavior and ensuring the robustness of models in practical applications.

Chuiko et al. [19] established constructive conditions for the solvability of linear boundary-value problems in difference-algebraic systems. Their work introduced a unified framework for constructing solutions and a classification scheme that elucidates the structural properties of such systems, with implications for engineering and physics. Stević et al. [20] studied a class of generalized hyperbolic–cotangent-type difference equations, showing that solvability is determined by algebraic rather than trigonometric relations, thereby simplifying the solution process and extending applicability. Research on higher-order difference equations has also progressed; for example, Mkhwanazi et al. [21] employed Lie symmetry analysis to derive nontrivial symmetries in higher-order systems, enabling exact solutions.

When analytical approaches are not feasible, numerical methods provide effective alternatives. Xing et al. [22] developed a fourth-order conservative difference scheme for Riesz space-fractional sine-Gordon equations, demonstrating both solvability and convergence, and highlighting the strength of computational approaches. Stability, a central aspect of solvability, was addressed by Alzabut et al. [23], who derived solvability and stability conditions for nonlinear hybrid Δ-difference equations of fractional order, emphasizing their practical significance. In control theory, Wang et al. [24] explored linear quadratic optimal control for time-delay stochastic systems, establishing solvability through Riccati equations. Their results illustrate the strong interplay between solvability theory and system optimization.

Numerous studies have examined specific nonlinear systems of difference equations. For example, Alayachi et al. [25] analyzed local and global attractivity, periodicity, and solutions of a sixth-order difference equation and provided several numerical examples to illustrate their findings. In another study, Sanbo and Elsayed [26] investigated the periodicity, stability, and particular solutions of a fifth-order recursive equation. Almatrafi and Alzubaidi [27] explored the dynamical behavior of an eighth-order difference relation, presenting two-dimensional graphical representations of their results. Furthermore, Ahmed et al. [28] introduced new solutions and performed a dynamical analysis of certain nonlinear fifteenth-order difference equations. Kurbanli et al. [17] studied the dynamics of positive solutions for the system:

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}}{y_n x_{n-1} + 1}, \\ y_{n+1} &= \frac{y_{n-1}}{x_n y_{n-1} + 1}, \quad n \text{ denotes a nonzero integer,} \end{aligned} \tag{1}$$

under the initial conditions $x_0, x_{-1}, y_0, y_{-1} \in [0, +\infty)$.

Kara et al. [15] studied the solvability and asymptotic behavior of the system

$$\begin{aligned} x_n &= \frac{x_{n-k}y_{n-k-l}}{y_{n-l}(a_n + b_n x_{n-k}y_{n-k-l})}, \\ y_n &= \frac{y_{n-k}x_{n-k-l}}{x_{n-l}(\alpha_n + \beta_n y_{n-k}x_{n-k-l})}, \quad n \text{ denotes a nonzero integer,} \end{aligned} \tag{2}$$

where $k, l \in \mathbb{N}$ and $(a_n), (b_n), (\alpha_n), (\beta_n)$ are sequences of real numbers.

Halim et al. [13] derived the general solution for the system

$$\begin{aligned} x_{n+1} &= \frac{y_{n-1}x_{n-2}}{y_n(a + by_{n-1}y_{n-2})}, \\ y_{n+1} &= \frac{x_{n-1}y_{n-2}}{x_n(a + bx_{n-1}y_{n-2})}, \quad n \text{ denotes a nonzero integer,} \end{aligned} \tag{3}$$

where $a, b \in \mathbb{R}$ and the initial conditions are real numbers.

El-Dessoky et al. [10] investigated the system

$$\begin{aligned} x_{n+1} &= \frac{x_{n-3}y_{n-4}}{y_n(\pm 1 \pm x_{n-3}y_{n-4})}, \\ y_{n+1} &= \frac{y_{n-3}x_{n-4}}{x_n(\pm 1 \pm y_{n-3}x_{n-4})}, \quad n \text{ denotes a nonzero integer,} \end{aligned} \tag{4}$$

where the initial conditions are nonzero real numbers.

Stević et al. [18] examined the equation

$$x_n = \frac{x_{n-2}x_{n-k-2}}{x_{n-k}(a_n + b_n x_{n-2}x_{n-k-2})}, \quad n \text{ denotes a nonzero integer,} \tag{5}$$

where $k \in \mathbb{N}$, the initial conditions are nonzero real numbers, and $(a_n), (b_n)$ are sequences of real numbers.

Elsayed et al. [11] explored the periodicity and structure of solutions of the system

$$\begin{aligned} x_{n+1} &= \frac{x_{n-2}y_{n-1}}{y_n(\pm 1 \pm x_{n-2}y_{n-1})}, \\ y_{n+1} &= \frac{y_{n-2}x_{n-1}}{x_n(\pm 1 \pm y_{n-2}x_{n-1})}, \quad n \text{ denotes a nonzero integer,} \end{aligned} \tag{6}$$

where the initial conditions are nonzero real numbers.

Kara and Yazlik [16] obtained the solution of the system

$$\begin{aligned} x_n &= \frac{x_{n-4}y_{n-5}}{y_{n-1}(a_n + b_n x_{n-2}y_{n-3}x_{n-4}y_{n-5})}, \\ y_n &= \frac{y_{n-4}x_{n-5}}{x_{n-1}(\alpha_n + \beta_n y_{n-2}x_{n-3}y_{n-4}x_{n-5})}, \quad n \text{ denotes a nonzero integer,} \end{aligned} \tag{7}$$

under nonzero real initial conditions.

This paper provides closed-form solutions for a class of difference equations, extending a system of such equations. It investigates their qualitative behavior, contributing to the understanding of difference equations in scientific applications. The main goal is to present closed-form solutions and explore their dynamics:

$$x_{n+1} = \frac{x_n y_{n-1}}{y_n(a + bx_n y_{n-1})}, \quad y_{n+1} = \frac{y_n z_{n-1}}{z_n(c + dy_n z_{n-1})}, \quad z_{n+1} = \frac{z_n x_{n-1}}{x_n(e + fz_n x_{n-1})}.$$

The study provides analytical expressions for the solutions and analyzes their qualitative behavior.

2. Main Results

In this section, we derive the solutions of the following nonlinear system of difference equations:

$$\begin{aligned} x_{n+1} &= \frac{x_n y_{n-1}}{y_n(a + b x_n y_{n-1})}, \\ y_{n+1} &= \frac{y_n z_{n-1}}{z_n(c + d y_n z_{n-1})}, \\ z_{n+1} &= \frac{z_n x_{n-1}}{x_n(e + f z_n x_{n-1})}, \end{aligned} \tag{8}$$

where n denotes a nonzero integer; the parameters a, b, c, d, e, f are real constants; and the initial conditions x_{-1}, x_0, y_{-1}, y_0, z_{-1}, and z_0 are nonzero real numbers.

To simplify the analysis, we introduce the following auxiliary variables:

$$u_n = x_n y_{n-1}, \quad v_n = y_n z_{n-1}, \quad w_n = z_n x_{n-1}, \quad n \text{ denotes a nonzero integer.} \tag{9}$$

With this change in variables, the original system (8) transforms into the following decoupled system:

$$u_{n+1} = \frac{u_{n-1}}{a + b u_n}, \; v_{n+1} = \frac{v_{n-1}}{c + d v_n}, \; w_{n+1} = \frac{w_{n-1}}{e + f w_n}, \; n \text{ denotes a nonzero integer.} \tag{10}$$

This transformation enables independent analysis of each equation in (10). In the subsequent analysis, we focus on solving each of these recursive relations, describing the behavior of their solutions, and reconstructing the original sequences $(x_n), (y_n), (z_n)$ using the inverse of the transformation (9).

Solutions of $x_{n+1} = \dfrac{x_n}{\alpha + \beta x_n}$

In this section, we derive the closed-form solution of the difference equation

$$x_{n+1} = \frac{x_n}{\alpha + \beta x_n}, \quad n \text{ denotes a nonzero integer.} \tag{11}$$

This equation can be transformed into a simpler form using by changing the variables. First, we define a new sequence w_n such that

$$x_n = \frac{1}{\beta}(w_n - \alpha), \quad n \text{ denotes a nonzero integer.} \tag{12}$$

Substituting this transformation into the original difference Equation (11) leads to the following equation for w_n:

$$w_{n+1} = \frac{(\alpha + 1)w_n - \alpha}{w_n}, \quad n \text{ denotes a nonzero integer.} \tag{13}$$

Now, we consider Equation (13) with the initial condition $w_0 \neq 0$.

To simplify this equation further, we introduce the transformation

$$w_n = \frac{t_n}{t_{n-1}}, \quad n \text{ denotes a nonzero integer,} \tag{14}$$

where t_{-1}, t_0 are nonzero real numbers. This transforms Equation (13) into the following recurrence relation for t_n:

$$t_{n+1} - (\alpha + 1)t_n + \alpha t_{n-1} = 0, \quad n \text{ denotes a nonzero integer.} \tag{15}$$

The characteristic equation associated with Equation (15) is

$$\lambda^2 - (1+\alpha)\lambda + \alpha = 0.$$

Here, we state the following lemmas without proof.

Lemma 1. *The solution of Equation (15) can be expressed as follows:*

1. *If* $\alpha \neq 1$, *then*

$$t_n = \frac{1}{1-\alpha}\left(t_0\left(1-\alpha^{n+1}\right) - \alpha t_{-1}(1-\alpha^n)\right), \quad n \text{ denotes a nonzero integer.} \tag{16}$$

2. *If* $\alpha = 1$, *then*

$$t_n = t_0(n+1) - t_{-1}n, \quad n \text{ denotes a nonzero integer.} \tag{17}$$

Lemma 2. *Let* $\{w_n\}_{n\geq -1}$ *be a solution to Equation (13). The following statements are true:*

1. *If* $\alpha \neq 1$, *then*

$$w_n = \frac{\alpha(1-\alpha^n) - w_0(1-\alpha^{n+1})}{\alpha(1-\alpha^{n-1}) - w_0(1-\alpha^n)}, \quad n \text{ denotes a nonzero integer.}$$

2. *If* $\alpha = 1$, *then*

$$w_n = \frac{n - w_0(n+1)}{(n-1) - w_0 n}, \quad n \text{ denotes a nonzero integer.}$$

Using the above arguments, we obtain the following results:

Theorem 1. *Let* $\{x_n\}_{n\geq -1}$ *be an admissible solution of Equation (11). The following statements are true:*

1. *If* $\alpha \neq 1$, *then*

$$x_n = \frac{x_0}{\alpha^n + \beta\left(\sum_{i=0}^{n-1}\alpha^i\right)x_0}, \quad n \text{ denotes a nonzero integer.}$$

2. *If* $\alpha = 1$, *then*

$$x_n = \frac{x_0}{1+\beta n x_0}, \quad n \text{ denotes a nonzero integer.}$$

Theorem 2. *Let* $\{u_n, v_n, w_n\}_{n\geq -1}$ *be an admissible solution of System (10). The following statements are true:*

1. *If* $\alpha \neq 1$, *then*

$$u_n = \frac{u_0}{a^n + b\left(\sum_{i=0}^{n-1}a^i\right)u_0}, \quad v_n = \frac{v_0}{c^n + d\left(\sum_{i=0}^{n-1}c^i\right)v_0}, w_n = \frac{w_0}{e^n + f\left(\sum_{i=0}^{n-1}e^i\right)w_0},$$

where n denotes a nonzero integer.

2. *If* $\alpha = 1$, *then*

$$u_n = \frac{u_0}{1+bnu_0}, \quad v_n = \frac{v_0}{1+dnv_0}, \quad w_n = \frac{w_0}{1+fnw_0},$$

where n denotes a nonzero integer.

The main result in this section is the following:

Corollary 1. *Let* $\{x_n, y_n, z_n\}_{n\geq -1}$ *be an admissible solution of system (8). Then, the following statements are true:*

1. *If* $a \neq 1$, $c \neq 1$ *and* $e \neq 1$, *then the solution of system (8) satisfies*

$$x_{6n} = x_0 \quad \times$$
$$\times \prod_{i=0}^{n-1} \left(\frac{\left(c^{6i-1} + d\left(\sum_{r=0}^{6i-2} c^r\right) y_0 z_{-1}\right)\left(a^{6i-3} + b\left(\sum_{r=0}^{6i-4} a^r\right) x_0 y_{-1}\right)\left(e^{6i-5} + f\left(\sum_{r=0}^{6i-6} e^r\right) z_0 x_{-1}\right)}{\left(a^{6i} + b\left(\sum_{r=0}^{6i-1} a^r\right) x_0 y_{-1}\right)\left(e^{6i-2} + f\left(\sum_{r=0}^{6i-3} e^r\right) z_0 x_{-1}\right)\left(c^{6i-4} + d\left(\sum_{r=0}^{6i-5} c^r\right) y_0 z_{-1}\right)} \right),$$

$$x_{6n-1} = \frac{x_{-1}}{e^{6n} + f\left(\sum_{r=0}^{6n-1} e^r\right) z_0 x_{-1}} \quad \times$$
$$\times \prod_{i=0}^{n-1} \left(\frac{\left(e^{6i} + f\left(\sum_{r=0}^{6i-1} e^r\right) z_0 x_{-1}\right)\left(c^{6i-2} + d\left(\sum_{r=0}^{6i-3} c^r\right) y_0 z_{-1}\right)\left(a^{6i-4} + b\left(\sum_{r=0}^{6i-5} a^r\right) x_0 y_{-1}\right)}{\left(a^{6i-1} + b\left(\sum_{r=0}^{6i-2} a^r\right) x_0 y_{-1}\right)\left(e^{6i-3} + f\left(\sum_{r=0}^{6i-4} e^r\right) z_0 x_{-1}\right)\left(c^{6i-5} + d\left(\sum_{r=0}^{6i-6} c^r\right) y_0 z_{-1}\right)} \right),$$

$$x_{6n+1} = \frac{x_0 y_{-1}}{y_0\left(a^{6n+1} + b\left(\sum_{r=0}^{6n} a^r\right) x_0 y_{-1}\right)} \quad \times$$
$$\times \prod_{i=0}^{n-1} \left(\frac{\left(c^{6i} + d\left(\sum_{r=0}^{6i-1} c^r\right) y_0 z_{-1}\right)\left(a^{6i-2} + b\left(\sum_{r=0}^{6i-3} a^r\right) x_0 y_{-1}\right)\left(e^{6i-4} + f\left(\sum_{r=0}^{6i-5} e^r\right) z_0 x_{-1}\right)}{\left(e^{6i-1} + f\left(\sum_{r=0}^{6i-2} e^r\right) z_0 x_{-1}\right)\left(c^{6i-3} + d\left(\sum_{r=0}^{6i-4} c^r\right) y_0 z_{-1}\right)\left(a^{6i-5} + b\left(\sum_{r=0}^{6i-6} a^r\right) x_0 y_{-1}\right)} \right),$$

$$x_{6n+2} = \frac{z_0 x_0 y_{-1}\left(c^{6n+1} + d\left(\sum_{r=0}^{6n} c^r\right) y_0 z_{-1}\right)}{y_0 z_{-1}\left(a^{6n+2} + b\left(\sum_{r=0}^{6n+1} a^r\right) x_0 y_{-1}\right)} \quad \times$$
$$\times \prod_{i=0}^{n-1} \left(\frac{\left(a^{6i-1} + b\left(\sum_{r=0}^{6i-2} a^r\right) x_0 y_{-1}\right)\left(e^{6i-3} + f\left(\sum_{r=0}^{6i-4} e^r\right) z_0 x_{-1}\right)\left(c^{6i-5} + d\left(\sum_{r=0}^{6i-6} c^r\right) y_0 z_{-1}\right)}{\left(e^{6i} + f\left(\sum_{r=0}^{6i-1} e^r\right) z_0 x_{-1}\right)\left(c^{6i-2} + d\left(\sum_{r=0}^{6i-3} c^r\right) y_0 z_{-1}\right)\left(a^{6i-4} + b\left(\sum_{r=0}^{6i-5} a^r\right) x_0 y_{-1}\right)} \right),$$

$$x_{6n+3} = \frac{z_0 y_{-1} x_{-1}\left(c^{6n+2} + d\left(\sum_{r=0}^{6n+1} c^r\right) y_0 z_{-1}\right)}{y_0 z_{-1}\left(a^{6n+3} + b\left(\sum_{r=0}^{6n+2} a^r\right) x_0 y_{-1}\right)\left(e^{6n+1} + f\left(\sum_{r=0}^{6n} e^r\right) z_0 x_{-1}\right)} \quad \times$$
$$\times \prod_{i=0}^{n-1} \left(\frac{\left(a^{6i} + b\left(\sum_{r=0}^{6i-1} a^r\right) x_0 y_{-1}\right)\left(e^{6i-2} + f\left(\sum_{r=0}^{6i-3} e^r\right) z_0 x_{-1}\right)\left(c^{6i-4} + d\left(\sum_{r=0}^{6i-5} c^r\right) y_0 z_{-1}\right)}{\left(c^{6i-1} + d\left(\sum_{r=0}^{6i-2} c^r\right) y_0 x_{-1}\right)\left(a^{6i-3} + b\left(\sum_{r=0}^{6i-4} a^r\right) x_0 y_{-1}\right)\left(e^{6i-5} + f\left(\sum_{r=0}^{6i-6} e^r\right) z_0 x_{-1}\right)} \right),$$

$$x_{6n+4} = \frac{z_0 x_{-1}\left(a^{6n+1} + b\left(\sum_{r=0}^{6n} a^r\right) x_0 y_{-1}\right)\left(c^{6n+3} + d\left(\sum_{r=0}^{6n+2} c^r\right) y_0 z_{-1}\right)}{z_{-1}\left(e^{6n+2} + f\left(\sum_{r=0}^{6n+1} e^r\right) z_0 x_{-1}\right)\left(a^{6n+4} + b\left(\sum_{r=0}^{6n+3} a^r\right) x_0 y_{-1}\right)} \quad \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(e^{6i-1} + f\left(\sum_{r=0}^{6i-2} e^r\right) z_0 x_{-1}\right)\left(c^{6i-3} + d\left(\sum_{r=0}^{6i-4} c^r\right) y_0 z_{-1}\right)\left(a^{6i-5} + b\left(\sum_{r=0}^{6i-6} a^r\right) x_0 y_{-1}\right)}{\left(c^{6i} + d\left(\sum_{r=0}^{6i-1} c^r\right) y_0 z_{-1}\right)\left(a^{6i-2} + b\left(\sum_{r=0}^{6i-3} a^r\right) x_0 y_{-1}\right)\left(e^{6i-4} + f\left(\sum_{r=0}^{6i-5} e^r\right) z_0 x_{-1}\right)}\right),$$

$$y_{6n} = y_0 \quad \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(e^{6i-1} + f\left(\sum_{r=0}^{6i-2} e^r\right) z_0 x_{-1}\right)\left(c^{6i-3} + d\left(\sum_{r=0}^{6i-4} c^r\right) y_0 z_{-1}\right)\left(a^{6i-5} + b\left(\sum_{r=0}^{6i-6} a^r\right) x_0 y_{-1}\right)}{\left(c^{6i} + d\left(\sum_{r=0}^{6i-1} c^r\right) y_0 z_{-1}\right)\left(a^{6i-2} + b\left(\sum_{r=0}^{6i-3} a^r\right) x_0 y_{-1}\right)\left(e^{6i-4} + f\left(\sum_{r=0}^{6i-5} e^r\right) z_0 x_{-1}\right)}\right),$$

$$y_{6n-1} = \frac{y_{-1}}{a^{6n} + b\left(\sum_{r=0}^{6n-1} a^r\right) x_0 y_{-1}} \quad \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(a^{6i} + b\left(\sum_{r=0}^{6i-1} a^r\right) x_0 y_{-1}\right)\left(e^{6i-2} + f\left(\sum_{r=0}^{6i-3} e^r\right) z_0 x_{-1}\right)\left(c^{6i-4} + d\left(\sum_{r=0}^{6i-5} c^r\right) y_0 z_{-1}\right)}{\left(c^{6i-1} + d\left(\sum_{r=0}^{6i-2} c^r\right) y_0 x_{-1}\right)\left(a^{6i-3} + b\left(\sum_{r=0}^{6i-4} a^r\right) x_0 y_{-1}\right)\left(e^{6i-5} + f\left(\sum_{r=0}^{6i-6} e^r\right) z_0 x_{-1}\right)}\right),$$

$$y_{6n+1} = \frac{y_0 z_{-1}}{z_0\left(c^{6n+1} + d\left(\sum_{r=0}^{6n} c^r\right) y_0 z_{-1}\right)} \quad \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(e^{6i} + f\left(\sum_{r=0}^{6i-1} e^r\right) z_0 x_{-1}\right)\left(c^{6i-2} + d\left(\sum_{r=0}^{6i-3} c^r\right) y_0 z_{-1}\right)\left(a^{6i-4} + b\left(\sum_{r=0}^{6i-5} a^r\right) x_0 y_{-1}\right)}{\left(a^{6i-1} + b\left(\sum_{r=0}^{6i-2} a^r\right) x_0 y_{-1}\right)\left(e^{6i-3} + f\left(\sum_{r=0}^{6i-4} e^r\right) z_0 x_{-1}\right)\left(c^{6i-5} + d\left(\sum_{r=0}^{6i-6} c^r\right) y_0 z_{-1}\right)}\right),$$

$$y_{6n+2} = \frac{x_0 y_0 z_{-1}\left(e^{6n+1} + f\left(\sum_{r=0}^{6n} e^r\right) z_0 x_{-1}\right)}{z_0 x_{-1}\left(c^{6n+2} + d\left(\sum_{r=0}^{6n+1} c^r\right) y_0 z_{-1}\right)} \quad \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(c^{6i-1} + d\left(\sum_{r=0}^{6i-2} c^r\right) y_0 x_{-1}\right)\left(a^{6i-3} + b\left(\sum_{r=0}^{6i-4} a^r\right) x_0 y_{-1}\right)\left(e^{6i-5} + f\left(\sum_{r=0}^{6i-6} e^r\right) z_0 x_{-1}\right)}{\left(a^{6i} + b\left(\sum_{r=0}^{6i-1} a^r\right) x_0 y_{-1}\right)\left(e^{6i-2} + f\left(\sum_{r=0}^{6i-3} e^r\right) z_0 x_{-1}\right)\left(c^{6i-4} + d\left(\sum_{r=0}^{6i-5} c^r\right) y_0 z_{-1}\right)}\right),$$

$$y_{6n+3} = \frac{x_0 y_{-1} z_{-1}\left(e^{6n+2} + f\left(\sum_{r=0}^{6n+1} e^r\right) z_0 x_{-1}\right)}{z_0 x_{-1}\left(a^{6n+1} + b\left(\sum_{r=0}^{6n} a^r\right) x_0 y_{-1}\right)\left(c^{6n+3} + d\left(\sum_{r=0}^{6n+2} c^r\right) y_0 z_{-1}\right)} \quad \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(c^{6i} + d\left(\sum_{r=0}^{6i-1} c^r\right) y_0 z_{-1}\right)\left(a^{6i-2} + b\left(\sum_{r=0}^{6i-3} a^r\right) x_0 y_{-1}\right)\left(e^{6i-4} + f\left(\sum_{r=0}^{6i-5} e^r\right) z_0 x_{-1}\right)}{\left(e^{6i-1} + f\left(\sum_{r=0}^{6i-2} e^r\right) z_0 x_{-1}\right)\left(c^{6i-3} + d\left(\sum_{r=0}^{6i-4} c^r\right) y_0 z_{-1}\right)\left(a^{6i-5} + b\left(\sum_{r=0}^{6i-6} a^r\right) x_0 y_{-1}\right)}\right),$$

$$y_{6n+4} = \frac{x_0 y_{-1}\left(c^{6n+1} + d\left(\sum_{r=0}^{6n} c^r\right) y_0 z_{-1}\right)\left(e^{6n+3} + f\left(\sum_{r=0}^{6n+2} e^r\right) z_0 x_{-1}\right)}{x_{-1}\left(a^{6n+2} + b\left(\sum_{r=0}^{6n+1} a^r\right) x_0 y_{-1}\right)\left(c^{6n+4} + d\left(\sum_{r=0}^{6n+3} c^r\right) y_0 z_{-1}\right)} \quad \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(a^{6i-1} + b\left(\sum_{r=0}^{6i-2} a^r\right) x_0 y_{-1}\right)\left(e^{6i-3} + f\left(\sum_{r=0}^{6i-4} e^r\right) z_0 x_{-1}\right)\left(c^{6i-5} + d\left(\sum_{r=0}^{6i-6} c^r\right) y_0 z_{-1}\right)}{\left(e^{6i} + f\left(\sum_{r=0}^{6i-1} e^r\right) z_0 x_{-1}\right)\left(c^{6i-2} + d\left(\sum_{r=0}^{6i-3} c^r\right) y_0 z_{-1}\right)\left(a^{6i-4} + b\left(\sum_{r=0}^{6i-5} a^r\right) x_0 y_{-1}\right)}\right),$$

$$z_{6n} = z_0 \quad \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(a^{6i-1} + b\left(\sum_{r=0}^{6i-2} a^r\right) x_0 y_{-1}\right)\left(e^{6i-3} + f\left(\sum_{r=0}^{6i-4} e^r\right) z_0 x_{-1}\right)\left(c^{6i-5} + d\left(\sum_{r=0}^{6i-6} c^r\right) y_0 z_{-1}\right)}{\left(e^{6i} + f\left(\sum_{r=0}^{6i-1} e^r\right) z_0 x_{-1}\right)\left(c^{6i-2} + d\left(\sum_{r=0}^{6i-3} c^r\right) y_0 z_{-1}\right)\left(a^{6i-4} + b\left(\sum_{r=0}^{6i-5} a^r\right) x_0 y_{-1}\right)}\right),$$

$$z_{6n-1} = \frac{z_{-1}}{c^{6n} + d\left(\sum_{r=0}^{6n-1} c^r\right) y_0 z_{-1}} \quad \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(c^{6i} + d\left(\sum_{r=0}^{6i-1} c^r\right) y_0 z_{-1}\right)\left(a^{6i-2} + b\left(\sum_{r=0}^{6i-3} a^r\right) x_0 y_{-1}\right)\left(e^{6i-4} + f\left(\sum_{r=0}^{6i-5} e^r\right) z_0 x_{-1}\right)}{\left(e^{6i-1} + f\left(\sum_{r=0}^{6i-2} e^r\right) z_0 x_{-1}\right)\left(c^{6i-3} + d\left(\sum_{r=0}^{6i-4} c^r\right) y_0 z_{-1}\right)\left(a^{6i-5} + b\left(\sum_{r=0}^{6i-6} a^r\right) x_0 y_{-1}\right)}\right),$$

$$z_{6n+1} = \frac{z_0 x_{-1}}{x_0\left(e^{6n+1} + f\left(\sum_{r=0}^{6n} e^r\right) z_0 x_{-1}\right)} \quad \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(a^{6i} + b\left(\sum_{r=0}^{6i-1} a^r\right) x_0 y_{-1}\right)\left(e^{6i-2} + f\left(\sum_{r=0}^{6i-3} e^r\right) z_0 x_{-1}\right)\left(c^{6i-4} + d\left(\sum_{r=0}^{6i-5} c^r\right) y_0 z_{-1}\right)}{\left(c^{6i-1} + d\left(\sum_{r=0}^{6i-2} c^r\right) y_0 x_{-1}\right)\left(a^{6i-3} + b\left(\sum_{r=0}^{6i-4} a^r\right) x_0 y_{-1}\right)\left(e^{6i-5} + f\left(\sum_{r=0}^{6i-6} e^r\right) z_0 x_{-1}\right)}\right),$$

$$z_{6n+2} = y_0 \frac{y_0 z_0 x_{-1}\left(a^{6n+1} + b\left(\sum_{r=0}^{6n} a^r\right) x_0 y_{-1}\right)}{x_0 y_{-1}\left(e^{6n+2} + f\left(\sum_{r=0}^{6n+1} e^r\right) z_0 x_{-1}\right)} \quad \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(e^{6i-1} + f\left(\sum_{r=0}^{6i-2} e^r\right) z_0 x_{-1}\right)\left(c^{6i-3} + d\left(\sum_{r=0}^{6i-4} c^r\right) y_0 z_{-1}\right)\left(a^{6i-5} + b\left(\sum_{r=0}^{6i-6} a^r\right) x_0 y_{-1}\right)}{\left(c^{6i} + d\left(\sum_{r=0}^{6i-1} c^r\right) y_0 z_{-1}\right)\left(a^{6i-2} + b\left(\sum_{r=0}^{6i-3} a^r\right) x_0 y_{-1}\right)\left(e^{6i-4} + f\left(\sum_{r=0}^{6i-5} e^r\right) z_0 x_{-1}\right)}\right),$$

$$z_{6n+3} = \frac{y_0 z_{-1} x_{-1}\left(a^{6n+2} + b\left(\sum_{r=0}^{6n+1} a^r\right) x_0 y_{-1}\right)}{x_0 y_{-1}\left(c^{6n+1} + d\left(\sum_{r=0}^{6n} c^r\right) y_0 z_{-1}\right)\left(e^{6n+3} + f\left(\sum_{r=0}^{6n+2} e^r\right) z_0 x_{-1}\right)} \quad \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(e^{6i} + f\left(\sum_{r=0}^{6i-1} e^r\right) z_0 x_{-1}\right)\left(c^{6i-2} + d\left(\sum_{r=0}^{6i-3} c^r\right) y_0 z_{-1}\right)\left(a^{6i-4} + b\left(\sum_{r=0}^{6i-5} a^r\right) x_0 y_{-1}\right)}{\left(a^{6i-1} + b\left(\sum_{r=0}^{6i-2} a^r\right) x_0 y_{-1}\right)\left(e^{6i-3} + f\left(\sum_{r=0}^{6i-4} e^r\right) z_0 x_{-1}\right)\left(c^{6i-5} + d\left(\sum_{r=0}^{6i-6} c^r\right) y_0 z_{-1}\right)}\right),$$

$$z_{6n+4} = \frac{y_0 z_{-1}\left(a^{6n+3} + b\left(\sum_{r=0}^{6n+2} a^r\right)x_0 y_{-1}\right)\left(e^{6n+1} + f\left(\sum_{r=0}^{6n} e^r\right)z_0 x_{-1}\right)}{y_{-1}\left(c^{6n+2} + d\left(\sum_{r=0}^{6n+1} c^r\right)y_0 z_{-1}\right)\left(e^{6n+4} + f\left(\sum_{r=0}^{6n+3} e^r\right)z_0 x_{-1}\right)} \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{\left(a^{6i} + b\left(\sum_{r=0}^{6i-1} a^r\right)x_0 y_{-1}\right)\left(e^{6i-2} + f\left(\sum_{r=0}^{6i-3} e^r\right)z_0 x_{-1}\right)\left(c^{6i-4} + d\left(\sum_{r=0}^{6i-5} c^r\right)y_0 z_{-1}\right)}{\left(c^{6i-1} + d\left(\sum_{r=0}^{6i-2} c^r\right)y_0 x_{-1}\right)\left(a^{6i-3} + b\left(\sum_{r=0}^{6i-4} a^r\right)x_0 y_{-1}\right)\left(e^{6i-5} + f\left(\sum_{r=0}^{6i-6} e^r\right)z_0 x_{-1}\right)}\right).$$

2. *If $a = 1$, $c = 1$, and $e = 1$, then the solution of system (8) satisfies*

$$x_{6n} = x_0\prod_{i=0}^{n-1}\left(\frac{(1+d(6i-1)y_0 z_{-1})(1+b(6i-3)x_0 y_{-1})(1+f(6i-5)z_0 x_{-1})}{(1+b(6i)x_0 y_{-1})(1+f(6i-2)z_0 x_{-1})(1+d(6i-4)y_0 z_{-1})}\right),$$

$$x_{6n-1} = \frac{x_{-1}}{1+f(6n)z_0 x_{-1}}\prod_{i=0}^{n-1}\left(\frac{(1+f(6i)z_0 x_{-1})(1+d(6i-2)y_0 z_{-1})(1+b(6i-4)x_0 y_{-1})}{(1+b(6i-1)x_0 y_{-1})(1+f(6i-3)z_0 x_{-1})(1+d(6i-5)y_0 z_{-1})}\right),$$

$$x_{6n+1} = \frac{x_0 y_{-1}}{y_0(1+b(6n+1)x_0 y_{-1})} \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{(1+d(6i)y_0 z_{-1})(1+b(6i-2)x_0 y_{-1})(1+f(6i-4)z_0 x_{-1})}{(1+f(6i-1)z_0 x_{-1})(1+d(6i-3)y_0 z_{-1})(1+b(6i-5)x_0 y_{-1})}\right),$$

$$x_{6n+2} = \frac{z_0 x_0 y_{-1}(1+d(6n+1)y_0 z_{-1})}{y_0 z_{-1}(1+b(6n+2)x_0 y_{-1})} \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{(1+b(6i-1)x_0 y_{-1})(1+f(6i-3)z_0 x_{-1})(1+d(6i-5)y_0 z_{-1})}{(1+f(6i)z_0 x_{-1})(1+d(6i-2)y_0 z_{-1})(1+b(6i-4)x_0 y_{-1})}\right),$$

$$x_{6n+3} = \frac{z_0 y_{-1} x_{-1}(1+d(6n+2)y_0 z_{-1})}{y_0 z_{-1}(1+b(6n+3)x_0 y_{-1})(1+f(6n+1)z_0 x_{-1})} \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{(1+b(6i)x_0 y_{-1})(1+f(6i-2)z_0 x_{-1})(1+d(6i-4)y_0 z_{-1})}{(1+d(6i-1)y_0 x_{-1})(1+b(6i-3)x_0 y_{-1})(1+f(6i-5)z_0 x_{-1})}\right),$$

$$x_{6n+4} = \frac{z_0 x_{-1}(1+b(6n+1)x_0 y_{-1})(1+d(6n+3)y_0 z_{-1})}{z_{-1}(1+f(6n+2)z_0 x_{-1})(1+b(6n+4)x_0 y_{-1})} \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{(1+f(6i-1)z_0 x_{-1})(1+d(6i-3)y_0 z_{-1})(1+b(6i-5)x_0 y_{-1})}{(1+d(6i)y_0 z_{-1})(1+b(6i-2)x_0 y_{-1})(1+f(6i-4)z_0 x_{-1})}\right),$$

$$y_{6n} = y_0\prod_{i=0}^{n-1}\left(\frac{(1+f(6i-1)z_0 x_{-1})(1+d(6i-3)y_0 z_{-1})(1+b(6i-5)x_0 y_{-1})}{(1+d(6i)y_0 z_{-1})(1+b(6i-2)x_0 y_{-1})(1+f(6i-4)z_0 x_{-1})}\right),$$

$$y_{6n-1} = \frac{y_{-1}}{1+b(6n)x_0 y_{-1}}\prod_{i=0}^{n-1}\left(\frac{(1+b(6i)x_0 y_{-1})(1+f(6i-2)z_0 x_{-1})(1+d(6i-4)y_0 z_{-1})}{(1+d(6i-1)y_0 x_{-1})(1+b(6i-3)x_0 y_{-1})(1+f(6i-5)z_0 x_{-1})}\right),$$

$$y_{6n+1} = \frac{y_0 z_{-1}}{z_0(1+d(6n+1)y_0 z_{-1})} \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{(1+f(6i)z_0 x_{-1})(1+d(6i-2)y_0 z_{-1})(1+b(6i-4)x_0 y_{-1})}{(1+b(6i-1)x_0 y_{-1})(1+f(6i-3)z_0 x_{-1})(1+d(6i-5)y_0 z_{-1})}\right),$$

$$y_{6n+2} = \frac{x_0 y_0 z_{-1}(1+f(6n+1)z_0 x_{-1})}{z_0 x_{-1}(1+d(6n+2)y_0 z_{-1})} \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{(1+d(6i-1)y_0 x_{-1})(1+b(6i-3)x_0 y_{-1})(1+f(6i-5)z_0 x_{-1})}{(1+b(6i)x_0 y_{-1})(1+f(6i-2)z_0 x_{-1})(1+d(6i-4)y_0 z_{-1})}\right),$$

$$y_{6n+3} = \frac{x_0 y_{-1} z_{-1}(1+f(6n+2)z_0 x_{-1})}{z_0 x_{-1}(1+b(6n+1)x_0 y_{-1})(1+d(6n+3)y_0 z_{-1})} \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{(1+d(6i)y_0 z_{-1})(1+b(6i-2)x_0 y_{-1})(1+f(6i-4)z_0 x_{-1})}{(1+f(6i-1)z_0 x_{-1})(1+d(6i-3)y_0 z_{-1})(1+b(6i-5)x_0 y_{-1})}\right),$$

$$y_{6n+4} = \frac{x_0 y_{-1}(1+d(6n+1)y_0 z_{-1})(1+f(6n+3)z_0 x_{-1})}{x_{-1}(1+b(6n+2)x_0 y_{-1})(1+d(6n+4)y_0 z_{-1})} \times$$
$$\times \prod_{i=0}^{n-1}\left(\frac{(1+b(6i-1)x_0 y_{-1})(1+f(6i-3)z_0 x_{-1})(1+d(6i-5)y_0 z_{-1})}{(1+f(6i)z_0 x_{-1})(1+d(6i-2)y_0 z_{-1})(1+b(6i-4)x_0 y_{-1})}\right),$$
$$z_{6n} = z_0 \prod_{i=0}^{n-1}\left(\frac{(1+b(6i-1)x_0 y_{-1})(1+f(6i-3)z_0 x_{-1})(1+d(6i-5)y_0 z_{-1})}{(1+f(6i)z_0 x_{-1})(1+d(6i-2)y_0 z_{-1})(1+b(6i-4)x_0 y_{-1})}\right),$$

$$z_{6n-1} = \frac{z_{-1}}{1+d(6n)y_0 z_{-1}} \prod_{i=0}^{n-1}\left(\frac{(1+d(6i)y_0 z_{-1})(1+b(6i-2)x_0 y_{-1})(1+f(6i-4)z_0 x_{-1})}{(1+f(6i-1)z_0 x_{-1})(1+d(6i-3)y_0 z_{-1})(1+b(6i-5)x_0 y_{-1})}\right),$$

$$z_{6n+1} = \frac{z_0 x_{-1}}{x_0(1+f(6n+1)z_0 x_{-1})} \times$$
$$\times \prod_{i=0}^{n-1}\left(\frac{(1+b(6i)x_0 y_{-1})(1+f(6i-2)z_0 x_{-1})(1+d(6i-4)y_0 z_{-1})}{(1+d(6i-1)y_0 x_{-1})(1+b(6i-3)x_0 y_{-1})(1+f(6i-5)z_0 x_{-1})}\right),$$
$$z_{6n+2} = y_0 \frac{y_0 z_0 x_{-1}(1+b(6n+1)x_0 y_{-1})}{x_0 y_{-1}(1+f(6n+2)z_0 x_{-1})} \times$$
$$\times \prod_{i=0}^{n-1}\left(\frac{(1+f(6i-1)z_0 x_{-1})(1+d(6i-3)y_0 z_{-1})(1+b(6i-5)x_0 y_{-1})}{(1+d(6i)y_0 z_{-1})(1+b(6i-2)x_0 y_{-1})(1+f(6i-4)z_0 x_{-1})}\right),$$
$$z_{6n+3} = \frac{y_0 z_{-1} x_{-1}(1+b(6n+2)x_0 y_{-1})}{x_0 y_{-1}(1+d(6n+1)y_0 z_{-1})(1+f(6n+3)z_0 x_{-1})} \times$$
$$\times \prod_{i=0}^{n-1}\left(\frac{(1+f(6i)z_0 x_{-1})(1+d(6i-2)y_0 z_{-1})(1+b(6i-4)x_0 y_{-1})}{(1+b(6i-1)x_0 y_{-1})(1+f(6i-3)z_0 x_{-1})(1+d(6i-5)y_0 z_{-1})}\right),$$
$$z_{6n+4} = \frac{y_0 z_{-1}(1+b(6n+3)x_0 y_{-1})(1+f(6n+1)z_0 x_{-1})}{y_{-1}(1+d(6n+2)y_0 z_{-1})(1+f(6n+4)z_0 x_{-1})} \times$$
$$\times \prod_{i=0}^{n-1}\left(\frac{(1+b(6i)x_0 y_{-1})(1+f(6i-2)z_0 x_{-1})(1+d(6i-4)y_0 z_{-1})}{(1+d(6i-1)y_0 x_{-1})(1+b(6i-3)x_0 y_{-1})(1+f(6i-5)z_0 x_{-1})}\right).$$

3. *If $a \neq 1$, $c = 1$, and $e \neq 1$, then the solution of system (8) satisfies*

$$x_{6n} = x_0 \prod_{i=0}^{n-1}\left(\frac{(1+d(6i-1)y_0 z_{-1})\left(a^{6i-3}+b\left(\sum_{r=0}^{6i-4}a^r\right)x_0 y_{-1}\right)\left(e^{6i-5}+f\left(\sum_{r=0}^{6i-6}e^r\right)z_0 x_{-1}\right)}{\left(a^{6i}+b\left(\sum_{r=0}^{6i-1}a^r\right)x_0 y_{-1}\right)\left(e^{6i-2}+f\left(\sum_{r=0}^{6i-3}e^r\right)z_0 x_{-1}\right)(1+d(6i-4)y_0 z_{-1})}\right),$$

$$x_{6n-1} = \frac{x_{-1}}{e^{6n}+f\left(\sum_{r=0}^{6n-1}e^r\right)z_0 x_{-1}} \times$$
$$\times \prod_{i=0}^{n-1}\left(\frac{\left(e^{6i}+f\left(\sum_{r=0}^{6i-1}e^r\right)z_0 x_{-1}\right)\left(a^{6i-4}+b\left(\sum_{r=0}^{6i-5}a^r\right)x_0 y_{-1}\right)(1+d(6i-2)y_0 z_{-1})}{\left(a^{6i-1}+b\left(\sum_{r=0}^{6i-2}a^r\right)x_0 y_{-1}\right)\left(e^{6i-3}+f\left(\sum_{r=0}^{6i-4}e^r\right)z_0 x_{-1}\right)(1+d(6i-5)y_0 z_{-1})}\right),$$

$$x_{6n+1} = \frac{x_0 y_{-1}}{y_0\left(a^{6n+1}+b\left(\sum_{r=0}^{6n}a^r\right)x_0 y_{-1}\right)} \times$$
$$\times \prod_{i=0}^{n-1}\left(\frac{(1+d(6i)y_0 z_{-1})\left(a^{6i-2}+b\left(\sum_{r=0}^{6i-3}a^r\right)x_0 y_{-1}\right)\left(e^{6i-4}+f\left(\sum_{r=0}^{6i-5}e^r\right)z_0 x_{-1}\right)}{\left(e^{6i-1}+f\left(\sum_{r=0}^{6i-2}e^r\right)z_0 x_{-1}\right)(1+d(6i-3)y_0 z_{-1})\left(a^{6i-5}+b\left(\sum_{r=0}^{6i-6}a^r\right)x_0 y_{-1}\right)}\right),$$

$$x_{6n+2} = \frac{z_0 x_0 y_{-1}(1+d(6n+1)y_0 z_{-1})}{y_0 z_{-1}\left(a^{6n+2}+b\left(\sum_{r=0}^{6n+1} a^r\right)x_0 y_{-1}\right)} \quad \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{\left(a^{6i-1}+b\left(\sum_{r=0}^{6i-2} a^r\right)x_0 y_{-1}\right)\left(e^{6i-3}+f\left(\sum_{r=0}^{6i-4} e^r\right)z_0 x_{-1}\right)(1+d(6i-5)y_0 z_{-1})}{\left(e^{6i}+f\left(\sum_{r=0}^{6i-1} e^r\right)z_0 x_{-1}\right)(1+d(6i-2)y_0 z_{-1})\left(a^{6i-4}+b\left(\sum_{r=0}^{6i-5} a^r\right)x_0 y_{-1}\right)}\right),$$

$$x_{6n+3} = \frac{z_0 y_{-1} x_{-1}(1+d(6n+2)y_0 z_{-1})}{y_0 z_{-1}\left(a^{6n+3}+b\left(\sum_{r=0}^{6n+2} a^r\right)x_0 y_{-1}\right)\left(e^{6n+1}+f\left(\sum_{r=0}^{6n} e^r\right)z_0 x_{-1}\right)} \quad \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{\left(a^{6i}+b\left(\sum_{r=0}^{6i-1} a^r\right)x_0 y_{-1}\right)\left(e^{6i-2}+f\left(\sum_{r=0}^{6i-3} e^r\right)z_0 x_{-1}\right)(1+d(6i-4)y_0 z_{-1})}{(1+d(6i-1)y_0 x_{-1})\left(a^{6i-3}+b\left(\sum_{r=0}^{6i-4} a^r\right)x_0 y_{-1}\right)\left(e^{6i-5}+f\left(\sum_{r=0}^{6i-6} e^r\right)z_0 x_{-1}\right)}\right),$$

$$x_{6n+4} = \frac{z_0 x_{-1}\left(a^{6n+1}+b\left(\sum_{r=0}^{6n} a^r\right)x_0 y_{-1}\right)(1+d(6n+3)y_0 z_{-1})}{z_{-1}\left(e^{6n+2}+f\left(\sum_{r=0}^{6n+1} e^r\right)z_0 x_{-1}\right)\left(a^{6n+4}+b\left(\sum_{r=0}^{6n+3} a^r\right)x_0 y_{-1}\right)} \quad \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{\left(e^{6i-1}+f\left(\sum_{r=0}^{6i-2} e^r\right)z_0 x_{-1}\right)(1+d(6i-3)y_0 z_{-1})\left(a^{6i-5}+b\left(\sum_{r=0}^{6i-6} a^r\right)x_0 y_{-1}\right)}{(1+d(6i)y_0 z_{-1})\left(a^{6i-2}+b\left(\sum_{r=0}^{6i-3} a^r\right)x_0 y_{-1}\right)\left(e^{6i-4}+f\left(\sum_{r=0}^{6i-5} e^r\right)z_0 x_{-1}\right)}\right),$$

$$y_{6n} = y_0\prod_{i=0}^{n-1}\left(\frac{\left(e^{6i-1}+f\left(\sum_{r=0}^{6i-2} e^r\right)z_0 x_{-1}\right)(1+d(6i-3)y_0 z_{-1})\left(a^{6i-5}+b\left(\sum_{r=0}^{6i-6} a^r\right)x_0 y_{-1}\right)}{(1+d(6i)y_0 z_{-1})\left(a^{6i-2}+b\left(\sum_{r=0}^{6i-3} a^r\right)x_0 y_{-1}\right)\left(e^{6i-4}+f\left(\sum_{r=0}^{6i-5} e^r\right)z_0 x_{-1}\right)}\right),$$

$$y_{6n-1} = \frac{y_{-1}}{a^{6n}+b\left(\sum_{r=0}^{6n-1} a^r\right)x_0 y_{-1}} \quad \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{\left(a^{6i}+b\left(\sum_{r=0}^{6i-1} a^r\right)x_0 y_{-1}\right)\left(e^{6i-2}+f\left(\sum_{r=0}^{6i-3} e^r\right)z_0 x_{-1}\right)(1+d(6i-4)y_0 z_{-1})}{(1+d(6i-1)y_0 x_{-1})\left(a^{6i-3}+b\left(\sum_{r=0}^{6i-4} a^r\right)x_0 y_{-1}\right)\left(e^{6i-5}+f\left(\sum_{r=0}^{6i-6} e^r\right)z_0 x_{-1}\right)}\right),$$

$$y_{6n+1} = \frac{y_0 z_{-1}}{z_0(1+d(6n+1)y_0 z_{-1})} \quad \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{\left(e^{6i}+f\left(\sum_{r=0}^{6i-1} e^r\right)z_0 x_{-1}\right)(1+d(6i-2)y_0 z_{-1})\left(a^{6i-4}+b\left(\sum_{r=0}^{6i-5} a^r\right)x_0 y_{-1}\right)}{\left(a^{6i-1}+b\left(\sum_{r=0}^{6i-2} a^r\right)x_0 y_{-1}\right)\left(e^{6i-3}+f\left(\sum_{r=0}^{6i-4} e^r\right)z_0 x_{-1}\right)(1+d(6i-5)y_0 z_{-1})}\right),$$

$$y_{6n+2} = \frac{x_0 y_0 z_{-1}\left(e^{6n+1}+f\left(\sum_{r=0}^{6n} e^r\right)z_0 x_{-1}\right)}{z_0 x_{-1}(1+d(6n+2)y_0 z_{-1})} \quad \times$$

$$\times\prod_{i=0}^{n-1}\left(\frac{(1+d(6i-1)y_0 x_{-1})\left(a^{6i-3}+b\left(\sum_{r=0}^{6i-4} a^r\right)x_0 y_{-1}\right)\left(e^{6i-5}+f\left(\sum_{r=0}^{6i-6} e^r\right)z_0 x_{-1}\right)}{\left(a^{6i}+b\left(\sum_{r=0}^{6i-1} a^r\right)x_0 y_{-1}\right)\left(e^{6i-2}+f\left(\sum_{r=0}^{6i-3} e^r\right)z_0 x_{-1}\right)(1+d(6i-4)y_0 z_{-1})}\right),$$

$$y_{6n+3} = \frac{x_0 y_{-1} z_{-1}\left(e^{6n+2} + f\left(\sum_{r=0}^{6n+1} e^r\right) z_0 x_{-1}\right)}{z_0 x_{-1}\left(a^{6n+1} + b\left(\sum_{r=0}^{6n} a^r\right) x_0 y_{-1}\right)(1 + d(6n+3) y_0 z_{-1})} \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{(1 + d(6i) y_0 z_{-1})\left(a^{6i-2} + b\left(\sum_{r=0}^{6i-3} a^r\right) x_0 y_{-1}\right)\left(e^{6i-4} + f\left(\sum_{r=0}^{6i-5} e^r\right) z_0 x_{-1}\right)}{\left(e^{6i-1} + f\left(\sum_{r=0}^{6i-2} e^r\right) z_0 x_{-1}\right)\left(a^{6i-5} + b\left(\sum_{r=0}^{6i-6} a^r\right) x_0 y_{-1}\right)}\right),$$

$$y_{6n+4} = \frac{x_0 y_{-1}(1 + d(6n+1) y_0 z_{-1})\left(e^{6n+3} + f\left(\sum_{r=0}^{6n+2} e^r\right) z_0 x_{-1}\right)}{x_{-1}\left(a^{6n+2} + b\left(\sum_{r=0}^{6n+1} a^r\right) x_0 y_{-1}\right)(1 + d(6n+4) y_0 z_{-1})} \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(a^{6i-1} + b\left(\sum_{r=0}^{6i-2} a^r\right) x_0 y_{-1}\right)\left(e^{6i-3} + f\left(\sum_{r=0}^{6i-4} e^r\right) z_0 x_{-1}\right)(1 + d(6i-5) y_0 z_{-1})}{\left(e^{6i} + f\left(\sum_{r=0}^{6i-1} e^r\right) z_0 x_{-1}\right)(1 + d(6i-2) y_0 z_{-1})\left(a^{6i-4} + b\left(\sum_{r=0}^{6i-5} a^r\right) x_0 y_{-1}\right)}\right),$$

$$z_{6n} = z_0 \prod_{i=0}^{n-1}\left(\frac{\left(a^{6i-1} + b\left(\sum_{r=0}^{6i-2} a^r\right) x_0 y_{-1}\right)\left(e^{6i-3} + f\left(\sum_{r=0}^{6i-4} e^r\right) z_0 x_{-1}\right)(1 + d(6i-5) y_0 z_{-1})}{\left(e^{6i} + f\left(\sum_{r=0}^{6i-1} e^r\right) z_0 x_{-1}\right)(1 + d(6i-2) y_0 z_{-1})\left(a^{6i-4} + b\left(\sum_{r=0}^{6i-5} a^r\right) x_0 y_{-1}\right)}\right),$$

$$z_{6n-1} = \frac{z_{-1}}{1 + d(6n) y_0 z_{-1}} \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{(1 + d(6i) y_0 z_{-1})\left(a^{6i-2} + b\left(\sum_{r=0}^{6i-3} a^r\right) x_0 y_{-1}\right)\left(e^{6i-4} + f\left(\sum_{r=0}^{6i-5} e^r\right) z_0 x_{-1}\right)}{\left(e^{6i-1} + f\left(\sum_{r=0}^{6i-2} e^r\right) z_0 x_{-1}\right)(1 + d(6i-3) y_0 z_{-1})\left(a^{6i-5} + b\left(\sum_{r=0}^{6i-6} a^r\right) x_0 y_{-1}\right)}\right),$$

$$z_{6n+1} = \frac{z_0 x_{-1}}{x_0\left(e^{6n+1} + f\left(\sum_{r=0}^{6n} e^r\right) z_0 x_{-1}\right)} \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(a^{6i} + b\left(\sum_{r=0}^{6i-1} a^r\right) x_0 y_{-1}\right)\left(e^{6i-2} + f\left(\sum_{r=0}^{6i-3} e^r\right) z_0 x_{-1}\right)(1 + d(6i-4) y_0 z_{-1})}{(1 + d(6i-1) y_0 x_{-1})\left(a^{6i-3} + b\left(\sum_{r=0}^{6i-4} a^r\right) x_0 y_{-1}\right)\left(e^{6i-5} + f\left(\sum_{r=0}^{6i-6} e^r\right) z_0 x_{-1}\right)}\right),$$

$$z_{6n+2} = y_0 \frac{y_0 z_0 x_{-1}\left(a^{6n+1} + b\left(\sum_{r=0}^{6n} a^r\right) x_0 y_{-1}\right)}{x_0 y_{-1}\left(e^{6n+2} + f\left(\sum_{r=0}^{6n+1} e^r\right) z_0 x_{-1}\right)} \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(e^{6i-1} + f\left(\sum_{r=0}^{6i-2} e^r\right) z_0 x_{-1}\right)(1 + d(6i-3) y_0 z_{-1})\left(a^{6i-5} + b\left(\sum_{r=0}^{6i-6} a^r\right) x_0 y_{-1}\right)}{(1 + d(6i) y_0 z_{-1})\left(a^{6i-2} + b\left(\sum_{r=0}^{6i-3} a^r\right) x_0 y_{-1}\right)\left(e^{6i-4} + f\left(\sum_{r=0}^{6i-5} e^r\right) z_0 x_{-1}\right)}\right),$$

$$z_{6n+3} = \frac{y_0 z_{-1} x_{-1}\left(a^{6n+2} + b\left(\sum_{r=0}^{6n+1} a^r\right) x_0 y_{-1}\right)}{x_0 y_{-1}(1 + d(6n+1) y_0 z_{-1})\left(e^{6n+3} + f\left(\sum_{r=0}^{6n+2} e^r\right) z_0 x_{-1}\right)} \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(e^{6i} + f\left(\sum_{r=0}^{6i-1} e^r\right) z_0 x_{-1}\right)(1 + d(6i-2) y_0 z_{-1})\left(a^{6i-4} + b\left(\sum_{r=0}^{6i-5} a^r\right) x_0 y_{-1}\right)}{\left(a^{6i-1} + b\left(\sum_{r=0}^{6i-2} a^r\right) x_0 y_{-1}\right)\left(e^{6i-3} + f\left(\sum_{r=0}^{6i-4} e^r\right) z_0 x_{-1}\right)(1 + d(6i-5) y_0 z_{-1})}\right),$$

$$z_{6n+4} = \frac{y_0 z_{-1}\left(a^{6n+3} + b\left(\sum_{r=0}^{6n+2} a^r\right) x_0 y_{-1}\right)\left(e^{6n+1} + f\left(\sum_{r=0}^{6n} e^r\right) z_0 x_{-1}\right)}{y_{-1}(1 + d(6n+2) y_0 z_{-1})\left(e^{6n+4} + f\left(\sum_{r=0}^{6n+3} e^r\right) z_0 x_{-1}\right)} \times$$

$$\times \prod_{i=0}^{n-1}\left(\frac{\left(a^{6i} + b\left(\sum_{r=0}^{6i-1} a^r\right) x_0 y_{-1}\right)\left(e^{6i-2} + f\left(\sum_{r=0}^{6i-3} e^r\right) z_0 x_{-1}\right)(1 + d(6i-4) y_0 z_{-1})}{(1 + d(6i-1) y_0 x_{-1})\left(a^{6i-3} + b\left(\sum_{r=0}^{6i-4} a^r\right) x_0 y_{-1}\right)\left(e^{6i-5} + f\left(\sum_{r=0}^{6i-6} e^r\right) z_0 x_{-1}\right)}\right).$$

Proof. Using Formula (9), we can write the following for all (n denotes a nonzero integer):

$$x_n = \frac{u_n}{y_{n-1}}, \tag{18}$$

$$y_n = \frac{v_n}{z_{n-1}}, \tag{19}$$

$$z_n = \frac{w_n}{x_{n-1}}. \tag{20}$$

Using Formulas (18)–(20), after some calculations, we obtain

$$x_{6n} = \frac{u_{6n} w_{6n-2} v_{6n-4}}{v_{6n-1} u_{6n-3} w_{6n-5}} x_{6n-6}, \tag{21}$$

$$y_{6n} = \frac{v_{6n} u_{6n-2} w_{6n-4}}{w_{6n-1} v_{6n-3} u_{6n-5}} y_{6n-6} \tag{22}$$

and

$$z_{6n} = \frac{w_{6n} v_{6n-2} u_{6n-4}}{u_{6n-1} w_{6n-3} v_{6n-5}} z_{6n-6}, \tag{23}$$

for any $n \in \mathbb{N}$. This implies that

$$x_{6n} = x_0 \prod_{i=0}^{n-1}\left(\frac{u_{6i} w_{6i-2} v_{6i-4}}{v_{6i-1} u_{6i-3} w_{6i-5}}\right), \tag{24}$$

$$y_{6n} = y_0 \prod_{i=0}^{n-1}\left(\frac{v_{6i} u_{6i-2} w_{6i-4}}{w_{6i-1} v_{6i-3} u_{6i-5}}\right), \tag{25}$$

$$z_{6n} = z_0 \prod_{i=0}^{n-1}\left(\frac{w_{6i} v_{6i-2} u_{6i-4}}{u_{6i-1} w_{6i-3} v_{6i-5}}\right). \tag{26}$$

Using Equations (18)–(20), we obtain

$$x_{6n-1} = \frac{w_{6n}}{z_{6n}} = \frac{w_{6n}}{z_0} \prod_{i=0}^{n-1}\left(\frac{u_{6i-1} w_{6i-3} v_{6i-5}}{w_{6i} v_{6i-2} u_{6i-4}}\right),$$

$$y_{6n-1} = \frac{u_{6n}}{x_{6n}} = \frac{u_{6n}}{x_0} \prod_{i=0}^{n-1}\left(\frac{v_{6i-1} u_{6i-3} w_{6i-5}}{u_{6i} w_{6i-2} v_{6i-4}}\right),$$

$$z_{6n-1} = \frac{v_{6n}}{y_{6n}} = \frac{v_{6n}}{y_0} \prod_{i=0}^{n-1}\left(\frac{w_{6i-1} v_{6i-3} u_{6i-5}}{v_{6i} u_{6i-2} w_{6i-4}}\right),$$

$$
\begin{aligned}
x_{6n+1} &= \frac{u_{6n+1}}{y_{6n}} = \frac{u_{6n+1}}{y_0}\prod_{i=0}^{n-1}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right),\\
y_{6n+1} &= \frac{v_{6n+1}}{z_{6n}} = \frac{v_{6n+1}}{z_0}\prod_{i=0}^{n-1}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right),\\
z_{6n+1} &= \frac{w_{6n+1}}{x_{6n}} = \frac{w_{6n+1}}{x_0}\prod_{i=0}^{n-1}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right),
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+2} &= \frac{u_{6n+2}}{y_{6n+1}} = z_0\frac{u_{6n+2}}{v_{6n+1}}\prod_{i=0}^{n-1}\left(\frac{w_{6i}v_{6i-2}u_{6i-4}}{u_{6i-1}w_{6i-3}v_{6i-5}}\right),\\
y_{6n+2} &= \frac{v_{6n+2}}{z_{6n+1}} = x_0\frac{v_{6n+2}}{w_{6n+1}}\prod_{i=0}^{n-1}\left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}}\right),\\
z_{6n+2} &= \frac{w_{6n+2}}{x_{6n+1}} = y_0\frac{w_{6n+2}}{u_{6n+1}}\prod_{i=0}^{n-1}\left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}}\right),
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+3} &= \frac{u_{6n+3}}{y_{6n+2}} = \frac{u_{6n+3}w_{6n+1}}{x_0v_{6n+2}}\prod_{i=0}^{n-1}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right),\\
y_{6n+3} &= \frac{v_{6n+3}}{z_{6n+2}} = \frac{u_{6n+1}v_{6n+3}}{y_0w_{6n+2}}\prod_{i=0}^{n-1}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right),\\
z_{6n+3} &= \frac{w_{6n+3}}{x_{6n+2}} = \frac{v_{6n+1}w_{6n+3}}{z_0u_{6n+2}}\prod_{i=0}^{n-1}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right),
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+4} &= \frac{u_{6n+4}}{y_{6n+3}} = y_0\frac{w_{6n+2}u_{6n+4}}{u_{6n+1}v_{6n+3}}\prod_{i=0}^{n-1}\left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}}\right),\\
y_{6n+4} &= \frac{v_{6n+4}}{z_{6n+3}} = z_0\frac{u_{6n+2}v_{6n+4}}{v_{6n+1}w_{6n+3}}\prod_{i=0}^{n-1}\left(\frac{w_{6i}v_{6i-2}u_{6i-4}}{u_{6i-1}w_{6i-3}v_{6i-5}}\right),\\
z_{6n+4} &= \frac{w_{6n+4}}{x_{6n+3}} = x_0\frac{v_{6n+2}w_{6n+4}}{u_{6n+3}w_{6n+1}}\prod_{i=0}^{n-1}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right).
\end{aligned}
$$

The result is supported by Theorem 2, where the following initial conditions are considered:

$$u_0 = x_0y_{-1}, \quad v_0 = y_0z_{-1}, \quad w_0 = z_0x_{-1}.$$

□

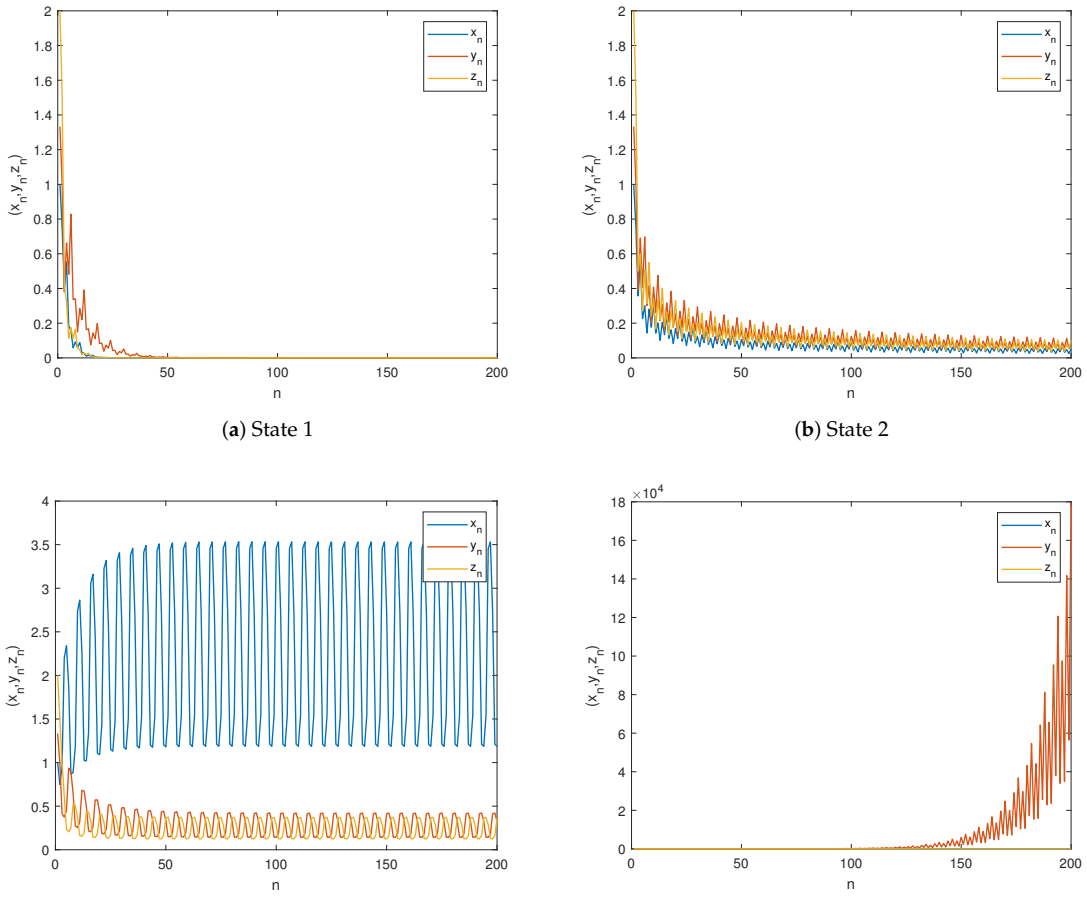
3. Numerical Simulation

In this section, we provide numerical examples to illustrate the theoretical results obtained for the nonlinear difference systems discussed earlier. The calculations are carried out using initial conditions that satisfy the assumptions stated in the previous sections. The simulations provide visual insight into the system's stability, oscillatory behavior, and divergence patterns.

Example 1. *To illustrate and validate the theoretical results, we perform numerical simulations of the system given in Equation (8) under the following parameter configurations:*

- *State 1:* $a = 1.5$, $b = 1.04$, $c = 1.5$, $d = 1$, $e = 1.80$ *and* $f = 2$.
- *State 2:* $a = 1$, $b = 1.8$, $c = 1$, $d = 1.2$, $e = 1$ *and* $f = 2$.
- *State 3:* $a = 0.95$, $b = 0.1$, $c = 0.94$, $d = 1.14$, $e = 0.59$, $f = 0.930$.
- *State 4:* $a = 2.1523$, $b = 6.83$, $c = 1.193$, $d = 2.84$, $e = 2.93$, $f = 2.32$.

The initial conditions are $x_{-k} = (5+k)/4$, $y_{-k} = (5+k)/3$, $z_{-k} = (5+k)/2$, *and* $k = 0, 1$. *The diagram of the system (8) is shown in Figure 1.*



(**a**) State 1 (**b**) State 2

(**c**) State 3 (**d**) State 4

Figure 1. This figure shows the solutions of the system (8).

4. Conclusions

In this paper, we have derived admissible solutions of the nonlinear system of rational difference equations of order two:

$$x_{n+1} = \frac{x_n y_{n-1}}{y_n(a + b x_n y_{n-1})}, \quad y_{n+1} = \frac{y_n z_{n-1}}{z_n(c + d y_n z_{n-1})}, \quad z_{n+1} = \frac{z_n x_{n-1}}{x_n(e + f z_n x_{n-1})},$$

Here, n denotes a nonzero integer, the parameters $a, b, c, d, e, f \in \mathbb{R}$, and the initial conditions x_{-1}, x_0, y_{-1}, y_0, z_{-1}, and z_0 represents nonzero real numbers.

By introducing a change in variables, we transformed the coupled system into a set of independent equations, which allowed for a detailed analysis of each sequence. We derived explicit recurrence relations and examined the qualitative behavior of the solutions, including their stability, boundedness, and periodicity. In addition, we discussed the conditions under which the system exhibits long-term dynamics and convergence.

This study improves the understanding of rational difference equations in multiple fields, including mathematical biology, physics, and economics, where systems of this form can be used to model interactions between multiple variables.

Future work could extend this analysis to higher-order systems or incorporate additional complexities, such as time delays or stochastic perturbations. Moreover, exploring the practical implications of solutions in real-world applications, such as population dynamics or economic modeling, presents an exciting avenue for further research.

5. Future Work

The results established in this paper open several promising avenues for future investigation. One natural extension is the study of more generalized and higher-dimensional nonlinear difference systems. In particular, we intend to analyze the following three-dimensional system:

$$x_{n+1} = \frac{x_{n-k}y_{n-(k+1)}}{y_n\left(a + bx_{n-k}y_{n-(k+1)}\right)},$$
$$y_{n+1} = \frac{y_{n-k}z_{n-(k+1)}}{z_n\left(c + dy_{n-k}z_{n-(k+1)}\right)},$$
$$z_{n+1} = \frac{z_{n-k}x_{n-(k+1)}}{x_n\left(c + dz_{n-k}x_{n-(k+1)}\right)}, \quad n = 0, 1, 2, \ldots,$$

where $k \in \mathbb{N}$, and the initial conditions $x_{-j}, y_{-j}, z_{-j} \in \mathbb{R} \setminus \{0\}$ for $j = 0, 1, \ldots, k+1$. Parameters $a, b, c, d \in \mathbb{R}$ are assumed to satisfy certain positivity or boundedness conditions to ensure a well-defined evolution of the system.

This system generalizes the class of difference equations studied in this work by incorporating an additional sequence and introducing cyclic coupling among the three variables. Such a system has the potential to exhibit significantly more intricate behavior, including higher-order periodicity. Several specific directions for future work include the following:

- Stability and Boundedness: This encompasses investigating sufficient conditions under which the solutions of the extended system remain bounded, converge to fixed points, or exhibit asymptotic stability.
- Exact Solutions: Attempting to derive closed-form or semi-closed-form solutions under special constraints on initial values or parameter sets will be key. This includes identifying invariant curves, symmetries, or conserved quantities.

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