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On the Unique Solution of a Partial Integro-Differential Equation on C^{∞}_{zwu} and Related Problems

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Abstract

We investigate uniqueness of the solution of a partial Duhamel equation in three dimensions in a subspace of infinite differentiable functions on the cube $[0,1] \times [0,1] \times [0,1]$. Some other related questions are also discussed.

Keywords: Duhamel product; Duhamel equation; unique solution; integro-differential equation

MSC: 46E35; 47B38

1. Introduction and Notations

Wigley [1,2] firstly gave deep analysis of the classical Duhamel product

$$(f \circledast g)(z) = \frac{d}{dz} \int_0^z f(z-t)g(t)dt \tag{1}$$

defined for analytic functions f and g on some regions of the complex plan $\mathbb{C}$ including the unit disc $\mathbb{D} = \{z \in \mathbb{C}, : \ |z| < 1\}$. He studied properties and applications of this product. In his investigation Wigley used the methods of formal power series and Banach algebras to study the solution of various differential equations including ordinary differential equations, partial differential equations and some equations of mathematical physics. Also in [2], Wigley proved the Banach algebra structure of the Hardy spaces $H^p(\mathbb{D})$ of analytic functions on the unit disc $\mathbb{D}$ via the Duhamel product (1). In [3], Karaev obtained a C^{∞} analogue of Wigley's results. Merryfield and Watson in [4] investigated similar results for the Duhamel product of the two variable functions. In [5], Tapdigoglu initiated the study uniqueness of Duhamel equations on the unit square $[0,1] \times [0,1]$.

For $f, g \in C^{\infty}$ their Duhamel product is defined by

$$(f \circledast g)(z,w,u) := \frac{\partial^3}{\partial z \partial w \partial u} \int_0^z \int_0^w \int_0^u f(z-x, w-y, u-t)g(x,y,t)dtdydx. \tag{2}$$

For functions $\varphi, f, g \in C^{\infty}_{zwu}$ the Duhamel equation has the form

$$\frac{\partial^3}{\partial z \partial w \partial u} \int_0^z \int_0^w \int_0^u \varphi((z-x)(w-y)(u-t))f(xyt)dtdydx = g(zwu), \tag{3}$$

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that is

$$\begin{aligned} g(zwu) &= \varphi_{|zwu=0} f(zwu) + \int_0^z \int_0^w \int_0^u [\varphi_u((z-x)(w-y)(u-t)) \\ &\quad +(z-x)\varphi_{zu}((z-x)(w-y)(u-t)) \\ &\quad +(w-y)\varphi_{wu}((z-x)(w-y)(u-t)) \\ &\quad +(z-x)(w-y)\varphi_{zwu}((z-x)(w-y)(u-t))] f(xyt), \end{aligned} \tag{4}$$

where $\varphi_u := \frac{\partial \varphi(zwu)}{\partial u}$, $\varphi_{zu} := \frac{\partial^2 \varphi(zwu)}{\partial z \partial u}$, $\varphi_{wu} := \frac{\partial^2 \varphi(zwu)}{\partial w \partial u}$ and $\varphi_{zwu} := \frac{\partial^3 \varphi(zwu)}{\partial z \partial w \partial u}$. In this article, we prove that if $\varphi_{|zwu=0} \neq 0$, then Equation (4) has a unique solution and it has a concrete representation via the Duhamel product in the subspace C^∞_{zwu} (see Theorem 2 in Section 3). Our proof mainly uses the Duhamel algebra [6] method and the commutant of the operator W_{zwu}. It is completely different from the proofs of the papers by Wigley [1,2], Merryfield and Watson [4] and Tapdigoglu [5]. Here, together with the Banach algebra technique, we also begin to use the commutant approach, which is apparently new.

We set $[0,1]^3 := [0,1] \times [0,1] \times [0,1]$ and

$$C^\infty_{zwu} := \{f \in C^\infty([0,1]^3) : \ f(z,w,u) = g(zwu) \ \text{ for some } \ g \in C^\infty([0,1])\}.$$

It is easy to verify that C^∞_{zwu} is a closed subspace of C^∞ which is invariant with respect to the Volterra triple integration operator on C^∞ defined by

$$(Wf)(z,w,u) := \int_0^z \int_0^w \int_0^u f(x,y,t)dtdydx.$$

Let us denote $W_{zwu} := W_{|C^\infty_{zwu}}$, that is $W_{zwu} f(zwu) = \int_0^z \int_0^w \int_0^u f(xyt)dtdydx$, for all $f \in C^\infty_{zwu}$.

2. Commutant and Bicommutant of W_{zwu}

Let $\mathcal{L}(C^\infty_{zwu})$ denote the algebra of linear continuous operators on C^∞_{zwu}. For any $A \in \mathcal{L}(C^\infty_{zwu})$, its commutant $\{A\}'$ is defined

$$\{A\}' := \{B \in \mathcal{L}(C^\infty_{zwu}) : \ BA = AB\}.$$

The commutant of the commutant $\{A\}'$ is bicommutant, that is

$$\{A\}'' := \{B \in \mathcal{L}(C^\infty_{zwu}) : \ BC = CB, \ \forall C \in \{A\}'\}.$$

In this section, we study the commutant of operator W_{zwu} on C^∞_{zwu} which plays a central role in investigations into the uniqueness of the solution of Equation (3). We also describe the bicommutant of W_{zwu}.

Note that the study of the commutant of a concrete operator A is one of the important questions of operator theory in topological vector spaces, including Hilbert and Banach spaces. Actually, it is enough to mention the famous Neumann and Lomouosov theorems on the existence of closed nontrivial invariant and hyperinvariant subspaces of completely continuous linear maps (i.e., compact operators). Recall that a closed subspace $M \subset \mathcal{X}$ is called hyperinvariant subspace for the operator A on the topological vector space $\mathcal{X}$ if it is invariant with respect to any operator $B \in \{A\}'$, that is $BM \subset M$ for any $B \in \{A\}'$. We also recall that the topology in C^∞ is given by the family of the semi-norms $\{P_n\}_{n\geq 0}$ defined as follows

$$P_n(f) := \max\left\{ \max_{(z,w,u)\in[0,1]^3} \left| \frac{\partial^{|\alpha|}}{\partial z^{\alpha_1}\partial w^{\alpha_2}\partial u^{\alpha_3}} f(z,w,u) \right| : \ |\alpha| = \alpha_1+\alpha_2+\alpha_3 = 0,1,...,n \right\}. \tag{5}$$

It is easy to see from formulas (2) and (5) that the Duhamel operator D_f, $D_f g := f \circledast g$, is the continuous operator on $C^\infty([0,1]^3)$ for any $f \in C^\infty([0,1]^3)$; in particular, for any $f \in C^\infty_{zwu}$ the Duhamel operator D_f, $D_f g(zwu) = (f \circledast g)(zwu)$, is continuous in C^∞_{zwu}. In general, by using the arguments of the paper [3], it can be shown that $(C^\infty, \circledast)$ and $(C^\infty_{zwu}, \circledast)$ are algebras (which is omitted).

The main result of this section is the following theorem which describes the commutant of operator W_{zwu}.

Theorem 1. *Let $T \in \mathcal{L}(C^\infty_{zwu})$ be an operator; then, $TW_{zwu} = W_{zwu}T$ if and only if there exists a function $\varphi \in C^\infty_{zwu}$ such that $T = D_\varphi$ is the Duhamel operator defined by the formula*

$$\begin{aligned}
& (D_\varphi f)(zwu) \\
= \ & (\varphi \circledast f)(zwu) \\
= \ & \frac{\partial^3}{\partial z\partial w\partial u}\int_0^z\int_0^w\int_0^u \varphi((z-x)(w-y)(u-t))f(xyt)dtdydx \\
= \ & \varphi_{|zwu=0} f(zwu) + \int_0^z\int_0^w\int_0^u [\varphi_u((z-x)(w-y)(u-t)) \\
& +(z-x)\varphi_{zu}((z-x)(w-y)(u-t)) + (w-y)\varphi_{wu}((z-x)(w-y)(u-t)) \\
& +(z-x)(w-y)\varphi_{zwu}((z-x)(w-y)(u-t))]f(xyt)dtdydx
\end{aligned}$$

where $\varphi_u := \frac{\partial\varphi(zwu)}{\partial u}$, $\varphi_{zu} := \frac{\partial^2\varphi(zwu)}{\partial z\partial u}$, $\varphi_{wu} := \frac{\partial^2\varphi(zwu)}{\partial w\partial u}$ and $\varphi_{zwu} := \frac{\partial^3\varphi(zwu)}{\partial z\partial w\partial u}$.

Proof. Let $T \in \mathcal{L}(C^\infty_{zwu})$ be such that $TW_{zwu} = W_{zwu}T$; then $TW_{zwu}(zwu)^k = W_{zwu}T(zwu)^k$ for all $k \geq 0$. Then

$$\begin{aligned}
TW_{zwu}(zwu)^k &= T\left[\int_0^z\int_0^w\int_0^u (xyt)^k dtdydx\right] \\
&= T\frac{(zwu)^k}{(k+1)^3}
\end{aligned}$$

Therefore

$$T(zwu)^{k+1} = (k+1)^3 W_{zwu}T(zwu)^k, \quad \forall k \geq 0 \tag{6}$$

From which we can deduce that

$$T(zwu)^n = W^n_{zwu}T\mathbb{I}\prod_{m=1}^n m^3, \quad \forall n \geq 1 \tag{7}$$

Indeed for $n = 1$, we have form (6), $T(zwu) = W_{zwu}(zwu) = W_{zwu}T\mathbb{I}$.

We assume that Equation (7) is verified, which we will show for $n+1$.

$$\begin{aligned}
T(zwu)^{n+1} &= (n+1)^3 W_{zwu}T(zwu)^n \\
&= (n+1)^3 W_{zwu}\left(W^n_{zwu}T\mathbb{I}\prod_{m=1}^n m^3\right) \\
&= W^{n+1}_{zwu}T\mathbb{I}\prod_{m=1}^{n+1} m^3.
\end{aligned}$$

Now, we will prove that

$$(W_{zwu}^{k}f)(zwu) = \int_0^z \int_0^w \int_0^u \frac{[(z-x)(w-y)(u-t)]^{k-1}}{(k-1)!^3} f(xyt)dtdydx. \tag{8}$$

Since

$$(W_{zwu}^{k}f)(zwu) := \frac{(zwu)^k}{k!^3} \circledast f(zwu), \tag{9}$$

we get

$$\begin{aligned}
(W_{zwu}^{k}f)(zwu) &= \frac{(zwu)^k}{k!^3} \circledast f(zwu) \\
&= \frac{\partial^3}{\partial z\partial w\partial u} \int_0^z \int_0^w \int_0^u \frac{[(z-x)(w-y)(u-t)]^{k}}{(k)!^3} f(xyt)dtdydx \\
&= \frac{1}{k!^3} \int_0^w \int_0^u \frac{\partial^3}{\partial z\partial w\partial u} \Big([(z-x)(w-y)(u-t)]^{k} \Big) f(xyt)dtdydx \\
&= \frac{1}{k!^3} \int_0^w \int_0^u k^3 \Big([(z-x)(w-y)(u-t)]^{k-1} \Big) f(xyt)dtdydx \\
&= \frac{k^3}{k!^3} \int_0^w \int_0^u \Big([(z-x)(w-y)(u-t)]^{k-1} \Big) f(xyt)dtdydx \\
&= \int_0^z \int_0^w \int_0^u \frac{[(z-x)(w-y)(u-t)]^{k-1}}{(k-1)!^3} f(xyt)dtdydx.
\end{aligned}$$

From (7), we can deduce for all $k \geq 0$ that

$$\begin{aligned}
T(zwu)^k &= \prod_{m=1}^{k} m^3 W_{zwu}^{k} T\mathbb{I} \\
&= \prod_{m=1}^{k} m^3 \int_0^z \int_0^w \int_0^u \frac{[(z-x)(w-y)(u-t)]^{k-1}}{(k-1)!^3} T\mathbb{I}dtdydx \\
&= \prod_{m=1}^{k} m^3 \frac{(zwu)^k}{k!^3} \circledast T\mathbb{I} \\
&= (zwu)^k \circledast T\mathbb{I}.
\end{aligned}$$

Therefore, $Tp(zwu) = T\mathbb{I} \circledast p(zwu)$ for all polynomials p. Since the polynomials are dens in C_{zwu}^{∞}, we have

$$\begin{aligned}
Tf(zwu) &= T\mathbb{I} \circledast f(zwu) \\
&= \frac{\partial^3}{\partial z\partial w\partial u} \int_0^z \int_0^w \int_0^u T\mathbb{I}((z-x)(w-y)(u-t)) f(xyt)dtdydx \\
&= \int_0^z \int_0^w \int_0^u \frac{\partial^3}{\partial z\partial w\partial u} T\mathbb{I}((z-x)(w-y)(u-t)) f(xyt)dtdydx + T\mathbb{I}(0)f(zwu) \\
&= \int_0^z \int_0^w \int_0^u \frac{\partial^2}{\partial z\partial w} \left[\frac{\partial}{\partial u} T\mathbb{I}((z-x)(w-y)(u-t)) \right] f(xyt)dtdydx + T\mathbb{I}(0)f(zwu).
\end{aligned}$$

Let us calculate

$$\begin{aligned}
&\int_0^z\int_0^w\int_0^u \frac{\partial^2}{\partial z\partial w}\left[\frac{\partial}{\partial u}T\mathbb{I}((z-x)(w-y)(u-t))\right]f(xyt)dtdydx\\
=\;&\int_0^z\int_0^w\int_0^u \frac{\partial}{\partial z}\left[\frac{\partial}{\partial w}((z-x)(w-y))\frac{\partial}{\partial u}T\mathbb{I}((z-x)(w-y)(u-t))\right]f(xyt)dtdydx\\
=\;&\int_0^z\int_0^w\int_0^u \frac{\partial}{\partial z}\Big[(z-x)\frac{\partial}{\partial u}T\mathbb{I}((z-x)(w-y)(u-t))\\
&+(z-x)(w-y)\frac{\partial^2}{\partial w\partial u}T\mathbb{I}((z-x)(w-y)(u-t))\Big]f(xyt)dtdydx\\
=\;&\int_0^z\int_0^w\int_0^u \Big[\frac{\partial}{\partial u}T\mathbb{I}((z-x)(w-y)(u-t))\\
&+(z-x)\frac{\partial^2}{\partial z\partial u}T\mathbb{I}((z-x)(w-y)(u-t))\\
&+(w-y)\frac{\partial^2}{\partial w\partial u}T\mathbb{I}((z-x)(w-y)(u-t))\\
&+(z-x)(w-y)\frac{\partial^3}{\partial z\partial w\partial u}T\mathbb{I}((z-x)(w-y)(u-t))\Big]f(xyt)dtdydx.
\end{aligned}$$

Then

$$\begin{aligned}
Tf(zwu) =\;&\int_0^z\int_0^w\int_0^u \Big[\frac{\partial}{\partial u}T\mathbb{I}((z-x)(w-y)(u-t))\\
&+(z-x)\frac{\partial^2}{\partial z\partial u}T\mathbb{I}((z-x)(w-y)(u-t))\\
&+(w-y)\frac{\partial^2}{\partial w\partial u}T\mathbb{I}((z-x)(w-y)(u-t))\\
&+(z-x)(w-y)\frac{\partial^3}{\partial z\partial w\partial u}T\mathbb{I}((z-x)(w-y)(u-t))\Big]f(xyt)dtdydx\\
&+T\mathbb{I}(0)f(zwu).
\end{aligned}$$

We set $\varphi = T\mathbb{I}$; then, $Tf(zwu) = \varphi \circledast f(zwu)$, as desired. □

Corollary 1. *Let* $\{W_{zwu}\}'' := \{X \in \mathcal{L}(C^\infty_{zwu}) : \quad XT = TX,\ \forall T \in \{W_{zwu}\}'\}$. *Then* $\{W_{zwu}\}'' = \{W_{zwu}\}'$.

Proof. To show that $\{W_{zwu}\}'' = \{W_{zwu}\}'$ it is enough to show that $T_1T_2 = T_2T_1$, for all $T_1, T_2 \in \{W_{zwu}\}'$. According to Theorem 1, there exist $\varphi_1, \varphi_2 \in C^\infty_{zwu}$ such that for all $f \in C^\infty_{zwu}$

$$\begin{aligned}
(T_1f)(zwu) &= (\varphi_1(0)I + K_{\varphi_1})f(zwu)\\
(T_1f)(zwu) &= (\varphi_2(0)I + K_{\varphi_2})f(zwu),
\end{aligned}$$

where for $i = 1, 2$

$$\begin{aligned}
K_{\varphi_i}f(zwu) &= (\varphi_i \circledast f)(zwu)\\
&= \int_0^z\int_0^w\int_0^u \Big[\frac{\partial}{\partial u}\varphi_i((z-x)(w-y)(u-t))\\
&\quad+(z-x)\frac{\partial^2}{\partial z\partial u}\varphi_i((z-x)(w-y)(u-t))\\
&\quad+(w-y)\frac{\partial^2}{\partial w\partial u}\varphi_i((z-x)(w-y)(u-t))\\
&\quad+(z-x)(w-y)\frac{\partial^3}{\partial z\partial w\partial u}\varphi_i((z-x)(w-y)(u-t))\Big]f(xyt)dtdydx.
\end{aligned}$$

Since $K_{\varphi_1}K_{\varphi_2} = K_{\varphi_2}K_{\varphi_1}$, we have

$$\begin{aligned} T_1T_2 &= (\varphi_1(0)I + K_{\varphi_1})(\varphi_2(0)I + K_{\varphi_2}) \\ &= (\varphi_2(0)I + K_{\varphi_2})(\varphi_1(0)I + K_{\varphi_1}) \\ &= T_2T_1. \end{aligned}$$

This proves the corollary. □

3. Uniqueness

In this section, we investigate uniqueness of the Duhamel integro-differential Equation (3) (or Equation (4)). The following lemma is important for our proof.

Lemma 1. *If $f \in (C^{\infty}_{zwu}, \circledast)$, then f is $\circledast$-invertible if and only if $f_{|zwu=0} \neq 0$.*

Proof. Assume that f is $\circledast$-invertible; then, there exists $g \in C^{\infty}_{zwu}$ such that $f \circledast g = 1$. This implies that

$$1 = (f \circledast g)_{|zwu=0} = f_{|zwu=0}g_{|zwu=0}.$$

Then $f_{|zwu=0} \neq 0$.

Conversely, under the fact that $f_{|zwu=0} \neq 0$ we will show f is $\circledast$-invertible. Without loss of generality, we will assume that $f_{|zwu=0} = 1$, and then $f(zwu) = 1 - h(zwu)$ where $h \in C^{\infty}_{zwu}$ such that $h_{|zwu=0} = 0$. Therefore, there exists $M > 0$ such that $\left|\frac{\partial^3}{\partial z\partial w\partial u}h(zwu)\right| \leq M$, then

$$|h(zwu)| = \left|\int_0^z \int_0^w \int_0^u \frac{\partial^3}{\partial z\partial w\partial u}h(xyt)dtdydx\right| \leq Mzwu.$$

Note $h^{[n]}(zwu) := \underbrace{h(zwu) \circledast ... \circledast h(zwu)}_{n}$, where $h^{[0]} := 1$. According to ([3] Theorem 2.1), we have

$$\begin{aligned} (f \circledast g)(z,w,u) &= \int_0^z \int_0^w \int_0^u \frac{\partial^3}{\partial z\partial w\partial u}f(z-x,w-y,u-t)g(x,y,t)dxdydt \\ &+ \int_0^z \int_0^w \frac{\partial^2}{\partial z\partial w}f(z-x,w-y,0)g(x,y,t)dxdy \\ &+ \int_0^z \int_0^u \frac{\partial^2}{\partial z\partial u}f(z-x,0,u-t)g(x,y,t)dxdt \\ &+ \int_0^w \int_0^u \frac{\partial^2}{\partial w\partial u}f(0,w-y,u-t)g(x,y,t)dydt \\ &+ \int_0^z \frac{\partial}{\partial z}f(z-x,0,0)g(x,y,t)dx \\ &+ \int_0^w \frac{\partial}{\partial w}f(0,w-y,0)g(x,y,t)dy \\ &+ \int_0^u \frac{\partial}{\partial u}f(0,0,u-t)g(x,y,t)dt \\ &+ f(0,0,0)g(z,w,u). \end{aligned}$$

Since $f_{|zwu=0} = 1$, then

$$(f \circledast g)(z,w,u) = \int_0^z \int_0^w \int_0^u \frac{\partial^3}{\partial z\partial w\partial u}f(z-x,w-y,u-t)g(x,y,t)dxdydt + f(0,0,0)g(z,w,u). \tag{10}$$

We start by showing that

$$\left|h^{[n]}(zwu)\right| \leq \frac{M^n((zwu)^n}{(n!)^3} \tag{11}$$

and

$$\left|\frac{\partial^3}{\partial z\partial w\partial u}h^{[n]}(zwu)\right| \leq \frac{M^n((zwu)^{n-1}}{((n-1)!)^3}, \tag{12}$$

for all $z, w, u \in [0,1]$.

In fact, assume that (11) and (12) hold for n, and prove that these are also true for $n+1$. Due to (10), we have

$$\begin{aligned}
\left|h^{[n+1]}(zwu)\right| &= \left|\int_0^z\int_0^w\int_0^u \frac{\partial^3}{\partial z\partial w\partial u}h((z-x)(w-y)(u-t))h^{[n]}(x,y,t)dxdydt\right. \\
&\quad \left. +h_{|zwu=0}h^{[n]}(zwu)\right| \\
&= \left|\int_0^z\int_0^w\int_0^u \frac{\partial^3}{\partial z\partial w\partial u}h((z-x)(w-y)(u-t))h^{[n]}(xyt)dxdydt\right| \\
&\leq \int_0^z\int_0^w\int_0^u \left|\frac{\partial^3}{\partial z\partial w\partial u}h((z-x)(w-y)(u-t))\right|\left|h^{[n]}(xyt)\right|dxdydt \\
&\leq \int_0^z\int_0^w\int_0^u M\frac{M^n(xyt)^n}{(n!)^3}dxdydt \\
&= \frac{M^{n+1}}{(n!)^3}\int_0^z\int_0^w\int_0^u (xyt)^n dxdydt \\
&= \frac{M^{n+1}}{(n!)^3}\frac{(zwu)^{n+1}}{(n+1)^3} \\
&= \frac{M^{n+1}(zwu)^{n+1}}{((n+1)!)^3},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\partial^3}{\partial z\partial w\partial u}h^{[n+1]}(zwu)\right| \\
= &\left|\int_0^z\int_0^w\int_0^u \frac{\partial^6}{\partial z^2\partial w^2\partial u^2}h((z-x)(w-y)(u-t))h^{[n]}(x,y,t)dxdydt\right. \\
&\left. +\frac{\partial^3}{\partial z\partial w\partial u}h_{|zwu=0}h^{[n]}(zwu)\right| \\
= &\left|\int_0^z\int_0^w\int_0^u \frac{\partial^3}{\partial z\partial w\partial u}h((z-x)(w-y)(u-t))\frac{\partial^3}{\partial z\partial w\partial u}h^{[n]}((z-x)(w-y)(u-t))dxdydt\right| \\
\leq &\int_0^z\int_0^w\int_0^u \left|\frac{\partial^3}{\partial z\partial w\partial u}h((z-x)(w-y)(u-t))\right|\left|\frac{\partial^3}{\partial z\partial w\partial u}h^{[n]}((z-x)(w-y)(u-t))\right|dxdydt \\
\leq &\int_0^z\int_0^w\int_0^u M\frac{M^n(xyt)^{n-1}}{((n-1)!)^3}dxdydt \\
= &\frac{M^{n+1}}{((n-1)!)^3}\int_0^z\int_0^w\int_0^u (xyt)^{n-1}dxdydt \\
= &\frac{M^{n+1}}{(n!)^3}\frac{(zwu)^n}{n^3} \\
= &\frac{M^{n+1}(zwu)^n}{(n!)^3}.
\end{aligned}$$

According to (11), we deduce that $\sum_{n=0}^{\infty}\left|h^{[n]}(zwu)\right| \leq \sum_{n=0}^{\infty}\frac{M^n(zwu)^n}{(n!)^3}$, that is if

$$g(zwu) := \sum_{n=0}^{\infty} h^{[n]}(zwu),$$

then $|g(zwu)| \leq \sum_{n=0}^{\infty}\frac{M^n}{(n!)^3} := L$, because of $z, w, u \in [0,1]$. Therefore $\sum_{n=0}^{\infty} h^{[n]}(zwu)$ converges uniformly in $[0,1]\times[0,1]\times[0,1]$. In order to show that $g \in C^{\infty}_{zwu}$ we will show that for all $k \geq 3$

$$\sum_{n=0}^{\infty}\frac{\partial^k}{\partial z^{\alpha}\partial w^{\beta}\partial u^{\gamma}}h^{[n]}(zwu),$$

where $k = \alpha+\beta+\gamma$ converges uniformly in $[0,1]\times[0,1]\times[0,1]$. In fact, let $N_n \in \mathbb{N}$ be such that for all $z, w, u \in [0,1]$

$$\left|\frac{\partial^k}{\partial z^{\alpha}\partial w^{\beta}\partial u^{\gamma}}h^{[n]}(zwu)\right| \leq N_n.$$

By using (11), (12) and the fact that $h_{|zwu=0} = 0$, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left|\frac{\partial^k}{\partial z^{\alpha}\partial w^{\beta}\partial u^{\gamma}}h^{[n]}(zwu)\right| \\
= &\sum_{n=0}^{k-1}\left|\frac{\partial^k}{\partial z^{\alpha}\partial w^{\beta}\partial u^{\gamma}}h^{[n]}(zwu)\right| + \sum_{n=k}^{\infty}\left|\frac{\partial^k}{\partial z^{\alpha}\partial w^{\beta}\partial u^{\gamma}}h^{[n]}(zwu)\right| \\
\leq &\sum_{n=0}^{k-1}N_n + \sum_{n=k}^{\infty}\left|\frac{\partial^k}{\partial z^{\alpha}\partial w^{\beta}\partial u^{\gamma}}\left[h^{[k]}\circledast h^{[n-k]}\right](zwu)\right| \\
\leq &\sum_{n=0}^{k-1}N_n + \sum_{n=k}^{\infty}\left|\int_0^z\int_0^w\int_0^u\frac{\partial^{2k}}{\partial z^{2\alpha}\partial w^{2\beta}\partial u^{2\gamma}}h^{[k]}(z-x)(w-y)(u-t)h^{[n-k]}(xyt)dtdydx\right| \\
\leq &\sum_{n=0}^{k-1}N_n + \sum_{n=k}^{\infty}\left|\int_0^z\int_0^w\int_0^u\frac{\partial^{k}}{\partial z^{\alpha}\partial w^{\beta}\partial u^{\gamma}}h^{[k]}(z-x)(w-y)(u-t)h^{[n-k]}(xyt)dtdydx\right| \\
\leq &\sum_{n=0}^{k-1}N_n + N_k\sum_{n=k}^{\infty}\int_0^z\int_0^w\int_0^u\left|h^{[n-k]}(xyt)dtdyx\right| \\
\leq &\sum_{n=0}^{k-1}N_n + N_k\sum_{n=k}^{\infty}\frac{M^{n-k}(zwu)^{n-k}}{((n-k)!)^3},
\end{aligned}$$

then $\sum_{n=0}^{\infty}\frac{\partial^k}{\partial z^{\alpha}\partial w^{\beta}\partial u^{\gamma}}h^{[n]}(zwu)$ converges uniformly in $[0,1]\times[0,1]\times[0,1]$. Therefore $g \in C^{\infty}_{zwu}$. Since

$$(f\circledast g)(zwu) = \left((1-h)\circledast\sum_{n=0}^{\infty}h^{[n]}\right)(zwu) = 1,$$

we deduce that f is $\circledast$-invertible. This proves the lemma. □

Theorem 2. *If $\varphi \in C^{\infty}_{zwu}$ such that $\varphi_{|zwu=0} \neq 0$ then the equation*

$$\varphi\circledast f = g \tag{13}$$

has a unique solution for any right-hand side $g \in C^{\infty}_{zwu}$.

Proof. Since $\varphi_{|zwu=0} \neq 0$, using Lemma 1, we get that φ is $\circledast$-invertible. Note that $\psi = \varphi^{-1_{\circledast}}$; then, from (13), we get

$$\begin{aligned} \psi \circledast g &= \psi \circledast (\varphi \circledast f) \\ &= (\psi \circledast \varphi) \circledast f \\ &= \mathbb{I} \circledast f \\ &= f. \end{aligned}$$

Then, Equation (13) has a unique solution in C^{∞}_{zwu}. The theorem is proved. □

Example 1. *The equation* $\exp(zwu) \circledast f = g$ *has a unique solution for any right-hand side* $g \in C^{\infty}_{zwu}$.

For related results, see, for instance, [7–15].

4. An Estimate for the Solutions of Triple Convolution Equation

In the present section, we give an estimate for the solution of the triple convolution equation (i.e., the Volterra integral equation)

$$(C_K f)(xyz) = \int_0^x \int_0^y \int_0^z K((x-t)(y-\tau)(z-r)) f(t\tau r) dr d\tau dt = g(xyz) \tag{14}$$

in terms of the kernel function $K(xyz)$ in $C^{(n)}_{xyz}$.

It is well known from the general theory of Volterra integral equations that Equation (14) possesses a solution in the subspace $C^{(n)}_{xyz}$ for any given function $g(xyz)|_{xyz=0} = 0$. Let us define the following set:

$$\mathcal{F}_g := \{f \in C^{(n)}_{xyz} : \ f(xyz) \text{ is the solution of (14)}\}$$

It is well known (and easy to prove) that C_K is the Volterra operator; that is, C_K is compact and $\sigma(C_k) = \{0\}$. Let $\sigma_p(C_K)$ denote the point spectrum of C_K (i.e., the set of eigenvalues of C_K). Since C_K is compact, we deduce that $\sigma_p(C_K) = \emptyset$, which implies that $g \notin \mathcal{F}_g$. Let us denote the unit sphere of the set $\mathcal{F}_g$ by $\mathcal{F}^{(1)}_g$,

$$\mathcal{F}^{(1)}_g := \{f \in \mathcal{F}_g : \ \|f\|_n = 1\}.$$

Here we will focus on calculating the distance between g and $\mathcal{F}^{(1)}_g$, that is to estimate $dist(g, \mathcal{F}^{(1)}_g)$.

The following result estimates $dist(g, \mathcal{F}^{(1)}_g)$ in terms of the kernel function $K(xyz)$.

Theorem 3. *We have*

$$dist(g, \mathcal{F}^{(1)}_g) \geq M_n^{-1} \left\| \left(-1 + \int_0^x \int_0^y \int_0^z K(t\tau r) dr d\tau dt \right)^{-1_{\circledast}} \right\|_n^{-1},$$

where $M_n > 0$ *is the constant and the symbol* $-1\circledast$ *denotes the* $\circledast$*-inverse in the algebra* $(C^{(n)}_{xyz}, \circledast)$.

Proof. Let us denote $G(xyz) := -1 + \int_0^x \int_0^y \int_0^z K(t\tau r) dr d\tau dt$.

Then the triple convolution equation

$$\int_0^x \int_0^y \int_0^z K((x-t)(y-\tau)(z-r)) f(t\tau r) dr d\tau dt = g(xyz)$$

can be rewritten as

$$\frac{\partial^3}{\partial x\partial y\partial z}\int_0^x\int_0^y\int_0^z G((x-t)(y-\tau)(z-r))f(t\tau r)drd\tau dt + f(xyz) = g(xyz),$$

or shortly as $G \circledast f = g - f$. Since $G|_{xyz=0} = -1(\neq 0)$, by Lemma 1 there is a function $h \in C^{(n)}_{xyz}$ such that $h \circledast G = 1$, which implies that

$$h \circledast G \circledast f = h \circledast (g - f),$$

therefore $f = h \circledast (g - f)$. By the same argument as in the proof of Lemma 2 in [16], it can be easily proved that $C^{(n)}_{xyz}$ is a Banach algebra with respect to the Duhamel product $\circledast$ (we omit the proof), and hence we obtain that there exists a constant $M_n > 0$ such that

$$1 = \|f\|_n = \|h \circledast (g - f)\|_n \leq M_n\|h\|_n\|g - f\|_n.$$

This shows that

$$\|g - f\|_n \geq M_n^{-1}\frac{1}{\|h\|_n}$$

for all $f \in \mathcal{F}^{(1)}_g$. By considering that $h = G^{-1\circledast}$, we infer that

$$\|g - f\|_n =\geq M_n^{-1}\frac{1}{\|G^{-1\circledast}\|_n} = M_n^{-1}\left\|\left(-1 + \int_0^x\int_0^y\int_0^z K(t\tau r)drd\tau dt\right)^{-1\circledast}\right\|_n^{-1}$$

for any $f \in \mathcal{F}^{(1)}_g$. Therefore

$$dist(g, \mathcal{F}^{(1)}_g) \geq M_n^{-1}\left\|\left(-1 + \int_0^x\int_0^y\int_0^z K(t\tau r)drd\tau dt\right)^{-1\circledast}\right\|_n^{-1}.$$

This completes the proof of the theorem. □

5. Conclusions

In the present article we mainly investigated the existence of a unique solution of the Duhamel Equation (3). Our approach is adapted only for this equation, but not for the more general Duhamel equation

$$\frac{\partial^3}{\partial z\partial w\partial u}\int_0^z\int_0^w\int_0^u \varphi(z-x, w-y, u-t)f(x,y,t)dtdydx = g(z,w,u). \tag{15}$$

Note that Equation (15) can be solved, of course, by using another methods, in particular, by triple Laplace transform (see, for example, Mousa and Elzaki [17]). If $N \geq 3$ is an integer, then it can be easily shown that $(C^{(N)}_{zwu}, \circledast)$ is a Banach algebra. Hence by the same argument as in the proof of Theorem 2, if we consider Equation (13) in the space $C^{(N)}_{zwu}$ we have under the condition $\varphi|_{zwu=0} \neq 0$ that the unique solution f has the representation

$$f = \psi \circledast g = \varphi^{-1\circledast} \circledast g.$$

Then by considering that $C^{(N)}_{zwu}$ is a Banach algebra with respect to the Duhamel product $\circledast$, we obtain the following estimate for the solution f

$$\|f\|_N \leq \|\varphi^{-1\circledast}\|_N\|g\|_N. \tag{16}$$

In our future research we are planning to investigate the equation $W_{zwu}X = \lambda X W_{zwu}$, where $\lambda \in \mathbb{C}$. For $\lambda = 1$ it is a characterization of the commutant of the restricted Volterra triple integration operator (see Theorem 1). It will be also interesting to solve the Duhamel Equation (3) and (15) numerically. We will try to return to these questions in our future works.

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