

Article

A New Extended Weibull Distribution: Estimation Methods and Applications in Engineering, Physics, and Medicine

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Abstract

Increasing the amount of data with complex dynamics requires the constant updating of statistical distributions. This study aimed to introduce a new three-parameter distribution, named the new exponentiated Weibull (NEW) distribution, by applying the logarithmic transformation to the exponentiated Weibull distribution. The exponentiated Weibull distribution is a powerful generalization of the Weibull distribution that includes several classical distributions as special cases—Weibull, exponential, Rayleigh, and exponentiated exponential—which make it capable of capturing diverse forms of hazard functions. By combining the advantages of the logarithmic transformation and exponentiated Weibull, the new distribution offers great flexibility in modeling different forms of hazard functions, including increasing, J-shaped, reverse-J-shaped, and bathtub-shaped functions. Some mathematical properties of the NEW distribution were studied. Moreover, four different methods of estimation—the maximum likelihood (ML), least squares (LS), Cramer–Von Mises (CVM), and percentile (PE) methods—were employed to estimate the distribution parameters. To assess the performance of the estimates, three simulation studies were conducted, showing the benefit of the ML method, followed by the PE method, in estimating the model parameters. Additionally, five datasets were used to evaluate the effectiveness of the new distribution in fitting real data. Compared with some Weibull-type extensions, the results demonstrate the superiority of the new distribution in modeling various forms of real data and provide evidence for the applicability of the new distribution.



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1. Introduction

In real phenomena, data may not always be well captured by classical distributions. Thus, statistical distributions are continuously updated and improved in an attempt to accommodate the growth and complexity of data and enhance their fit. Researchers have considered proposing new methods to introduce new distributions with a good ability to fit various types of data in different applications. Some methods have focused on adding a new parameter to traditional distributions, such as the Marshall and Olkin [1], exponentiated [2], and alpha-power [3] methods, while others use any statistical distribution as a generator for new ones, such as the transformed-transformer (T-X) method [4]. These approaches have been utilized widely to generate highly adaptable generalizations of many well-known distributions. Examples include the exponentiated Chen distribution [5],

the alpha-power Kumaraswamy distribution [6], the Marshall–Olkin odd Burr III exponential distribution [7], the exponentiated Gompertz distribution [8], the Marshall–Olkin Bilal distribution [9], the extended Weibull distribution [10], the exponentiated XLindley distribution [11], the alpha-power modified Weibull geometric distribution [12], and the alpha-power Erlang distribution [13], among others.

The Weibull distribution is the most commonly applied distribution in lifetime analyses and reliability engineering. Although it has great flexibility in modeling the monotone failure rates for many phenomena, it performs poorly when modeling non-monotonic hazard rate functions, which are common in medical applications and reliability. This limitation has motivated researchers to introduce different extensions of the Weibull distribution to enhance its ability to capture non-monotone hazards. For example, Ref. [14] presents the modified Weibull distribution, Ref. [15] proposes the Kumaraswamy Weibull distribution, Ref. [16] provides the generalized inverse Weibull distribution, Ref. [17] introduces the exponential Weibull distribution, Ref. [18] presents the odd Weibull Weibull distribution, Ref. [19] formulated the odd generalized exponential Weibull distribution, and Ref. [20] developed the gull alpha-power Weibull distribution.

One of the well-known distributions for generalizing the two-parameter Weibull distribution is called the exponentiated Weibull distribution, which was introduced in Ref. [21]. This generalization of the Weibull distribution includes the Weibull, exponential, Rayleigh, and exponentiated exponential distributions as special cases and produces a broad versatility in modeling lifetime data. This extension of the Weibull distribution presents an additional shape parameter that assists in capturing a wider range of hazard rate behaviors, including unimodal, bathtub, increasing, and decreasing shapes. These characteristics have attracted authors' attention toward introducing more extended forms of this distribution; e.g., see [22–25]. The probability density function (pdf) and cumulative distribution function (cdf) of the exponentiated Weibull distribution are given, respectively, as follows:

$$f(y) = \alpha\beta\lambda^\beta y^{\beta-1} e^{-(\lambda y)^\beta} \left(1 - e^{-(\lambda y)^\beta}\right)^{\alpha-1}; y > 0, \quad (1)$$

$$F(y) = \left(1 - e^{-(\lambda y)^\beta}\right)^\alpha, \quad (2)$$

where α, β , and $\lambda > 0$.

In recent years, the parsimonious transformation technique has been considered as a powerful method for creating new flexible distributions that are able to model numerous types of data. This method applies specific mathematical transformations, such as exponentiation, logarithmic, power, or trigonometric functions, to an existing baseline distribution without incorporating additional parameters. These transformations make the distributions more flexible in capturing skewness, kurtosis, heavy tails, or non-monotonic hazard functions without increasing the model complexity. A considerable number of studies have proposed novel transformation techniques. For example, Ref. [26] proposes the sin function transformation, Ref. [27] presents the DUS transformation, Ref. [28] introduces the log transformation, Ref. [29] provides the MG transformation, Ref. [30] suggests the PCM transformation, Ref. [31] proposes the Kavya and Manoharan (KM) transformation, and Ref. [32] presents the exponentiated transformation. These transformations have been utilized in the literature to introduce new distributions, such as the logarithmically transformed exponential distribution [28], the DUS transformation of the inverse Weibull distribution [33], the PCM exponential distribution [30], the KM exponentiated Weibull distribution [34], the KM Kumaraswamy distribution [35], and the modified log-logistic distribution [36].

In this study, the exponentiated Weibull distribution was chosen as a baseline due to its versatility and capability in capturing a wide range of hazard forms and tail behaviors. Applying a logarithmic transformation, particularly $\log(1 + x)$, will enhance its ability to model heavy-tailed data and enrich the variety of possible hazard rate forms with no

added parameters. The term heavy-tailed here was adopted in a practical sense, denoting the greater flexibility of the NEW distribution in fitting moderately heavy-tailed data. Moreover, it is advantageous in applications when the data include zero or near-zero values, which makes it suitable for modeling the time to failure and income [37]. Therefore, this type of transformation is commonly used to represent different types of data and address problems related to asymmetric distributions [38]. Thus, the main aim of this study was to introduce a new distribution that enhances the flexibility of the exponentiated Weibull distribution while maintaining the same number of parameters.

The remaining sections of this paper are organized as follows: Section 2 defines the pdf and cdf of the NEW distribution along with its relation to some existing distributions. Section 3 discusses the mathematical characteristics of the proposed distribution. Section 4 presents four methods of estimating the new model parameters. Section 5 presents a Monte Carlo simulation used to evaluate the accuracy of the estimated parameters. Five real data applications are presented in Section 6 to illustrate the adaptability of the NEW distribution. Finally, Section 6 provides the concluding remarks.

2. The New Exponentiated Weibull Distribution

Let Y be a random variable from the exponentiated Weibull distribution defined in Equations (1) and (2). Then, the NEW distribution can be introduced by taking the transformation $X = \log(1 + Y)$. Thus, the random variable X is said to have a NEW distribution with the following pdf and cdf, respectively:

$$f(x) = \alpha\beta\lambda^\beta e^x (e^x - 1)^{\beta-1} e^{-(\lambda(e^x-1))^\beta} \left(1 - e^{-(\lambda(e^x-1))^\beta}\right)^{\alpha-1}, \quad (3)$$

$$F(x) = \left(1 - e^{-(\lambda(e^x-1))^\beta}\right)^\alpha, \quad (4)$$

where $\alpha, \beta, \lambda > 0$. Moreover, the hazard rate function takes the following form:

$$h(x) = \frac{\alpha\beta\lambda^\beta e^x (e^x - 1)^{\beta-1} e^{-(\lambda(e^x-1))^\beta} \left(1 - e^{-(\lambda(e^x-1))^\beta}\right)^{\alpha-1}}{1 - \left(1 - e^{-(\lambda(e^x-1))^\beta}\right)^\alpha}. \quad (5)$$

Figures 1 and 2 present the pdf and hazard function for the NEW distribution.

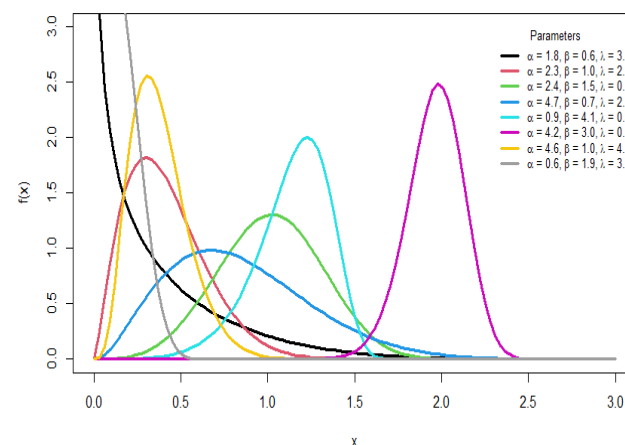


Figure 1. Plots of the pdf for different parameter values.

Various shapes are shown in the pdf, including symmetric, left-skewed, right-skewed, and reverse J-shaped forms, whereas the hazard function provides different forms, such as

increasing, bathtub, J-shaped, and reverse J-shaped forms. This illustrates the flexibility of the distribution in modeling different forms of data.

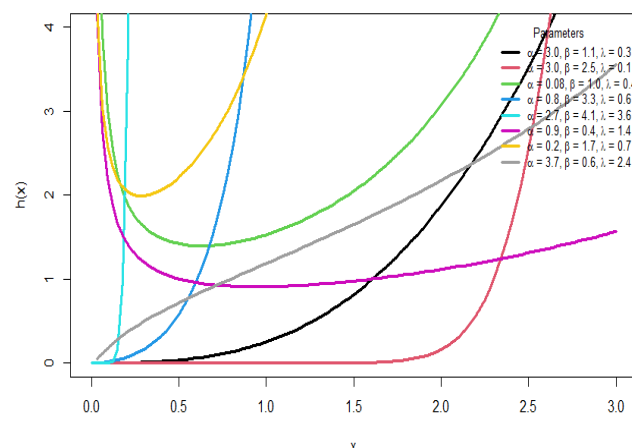


Figure 2. Plots of the HRF for different parameter values.

Special Cases of the NEW Distribution

It is noteworthy to mention that some well-known distributions are considered as special cases of the proposed distribution, such as the following:

1. The exponentiated Gompertz distribution proposed by Ref. [8], when $\alpha = 1$.
2. The exponentiated Chen distribution introduced by Ref. [5], when $\beta = 1$.
3. The odd Weibull Weibull distribution presented by Ref. [18], when $\lambda = \gamma = 1$.
4. The odd generalized exponential Weibull distribution suggested by [19], when $\beta = \theta = 1$.

3. Mathematical Properties

3.1. Quantile Function

The quantile function for the NEW distribution is easily obtained by inverting the cdf in Equation (4), and is given by

$$x_p = \log \left[1 + \frac{1}{\lambda} \left(-\log(1 - p^{\frac{1}{\alpha}}) \right)^{\frac{1}{\beta}} \right]; 0 < p < 1 \quad (6)$$

3.2. Moments

The r th moment of a random variable X is a measure that gives information on a distribution's shape and provides details of its central tendency and variations. Let X be a random variable with the pdf of the NEW distribution defined in Equation (3); then, the r th moment about the origin of X is given as

$$\begin{aligned} \mu_r &= E(X^r) = \int_0^\infty x^r f(x) dx \\ &= \alpha \beta \lambda^\beta \int_0^\infty x^r e^x (e^x - 1)^{\beta-1} e^{-(\lambda(e^x-1))^\beta} \left(1 - e^{-(\lambda(e^x-1))^\beta} \right)^{\alpha-1} dx. \end{aligned}$$

Using the binomial expansion

$$(1 - z)^b = \sum_{i=0}^{\infty} \binom{b}{i} (-1)^i z^i; |z| < 1, \quad (7)$$

we obtain

$$\mu_r = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \alpha \beta \lambda^\beta \int_0^\infty x^r e^x (e^x - 1)^{\beta-1} e^{-(i+1)(\lambda(e^x-1))^\beta} dx.$$

We apply the following substitution:

Letting $u = (\lambda(e^x - 1))^\beta \implies du = \beta \lambda^\beta (e^x - 1)^{\beta-1} e^x dx$, we obtain

$$\mu_r = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \alpha \int_0^\infty \left[\log \left(\frac{u^{1/\beta}}{\lambda} + 1 \right) \right]^r e^{-(i+1)u} du. \quad (8)$$

For r in general, Equation (8) can be solved numerically by applying statistical software for fixed values of the distribution's parameters. When $r = 1$, the first moment about the origin is

$$\mu = E(X) = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \alpha \int_0^\infty \log \left(\frac{u^{1/\beta}}{\lambda} + 1 \right) e^{-(i+1)u} du$$

For $|\frac{u^{1/\beta}}{\lambda}| < 1$, using the expansion

$$\log(1+z) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^k}{k}; |z| < 1, \quad (9)$$

we obtain

$$\begin{aligned} \mu = E(X) &= \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \alpha \int_0^\infty \sum_{k=1}^{\infty} (-1)^{k-1} \frac{u^{k/\beta}}{k \lambda^k} e^{-(i+1)u} du, \\ &= \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} (-1)^{i+k-1} \binom{\alpha-1}{i} \frac{\alpha}{k \lambda^k} \int_0^\infty u^{k/\beta} e^{-(i+1)u} du, \\ &= \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} (-1)^{i+k-1} \binom{\alpha-1}{i} \frac{\alpha}{k \lambda^k} \frac{\Gamma(\frac{k}{\beta} + 1)}{(i+1)^{\frac{k}{\beta} + 1}}. \end{aligned} \quad (10)$$

3.3. Moment-Generating Function

The moment-generating function (MGF) is a method that is used to derive moments and understand the characteristics of a random variable for any distribution. The MGF is defined as

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x) dx; t \in R \quad (11)$$

By substituting Equation (3) into Equation (11) and applying the binomial expansion in Equation (7), we have the following:

$$M_X(t) = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \alpha \beta \lambda^\beta \int_0^\infty e^{tx} e^x (e^x - 1)^{\beta-1} e^{-(i+1)(\lambda(e^x-1))^\beta} dx; t \in R$$

Let $u = (\lambda(e^x - 1))^\beta \implies du = \beta \lambda^\beta (e^x - 1)^{\beta-1} e^x dx$. Then,

$$\begin{aligned} M_X(t) &= \alpha \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \int_0^\infty e^{t \log(\frac{u^{1/\beta}}{\lambda} + 1)} e^{-(i+1)u} du, \\ &= \alpha \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \int_0^\infty \left(\frac{u^{1/\beta}}{\lambda} + 1 \right)^t e^{-(i+1)u} du. \end{aligned} \quad (12)$$

For $|\frac{u^{1/\beta}}{\lambda}| < 1$, by applying the following expansion in Equation (12)

$$(1+z)^t = \sum_{k=0}^{\infty} \binom{t}{k} z^k, |z| < 1, \quad (13)$$

we have

$$\begin{aligned} M_X(t) &= \alpha \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \int_0^{\infty} \sum_{k=0}^{\infty} \binom{t}{k} \frac{u^{k/\beta}}{\lambda^k} e^{-(i+1)u} du \\ &= \alpha \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \binom{\alpha-1}{i} \binom{t}{k} \frac{(-1)^i}{\lambda^k} \int_0^{\infty} u^{k/\beta} e^{-(i+1)u} du \\ &= \alpha \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \binom{\alpha-1}{i} \binom{t}{k} \frac{(-1)^i}{\lambda^k} \frac{\Gamma(\frac{k}{\beta} + 1)}{(i+1)^{\frac{k}{\beta} + 1}} \end{aligned}$$

3.4. Rényi Entropy

The entropy of a random variable X captures how much its uncertainty can vary. Let X be a random variable that follows the NEW distribution with the pdf defined in Equation (3). Then, the Rényi entropy can be obtained as follows:

$$RE_X(\eta) = \frac{1}{1-\eta} \log \left(\int_0^{\infty} f(x)^\eta dx \right); \eta > 0, \eta \neq 1. \quad (14)$$

By applying Equation (3) in Equation (14), we have

$$RE(\eta) = \frac{1}{1-\eta} \log \left((\alpha\beta\lambda^\beta)^\eta \int_0^{\infty} (e^x - 1)^{\eta(\beta-1)} (1 - e^{-(\lambda(e^x-1))^\beta})^{\eta(\alpha-1)} e^{-\eta(\lambda(e^x-1))^\beta} e^{\eta x} dx \right)$$

Since $|e^{-(\lambda(e^x-1))^\beta}| < 1$, we apply the binomial expansion in Equation (7):

$$(1 - e^{-(\lambda(e^x-1))^\beta})^{\eta(\alpha-1)} = \sum_{q=0}^{\infty} (-1)^q \binom{\eta(\alpha-1)}{q} e^{-q(\lambda(e^x-1))^\beta}.$$

Therefore,

$$RE(\eta) = \frac{1}{1-\eta} \log \left((\alpha\beta\lambda^\beta)^\eta \sum_{q=0}^{\infty} (-1)^q \binom{\eta(\alpha-1)}{q} \int_0^{\infty} (e^x - 1)^{\eta(\beta-1)} e^{-(q+\eta)(\lambda(e^x-1))^\beta} e^{\eta x} dx \right)$$

Let $u = (\lambda(e^x - 1))^\beta$. Then, $x = \log \left(\frac{u^{1/\beta}}{\lambda} + 1 \right)$, and $dx = \frac{u^{1/\beta-1}}{\lambda\beta(\frac{u^{1/\beta}}{\lambda} + 1)} du$.

Hence,

$$RE(\eta) = \frac{1}{1-\eta} \log \left(\frac{A}{\beta\lambda^{\eta(\beta-1)+1}} \int_0^{\infty} u^{\eta(1-\frac{1}{\beta})} e^{-(q+\eta)u} \left(\frac{u^{1/\beta}}{\lambda} + 1 \right)^{\eta-1} u^{1/\beta-1} du \right), \quad (15)$$

where $A = (\alpha\beta\lambda^\beta)^\eta \sum_{q=0}^{\infty} \binom{\eta(\alpha-1)}{q} (-1)^q$. When $|\frac{u^{1/\beta}}{\lambda}| < 1$, applying the expansion in Equation (13) to Equation (15) gives

$$\begin{aligned} RE(\eta) &= \frac{1}{1-\eta} \log \left(\frac{A}{\beta\lambda^{\eta(\beta-1)+1}} \int_0^{\infty} u^{\eta(1-\frac{1}{\beta})} e^{-(q+\eta)u} \sum_{m=0}^{\infty} \binom{\eta-1}{m} \left(\frac{u^{1/\beta}}{\lambda} \right)^m u^{1/\beta-1} du \right) \\ &= \frac{1}{1-\eta} \log \left(\frac{A}{\beta\lambda^{\eta(\beta-1)+1}} \sum_{m=0}^{\infty} \binom{\eta-1}{m} \left(\frac{1}{\lambda} \right)^m \frac{\Gamma(\eta - \frac{1}{\beta}(\eta - m - 1))}{(q+\eta)^{\eta - \frac{1}{\beta}(\eta - m - 1)}} \right). \end{aligned}$$

3.5. Order Statistics

Let $x_1, x_2, \dots, x_n$ be an independent and identically distributed (iid) random sample from the NEW distribution with the parameters α , β , and λ , which have the pdf and cdf defined in Equations (3) and (4). Then, the pdf of the k th order statistic $X_{k:n}$ is

$$f_{k:n}(x) = \sum_{m=0}^{n-k} \frac{n!}{(k-1)!(n-k)!} (-1)^m \binom{n-k}{m} f(x) [F(x)]^{m+k-1} \quad (16)$$

By substituting Equations (3) and (4) into Equation (16), we have

$$f_{k:n}(x) = A e^x (e^x - 1)^{\beta-1} e^{-(\lambda(e^x-1))^\beta} (1 - e^{-(\lambda(e^x-1))^\beta})^{\alpha(k+m)-1}, \quad (17)$$

where $A = \sum_{m=0}^{n-k} \frac{n!}{(k-1)!(n-k)!} (-1)^m \binom{n-k}{m} \alpha \beta \lambda^\beta$.

4. Methods of Estimation

As part of assessing the efficiency of the NEW distribution, its parameters were estimated using different techniques, each with complementary advantages. These techniques included the maximum likelihood (ML) method, the least squares (LS) method, the Cramer-Von Mises (CVM) method, and the percentile (PE) method. These methods were chosen because they provide a comprehensive comparison of both distance- and likelihood-based estimation techniques.

4.1. Maximum Likelihood Method

The ML estimation is the most commonly utilized method of estimating the parameters in the statistical literature. It focuses on maximizing the log-likelihood function to obtain an estimate of the unknown parameters. Let $x_1, x_2, \dots, x_n$ be a sample obtained from the NEW distribution with parameters α , β , and λ ; then, the log-likelihood function is given as

$$\begin{aligned} \ell = & n \log(\alpha) + n \log(\beta) + n\beta \log(\lambda) + (\beta - 1) \log \sum_{i=1}^n (e^{x_i} - 1) - \lambda^\beta \sum_{i=1}^n (e^{x_i} - 1)^\beta + \sum_{i=1}^n x_i \\ & + (\alpha - 1) \sum_{i=1}^n \log \left(1 - e^{-(\lambda(e^{x_i}-1))^\beta} \right). \end{aligned} \quad (18)$$

Therefore, an estimate of the distribution parameters can be obtained by taking the first derivative of Equation (18) with respect to each parameter as follows:

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log \left(1 - e^{-(\lambda(e^{x_i}-1))^\beta} \right), \quad (19)$$

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} = & \frac{n}{\beta} + n \log(\lambda) + \sum_{i=1}^n \log(e^{x_i} - 1) - (\lambda(e^{x_i} - 1))^\beta \sum_{i=1}^n [\log(\lambda) + \log(e^{x_i} - 1)] \\ & + (\alpha - 1) \sum_{i=1}^n \frac{e^{-(\lambda(e^{x_i}-1))^\beta}}{1 - e^{-(\lambda(e^{x_i}-1))^\beta}} (\lambda(e^{x_i} - 1))^\beta [\log(\lambda) + \log(e^{x_i} - 1)], \end{aligned} \quad (20)$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n\beta}{\lambda} - \beta \lambda^{\beta-1} \sum_{i=1}^n (e^{x_i} - 1)^\beta + \beta \lambda^{\beta-1} (\alpha - 1) \sum_{i=1}^n \frac{(e^{x_i} - 1)^\beta e^{-(\lambda(e^{x_i}-1))^\beta}}{1 - e^{-(\lambda(e^{x_i}-1))^\beta}}, \quad (21)$$

Equating Equations (19)–(21) to zero and solving them simultaneously produces the estimates $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\lambda}$. As an alternative, optimizing Equation (18) directly using any optimization function in any statistical software produces the ML estimates.

4.2. Least Squares Method

The LS method is one of the oldest estimation techniques, introduced in Ref. [39]. It estimates the parameters by minimizing the difference between the empirical and theoretical cdf. Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be the order statistics of a random sample from the NEW distribution; the least squares method aims to minimize the following objective function

$$LS = \sum_{i=1}^n \left[\left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta} \right)^\alpha - \left(\frac{i}{n+1} \right) \right]^2. \quad (22)$$

Then, taking the first derivative of the objective function with respect to each parameter leads to the following:

$$\frac{\partial LS}{\partial \alpha} = 2 \sum_{i=1}^n \left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta} \right)^\alpha \log \left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta} \right) \left[\left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta} \right)^\alpha - \left(\frac{i}{n+1} \right) \right] \quad (23)$$

$$\begin{aligned} \frac{\partial LS}{\partial \beta} = 2\alpha\lambda^\beta \sum_{i=1}^n e^{-(\lambda(e^{x(i)}-1))^\beta} (e^{x(i)}-1)^\beta \log(\lambda(e^{x(i)}-1)) \left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta} \right)^{\alpha-1} \\ \left[\left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta} \right)^\alpha - \left(\frac{i}{n+1} \right) \right] \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{\partial LS}{\partial \lambda} = 2\alpha\beta\lambda^{\beta-1} \sum_{i=1}^n e^{-(\lambda(e^{x(i)}-1))^\beta} (e^{x(i)}-1)^\beta \left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta} \right)^{\alpha-1} \\ \left[\left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta} \right)^\alpha - \left(\frac{i}{n+1} \right) \right] \end{aligned} \quad (25)$$

Thus, by solving Equations (23)–(25) analytically or minimizing Equation (22) numerically, the LS estimates are obtained.

4.3. Cramer–Von Mises Method

The CVM approach was formulated in Ref. [40] to estimate parameters by comparing the empirical and theoretical distribution functions in order to ensure an effective overall fit to the cdf. If $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ is the order statistics of the random sample from the NEW distribution, the CVM method obtains an estimate of the parameters α , β , and λ by minimizing the following equation:

$$CV = \sum_{i=1}^n \left[\left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta} \right)^\alpha - \left(\frac{2i-1}{2n} \right) \right]^2. \quad (26)$$

Minimizing Equation (26) using a numerical optimization method or, alternatively, by solving the resulting equations gives the CVM estimates.

$$\frac{\partial CV}{\partial \alpha} = 2 \sum_{i=1}^n \left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta} \right)^\alpha \log \left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta} \right) \left[\left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta} \right)^\alpha - \left(\frac{2i-1}{2n} \right) \right] \quad (27)$$

$$\frac{\partial CV}{\partial \beta} = 2\alpha\lambda^\beta \sum_{i=1}^n e^{-(\lambda(e^{x(i)}-1))^\beta} (e^{x(i)}-1)^\beta \log(\lambda(e^{x(i)}-1)) \left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta}\right)^{\alpha-1} \left[\left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta}\right)^\alpha - \left(\frac{2i-1}{2n}\right) \right] \quad (28)$$

$$\frac{\partial CV}{\partial \lambda} = 2\alpha\beta\lambda^{\beta-1} \sum_{i=1}^n e^{-(\lambda(e^{x(i)}-1))^\beta} (e^{x(i)}-1)^\beta \left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta}\right)^{\alpha-1} \left[\left(1 - e^{-(\lambda(e^{x(i)}-1))^\beta}\right)^\alpha - \left(\frac{2i-1}{2n}\right) \right] \quad (29)$$

4.4. Percentile Method

The PE method was proposed in Ref. [41] as a simple estimation method. The PE approach compares the percentile points of the sample with their corresponding percentiles in the population by minimizing the Euclidean distance between them. In many scenarios, the PE estimators can be derived in a closed form, which makes it an attractive estimation method. For the order statistics $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ obtained from a random sample from the NEW distribution, the PE approach aims to minimize the equation given below.

$$PE = \sum_{i=1}^n \left[x_{(i)} - \log \left(1 + \frac{1}{\lambda} \left(-\log(1 - u_i^{\frac{1}{\alpha}}) \right)^{\frac{1}{\beta}} \right) \right]^2, \quad (30)$$

where $u_i = \frac{i}{n+1}$ is an estimate of $F(x_{(i)})$. Similarly, percentile estimates are derived by minimizing Equation (30) or solving the following derivatives:

$$\frac{\partial PE_i}{\partial \alpha} = 2 \frac{1}{\alpha^2 \beta \lambda} \sum_{i=1}^n \frac{u_i^{\frac{1}{\alpha}} \log(u_i)}{1 - u_i^{\frac{1}{\alpha}}} \frac{\left(-\log(1 - u_i^{\frac{1}{\alpha}}) \right)^{\frac{1}{\beta}-1}}{\left[1 + \frac{1}{\lambda} \left(-\log(1 - u_i^{\frac{1}{\alpha}}) \right)^{\frac{1}{\beta}} \right]} \left[x_{(i)} - \log \left(1 + \frac{1}{\lambda} \left(-\log(1 - u_i^{\frac{1}{\alpha}}) \right)^{\frac{1}{\beta}} \right) \right] \quad (31)$$

$$\frac{\partial PE_i}{\partial \beta} = 2 \frac{1}{\beta^2 \lambda} \sum_{i=1}^n \log \left(-\log(1 - u_i^{\frac{1}{\alpha}}) \right) \frac{\left(-\log(1 - u_i^{\frac{1}{\alpha}}) \right)^{\frac{1}{\beta}}}{\left[1 + \frac{1}{\lambda} \left(-\log(1 - u_i^{\frac{1}{\alpha}}) \right)^{\frac{1}{\beta}} \right]} \left[x_{(i)} - \log \left(1 + \frac{1}{\lambda} \left(-\log(1 - u_i^{\frac{1}{\alpha}}) \right)^{\frac{1}{\beta}} \right) \right] \quad (32)$$

$$\frac{\partial PE_i}{\partial \lambda} = 2 \frac{1}{\lambda^2} \sum_{i=1}^n \frac{\left(-\log(1 - u_i^{\frac{1}{\alpha}}) \right)^{\frac{1}{\beta}}}{\left[1 + \frac{1}{\lambda} \left(-\log(1 - u_i^{\frac{1}{\alpha}}) \right)^{\frac{1}{\beta}} \right]} \left[x_{(i)} - \log \left(1 + \frac{1}{\lambda} \left(-\log(1 - u_i^{\frac{1}{\alpha}}) \right)^{\frac{1}{\beta}} \right) \right] \quad (33)$$

To obtain the estimates of the NEW distribution, the aforementioned methods were executed using the statistical software R (version 4.3.2). In particular, the optimization function `optim` with the Nelder–Mead algorithm was utilized to derive the estimates of the parameters.

5. Simulation

In an attempt to investigate the capability of the estimation methods in estimating the distribution parameters $\Omega = (\alpha, \beta, \lambda)$, simulation-based studies were conducted. By using the number of simulations $m = 10,000$ and sample sizes of 30, 100, 200, 300, and 500, the following three cases of the true value of the parameters were utilized:

Case 1: $\alpha = 1.7, \beta = 2.9, \lambda = 2.3$

Case 2: $\alpha = 1.7, \beta = 1.1, \lambda = 0.3$

Case 3: $\alpha = 1.5, \beta = 2.8, \lambda = 5.6$

As measures of accuracy, the bias and mean square error (MSE) were calculated for each estimate as follows:

$$\text{Bias}(\hat{\Omega}) = \frac{1}{m} \sum_{v=0}^m (\hat{\Omega}_v - \Omega)$$

$$\text{MSE}(\hat{\Omega}) = \frac{1}{m} \sum_{v=0}^m (\hat{\Omega}_v - \Omega)^2$$

The results are listed in Tables 1–3 and Figures 3–5.

Table 1. Estimates, bias, and MSE of the parameters α, β , and λ for case 1.

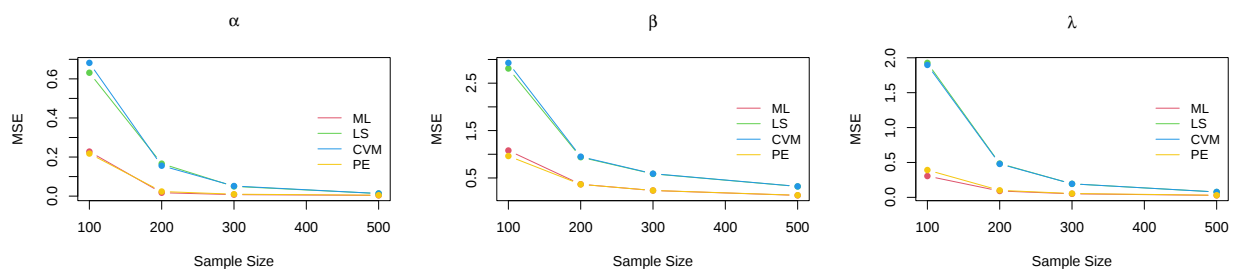
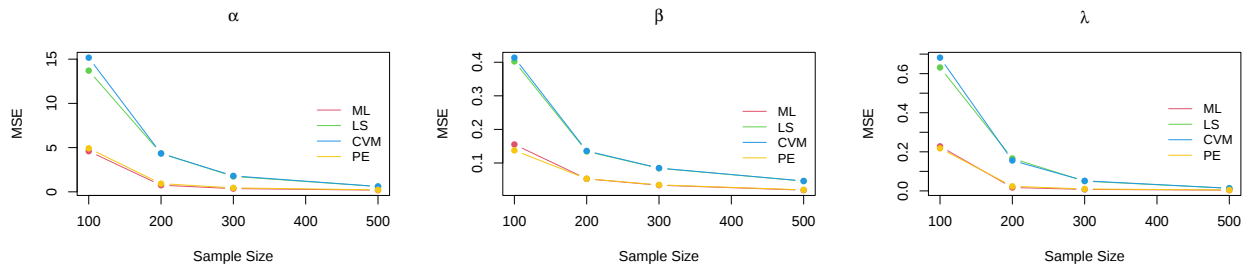
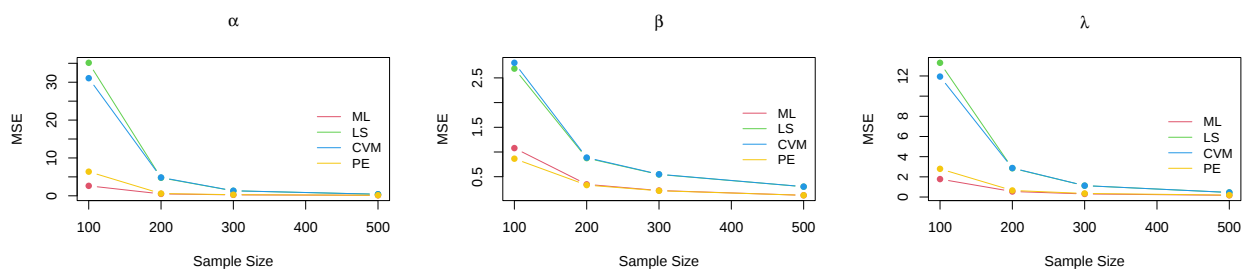
n	Parameter	ML			LS			CVM			PE		
		Estimate	Bias	MSE	Estimate	Bias	MSE	Estimate	Bias	MSE	Estimate	Bias	MSE
$n = 30$	α	21.6842	19.9842	24,196.8073	11.2838	9.5838	776.9405	12.2995	10.5995	940.8378	10.2026	8.5026	1641.7255
	β	4.5681	1.6681	32.5739	4.6948	1.7948	35.5991	4.9269	2.0269	42.1780	3.7051	0.8051	11.3993
	λ	3.9237	1.6237	90.2580	4.0191	1.7191	24.3403	3.9661	1.6661	20.7270	3.5755	1.2755	23.4206
$n = 100$	α	2.1361	0.4361	4.6354	3.3866	1.6866	51.4324	3.4553	1.7553	53.4732	2.1784	0.4784	8.1855
	β	3.1230	0.2230	1.0794	3.2423	0.3423	2.8096	3.2686	0.3686	2.9283	3.0362	0.1362	0.9626
	λ	2.4057	0.1057	0.3051	2.6682	0.3682	1.9271	2.6746	0.3746	1.8977	2.4223	0.1223	0.3914
$n = 200$	α	1.8736	0.1736	0.7476	2.3286	0.6286	7.6210	2.3475	0.6475	7.5905	1.8789	0.1789	0.7918
	β	2.9856	0.0856	0.3668	3.0229	0.1229	0.9380	3.0338	0.1338	0.9493	2.9367	0.0367	0.3651
	λ	2.3470	0.0470	0.0910	2.4631	0.1631	0.4845	2.4669	0.1669	0.4788	2.3524	0.0524	0.1005
$n = 300$	α	1.7965	0.0965	0.3625	2.0297	0.3297	2.1313	2.0413	0.3413	2.1608	1.8003	0.1003	0.3853
	β	2.9617	0.0617	0.2350	2.9835	0.0835	0.5860	2.9904	0.0904	0.5903	2.9262	0.0262	0.2365
	λ	2.3270	0.0270	0.0508	2.3932	0.0932	0.1935	2.3961	0.0961	0.1937	2.3305	0.0305	0.0550
$n = 500$	α	1.7553	0.0553	0.1898	1.8548	0.1548	0.5989	1.8611	0.1611	0.6069	1.7587	0.0587	0.1990
	β	2.9368	0.0368	0.1328	2.9539	0.0539	0.3211	2.9579	0.0579	0.3225	2.9125	0.0125	0.1356
	λ	2.3161	0.0161	0.0284	2.3462	0.0462	0.0768	2.3480	0.0480	0.0770	2.3186	0.0186	0.0302

Table 2. Estimates, bias, and MSE of the parameters α, β , and λ for case 2.

n	Parameter	ML			LS			CVM			PE		
		Estimate	Bias	MSE	Estimate	Bias	MSE	Estimate	Bias	MSE	Estimate	Bias	MSE
$n = 30$	α	9.4673	7.7673	3272.9569	5.8017	4.1017	128.0415	6.4569	4.7569	177.5209	5.4268	3.7268	153.6565
	β	1.7316	0.6316	4.6288	1.7922	0.6922	5.0605	1.87900	0.7790	5.9945	1.3984	0.2984	1.6614
	λ	2.9256	2.6256	278.7868	1.3481	1.0481	8.3484	1.3494	1.0494	8.4257	1.4719	1.1719	19.2303
$n = 100$	α	2.1354	0.4354	4.5938	2.8531	1.1531	13.7024	2.9255	1.2255	15.1697	2.2514	0.5514	4.9204
	β	1.1846	0.0846	0.1551	1.2323	0.1323	0.4025	1.2417	0.1417	0.4137	1.1369	0.0369	0.1374
	λ	0.3804	0.0804	0.2278	0.5381	0.2381	0.6314	0.5452	0.2452	0.6818	0.4079	0.1079	0.2178
$n = 200$	α	1.8735	0.1735	0.7475	2.2564	0.5564	4.3515	2.2737	0.5737	4.3355	1.9418	0.2418	0.92912
	β	1.1325	0.0325	0.0528	1.1471	0.0471	0.1344	1.1512	0.0512	0.1359	1.1026	0.0026	0.0531
	λ	0.3279	0.0279	0.0173	0.4039	0.1039	0.1661	0.4044	0.1044	0.1556	0.3400	0.0400	0.0235
$n = 300$	α	1.7963	0.0964	0.3624	2.0204	0.3204	1.7382	2.0331	0.3331	1.8022	1.8475	0.1475	0.4452
	β	1.1234	0.0234	0.0338	1.1318	0.0318	0.0842	1.1343	0.0343	0.0849	1.1007	0.0007	0.0348
	λ	0.3156	0.0156	0.0076	0.3558	0.0558	0.0497	0.3571	0.0571	0.05167	0.3239	0.0239	0.0099
$n = 500$	α	1.7553	0.0553	0.1898	1.8548	0.1548	0.5988	1.8609	0.1609	0.6047	1.7915	0.0915	0.2261
	β	1.1139	0.0139	0.0191	1.1204	0.0204	0.0462	1.1219	0.0219	0.0464	1.0978	−0.0022	0.0201
	λ	0.3091	0.0091	0.0039	0.3257	0.0257	0.0135	0.3263	0.0263	0.0134	0.3146	0.0146	0.0047

Table 3. Estimates, bias, and MSE of the parameters α , β , and λ for case 3.

n	Parameter	ML			LS			CVM			PE		
		Estimate	Bias	MSE	Estimate	Bias	MSE	Estimate	Bias	MSE	Estimate	Bias	MSE
$n = 30$	α	16.0887	14.5887	12,013.8878	9.8742	8.3742	642.3965	10.7201	9.2201	778.0744	8.7871	7.2871	1289.0428
	β	4.4568	1.6568	29.2649	4.5318	1.7318	32.1369	4.7609	1.9609	38.2404	3.5382	0.7382	10.3504
	λ	9.8292	4.2292	687.7262	10.4537	4.8537	222.1804	10.3303	4.7303	196.5355	9.3227	3.7227	258.3909
$n = 100$	α	1.8323	0.3323	2.6321	2.8135	1.3135	35.1296	2.8535	1.3535	31.0659	1.9478	0.4478	6.3721
	β	3.0277	0.2277	1.0791	3.1404	0.3404	2.6873	3.1649	0.3649	2.8055	2.8926	0.0926	0.8644
	λ	5.8509	0.2509	1.7772	6.5093	0.9093	13.3048	6.5221	0.9221	11.9380	5.9611	0.3611	2.7939
$n = 200$	α	1.6363	0.1363	0.5082	1.9857	0.4857	4.7559	2.006	0.5056	4.8449	1.6799	0.1799	0.5961
	β	2.8870	0.0870	0.3450	2.9224	0.1224	0.8761	2.9323	0.1323	0.8869	2.8079	0.0079	0.3302
	λ	5.7118	0.1118	0.5469	5.9888	0.3888	2.8507	6.0024	0.4024	2.8755	5.7635	0.1635	0.6512
$n = 300$	α	1.5758	0.0758	0.2546	1.7579	0.2579	1.3150	1.7682	0.2682	1.3339	1.6088	0.1088	0.2939
	β	2.8622	0.0622	0.2202	2.8829	0.0829	0.5450	2.8891	0.0891	0.5490	2.8023	0.0023	0.2156
	λ	5.6643	0.0643	0.3071	5.8224	0.2224	1.1262	5.8309	0.2309	1.1292	5.7032	0.1032	0.3537
$n = 500$	α	1.5436	0.0436	0.1349	1.6228	0.1228	0.4085	1.6284	0.1284	0.4135	1.5682	0.0682	0.1506
	β	2.8368	0.0368	0.1236	2.8531	0.0531	0.2975	2.8568	0.0568	0.2988	2.7939	−0.0061	0.1238
	λ	5.6385	0.0385	0.1718	5.7107	0.1107	0.4537	5.7158	0.1158	0.4554	5.6674	0.0674	0.1914

**Figure 3.** Plot of the MSE for each parameter for all methods of estimation—case 1.**Figure 4.** Plot of the MSE for each parameter for all methods of estimation—case 2.**Figure 5.** Plot of the MSE for each parameter for all methods of estimation—case 3.

The above tables and figures show that, for small samples, the bias and MSE were high for all approaches, particularly for the parameter α . For β and λ , however, all methods performed better, with a slightly higher MSE for the ML method. As the sample size increased, there was a dramatic improvement in all the estimation methods, with a considerable reduction in both the bias and MSE for all parameters. By $n = 500$, the consistency property was achieved across all the estimation techniques, with a noticeable decrease in the MSE

for all parameters. The ML method became much more stable, followed by the PE method. The LS and CVM methods were further enhanced with a high bias and MSE compared with ML and PE in most scenarios. Thus, they can be considered acceptable alternatives to the ML and PE approaches, despite being less efficient.

6. Application

For the purpose of assessing the distribution adequacy in fitting real data, seven statistical distributions were used. The considered distributions were the Weibull distribution along with six of its extensions that were built using different techniques. These Weibull-type extensions are widely employed in the literature due to their flexibility and applicability in modeling various forms of data. The aim was to investigate the flexibility of the NEW distribution compared with the Weibull distribution and its different extensions. The considered distributions were as follows:

- The Weibull (W) distribution:

$$F(x) = 1 - e^{-(\lambda x)^\beta}; x > 0, \beta, \lambda > 0$$

- The exponentiated Weibull (EW) distribution [21]:

$$F(x) = \left(1 - e^{-(\lambda x)^\beta}\right)^\alpha; x > 0, \alpha, \beta, \lambda > 0$$

- The modified Weibull (MW) distribution [14]:

$$F(x) = 1 - e^{(-\lambda x^\beta e^{\alpha x})}; x > 0, \alpha, \beta, \lambda > 0$$

- The gull alpha-power Weibull (GAPW) distribution [20]:

$$F(x) = \frac{\alpha(1 - e^{-\lambda x^\beta})}{\alpha(1 - e^{-\lambda x^\beta})}; x > 0, \alpha, \beta, \lambda > 0$$

- The generalized inverse Weibull (GIW) distribution [16]:

$$F(x) = 1 - \left(1 - e^{-\lambda x^{-\beta}}\right)^\alpha; x > 0, \alpha, \beta, \lambda > 0$$

- The odd Weibull Weibull (OWW) distribution [18]:

$$F(x) = 1 - e^{-\lambda(e^{\alpha x^\beta} - 1)}; x > 0, \alpha, \beta, \lambda > 0$$

- The KM exponentiated Weibull (KMEW) distribution [34]:

$$F(x) = \frac{e}{e-1} \left[1 - e^{-\left(1 - e^{-(\alpha x)^\beta}\right)^\lambda} \right]; x > 0, \alpha, \beta, \lambda > 0$$

The comparison between the aforementioned distributions and the proposed NEW distribution was carried out using five different datasets to investigate the goodness of fit and adequacy in describing the data behavior.

6.1. Dataset Description

6.1.1. Dataset I: Blood Cancer Dataset

This dataset represents the lifetime, from diagnosis until death (in years), for 40 patients who suffered from blood cancer (leukemia). The data are complete (no censoring) and were provided by a hospital of the Ministry of Health in Saudi Arabia. The following data values have been used by many authors, such as Refs. [42–44]: 0.315, 0.496,

0.616, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036, 2.162, 2.211, 2.370, 2.532, 2.693, 2.805, 2.910, 2.912, 3.192, 3.263, 3.348, 3.348, 3.427, 3.499, 3.534, 3.767, 3.751, 3.858, 3.986, 4.049, 4.244, 4.323, 4.381, 4.392, 4.397, 4.647, 4.753, 4.929, 4.973, 5.074, 5.381. The data are summarized in Table 4.

Table 4. Descriptive statistics for blood cancer data.

	Min	Q ₁	Mean	Q ₃	Max	SD
Value	0.315	2.199	3.141	4.264	5.381	1.358

6.1.2. Dataset II: Single Carbon Fibers

The data are presented in Ref. [45] and contain tensile strength measurements in gigapascals (GPa) for single carbon fibers and impregnated bundles of 1000 fibers, tested at a gauge length of 20 mm. The dataset includes 69 recorded values as follows:

1.312, 1.314, 1.479, 1.552, 1.7, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.24, 2.253, 2.27, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.49, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.77, 2.773, 2.8, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.09, 3.096, 3.128, 3.233, 3.433, 3.585, 3.858. Table 5 provides a summary of the data.

Table 5. Descriptive statistics for the single carbon fiber dataset.

	Min	Q ₁	Mean	Q ₃	Max	SD
Value	1.312	2.478	2.455	2.773	3.858	0.505

6.1.3. Dataset III: Aircraft Windshield Dataset

This aircraft windshield dataset was used in Ref. [46] and represents the lifetimes of aircraft windshields (in units of 1000 h). A windshield is a multi-layered structure consisting of a strong outer skin and inner layers. If a malfunction of the heating system or the delamination of the outer ply occurs, a failure will occur and replacement of the windshield is required. The failure times for 63 aircraft windshields are as follows: 0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140. Some descriptive statistics for the data are given in Table 6.

Table 6. Descriptive statistics for the aircraft windshield dataset.

	Min	Q ₁	Mean	Q ₃	Max	SD
Value	0.046	1.122	2.085	2.820	5.14	1.245

6.1.4. Dataset IV: Aluminum Fracture Toughness Dataset

This dataset denotes the fracture toughness of Aluminum₂O₃ and is reported in [47]. The values are as follows: 7.060066, 6.418242, 6.289877, 8.215349, 6.546606, 6.674971, 6.674971, 6.418242, 6.033147, 5.134593, 5.776417, 5.391323, 5.262958, 5.853436, 6.431078, 6.033147, 4.017819, 4.004983, 3.440177, 3.555706, 3.465850, 3.029410, 5.622380, 7.355305, 5.583870, 8.741645, 2.451768, 3.414504, 3.350322, 2.156529, 2.618643, 2.669988, 2.734171, 4.877864, 4.788008, 4.762335, 4.210366, 5.006228, 5.134593, 4.877864, 5.262958, 5.006228, 5.198776, 5.134593, 5.070411, 5.134593, 5.776417, 5.776417, 5.391323, 5.840600, 5.968965,

5.262958, 5.455505, 5.519688, 5.776417, 6.033147, 6.610789, 5.519688, 5.776417, 6.289877, 6.418242, 6.867518, 6.610789, 6.739154, 7.445160, 7.509343, 7.573525, 7.380978, 8.022802, 7.766072, 7.573525, 4.621134, 5.262958, 5.776417, 6.803336, 6.225694, 6.803336, 6.995883, 6.546606, 6.803336, 6.674971, 6.803336, 6.739154, 6.097329, 5.776417, 5.391323, 5.134593, 5.327140, 5.455505, 5.519688, 4.813681, 5.070411, 4.505606, 5.301467, 6.931701, 6.418242, 2.695661, 5.904782, 4.107675, 3.209121, 5.262958, 4.492769, 4.107675, 4.236039, 5.904782, 5.519688, 5.519688, 5.776417, 7.060066, 5.904782, 6.289877, 5.519688, 3.850945, 4.364404, 4.749499, 5.648053, 6.289877, 6.289877, 6.418242, 10.654281, 9.370633, 7.188430, 8.728808, 7.958619, 9.884092, 9.370633, 10.525916, 10.140822, 8.343714, 8.600444. Table 7 summarizes the data.

Table 7. Descriptive statistics for the aluminum fracture toughness dataset.

	Min	Q_1	Mean	Q_3	Max	SD
Value	2.157	5.070	5.858	6.675	10.654	1.635

6.1.5. Dataset V: Reliability Dataset

This dataset was presented in Ref. [48] and reported in Ref. [49], and it focuses on the failure and running times for a large system with 30 units. The values are as follows: 2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66. A summarization of this dataset is given in Table 8.

Table 8. Descriptive statistics for the reliability dataset.

	Min	Q_1	Mean	Q_3	Max	SD
Value	0.0200	0.6875	1.7703	2.9825	3.0000	1.1499

6.2. Distribution Comparison

This section compares the adequacy of the NEW distribution in fitting real data with the selected competitors by focusing on the relative performance with respect to goodness-of-fit indices and the predictive accuracy. The acquired information criteria are the following: the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), corrected Akaike information criterion (AICc), consistent Akaike information criterion (CAIC), and Kolmogorov–Smirnov (KS) test with its p -value. It is worth mentioning that the KS statistic is calculated using the estimated parameters; thus, its p -value should be interpreted as approximate. The utilization of these criteria is motivated by their distinct objectives and penalty structures, which allow for a more comprehensive model assessment. The results are shown in Tables 9–13 and Figures 6–10.

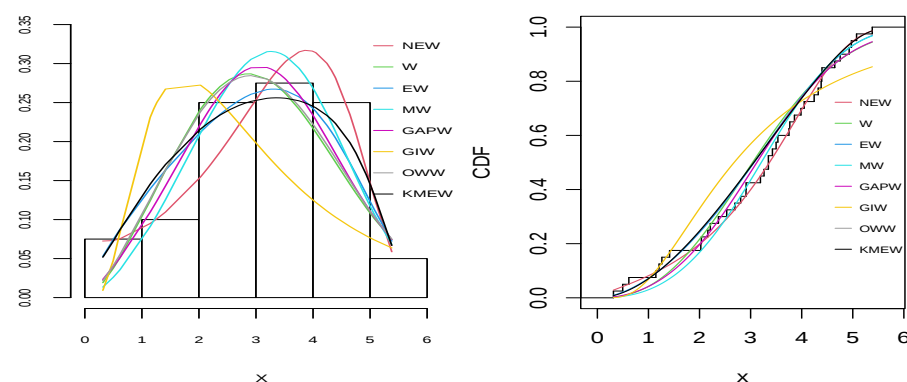


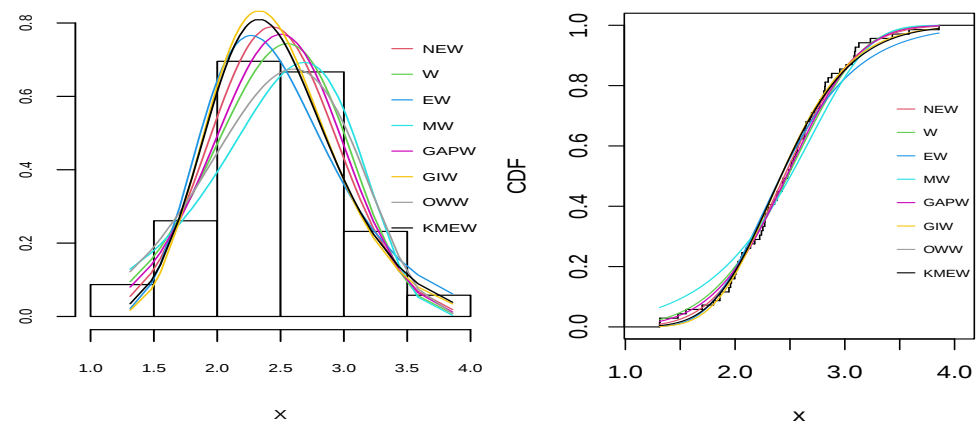
Figure 6. The empirical pdf (right) and cdf (left) for all distributions for the blood cancer dataset.

Table 9. The ML estimates with the standard error in parentheses and the goodness-of-fit measures for the blood cancer dataset

Model	NEW	W	EW	MW	GAPW	GIW	OWW	KMEW
	$\hat{\alpha} = 0.6257$ (0.5026)	$\hat{\alpha} = 2.4991$ (0.3370)	$\hat{\alpha} = 0.3537$ (0.1420)	$\hat{\alpha} = 0.1750$ (0.3029)	$\hat{\alpha} = 0.4367$ (0.3408)	$\hat{\alpha} = 12.0487$ (6.5362)	$\hat{\alpha} = 0.0041$ (0.0020)	$\hat{\alpha} = 0.1959$ (0.0045)
	$\hat{\beta} = 1.1034$ (0.6004)	$\hat{\beta} = 0.2841$ (0.0187)	$\hat{\beta} = 4.9594$ (1.5843)	$\hat{\beta} = 2.3580$ (1.0089)	$\hat{\beta} = 2.1547$ (0.4312)	$\hat{\beta} = 0.5865$ (0.0940)	$\hat{\beta} = 2.4139$ (0.3342)	$\hat{\beta} = 12.5189$ (0.0055)
	$\hat{\lambda} = 0.0155$ (0.0106)	–	$\hat{\lambda} = 0.2229$ (0.0178)	$\hat{\lambda} = 0.0252$ (0.0140)	$\hat{\lambda} = 0.0932$ (0.0714)	$\hat{\lambda} = 5.1389$ (0.7615)	$\hat{\lambda} = 10.9544$ (8.6834)	$\hat{\lambda} = 0.1513$ (0.0216)
Information Criteria and Fit Statistics								
AIC	137.1651	143.1159	138.9426	143.2146	143.6468	164.4864	143.8760	137.3035
BIC	142.2317	146.4937	144.0092	148.2813	148.7134	169.5530	148.9427	142.3702
HQIC	138.9970	144.3372	140.7745	145.0466	145.4787	166.3183	145.7080	139.1355
AICc	137.8317	143.4403	139.6093	143.8813	144.3135	165.1530	144.5427	137.9702
CAIC	142.2317	146.4937	144.0092	148.2813	148.7134	169.5530	148.9427	142.3702
KS	0.0526	0.1183	0.1080	0.1042	0.0958	0.1798	0.1175	0.1080
p-value	0.9998	0.6298	0.7391	0.7769	0.8555	0.1501	0.6380	0.7382

Table 10. The ML estimates with the standard error in parentheses and the goodness-of-fit measures for the single carbon fibers dataset.

Model	NEW	W	EW	MW	GAPW	GIW	OWW	KMEW
	$\hat{\alpha} = 10.2889$ (10.7520)	$\hat{\alpha} = 5.2711$ (0.4688)	$\hat{\alpha} = 32.2677$ (29.2769)	$\hat{\alpha} = 1.7008$ (0.7221)	$\hat{\alpha} = 1.7884$ (0.5257)	$\hat{\alpha} = 14.7747$ (11.2416)	$\hat{\alpha} = 0.0020$ (0.0001)	$\hat{\alpha} = 0.5222$ (0.2405)
	$\hat{\beta} = 0.7366$ (0.2538)	$\hat{\beta} = 0.3761$ (0.0091)	$\hat{\beta} = 1.3159$ (0.2304)	$\hat{\beta} = 0.4892$ (1.9075)	$\hat{\beta} = 5.8641$ (0.2574)	$\hat{\beta} = 1.8116$ (0.4163)	$\hat{\beta} = 4.4884$ (0.2149)	$\hat{\beta} = 2.4747$ (1.5662)
	$\hat{\lambda} = 0.3701$ (0.3670)	–	$\hat{\lambda} = 1.1526$ (0.4072)	$\hat{\lambda} = 0.0063$ (0.0023)	$\hat{\lambda} = 0.0021$ (0.0002)	$\hat{\lambda} = 15.2675$ (2.5918)	$\hat{\lambda} = 5.3847$ (1.5457)	$\hat{\lambda} = 5.2512$ (7.6128)
Information Criteria and Fit Statistics								
AIC	106.5551	107.4331	113.3200	119.5249	107.5790	110.5007	113.2115	110.5007
BIC	113.2574	111.9013	120.0223	126.2273	114.2813	117.2030	119.9138	117.2030
HQIC	109.2141	109.2058	115.9790	122.1840	110.2380	113.1597	115.8705	113.1597
AICc	106.9243	107.6149	113.6892	119.8942	107.9482	110.8699	113.5807	110.8699
CAIC	113.2574	111.9013	120.0223	126.2273	114.2813	117.2030	119.9138	117.2030
KS	0.0445	0.0664	0.0772	0.0951	0.0562	0.0569	0.0921	0.0569
p-value	0.9992	0.9209	0.8060	0.5606	0.9812	0.9788	0.6011	0.9787

**Figure 7.** The empirical pdf (right) and cdf (left) for all distributions for the single carbon fiber dataset.**Table 11.** The ML estimates with the standard error in parentheses and the goodness-of-fit measures for the aircraft windshield dataset.

Model	NEW	W	EW	MW	GAPW	GIW	OWW	KMEW
	$\hat{\alpha} = 3.3537$ (2.6480)	$\hat{\alpha} = 1.6291$ (0.1683)	$\hat{\alpha} = 0.3716$ (0.1802)	$\hat{\alpha} = 0.0259$ (0.1615)	$\hat{\alpha} = 0.4100$ (0.3252)	$\hat{\alpha} = 77.6441$ (48.3211)	$\hat{\alpha} = 0.4005$ (0.5236)	$\hat{\alpha} = 0.2480$ (0.0319)
	$\hat{\beta} = 0.3771$ (0.1298)	$\hat{\beta} = 0.4330$ (0.0350)	$\hat{\beta} = 3.1737$ (1.1407)	$\hat{\beta} = 1.7505$ (0.4095)	$\hat{\beta} = 1.3709$ (0.2532)	$\hat{\beta} = 0.2880$ (0.0393)	$\hat{\beta} = 1.0629$ (0.4131)	$\hat{\beta} = 4.3875$ (1.9824)
	$\hat{\lambda} = 0.6278$ (1.0550)	–	$\hat{\lambda} = 0.2916$ (0.0454)	$\hat{\lambda} = 0.2063$ (0.0546)	$\hat{\lambda} = 0.4608$ (0.2091)	$\hat{\lambda} = 5.4412$ (0.6801)	$\hat{\lambda} = 0.5379$ (0.8457)	$\hat{\lambda} = 0.2969$ (0.1623)

Table 11. Cont.

Model	NEW	W	EW	MW	GAPW	GIW	OWW	KMEW
Information Criteria and Fit Statistics								
AIC	202.1731	204.6354	202.6543	207.0139	205.0444	222.2806	202.5313	203.1560
BIC	208.6025	208.9217	209.0837	213.4433	211.4738	228.7100	208.9607	209.5854
HQIC	204.7018	206.3212	205.1830	209.5427	207.5731	224.8093	205.0600	205.6847
AICc	202.5799	204.8354	203.0611	207.4207	205.4512	222.6874	202.9381	203.5628
CAIC	208.6025	208.9217	209.0837	213.4433	211.4738	228.7100	208.9607	209.5854
KS	0.0672	0.1087	0.0760	0.0787	0.0921	0.1626	0.0694	0.0842
p-value	0.9203	0.4164	0.8324	0.8008	0.6263	0.0635	0.9004	0.7307

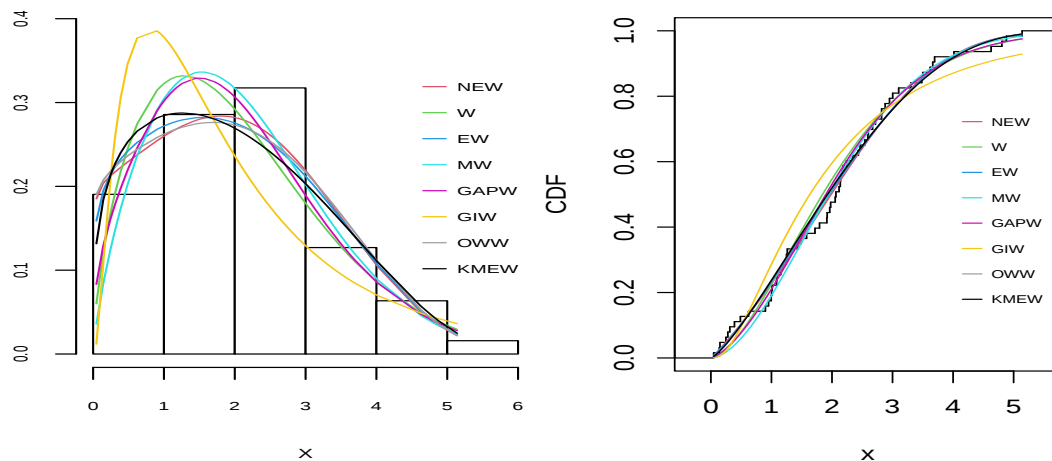


Figure 8. The empirical pdf (right) and cdf (left) for all distributions for the aircraft windshield dataset.

Table 12. The ML estimates with the standard error in parentheses and the goodness-of-fit measures for the aluminum fracture toughness dataset.

Model	NEW	W	EW	MW	GAPW	GIW	OWW	KMEW
$\hat{\alpha}$	23.3019 (18.0119)	3.8040 (0.2446)	4.5830 (6.1549)	0.3146 (0.0942)	0.5575 (0.1767)	54.4289 (37.1112)	0.6762 (0.1521)	0.2183 (0.1224)
$\hat{\beta}$	0.2020 (0.0404)	0.1547 (0.0038)	1.9154 (1.0652)	1.4976 (0.4971)	3.3734 (0.0675)	0.9082 (0.1428)	0.9068 (0.0750)	2.0311 (1.3359)
$\hat{\lambda}$	1.5290 (3.5645)	–	0.2489 (0.1365)	0.0076 (0.0029)	0.0025 (0.0003)	21.2000 (2.6427)	0.0237 (0.0091)	4.1129 (5.8167)
Information Criteria and Fit Statistics								
AIC	499.9850	503.7075	501.2128	522.2212	505.7049	502.3039	532.3287	501.8669
BIC	508.5876	509.4425	509.8154	530.8238	514.3075	510.9065	540.9313	510.4695
HQIC	503.4806	506.0378	504.7083	525.7167	509.2004	505.7995	535.8242	505.3624
AICc	500.1755	503.8019	501.4033	522.4116	505.8953	502.4944	532.5192	502.0574
CAIC	508.5876	509.4425	509.8154	530.8238	514.3075	510.9065	540.9313	510.4695
KS	0.0714	0.0964	0.0939	0.1207	0.0908	0.1014	0.1356	0.0937
p-value	0.5206	0.1785	0.2017	0.0452	0.2343	0.1381	0.0167	0.2032

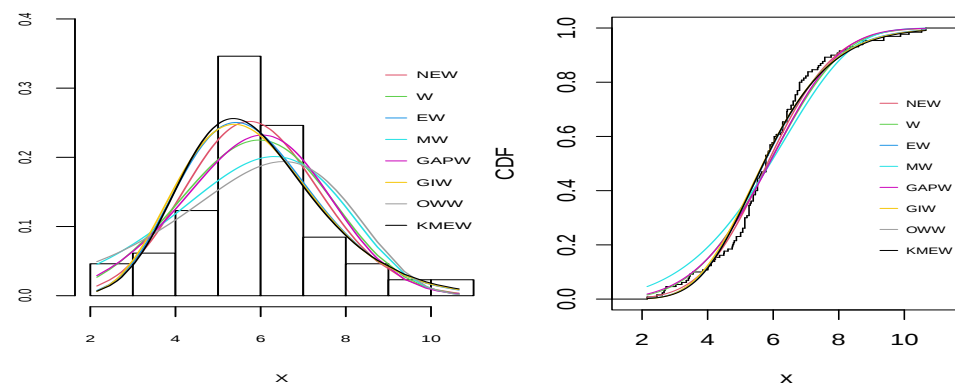
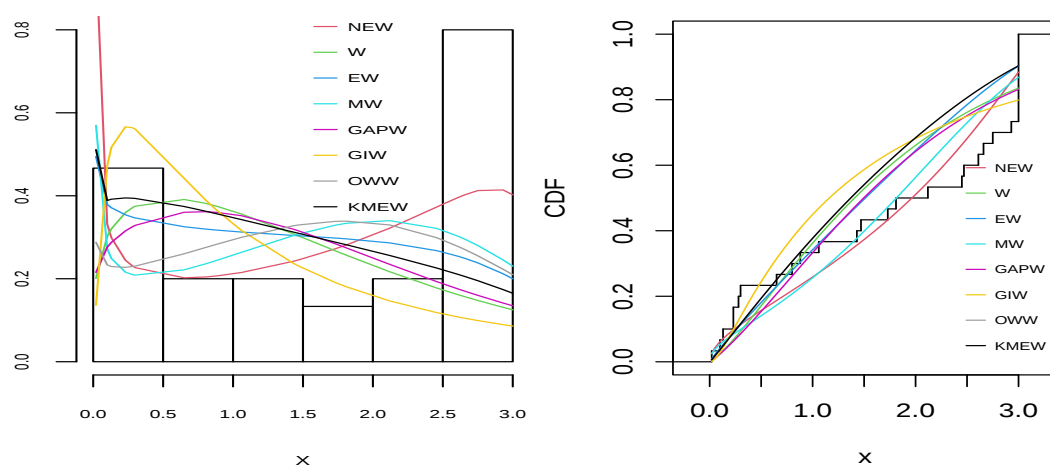


Figure 9. The empirical pdf (right) and cdf (left) for all distributions for the aluminum fracture toughness dataset.

Table 13. The ML estimates with the standard error in parentheses and the goodness-of-fit measures for the reliability dataset.

Model	NEW	W	EW	MW	GAPW	GIW	OWW	KMEW
	$\hat{\alpha} = 0.1004$ (0.0185)	$\hat{\alpha} = 1.2649$ (0.2044)	$\hat{\alpha} = 0.1314$ (0.0240)	$\hat{\alpha} = 0.7132$ (0.2111)	$\hat{\alpha} = 0.5369$ (0.3685)	$\hat{\alpha} = 55.4867$ (49.2265)	$\hat{\alpha} = 1.1340$ (0.7016)	$\hat{\alpha} = 0.2926$ (0.0025)
	$\hat{\beta} = 5.1482$ (0.0025)	$\hat{\beta} = 0.5317$ (0.0797)	$\hat{\beta} = 6.9845$ (0.0040)	$\hat{\beta} = 0.4535$ (0.2199)	$\hat{\beta} = 1.1494$ (0.2397)	$\hat{\beta} = 0.2214$ (0.0447)	$\hat{\beta} = 0.8019$ (0.2543)	$\hat{\beta} = 7.0663$ (0.0026)
	$\hat{\lambda} = 0.0426$ (0.0023)	–	$\hat{\lambda} = 0.3109$ (0.0039)	$\hat{\lambda} = 0.1451$ (0.0714)	$\hat{\lambda} = 0.6249$ (0.2455)	$\hat{\lambda} = 4.5377$ (0.9407)	$\hat{\lambda} = 0.1397$ (0.1554)	$\hat{\lambda} = 0.1498$ (0.0248)
Information Criteria and Fit Statistics								
AIC	73.5482	96.3174	83.4906	85.8168	97.2778	107.8242	88.0998	87.0004
BIC	77.7518	99.1198	87.6942	90.0204	101.4814	112.0278	92.3034	91.2040
HQIC	74.8930	97.2139	84.8354	87.1616	98.6226	109.1690	89.4445	88.3451
AICc	74.4713	96.7619	84.4137	86.7399	98.2009	108.7473	89.0228	87.9234
CAIC	77.7518	99.1198	87.6942	90.0204	101.4814	112.0278	92.3034	91.2040
KS	0.1556	0.2193	0.2380	0.1803	0.2084	0.2099	0.1919	0.2615
p-value	0.4616	0.1113	0.0668	0.2830	0.1475	0.1418	0.2190	0.0329

**Figure 10.** The empirical pdf (right) and cdf (left) for all distributions for the reliability dataset.

As can be seen from Tables 9–13 and Figures 6–10, the NEW distribution achieves a minimum value of all information criteria and a higher p -value among all distributions. This indicates the good ability of the NEW distribution to fit the data accurately compared with other extensions of the Weibull distribution.

7. Conclusions

Researchers' interest in introducing new statistical distributions has resulted in an influx of data modeling in a wide range of applications. This study aimed to introduce a new, adaptable extension of the Weibull distribution by applying a logarithmic transformation to the exponentiated Weibull distribution. This new distribution, called the NEW distribution, has the ability to model different forms of pdfs and hazard functions, making it flexible in fitting complex behaviors. Some statistical properties of the NEW distribution were studied. Four estimation methods—the MLE, LS, CVM, and PE methods—were utilized to estimate the distribution parameters. Moreover, the efficiency of the estimation methods was compared using three simulation cases. The results demonstrated that the ML approach, followed by the PE method, achieved a higher accuracy in estimating the NEW parameters in comparison with the other methods. All the methods achieved the consistency property when the sample size increased. Additionally, five datasets were modeled with the NEW distribution and other formerly specified extensions of the Weibull distribution, and their results were compared. The obtained results evidenced that the NEW distribution is superior in terms of fitting the data; thus, it is considered a good alternative to other Weibull extensions in modeling various forms of data.

Although the present study indicates the applicability and usefulness of the proposed model, future research may be able to extend these findings in various directions. For example, using the penalized likelihood method or Bayesian technique with informative prior can help stabilize the estimators, especially in the case of small sample settings. Moreover, constructing a regression model for the NEW distribution would have considerable research value. Furthermore, the NEW distribution can be applied to censored data, which are popular in the fields of reliability engineering, economics, and finance.

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