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Coupled Fixed Points in (q_1, q_2)-Quasi-Metric Spaces

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Abstract

This paper presents a new coupled fixed-point theorem for a pair of set-valued mappings acting on the Cartesian product of (m_1, m_2)- and (n_1, n_2)-quasi-metric spaces. Within the general, non-symmetric quasi-metric setting, we establish the existence of an approximate coupled fixed point. Moreover, under the additional assumption of q_0-symmetry, we guarantee the existence of a coupled fixed point. Together, these results extend and unify several known theorems in fixed-point theory for quasi-metric and asymmetric spaces. We illustrate the obtained results regarding fixed points when the underlying space is equipped with a graph structure and, thus, sufficient conditions are found to guarantee the existence of a subgraph with a loop with a length greater than or equal to 2.

Keywords: coupled fixed point; set-valued mapping; quasi-metric space; q_0-symmetry; approximate fixed point; nonlinear analysis

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1. Introduction

A central tool for analyzing mappings between metric spaces is Banach's contraction principle [1], which guarantees the existence of a fixed point for a contractive mapping. Such mappings occur across both pure and applied mathematics; recent examples include advances inn systems of nonlinear matrix equations [2] and studies of market equilibrium in oligopoly settings [3]. The classical theorem of Banach [1] has spawned an enormous variety of generalizations—too many to list comprehensively—so we focus on those most relevant to our investigation.

One line of generalization alters the underlying space. Working in b-metric spaces [4], modular function spaces [5], partially ordered metric spaces [6], or quasi-metric spaces [7] allows one to relax the usual completeness assumptions; see also [8–10] for developments within quasi-metric frameworks. A second direction alters the notion of a fixed point. Instead of a point $x \in X$ satisfying $x = Tx$, one considers a bivariate mapping $T : X \times X \to X$ and calls an ordered pair $(x, y) \in X \times X$ a coupled fixed point of T if $x = T(x, y)$ and $y = T(y, x)$ [11]. In [11], the setting is a normed space partially ordered by a cone; subsequently, this cone-ordered normed framework was replaced by a partially ordered metric space in [6]. Since the appearance of [6], the concept of coupled fixed points has been extensively studied. A known limitation of this framework is that a coupled fixed point (x, y) often collapses to the diagonal, i.e., $x = y$, because the definition effectively solves the symmetric pair of equations $x = T(x, y)$ and $y = T(y, x)$. To address genuinely non-symmetric systems, ref. [12] proposed modifying the notion by replacing a single bivariate self-map with an ordered pair of mappings $F, G : X \times X \to X$ and declaring (x, y) to be a coupled fixed point of (F, G) when

$$x = F(x, y) \quad \text{and} \quad y = G(x, y).$$

This formulation arises naturally in studies of market equilibria for duopoly models [3]. Observe that when $G(x, y) = F(y, x)$, one recovers the classical coupled fixed-point notion from [6,11].

Another influential direction equips the underlying space with a graph structure, a viewpoint initiated in [13]. Following their work, a growing body of literature has developed fixed point resulting in graph-based settings, including multi-valued mappings in b-metric spaces [14], mappings in metric spaces endowed with a directed graph [15], multi-valued mappings in cone metric spaces with a directed graph [16], and monotone mappings in modular function spaces [17].

It is worth mentioning that the first systematic investigation, which introduced the term "quasi-metric spaces," can be found in [18].

2. Materials and Methods

We begin by recalling the fundamental concepts and notation used in the theory of quasi-metric spaces. Throughout, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural numbers and real numbers, respectively. We use the capital Latin letters X, Y, and Z for arbitrary sets, while the lowercase letters x, y, z, u, v, w represent elements of these sets.

Our presentation follows the treatments in [7–10], whose terminology and notation are mutually consistent and will be adopted here.

Definition 1 ([7])**.** *Let X be a nonempty set, $q_1, q_2 \geq 1$, and the mapping $d : X \times X \to [0, \infty)$, satisfying the following for all $x, y, z \in X$:*

- *(identity axiom): $d(x, y) = 0$ if and only if $x = y$ for any $x, y \in X$*
- *(relaxed triangle inequality): there holds the inequality*

$$d(x, y) \leq q_1 d(x, z) + q_2 d(z, y)$$

The function $d(\cdot, \cdot)$, which satisfies the identity axiom and the relaxed triangle inequality, is called a (q_1, q_2)-quasi-metric.

If in Definition 1 we set $q_1 = q_2 = 1$, we obtain the classical definition of a quasi-metric space introduced in [18], though with different notation. If $q_1 = q_2 > 1$ we get the quasi-b-metric space introduced in [19]. Although the definitions of (q_1, q_2)-quasi-metric spaces appear more general than that of quasi-b-metric space, they coincide. The results formulated in the context of $(q1, q2)$-quasi-metric spaces enable a better estimate of convergence and therefore are preferred in investigations.

Definition 2 ([7])**.** *Let X be a nonempty set, $q_1, q_2 \geq 1$ and a mapping, and $d : X \times X \to [0, \infty)$ be a (q_1, q_2)-quasi-metric. If $d(\cdot, \cdot)$ satisfies*

- *(symmetry axiom): $d(x, y) = d(y, x)$ for every $x, y \in X$*

then $d(\cdot, \cdot)$ is referred to as a symmetric (q_1, q_2)-quasi-metric.

It is possible to relax the symmetry axiom.

Definition 3 ([7]). *Let X be a nonempty set, $q_1, q_2 \geq 1$, and the mapping $d : X \times X \to [0, \infty)$ be a (q_1, q_2)-quasi-metric. If $d(\cdot, \cdot)$ satisfies*

- *(weaker symmetry axiom) there exists $q_0 \geq 1$ so that the inequality $d(x, y) \leq q_0 d(y, x)$ holds for all $x, y \in X$*

then it is referred to as a q_0-symmetric (q_1, q_2)-quasi-metric.

Let X be a nonempty set, let $q_1, q_2 \geq 1$, and let $d \colon X \times X \to [0, \infty)$. If d is a (q_1, q_2)-quasi-metric, we refer to (X, d) as a (q_1, q_2)-quasi-metric space. If, in addition, $d(x, y) = d(y, x)$ for all $x, y \in X$, then (X, d) is called a symmetric (q_1, q_2)-quasi-metric space. If d satisfies the weaker symmetry condition $d(x, y) \leq q_0\, d(y, x)$ for some $q_0 > 0$, we call (X, d) a q_0-symmetric (q_1, q_2)-quasi-metric space. In particular, when $q_0 = 1$ and $q_1 = q_2 > 1$, a symmetric (q_1, q_1)-quasi-metric space is precisely a b-metric space. Extensive historical notes, including interesting and little-known references as well as up-to-date results, can be found in [20].

Note that for $q_0 = 1$, any q_0-symmetric (q_1, q_2)-quasi-metric space becomes symmetric; and for $q_0 = q_1 = q_2 = 1$, (X, d) is a (standard) metric space. Given any quasi-metric d, its conjugate $\bar{d}(x, y) := d(y, x)$ is a (q_2, q_1)-quasi-metric.

Definition 4 ([7]). *A (q_1, q_2)-quasi-metric space (X, d) is said to be weakly symmetric whenever the following holds:*

- *(weakly symmetry axiom): if $\lim_{n\to\infty} d(\xi, x_n) = 0$ implies $\lim_{n\to\infty} d(x_n, \xi) = 0$.*

Any q_0-symmetric (q_1, q_2)-quasi-metric space is weakly symmetric. The converse fails.

Definition 5 ([7]). *Let (X, d) be a (q_1, q_2)-quasi-metric space.*

- *An open ball centered at a point $x_0 \in X$ with radius $r > 0$ is defined by*

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}.$$

- *A closed ball centered at x_0 with radius $r > 0$ is given by*

$$B[x_0, r] = \{x \in X : d(x_0, x) \leq r\}.$$

A subset $U \subset X$ is considered open if for every $u \in U$ there exists $\varepsilon > 0$ such that $B(u, \varepsilon) \subset U$. A family of open sets determines a topology on any (q_1, q_2)-quasi-metric space (X, d). As usual, a set is closed if its complement is open.

A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said to converge to $x_0 \in X$ in the (q_1, q_2)-quasi-metric space (X, d) if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $x_n \in B(x_0, \varepsilon)$ for all $n \geq N$; we write $\lim_{n\to\infty} x_n = x_0$. It is straightforward to verify that, in a (q_1, q_2)-quasi-metric space, this is equivalent to $\lim_{n\to\infty} d(x_0, x_n) = 0$.

In a weakly symmetric (q_1, q_2)-quasi-metric space, every convergent sequence has a unique limit. By contrast, uniqueness of limits may fail in a general (q_1, q_2)-quasi-metric space (see, e.g., [9], where Examples 3.5 and 3.6 are some examples of quasi-metric spaces where some convergent sequences have a continuum of limits, and [18]).

Definition 6 ([7]). *A sequence $\{x_n\}$ in a (q_1, q_2)-quasi-metric space (X, d) is called a fundamental sequence, or a Cauchy sequence, if for every $\varepsilon > 0$ there is an N such that for all $m > n \geq N$, we have $d(x_m, x_n) < \varepsilon$.*

A (q_1, q_2)-quasi-metric space (X, d) is said to be complete if each of its fundamental sequences has a limit.

When $q_1 = q_2$, the pair (X, d) specializes to a quasi-metric space, which—depending on the context—is also termed a b-metric space [4,21]. The framework of (q_1, q_2)-quasi-metric spaces was introduced in [7] and further developed in [8–10] in connection with covering mappings, where sufficient conditions were obtained for the existence of coincidence points of two mappings (one a covering map and the other Lipschitz) defined in (q_1, q_2)-quasi-metric spaces.

In what follows we work exclusively within the class of (q_1, q_2)-quasi-metric spaces; whenever q_0-symmetry is needed, this assumption will be stated explicitly.

Assume that X and Y are endowed with the same quasi-metric d. For a point $x \in X$ and a subset $A \subseteq X$, define

$$\mathrm{dist}(x, A) := \inf_{a \in A} d(x, a),$$

with the convention $\mathrm{dist}(x, \varnothing) = +\infty$. For $\varepsilon \geq 0$, the ε-neighborhood of A is

$$A(\varepsilon) := \{\, x \in X : \mathrm{dist}(x, A) \leq \varepsilon \,\}.$$

A set-valued mapping $F : X \rightrightarrows Y$ assigns to each $x \in X$ a (possibly empty) subset $F(x) \subset Y$. Its graph and inverse are, respectively,

$$\mathrm{Gr}(F) := \{(x, y) \in X \times Y : \ y \in F(x)\}, \qquad F^{-1}(y) := \{x \in X : \ y \in F(x)\}.$$

We say F is closed-valued if $F(x)$ is closed in Y for every $x \in X$, and closed if $\mathrm{Gr}(F)$ is a closed subset of $X \times Y$. Every closed mapping is closed-valued, though the converse need not hold.

As an immediate consequence of Lemma 1, if $p_1 = p_2$ and $q_1 = q_2$, then $(X_1 \times X_2, d)$ is a (p_1, p_2)-quasi-metric space with respect to d.

Let X be a (q_1, q_2)-quasi-metric space, $F : X \rightrightarrows X$ a set-valued mapping, and $\varepsilon > 0$. A point $x \in X$ is called a fixed point of F if $x \in F(x)$; the set of all fixed points is

$$\mathrm{Fix}F := \{\, x \in X : \ x \in F(x) \,\}.$$

An approximate (or ε–) fixed point of F is a point x with $\mathrm{dist}(x, F(x)) \leq \varepsilon$. The corresponding set is

$$\mathrm{Fix}F(\varepsilon) := \{\, x \in X : \ \mathrm{dist}(x, F(x)) \leq \varepsilon \,\}.$$

For completeness, we also recall the extension of the coupled fixed-point notion to multi-valued maps.

Definition 7 ([22]). *A point $(x, y) \in X \times X$ is said to be a coupled fixed point of the set-valued map $F : X \times X \rightrightarrows X$ if $x \in F(x, y)$ and $y \in F(y, x)$.*

Subsequently, Definition 7 was extended to encompass an ordered pair of multi-valued mappings, leading to a notion of coupled fixed points for (F_1, F_2).

Definition 8 ([23]). *A point $(x, y) \in X \times X$ is said to be a generalized coupled fixed point of the ordered pair of set-valued maps $F_1 : X \times Y \rightrightarrows X$ and $F_2 : X \times Y \to Y$, provided that $x \in F_1(x, y)$ and $y \in F_2(x, y)$.*

3. Results

In this section, let (X, d) and (Y, σ) be two quasi-metric spaces. We consider an ordered pair of set-valued mappings, $F_1 : X \times Y \rightrightarrows X$ and $F_2 : X \times Y \rightrightarrows Y$, and we are interested in the existence of a generalized coupled fixed point of (F_1, F_2), that is, a pair $(x, y) \in X \times Y$ satisfying

$$x \in F_1(x, y) \qquad \text{and} \qquad y \in F_2(x, y).$$

In parallel with the usual notion of an approximate fixed point for a single multi-valued map, we will also introduce an approximate coupled fixed point adapted to the ordered pair (F_1, F_2).

Definition 9. *Let $\varepsilon, \mu > 0$. An approximate or ε, μ-fixed point of the ordered pair $F = (F_1, F_2)$ is a point (x, y) such that $\operatorname{dist}(x, F_1(x, y)) \le \varepsilon$ and $\operatorname{dist}(y, F_2(x, y)) \le \mu$. A set of such points is denoted by*

$$\operatorname{Fix} F(\varepsilon, \mu) := \{x \in X, y \in Y : \operatorname{dist}(x, F_1(x, y)) \le \varepsilon, \operatorname{dist}(y, F_2(x, y)) \le \mu\}.$$

Definition 10. *Let $F_1 : X \times Y \rightrightarrows X$ and $F_2 : X \times Y \rightrightarrows Y$. A sequence $(x_k, y_k) \cup X \times Y$ is called a sequence of successive approximations of (F_1, F_2) if $x_{k+1} \in F_1(x_k, y_k)$ and $y_{k+1} \in F_2(x_k, y_k)$ for all $k \in \mathbb{N}$.*

Theorem 1. *Let (X, d) be a (m_1, m_2)-quasi-metric space, (Y, σ) be a (n_1, n_2)-quasi-metric space with constants $m_1, m_2, n_1, n_2 \ge 1$, and $q_1 = \max\{m_1, n_1\}, q_2 = \max\{m_2, n_2\}$. Let U be an open subset of X; V be an open subset of Y; $\bar{x} \in U$; $\bar{y} \in V$; and $F_1 : X \times Y \rightrightarrows X$ and $F_2 : X \times Y \rightrightarrows Y$ be set-valued mappings. Suppose there exist constants $\alpha, \beta > 0$ and $\lambda \in (\max\{\alpha, \beta\}, 1/q_2)$ such that*

(a) $\operatorname{dist}(\bar{x}, F_1(\bar{x}, \bar{y})) + \operatorname{dist}(\bar{y}, F_2(\bar{x}, \bar{y})) < \dfrac{1 - q_2\lambda}{q_1} \min\left\{\operatorname{dist}(\bar{x}, X \setminus U), \operatorname{dist}(\bar{y}, Y \setminus V)\right\}$,

(b) $\operatorname{dist}(x, F_1(x, y)) + \operatorname{dist}(y, F_2(x, y)) \le \alpha\rho(u, x) + \beta\sigma(v, y)$ *for all $(x, y), (u, v) \in U \times V$ such that $x \in F_1(u, v)$, $y \in F_2(u, v)$, and*

$$\alpha d(u, x) < \operatorname{dist}(x, X \setminus U), \;\; \beta\sigma(v, y) < \operatorname{dist}(y, Y \setminus V).$$

Then there is a sequence $(x_k, y_k) \subset X \times Y$ of successive approximations of (F_1, F_2), starting from $(\bar{x}, \bar{y})$, such that the following hold:

(A) *For every $\varepsilon > 0$, $\mu > 0$, $(x_k, y_k) \in \operatorname{Fix} F(\varepsilon, \mu) \cap U \times V$.*

$$d(\bar{x}, x_k) + \sigma(\bar{y}, y_k) \le \frac{q_1}{1 - q_2\lambda}\Big(\operatorname{dist}(\bar{x}, F_1(\bar{x}, \bar{y})) + \operatorname{dist}(\bar{y}, F_2(\bar{x}, \bar{y}))\Big)$$

(B) *If, moreover, X and Y are complete, both F_1 and F_2 have closed graphs in $X \times Y \times X$ and $X \times Y \times Y$, respectively, and if X is p_0-symmetric and Y is q_0-symmetric, respectively, then there exist the elements $x^* \in X$ and $y^* \in Y$ such that $\{x_k\}_{k=0}^{\infty}$ converges to x^*, $\{y_k\}_{k=0}^{\infty}$ converges to y^*, and*

$$(x^*, y^*) \in \operatorname{Fix}(F_1 \times F_2) \cap (U \times V),$$

$$d(\bar{x}, x^*) + \sigma(\bar{y}, y^*) \le \frac{q_1}{1 - q_2\lambda}\big(\operatorname{dist}(\bar{x}, F_1(\bar{x}, \bar{y})) + \operatorname{dist}(\bar{y}, F_2(\bar{x}, \bar{y}))\big).$$

Proof. Let us choose $\overline{x}$ and $\overline{y}$ values that satisfy the assumptions (a) and (b). For brevity, we denote $S(x, y) = \operatorname{dist}(x, F_1(x, y)) + \operatorname{dist}(y, F_2(x, y))$.

We distinguish two cases: $S(\overline{x}, \overline{y}) = 0$ and $S(\overline{x}, \overline{y}) > 0$.

If $S(\overline{x}, \overline{y}) = 0$, then $(\overline{x}, \overline{y})$ is the generalized coupled fixed point, and the proof is finished.

Let us assume that $S(\overline{x}, \overline{y}) > 0$.

According to (a), there is $l > 0$ such that the following inequalities hold:

$$\frac{q_1}{1 - q_2\lambda} S(\overline{x}, \overline{y}) < l < \min\{\mathrm{dist}(\overline{x}, X \setminus U), \mathrm{dist}(\overline{y}, Y \setminus V)\}. \quad (1)$$

Through induction, we construct two sequences, $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$, starting with

$$x_0 = \overline{x} \text{ and } y_0 = \overline{y}.$$

From (1) we can pick up $x_1 \in F_1(x_0, y_0)$ and $y_1 \in F_2(x_0, y_0)$, satisfying

$$\frac{q_1}{1 - q_2\lambda}(d(x_0, x_1) + \sigma(y_0, y_1)) < l.$$

Thus, we can write the following chain of inequalities:

$$d(x_0, x_1) + \sigma(y_0, y_1) < \frac{(1 - q_2\lambda)l}{q_1} < \frac{1 - q_2\lambda}{q_1} \min\{\mathrm{dist}(x_0, X \setminus U), \mathrm{dist}(y_0, Y \setminus V)\}. \quad (2)$$

Since $q_1 \geq 1$ and $q_2\lambda \in (0, 1)$, it follows that

$$d(x_0, x_1) < \mathrm{dist}(x_0, X \setminus U) \text{ and } \sigma(y_0, y_1) < \mathrm{dist}(y_0, Y \setminus V),$$

and hence, $x_1 \in U$, $y_1 \in V$.

Using the relaxed triangular inequality and the inclusions $x_1 \in U$ and $y_1 \in V$, we get

$$\mathrm{dist}(x_0, X \setminus U) \leq q_1 d(x_0, x_1) + q_2 \mathrm{dist}(x_1, X \setminus U)$$

and

$$\mathrm{dist}(y_0, Y \setminus V) \leq q_1 \sigma(y_0, y_1) + q_2 \mathrm{dist}(y_1, Y \setminus V).$$

In order to fit the next inequalities into the text field let us use the notation $\rho_n = d(x_n, x_{n+1}) + \sigma(y_n, y_{n+1})$, we can write the chain of inequalities

$$\begin{aligned}
\rho_0 &< \tfrac{(1-q_2\lambda)l}{q_1} \\
&< \tfrac{1-q_2\lambda}{q_1} \min\{\mathrm{dist}(x_0, X \setminus U), \mathrm{dist}(y_0, Y \setminus V)\} \\
&\leq \tfrac{1-q_2\lambda}{q_1} \min\{(q_1 d(x_0, x_1) + q_2 \mathrm{dist}(x_1, X \setminus U), q_1\sigma(y_0, y_1) + q_2 \mathrm{dist}(y_1, Y \setminus V)\} \\
&\leq \tfrac{1-q_2\lambda}{q_1} q_1\rho_0 + \tfrac{1-q_2\lambda}{q_1} q_2 \min\{\mathrm{dist}(x_1, X \setminus U), \mathrm{dist}(y_1, Y \setminus V)\}.
\end{aligned}$$

Thus, the following holds:

$$(1 - (1 - q_2\lambda))\rho_0 < \frac{1 - q_2\lambda}{q_1} q_2 \min\{\mathrm{dist}(x_1, X \setminus U), \mathrm{dist}(y_1, Y \setminus V)\}$$

Consequently we end with the following inequality, keeping in mind the assumptions $q_1 \geq 1$ and $q_2\lambda \in (0, 1)$:

$$\begin{aligned}
\lambda(d(x_0, x_1) + \sigma(y_0, y_1)) &< \tfrac{1-q_2\lambda}{q_1} \min\{\mathrm{dist}(x_1, X \setminus U), \mathrm{dist}(y_1, Y \setminus V)\} \\
&< \min\{\mathrm{dist}(x_1, X \setminus U), \mathrm{dist}(y_1, Y \setminus V)\}.
\end{aligned}$$

Since $\lambda > \max\{\alpha, \beta\}$, we have

$$\begin{aligned}
&\alpha d(x_0, x_1) < \mathrm{dist}(x_1, X \backslash U), \\
&\beta\sigma(y_0, y_1) < \mathrm{dist}(y_1, Y \backslash V).
\end{aligned} \quad (3)$$

Using (b) and (2), we get

$$\begin{aligned} \operatorname{dist}(x_1, F_1(x_1, y_1)) + \operatorname{dist}(y_1, F_2(x_1, y_1)) &\leq \alpha d(x_0, x_1) + \beta\sigma(y_0, y_1) \\ &< \lambda(d(x_0, x_1) + \sigma(y_0, y_1)) \\ &< l\lambda\frac{(1-q_2\lambda)}{q_1} \end{aligned} \tag{4}$$

Hence,

$$S(x_1, y_1) < \min\left\{\frac{1-q_2\lambda}{q_1}\operatorname{dist}(x_1, X \setminus U), \frac{1-q_2\lambda}{q_1}\operatorname{dist}(y_1, Y \setminus V),\ l\lambda\frac{1-q_2\lambda}{q_1}\right\}. \tag{5}$$

From (4) and (5), it follows that the possibility of choosing $x_2 \in F_1(x_1, y_1)$ and $y_2 \in F_2(x_1, y_1)$ simultaneously satisfies

$$\rho_1 < \min\left\{\lambda\rho_0, \frac{1-q_2\lambda}{q_1}\operatorname{dist}(x_1, X \setminus U), \frac{1-q_2\lambda}{q_1}\operatorname{dist}(y_1, Y \setminus V),\ l\lambda\frac{1-q_2\lambda}{q_1}\right\}.$$

Let us denote $\rho_{n,m} = d(x_n, x_m) + \sigma(y_n, y_m)$ and

$$W_n = \min\{\operatorname{dist}(x_n, X \setminus U), \operatorname{dist}(y_n, Y \setminus V)\}.$$

Thus, $\rho_n = \rho_{n,n+1}$. It is easy to observe that for any $x > 1$ and $a \in (0, 1)$, the inequality $(x + a)\frac{1-a}{x} < 1$ holds. By using the relaxed triangular inequality and the last observation with $x = q_1$ and $a = \lambda q_2$ we get an upper estimate:

$$\begin{aligned} \rho_{0,2} &\leq q_1 d(x_0, x_1) + q_2 d(x_1, x_2) + q_1\sigma(y_0, y_1) + q_2\sigma(y_1, y_2) \\ &= q_1(d(x_0, x_1) + \sigma(y_0, y_1)) + q_2(d(x_1, x_2) + \sigma(y_1, y_2)) \\ &= (q_1 + \lambda q_2)\rho_0 \\ &< (q_1 + \lambda q_2)\frac{1-q_2\lambda}{q_1}W_0 < W_0 \\ &= \min\{\operatorname{dist}(x_0, X \setminus U), \operatorname{dist}(y_0, Y \setminus V)\}. \end{aligned}$$

Hence, $x_2 \in U$ and $y_2 \in V$.

The inequality

$$\begin{aligned} \rho_1 &< \frac{1-q_2\lambda}{q_1}W_1 \\ &\leq \frac{1-q_2\lambda}{q_1}\min\{q_1 d(x_1, x_2) + q_2\operatorname{dist}(x_2, X \setminus U)), q_1\sigma(y_1, y_2) + q_2\operatorname{dist}(y_2, Y \setminus V))\} \\ &\leq \frac{1-q_2\lambda}{q_1}q_1(d(x_1, x_2) + \sigma(y_1, y_2)) + \frac{1-q_2\lambda}{q_1}q_2 W_2 \end{aligned}$$

yields

$$\lambda\rho_1 < \frac{1-q_2\lambda}{q_1}W_2.$$

Combining the conditions x_n and y_n for $n = 0, 1, 2$, we will choose $\{x_n\}_{n=3}^{\infty}$ and $\{y_n\}_{n=3}^{\infty}$ as the remaining sequences to verify the following assumptions:

$$(x_n, y_n) \in U \times V, \tag{6}$$

$$\begin{aligned} x_n &\in F_1(x_{n-1}, y_{n-1}) \\ y_n &\in F_2(x_{n-1}, y_{n-1}), \end{aligned} \tag{7}$$

$$d(x_{n-1}, x_n) + \sigma(y_{n-1}, y_n) < \lambda(d(x_{n-2}, x_{n-1}) + \sigma(y_{n-2}, y_{n-1})), \tag{8}$$

and

$$\lambda(d(x_{n-1}, x_n) + \sigma(y_{n-1}, y_n)) < \frac{1 - q_2\lambda}{q_1} \min\{\text{dist}(x_n, X \setminus U), \text{dist}(y_n, Y \setminus V)\}. \quad (9)$$

Suppose that $\{x_k\}_{k=0}^n$ and $\{y_k\}_{k=0}^n$ have been defined to satisfy (6)–(9). We will show that we can choose x_{n+1} and y_{n+1} values that will verify the same conditions.

From (9), we get

$$\begin{aligned} S(x_n, y_n) &\leq \alpha d(x_{n-1}, x_n) + \beta\sigma(y_{n-1}, y_n) \\ &< \lambda\rho_{n-1} < \lambda^n \rho_0 \\ &< l\lambda^n \tfrac{1-q_2\lambda}{q_1} \end{aligned}$$

and hence,

$$S(x_n, y_n) < \frac{1 - q_2\lambda}{q_1} \min\{\text{dist}(x_n, X \setminus U), \text{dist}(y_n, Y \setminus V),\ l\lambda^n\}.$$

Thus, we can choose $x_{n+1} \in F_1(x_n, y_n)$ and $y_{n+1} \in F_2(x_n, y_n)$ so that

$$\rho_n < \min\left\{\lambda\rho_{n-1}, \frac{1 - q_2\lambda}{q_1}\text{dist}(x_n, X \setminus U), \frac{1 - q_2\lambda}{q_1}\text{dist}(y_n, Y \setminus V),\ l\lambda^n \frac{1 - q_2\lambda}{q_1}\right\}.$$

We estimate

$$\begin{aligned} \rho_{0,n+1} &= d(x_0, x_{n+1}) + \sigma(y_0, y_{n+1}) \\ &\leq q_1 d(x_0, x_1) + q_2 d(x_1, x_{n+1}) + q_1\sigma(y_0, y_1) + q_2\sigma(y_1, y_{n+1}) \\ &= q_1(d(x_0, x_1) + \sigma(y_0, y_1)) + q_2(d(x_1, x_{n+1}) + \sigma(y_1, y_{n+1})) \\ &= q_1(d(x_0, x_1) + \sigma(y_0, y_1)) \\ &\quad + q_1 q_2(d(x_1, x_2) + \sigma(y_1, y_2)) + q_2^2(d(x_2, x_{n+1}) + \sigma(y_2, y_{n+1})) \\ &\leq \cdots \leq q_1(d(x_0, x_1) + \sigma(y_0, y_1)) \textstyle\sum_{j=0}^{n} (q_2\lambda)^j \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &= q_1(d(x_0, x_1) + \sigma(y_0, y_1)) \tfrac{1-(q_2\lambda)^{n+1}}{1-q_2\lambda} < \tfrac{q_1}{1-q_2\lambda}(d(x_0, x_1) + \sigma(y_0, y_1)) \\ &< \tfrac{q_1}{1-q_2\lambda} \cdot \tfrac{1-q_2\lambda}{q_1} \min\{\text{dist}(x_0, X \setminus U), \text{dist}(y_0, Y \setminus V)\} \\ &= \min\{\text{dist}(x_0, X \setminus U), \text{dist}(y_0, Y \setminus V)\}. \end{aligned}$$

Hence,

$$d(x_0, x_{n+1}) + \sigma(y_0, y_{n+1}) < \min\{\text{dist}(x_0, X \setminus U), \text{dist}(y_0, Y \setminus V)\}. \quad (10)$$

Thus, $x_{n+1} \in U$ and $y_{n+1} \in V$. Also, the chain of inequalities

$$\begin{aligned} \rho_n &= d(x_n, x_{n+1}) + \sigma(y_n, y_{n+1}) \\ &< \tfrac{1-q_2\lambda}{q_1} \min\{\text{dist}(x_n, X \setminus U), \text{dist}(y_n, Y \setminus V)\} \\ &\leq \tfrac{1-q_2\lambda}{q_1} \min\{q_1 d(x_n, x_{n+1}) + q_2\text{dist}(x_{n+1}, X \setminus U)), q_1\sigma(y_n, y_{n+1}) + q_2\text{dist}(y_{n+1}, Y \setminus V))\} \\ &\leq \tfrac{1-q_2\lambda}{q_1} q_1(d(x_n, x_{n+1}) + \sigma(y_n, y_{n+1})) \\ &\quad + \tfrac{1-q_2\lambda}{q_1} q_2 \min\{\text{dist}(x_{n+1}, X \setminus U), \text{dist}(y_{n+1}, Y \setminus V)\}, \end{aligned}$$

leads to the inequality

$$\lambda(d(x_n, x_{n+1}) + \sigma(y_n, y_{n+1})) < \frac{1 - q_2\lambda}{q_1} \min\{\text{dist}(x_{n+1}, X \setminus U), \text{dist}(y_{n+1}, Y \setminus V)\}.$$

Through induction, the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ are constructed, satisfying (6)–(9).

For all $m > n$, the following holds:

$$\begin{aligned}\rho_{n,m} &= d(x_n, x_m) + \sigma(y_n, y_m)\\ &\le q_1\big(d(x_n, x_{n+1}) + \sigma(y_n, y_{n+1})\big) + q_1 q_2\big(d(x_{n+1}, x_{n+2}) + \sigma(y_{n+1}, y_{n+2})\big)\\ &\quad + \cdots + q_1 q_2^{m-n-1}\big(d(x_{m-1}, x_m) + \sigma(y_{m-1}, y_m)\big)\\ &= q_1\lambda^n\big(d(x_0, x_1) + \sigma(y_0, y_1)\big)\big) \textstyle\sum_{j=0}^{m-n-1}(q_2\lambda)^j\\ &= q_1\lambda^n\big(d(x_0, x_1) + \sigma(y_0, y_1)\big)\big) \cdot \frac{1-(q_2\lambda)^{m-n}}{1-q_2\lambda}.\end{aligned}$$

Hence,

$$d(x_n, x_m) + \sigma(y_n, y_m) \le q_1\lambda^n\big(d(x_0, x_1) + \sigma(y_0, y_1)\big)\big) \cdot \frac{1-(q_2\lambda)^{m-n}}{1-q_2\lambda}. \tag{11}$$

(A) By putting $n = 0$ in (11), and from (2), we get the inequality for every $m \in \mathbb{N}$:

$$d(x_0, x_m) + \sigma(y_0, y_m) < l < \min\{\mathrm{dist}(x_0, X \setminus U), \mathrm{dist}(y_0, Y \setminus V)\}. \tag{12}$$

Thus, we conclude that $x_m \in U$ and $y_m \in V$. We have the chain of inequalities

$$\begin{aligned}\lim_{m\to\infty} S(x_m, y_m) &= \lim_{m\to\infty}(\mathrm{dist}(x_m, F_1(x_m, y_m)) + \mathrm{dist}(y_m, F_2(x_m, y_m))\\ &\le \lim_{m\to\infty}(d(x_m, x_{m+1}) + \sigma(y_m, y_{m+1}))\\ &\le \lim_{m\to\infty} \lambda(d(x_{m-1}, x_m) + \sigma(y_{m-1}, y_m))\\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\\ &\le \lim_{m\to\infty} \lambda^m(d(x_0, x_1) + \sigma(y_0, y_1)) = 0.\end{aligned}$$

Hence, for every $\varepsilon, \mu > 0$, there is $M \in \mathbb{N}$ so that for every $m \ge M$, the following holds: $(x_m, y_m) \in \mathrm{Fix}F(\varepsilon, \mu)$.

Moreover, from (12) and $d(\bar{x}, x_m) + \sigma(\bar{y}, y_m) < l$, we determine that

$$d(\bar{x}, x_m) + \sigma(\bar{y}, y_m) \le \frac{q_1}{1-q_2\lambda}\Big(\mathrm{dist}(\bar{x}, F_1(\bar{x}, \bar{y})) + \mathrm{dist}(\bar{y}, F_2(\bar{x}, \bar{y}))\Big). \tag{13}$$

(B) Let X and Y be complete, and let them be p_0-symmetric and q_0-symmetric, respectively. Let $\mathrm{Gr}(F_1)$ and $\mathrm{Gr}(F_2)$ be closed. Let us assume that $c_0 = \max\{p_0, 1/p_0, q_0, 1/q_0\}$.

We have proven in (11) that for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ so that for all $N \le n < m$ the following inequality holds:

$$d(x_n, x_m) + \sigma(y_n, y_m) < \varepsilon.$$

Since we have assumed that the two quasi-metric spaces are p_0- and q_0-symmetric, we can write the inequality $d(x_m, x_n) + \sigma(y_m, y_n) \le c_0(d(x_n, x_m) + \sigma(y_n, y_m)) \le c_0\varepsilon$. Therefore, both sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are Cauchy ones in the considered quasi-metric spaces. According to the assumption that both spaces are complete, it follows that $\lim_{n\to\infty} x_n \to x^* \in X$ and $\lim_{n\to\infty} y_n \to y^* \in Y$. Passing to the limit for $m \to \infty$ in (12) gives us

$$d(x_0, x^*) + \sigma(y_0, y^*) \le l < \min\{\mathrm{dist}(x_0, X\backslash U), \mathrm{dist}(y_0, Y\backslash V)\},$$

and consequently, $x^* \in U$ and $y^* \in V$. Once again applying $m \to \infty$ in (13), we get

$$d(x_0, x^*) + \sigma(y_0, y^*) \leq \frac{q_1}{1 - q_2\lambda}(\text{dist}(\bar{x}, F_1(\bar{x}, \bar{y})) + \text{dist}(\bar{y}, F_2(\bar{x}, \bar{y}))).$$

From $(x_{n-1}, y_{n-1}, x_n) \in \text{Gr}(F_1), (x_{n-1}, y_{n-1}, y_n) \in \text{Gr}(F_2)$, based on the closeness of $\text{Gr}(F_i)$, with $i = 1, 2$, to $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$, we conclude that the inclusions $x^* \in F_1(x^*, y^*) \cap U$ and $y^* \in F_2(x^*, y^*) \cap V$ hold true. □

4. Application

We follow the notation and terminology from [13].

Let (Z, d) be a (q_1, q_2)-quasi-metric space and G be a weighted directed graph with a set of vertices $V(G) = Z$ and an edge set $E(G) \subseteq Z \times Z$, where the weights of the edges will be calculated as the quasi-metric distance between their endpoints. We set the edge weight $w(u, v) := d(u, v)$ for each $(u, v) \in E(G)$.

A subgraph of G is called a graph (V', E') such that $V' \subseteq V(G)$, $E' \subseteq E(G)$, and for each edge $(x, y) \in E'$, it holds that $x, y \in V'$.

If x and y are vertices of G, then a path of length n, where $n \in \mathbb{N} \cup \{0\}$, is a sequence of vertices $\{x_i\}_{i=0}^n$ such that

$$x_0 = x, \quad x_n = y, \quad (x_{i-1}, x_i) \in E(G) \text{ for } i = 1, 2, \ldots, n.$$

In what follows "path" means a directed path of length ≥1. We assume in the set of all "paths" that there are no loops (or self-loops), i.e., an edge that connects a vertex to itself.

A graph is said to be connected if there is a path between any two vertices. Given that $\widetilde{G}$ is connected, G is weakly connected. Here, $\widetilde{G}$ is the underlying undirected graph.

If the edge set $E(G)$ of a graph G is symmetric, then the component of G containing a vertex x is defined as the subgraph G_x that includes all vertices and edges that lie on a path starting from x. For a general directed graph, strongly connected components play the analogous role.

According to $[x]_G$, we will denote the equivalence class induced by the relation R, defined on $V(G)$ as

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

Note that R need not be symmetric; for an equivalence relation one may use paths in both directions.

It follows that $V(G_x) = [x]_G$. We will assume that $(z, z) \notin E(G)$, i.e., there is no path with a length 1 from z to z.

Let us define a multi-valued map $H : V(G) \rightrightarrows V(G)$ that assigns to any $z \in V(G)$ the set of all z' such that there exists a directed path of length ≥1 from z to z'. If $z \in Hz$, then z is a fixed point for the multi-valued map H, and there is a path from z to z, i.e., the relation zRz holds.

Let $G' = (V', E')$ be a subgraph of G. Based on the distance between $v \notin V'$ and V', we assume the directed shortest-path distance

$$\text{dist}(v, V') := \inf\left\{\sum_{i=1}^{k} w(u_{i-1}, u_i) \ : \ v = u_0 \to u_1 \to \cdots \to u_k,\ u_k \in V',\ k \geq 1\right\},$$

and will denote it as $\text{dist}(v, V')$. If no such directed path exists, set $\text{dist}(v, V') := +\infty$.

If there is not any $v' \in V'$ so that $(v, v') \in E(G)$, then we will assume that $\text{dist}(v, V') = +\infty$. If we assume that a graph G is connected, then for any $v \in V$ and

$V' \subset V$, the following holds: $\operatorname{dist}(v, V') < +\infty$ (for the undirected distance in $\widetilde{G}$, weak connectivity suffices; for the directed distance above, strong connectivity yields finiteness).

Let (X, d) be a (m_1, m_2)-quasi-metric space, and let (Y, σ) be a (n_1, n_2)-quasi-metric space with constants $m_1, m_2, n_1, n_2 \geq 1$ and $q_1 = \max\{m_1, n_1\}, q_2 = \max\{m_2, n_2\}$. Let us assume that $Z = X \times Y$ and endow Z with the (q_1, q_2)-quasi-metric

$$\rho((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + \sigma(x_2, y_2).$$

When $Z = X \times Y$ with quasi-metric ρ, we use the graph weight $w(u, v) := \rho(u, v)$ for $(u, v) \in E(G)$.

Let us assume that

$$F_1(x, y) := \{\, x' \in X : \exists y' \in Y \text{ with } (x', y') \in H(x, y) \,\},$$

which is the projection of H on X, and

$$F_2(x, y) := \{\, y' \in Y : \exists x' \in X \text{ with } (x', y') \in H(x, y) \,\},$$

which is the projection of H on Y. Thus, we can consider $Hz = H(x, y) = (F_1(x, y), F_2(x, y))$ for $z = (x, y) \in Z = X \times Y$.

Definition 11. *Let (X, d) be a (m_1, m_2)-quasi-metric space, and let (Y, σ) be a (n_1, n_2)-quasi-metric space with constants $m_1, m_2, n_1, n_2 \geq 1$ and $q_1 = \max\{m_1, n_1\}, q_2 = \max\{m_2, n_2\}$. Let us assume that $Z = X \times Y$ and endow Z with the (q_1, q_2)-quasi-metric ρ. Let the graph G be a directed graph, consisting of vertices $V(G) = Z$ and edges $E(G)$. Let $H : V(G) \rightrightarrows V(G)$ be a multi valued map that assigns to every $z \in V(G)$ all $z' \in V(G)$ such that there is a path from z to z'. Let us denote as F_1 the projection of H onto X, and as F_2 its projection onto Y, as explicitly defined above. We will call the map $H = (F_1, F_2)$ a path map for the graph G.*

Theorem 2. *Let (X, d) be a complete, p_0-symmetric (m_1, m_2)-quasi-metric space, and let (Y, σ) be a complete, q_0-symmetric (n_1, n_2)-quasi-metric space with constants $m_1, m_2, n_1, n_2 \geq 1$ and $q_1 = \max\{m_1, n_1\}, q_2 = \max\{m_2, n_2\}$. Let G be a directed graph on $X \times Y$ with edge set $E(G)$. Let $H : V(G) \rightrightarrows V(G)$ be a multi-valued map that assigns to every $z \in V(G)$ all $z' \in V(G)$ such that there is a path from z to z'. Let us denote as F_1 the projection of H onto X, and as F_2 its projection onto Y, i.e., $H(z) = H(x, y) = (F_1(x, y), F_2(x, y))$, and assume that the maps F_1 and F_2 have closed graphs in $X \times Y \times X$ and $X \times Y \times Y$, respectively.*

Let $U \subset X$ be an open subset, $V \subset Y$ be an open subset, $\bar{x} \in U$, and $\bar{y} \in V$. Suppose there exist constants $\alpha, \beta > 0$ and $\lambda \in (\max\{\alpha, \beta\}, 1/q_2)$ such that

(a) $$\operatorname{dist}(\bar{x}, F_1(\bar{x}, \bar{y})) + \operatorname{dist}(\bar{y}, F_2(\bar{x}, \bar{y})) < \frac{1 - q_2\lambda}{q_1} \min\{\operatorname{dist}(\bar{x}, X \setminus U), \operatorname{dist}(\bar{y}, Y \setminus V)\},$$

(b)
$$\operatorname{dist}(x, F_1(x, y)) + \operatorname{dist}(y, F_2(x, y)) \leq \alpha d(u, x) + \beta\sigma(v, y) \tag{14}$$

for all $(x, y), (u, v) \in U \times V$ such that

$$x \in F_1(u, v), \qquad y \in F_2(u, v)$$

and

$$\alpha d(u, x) < \operatorname{dist}(x, X \backslash U), \quad \beta\sigma(v, y) < \operatorname{dist}(y, Y \backslash V).$$

Then there exist elements $x^* \in X$ and $y^* \in Y$ such that the sequence $\{x_k\}_{k=0}^{\infty}$ converges to x^*, and the sequence $\{y_k\}_{k=0}^{\infty}$ converges to y^* and

$$(x^*, y^*) \in (F_1(x^*, y^*), F_2(x^*, y^*)) \cap (U \times V) = H(x^*, y^*) \cap (U \times V),$$

i.e., there exists a directed path of length ≥ 1 connecting $z^* = (x^*, y^*)$ with z^*.

Graph-Theoretic Interpretation of the Assumptions

(i) If $E(G)$ is generated by the one-step multimap (F_1, F_2) via $z \to z'$ whenever $z' \in (F_1 \times F_2)(z)$, then sequences of successive approximations $z_{k+1} \in (F_1 \times F_2)(z_k)$ are precisely directed paths in G. (ii) Condition (a) guarantees that the path starting at $(\bar{x}, \bar{y})$ remains in $U \times V$: the "margin to the boundary" dominates the first step and, due to the relaxed triangle inequality, all subsequent steps. (iii) Condition (b) encodes a contractive behavior along the path: the one-step error is bounded by $\alpha d(u, x) + \beta\sigma(v, y)$, and choosing $\lambda \in (\max\{\alpha, \beta\}, 1/q_2)$ yields geometric decay of consecutive increments. (iv) Completeness of (X, d) and (Y, σ) together with p_0- and q_0-symmetry ensures the Cauchy path converges to some $z^* = (x^*, y^*) \in U \times V$. (v) Closedness of the graphs of F_1 and F_2 turns the limit into a fixed point, $z^* \in (F_1 \times F_2)(z^*)$, which in graph language is a self-reachable node (a directed cycle of positive length).

5. Discussion

The results obtained in this paper demonstrate how the concept of coupled fixed points can be meaningfully extended to the framework of (q_1, q_2)-quasi-metric spaces. In particular, the use of approximate coupled fixed points addresses the limitations that occur when an exact solution can not be obtained. The notion of generalized coupled fixed points for ordered pairs of maps, proposed in [12] and further developed in [23] for multi-valued maps and in [3] in an investigation of market equilibrium in oligopoly markets, excludes the often appearing diagonal case for the solutions. The obtained result shows that asymmetry does not lead to fixed-point results in the classical sense, but only approximate ones. By introducing q_0-symmetry as an auxiliary condition, the theorems unify existing results from symmetric and b-metric contexts while allowing for genuinely non-symmetric systems.

We note that fixed-point results have recently been applied in fields not traditionally associated with them: fixed points of principal bundles over algebraic curves [24]; fixed points in Higgs bundles over a compact and connected Riemann surface [25,26] and the Hitchin integrable system [25]; and fixed points with applications to physics [27]. Following the important observations in [13] regarding the relation between fixed points in partially ordered metric spaces and metric spaces equipped with a graph, one aspect of this work that we would like to point out is the graph-theoretic interpretation of multi-valued maps, which translates analytic assumptions into conditions guaranteeing the existence of directed cycles. This establishes a bridge between nonlinear analysis and discrete mathematics, extending earlier graph-based studies of fixed points [13,15–17]. This perspective is particularly relevant for applications in networked systems, where asymmetry and directionality are inherent.

The broader significance of these contributions lies in their potential applications. The proposed ideas suggest that the applications of coupled and tripled fixed points presented in [3,23] can be extended in economics and game theory, as quasi-metric asymmetry naturally models situations with unequal information or sequential decision-making. In applied sciences, coupled fixed-point results underpin the analysis of nonlinear matrix equations and ecosystem dynamics [2,23]. The flexibility of the quasi-metric setting thus enlarges the scope of problems for which rigorous existence results can be established.

We would like to highlight an important observation from [28], which establishes a connection between coupled fixed points and fixed points. As noted in [28], the ordered pair (x, y) is a coupled fixed point for a mapping $F : X \times X \to X$ if and only if it is a fixed point of the operator $T(x, y) = \big(F(x, y), F(y, x)\big)$. This idea was further developed in [29], where generalized coupled fixed points, i.e., solutions to the systems $x = F(x, y)$ and $y = G(x, y)$, were associated with fixed points of the operator $T(x, y) = \big(F(x, y), G(x, y)\big)$. However, due to the nature of the conditions involved in the study of multi-valued maps, additional results concerning the geometry of the Cartesian product of two quasi-metric spaces are required in order to extend these techniques by applying the result from [30].

Lemma 1 ([30])**.** *Let (X_1, d_1) be a symmetric (p_1, q_1)-quasi-metric space, and let (X_2, d_2) be a (p_2, q_2)-quasi-metric space. Then, the Cartesian product $X_1 \times X_2$ endowed with the metric $d((x_1, y_1),(x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$ is a $(\max\{p_1, p_2\}, \max\{q_1, q_2\})$-quasi-metric space for $d(\cdot, \cdot)$.*

This observation poses an open question for further investigations in multi-valued maps in the different types of quasi-metric spaces.

6. Conclusions

This paper established new coupled fixed-point theorems for ordered pairs of set-valued mappings in (q_1, q_2)-quasi-metric spaces. The main contributions can be summarized as follows: (i) we introduce approximate coupled fixed points, which provide tools for situations where exact solutions may not exist; (ii) under q_0-symmetry and completeness assumptions, the existence of exact coupled fixed points is guaranteed, extending and unifying several known results in fixed-point theory; (iii) we establish a graph-theoretic formulation offering a combinatorial interpretation of the analytic conditions and ensuring the existence of cycles in associated graphs.

These contributions form a foundation for further research. Promising directions include extension to stochastic and fuzzy quasi-metric environments, the development of computational methods based on successive approximations, and the exploration of applications in economics, networked systems, and nonlinear analysis.

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