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Finite and Infinte Time Blow Up of Solutions to Wave Equations with Combined Logarithmic and Power-Type Nonlinearities

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Abstract: In this paper, we investigate the global behavior of the weak solutions to the initial boundary value problem for the nonlinear wave equation in a bounded domain. The nonlinearity includes a logarithmic term and several power-type terms with nonnegative variable coefficients. Two new necessary and sufficient conditions for blow up of the weak solutions are established. The first one addresses the blow up of the global weak solutions at infinity. The second necessary and sufficient condition is obtained in the case of strong superlinearity and concerns blow up of the weak solutions for a finite time. Additionally, we derive new sufficient conditions on the initial data that guarantee blow up for either finite or infinite time. A comparison with previous results is also given.

Keywords: nonlinear wave equation; combined logarithmic nonlinearities; blow up at infinity; blow up for a finite time

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1. Introduction

We consider the following initial boundary value problem for the nonlinear wave equation:

$$u_{tt} - \Delta u = a_0(x) u \ln |u|^k + \sum_{i=1}^{r} a_i(x) |u|^{p_i - 1} u = f(x,u), \quad t > 0, \ \ x \in \Omega, \tag{1}$$

$$\begin{cases} u(0,x) = u_0(x), & u_t(0,x) = u_1(x), \quad x \in \Omega, \\ u(t,x) = 0, & t \geq 0, \ \ x \in \partial\Omega, \\ u_0(x) \in \mathrm{H}_0^1(\Omega), & u_1(x) \in \mathrm{L}^2(\Omega), \end{cases} \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$. We assume that

$$k > 0, \ \ a_i(x) \in \mathrm{C}(\overline{\Omega}), \ \ 0 < a \leq a_0(x) \leq A, \ \ 0 \leq a_i(x) \leq A_i, \ \ i = 1, \ldots, r, \ \ \forall x \in \overline{\Omega} \tag{3}$$

$$1 < p_1 < \cdots < p_r, \quad p_r < \infty \ \text{ if } \ n = 1,2; \qquad p_r \leq \frac{n}{n-2} \ \text{ if } \ n \geq 3. \tag{4}$$

At a later stage, we impose the following additional condition on the coefficients $a_i(x)$, $i = 1, \dots, r$:

$$\sum_{i=1}^{r} a_i(x) \geq K > 0 \ \ \forall x \in \overline{\Omega}. \tag{5}$$

Assumption (5) guarantees the power-type superlinear growth of the nonlinearity $f(x,u)$. This assumption is satisfied, for instance, by a nonlinearity $f(x,u)$ involving a single power-type term with a strictly positive variable coefficient, i.e., $r = 1$ and $a_1(x) > 0$ for all $x \in \overline{\Omega}$.

Logarithmic nonlinearity has applications in many fields such as nuclear physics, inflation, cosmology, quantum optics, fluid dynamic, transport phenomena, and more; see e.g., refs. [1–8]. For this reason, partial differential equations with logarithmic nonlinearities have been the subject of intensive research in the last few decades.

After the pioneering papers of Gazenave [9] and Górka [10], the global behavior of the weak solution to the wave and Klein–Gordon equations with the logarithmic nonlinearity $f(x,u) = u \ln |u|^k$, $k > 0$ is considered in [11–14]. By means of the potential well method, the global existence and infinite time blow up of the weak solutions are proved in [12,13] for the wave equation and the Klein–Gordon equation, respectively. In the case of arbitrary positive initial energy, an infinite time blow up result is given in [12]. Moreover, a growth estimate of the infinite time blowup solution is obtained in [14] for any initial energy level.

Recently, a wave equation with combined logarithmic and power-type nonlinearity with variable coefficients was studied in our previous paper [15]. In [15], we established a necessary and sufficient condition for blow up at infinity of the global weak solutions to (1)–(4), applicable to arbitrary initial energy. Furthermore, we derive a growth estimate for the blowing up global solutions.

Let us also note that logarithmic nonlinearities have been studied in various types of equations, including the wave and the Klein–Gordon equations with damping terms [16–20], Boussinesq-type equations [21,22], plate equations [23,24], parabolic equations [25,26], wave equations with fractional boundary dissipation [27], etc.

The aim of this paper is twofold. First, we extend the research in [15] by deriving new necessary and sufficient conditions for infinite time blow up of the global weak solutions to (1)–(4); see Theorem 2. The newly proposed necessary and sufficient conditions require that the scalar product $(u(t), u_t(t))$ has a nonnegative sign at some time b.

The second goal of the paper is to investigate the impact of Assumption (5) on the behavior of the solutions. We demonstrate that Assumption (5) leads to the nonexistence of global solutions to problem (1)–(5). More precisely, we prove that the maximal existence time T_m of the weak solutions is finite; see Theorems 3 and 4. However, when $T_m < \infty$ we do not know whether $u(t,x)$ blows up at T_m, i.e., $\|u(t)\|_{\mathrm{L}^2(\Omega)} \to \infty$ as $t \to T_m$ or $\|\nabla u(t)\|_{\mathrm{L}^2(\Omega)} \to \infty$ as $t \to T_m$. To exclude the second case, when $\|u(t)\|_{\mathrm{L}^2(\Omega)}$ is bounded in $[0, T_m)$ while $\|\nabla u(t)\|_{\mathrm{L}^p(\Omega)}$ blows up at T_m, we impose more restrictive conditions (41) on the power exponents in $f(x,u)$. In this manner, novel necessary and sufficient conditions for the finite time blow up of every local weak solution are established; see Theorems 5 and 6. This indicates that the combination of conditions (5) and (41) significantly alters the nature of the blow up type of the weak solutions: from infinite time blow up (under (3) and (4)) to finite time blow up (under (3)–(5) and (41)).

Indeed, for the pure logarithmic nonlinearity $u \ln |u|^k, k > 0$, there is always an infinite time blow up of the weak solutions; see [12–14]. Conversely, for the nonlinearity with superlinear terms (such as $a(x)|u|^p \ln |u|^k$ or $a(x)|u|^{p-1}u$, $k > 0$, $p > 1$, $a(x) > 0 \, \forall x \in \overline{\Omega}$), finite time blow up of the local weak solutions has been observed; see [28,29]. It is important to note that if Assumption (5) fails in some subdomains of Ω, then the question whether

problem (1)–(4) has a global weak solution that blows up at infinity or local weak solutions that blow up for a finite time, remains open.

The structure of the paper is as follows. In Section 2, some definitions and preliminary results are introduced. The main results are formulated and proved in Sections 3 and 4. Section 3 addresses the infinite time blow up of the global weak solutions. Section 4 deals with the nonexistence of global weak solutions as well as the finite time blow up of local weak solutions. In Section 5, a comparison with previous results is provided.

2. Preliminary

Throughout the paper, we utilize the following short notation for the functions that are dependent on both t and x:

$$\|u(t)\|_p = \|u(t,\cdot)\|_{\mathrm{L}^p(\Omega)},\ 1 < p \le \infty, \quad (u(t), v(t)) = \int_\Omega u(t,x)v(t,x)\,dx.$$

For the sake of simplicity, we write $\|\cdot\|$ instead of $\|\cdot\|_2$.

First, we recall the important functionals $J(w)$ and $I(w)$, defined for every $w(x) \in \mathrm{H}_0^1(\Omega)$:

$$\begin{aligned} J(w) :=& \frac{1}{2}\|\nabla w\|^2 - \int_\Omega \int_0^{w(x)} f(x,z)\,dz\,dx \\ =& \frac{1}{2}\|\nabla w\|^2 - \frac{k}{2}\int_\Omega a_0(x)w^2 \ln|w|\,dx \\ &+ \frac{k}{4}\int_\Omega a_0(x)w^2\,dx - \sum_{i=1}^{r}\frac{1}{p_i+1}\int_\Omega a_i(x)|w|^{p_i+1}\,dx, \end{aligned}$$

$$\begin{aligned} I(w) :=& \|\nabla w\|^2 - \int_\Omega w f(x,w)dx \\ =& \|\nabla w\|^2 - k\int_\Omega a_0(x)w^2\ln|w|\,dx - \sum_{i=1}^{r}\int_\Omega a_i(x)|w|^{p_i+1}\,dx. \end{aligned}$$

For every $p \in [1, p_1]$, we have the following relation between the functionals $I(w)$ and $J(w)$:

$$\begin{aligned} J(w) =& \frac{1}{p+1}I(w) + \frac{p-1}{2(p+1)}\|\nabla w\|^2 + \frac{k}{4}\int_\Omega a_0(x)|w|^2\,dx \\ &- \frac{k(p-1)}{2(p+1)}\int_\Omega a_0(x)w^2\ln|w|\,dx + \sum_{i=1}^{r}\frac{p_i - p}{(p_i+1)(p+1)}\int_\Omega a_i(x)|w|^{p_i+1}\,dx. \end{aligned} \tag{6}$$

Formula (6) generalizes the relationship between the functionals $I(w)$ and $J(w)$ in the case of a combined logarithmic and power-type nonlinearity. We remark that expression (6) is applied with $p = 1$ for a pure logarithmic nonlinearity; see, e.g., [12,15]. In contrast, for a pure power-type nonlinearity ($a_0(x) \equiv 0$), relation (6) is used with $p = p_1$; see, e.g., [29] and references therein.

In order to introduce the main results of the paper, let us recall some definitions.

Definition 1. *The function $u(t,x)$ is a weak solution to problem (1)–(4) if*

$$u(t,x) \in \mathrm{C}((0,T_m);\mathrm{H}_0^1(\Omega)) \cap \mathrm{C}^1((0,T_m);\mathrm{L}^2(\Omega)) \cap \mathrm{C}^2((0,T_m);\mathrm{H}^{-1}(\Omega))$$

and the identity

$$\int_\Omega u_t(t,x)\mu(x)\,dx + \int_0^t \int_\Omega \nabla u(\tau,x)\nabla\mu(x)\,dx\,d\tau = \int_0^t \int_\Omega f(x,u(\tau,x))\mu(x)\,dx\,d\tau + \int_\Omega u_1(x)\mu(x)\,dx$$

holds for every $\mu(x) \in \mathrm{H}_0^1(\Omega)$ *and every* $t \in [0, T_m)$.

Note that $u(t,x) \in \mathrm{L}^p(\Omega)$ for every $t \in [0, T_m)$ according to the Sobolev inequality (13) and Assumption (4). Thus, the above definition is correct.

Definition 2. *Suppose* $u(t,x)$ *is a weak solution to problem* (1)–(4) *in the maximal existence time interval* $[0, T_m)$, $0 < T_m \le \infty$. *Then, solution* $u(t,x)$ *blows up at* T_m *if*

$$\limsup_{t \to T_m, t < T_m} \|u(t)\| = \infty.$$

If $u(t,x)$ is a weak solution to problem (1)–(4) in the maximal existence time interval $[0, T_m)$, $0 < T_m \le \infty$, we define the energy functional as

$$\begin{aligned} E(u(t,\cdot)) :&= \frac{1}{2}\|u_t(t)\|^2 + J(u(t,\cdot)) = \\ &= \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 - \frac{k}{2}\int_\Omega a_0(x)u^2(t,x)\ln|u(t,x)|\,dx \\ &\quad + \frac{k}{4}\int_\Omega a_0(x)u^2(t,x)\,dx - \sum_{i=1}^r \int_\Omega \frac{a_i(x)}{p_i+1}|u(t,x)|^{p_i+1}\,dx. \end{aligned} \tag{7}$$

Moreover, the following conservation law is valid:

$$E(0) = E(u(t,\cdot)) \quad \forall\, t \in [0, T_m). \tag{8}$$

For simplicity, we use the short notation $I(t) = I(u(t)) = I(u(t,\cdot))$, $J(t) = J(u(t)) = J(u(t,\cdot))$, $E(t) = E(u(t)) = E(u(t,\cdot))$.

Combining (6)–(8) for $p = 1$, we obtain the following useful relation:

$$\begin{aligned} \frac{1}{2}I(t) &= E(0) - \frac{ka}{4}\|u(t)\|^2 - \frac{1}{2}\|u_t(t)\|^2 - B(t) \\ &= E(0) - \frac{ka}{4}\|u(t)\|^2 - \frac{1}{2}\frac{(u(t),u_t(t))^2}{\|u(t)\|^2} - \frac{1}{2}\left(\|u_t(t)\|^2 - \frac{(u(t),u_t(t))^2}{\|u(t)\|^2}\right) - B(t), \end{aligned} \tag{9}$$

where

$$B(t) = \frac{k}{4}\int_\Omega (a_0(x) - a)|u|^2\,dx + \sum_{i=1}^r \frac{p_i - 1}{2(p_i+1)}\int_\Omega a_i(x)|u|^{p_i+1}\,dx \ge 0 \ \ \forall t \in [0, T_m). \tag{10}$$

The nonnegativity of $B(t)$ is due to the conditions (3) and (4) on the functions $a_i(x)$ and the power exponents p_i in $f(x,u)$.

The proofs of the main results are based on the behavior of the nonnegative smooth function $\psi(t) := \|u(t)\|^2$. In the following, we recall some preliminary results for $\psi(t)$.

Definition 3. *We say that a nonnegative function* $\psi(t) \in \mathrm{C}^2([0, T_m))$, $0 < T_m \le \infty$ *blows up at* T_m *if*

$$\limsup_{t \to T_m, t < T_m} \psi(t) = \infty.$$

Lemma 1 (Lemma 2.2 in [30]). *Suppose $\psi(t) \in C^2([0, T_m))$, $0 < T_m \leq \infty$, is a nonnegative function and M is an arbitrary nonnegative constant. If $\psi(t)$ blows up at T_m, then there exists $t_0 \in [0, T_m)$ such that $\psi(t_0) > M$ and $\psi'(t_0) > 0$.*

The following lemma is a modified version of Theorem 3.2 in [31], formulated for a function $\psi(t)$ defined in $[t_0, T_m)$, $t_0 \geq 0$ instead of $[0, T_m)$.

Lemma 2. *Suppose $\psi(t) \in \mathrm{C}^2([t_0, T_m))$, $t_0 \geq 0$ is a nonnegative solution to the problem*

$$\psi''(t)\psi(t) - \gamma\psi'^2(t) = Q(t), \quad t \in [t_0, T_m), \quad t_0 < T_m \leq \infty,$$
$$\gamma > 1,\ Q(t) \in \mathrm{C}([t_0, T_m)),\ Q(t) \geq 0 \text{ for } t \in [t_0, T_m).$$

If $\psi(t)$ blows up at T_m, then $T_m < \infty$.

Let us recall the necessary and sufficient condition for blow up at infinity of the global solutions to problem (1)–(4), proved in [15].

Theorem 1 (Theorem 1 in [15]). *If $u(t, x)$ is a global weak solution to problem (1)–(4), then*

(i) $u(t, x)$ blows up at infinity if and only if

$$\text{there exists} \quad b \geq 0 \quad \text{such that} \quad E(0) < \frac{ka}{4}\|u(b)\|^2 + \frac{\sqrt{ka}}{2}(u(b), u_t(b)); \tag{11}$$

(ii) if (11) fails, then the estimate

$$\|u(t)\|^2 \leq \left(\|u_0\|^2 - \frac{4E(0)}{ka}\right)e^{-\sqrt{ka}t} + \frac{4E(0)}{ka} \tag{12}$$

holds for every $t \geq 0$.

Below, we give some inequalities which are important for the proof of the main results. For $w \in \mathrm{H}_0^1(\Omega)$, we employ the Sobolev embedding theorem,

$$\|w\|_q \leq C_q\|\nabla w\|, \quad \text{where} \begin{cases} 1 \leq q \leq \infty & \text{if } n = 1; \\ 1 \leq q < \infty & \text{if } n = 2; \\ 1 \leq q \leq \frac{2n}{n-2} & \text{if } n \geq 3 \end{cases} \tag{13}$$

and the Gagliardo–Nirenberg inequality:

$$\|D^j w\|_p \leq C\|D^m w\|_r^{\alpha}\ \|w\|_q^{1-\alpha} + C\|w\|_q. \tag{14}$$

Inequality (14) is valid for $1 \leq q, r \leq \infty$, $m, j \in \mathbb{N}$, $j < m$, and real numbers α and p which satisfy relations

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\alpha + \frac{1-\alpha}{q}, \quad \frac{j}{m} \leq \alpha \leq 1.$$

The constant C in (14) depends on Ω and the parameters n, j, m, r, q, and α but not on w. For more details, we refer the reader to [32,33].

3. Infinite Time Blow Up

In this section, we formulate and prove a new necessary and sufficient condition for blow up at infinity of the global weak solutions to problem (1)–(4). In contrast to the necessary and sufficient condition for blow up at infinity in Theorem 1, this condition requires a

nonnegative sign of the scalar product $(u(b), u_t(b))$ at some time b. As a consequence, new sufficient conditions for infinite time blow up are derived.

Theorem 2. *Suppose* (3), (4) *hold and* $u(t,x)$ *is a global weak solution of problem* (1), (2).

(i) *If* $E(0) > 0$, *then*

(i_1) $u(t,x)$ *blows up at infinity if and only if*

$$\begin{aligned} &\text{there exists} \quad b \geq 0 \quad \text{such that} \\ &\|u(b)\| > 0, \quad (u(b), u_t(b)) \geq 0, \quad E(0) < \frac{ka}{4}\|u(b)\|^2 + \frac{1}{2}\frac{(u(b), u_t(b))^2}{\|u(b)\|^2}; \end{aligned} \tag{15}$$

(i_2) *if* (15) *fails, then* $\|u(t)\|$ *is globally bounded and the estimate*

$$\|u(t)\|^2 \leq \max\left\{\frac{4E(0)}{ka}, \|u_0\|^2\right\} \tag{16}$$

holds for $t \in [0, \infty)$.

(ii) *If* $E(0) = 0$, *then*

(ii_1) $u(t,x)$ *blows up at infinity if and only if*

$$\text{there exists} \quad b \geq 0 \quad \text{such that} \quad (u(b), u_t(b)) > 0; \tag{17}$$

(ii_2) *If* (17) *fails, then* $\|u(t)\|$ *is globally bounded and the estimate*

$$\|u(t)\|^2 \leq \|u_0\|^2 \tag{18}$$

holds for $t \in [0, \infty)$.

(iii) *If* $E(0) < 0$, *then every global weak solution* $u(t,x)$ *blows up at infinity.*

Proof. (i) Let $E(0) > 0$.

(i_1) **Sufficiency.** The proof of sufficiency is similar to the proof of Theorem 6.2 in [29], but for the reader's convenience we provide it below. Suppose that (15) holds. The proof of the infinite time blow up of $u(t,x)$ consists of three steps.

Step 1. Here, we prove that if $I(t) < 0$ for $t \in [b, T)$, $b < T \leq \infty$, and $(u(b), u_t(b)) \geq 0$, then the functions

$$\psi(t) = \|u(t)\|^2 \quad \text{and} \quad \phi(t) = \frac{(u(t), u_t(t))^2}{\|u(t)\|^2} = \frac{1}{4}\frac{(\psi'(t))^2}{\psi(t)}$$

are strictly increasing for $t \in (b, T)$. Simple computations give us

$$\psi'(t) = 2(u(t), u_t(t)), \quad \psi''(t) = 2\|u_t(t)\|^2 - 2I(t) > 0 \text{ for } t \in [b, T). \tag{19}$$

Therefore, $\psi'(t) > \psi'(b) \geq 0$ for $t \in (b, T)$, i.e., $\psi(t)$ is strictly increasing in (b, T). The monotonicity of $\psi(t)$ guarantees that $\phi(t)$ is also a strictly increasing function in (b, T). Indeed, from the Cauchy–Schwarz inequality, the negative sign of $I(t)$, and the positive sign of $\psi'(t)$, we obtain

$$\begin{aligned}\phi'(t) =& \frac{1}{4}\frac{\psi'(t)}{\psi^2(t)}\left[2\psi(t)\psi''(t) - (\psi'(t))^2\right]\\ =& \frac{1}{4}\frac{\psi'(t)}{\psi^2(t)}\left\{4\|u(t)\|^2\left[\|u_t(t)\|^2 - I(t)\right] - 4(u(t), u_t(t))^2\right\}\\ =& \frac{\psi'(t)}{\psi^2(t)}\left\{\left[\|u(t)\|^2\|u_t(t)\|^2 - (u(t), u_t(t))^2\right] - \|u(t)\|^2 I(t)\right\} > 0\end{aligned}$$

for $t \in (b, T)$. Thus, **Step 1** is proved.

Step 2. Now, we prove that if (15) holds, then $I(t) < 0$ for every $t \in [b, \infty)$. First, from (9) and (10) and the Cauchy–Schwarz inequality we obtain that $I(b) < 0$. Suppose by contradiction that there exists $T < \infty$ such that $I(t) < 0$ for $t \in [b, T)$ and $I(T) = 0$. Relations (9) and (10) and the monotonicity of $\psi(t)$ and $\phi(t)$ in (b, T) give us

$$\frac{1}{2}I(t) \leq E(0) - \frac{ka}{4}\|u(b)\|^2 - \frac{1}{2}\frac{(u(b), u_t(b))^2}{\|u(b)\|^2} := -\frac{c_1}{2} < 0 \tag{20}$$

for $t \in [b, T)$. Finally, by the following impossible chain of inequalities

$$\begin{aligned}0 = \frac{1}{2}I(T) &\leq E(0) - \frac{ka}{4}\|u(T)\|^2 - \frac{1}{2}\frac{(u(T), u_t(T))^2}{\|u(T)\|^2}\\ &\leq E(0) - \frac{ka}{4}\|u(b)\|^2 - \frac{1}{2}\frac{(u(b), u_t(b))^2}{\|u(b)\|^2} < 0\end{aligned}$$

we arrive at a contradiction. Hence, $I(t) < 0$ for every $t \in [b, \infty)$.

Step 3. In (20), from **Step 2**, we prove that there exists a constant c_1 such that

$$I(t) \leq -c_1 < 0 \ \text{ for } \ t \in [b, \infty). \tag{21}$$

From (19) and (21), we have

$$\psi''(t) \geq -2I(t) \geq 2c_1 > 0 \ \text{ for } \ t \in [b, \infty). \tag{22}$$

Integrating (22) twice, it follows that

$$\psi(t) \geq c_1(t-b)^2 + 2(t-b)(u(b), u_t(b)) + \|u(b)\|^2.$$

Hence, $\lim_{t\to\infty}\psi(t) = \infty$, i.e., $u(t, x)$ blows up at infinity.

Necessity. Suppose that $u(t, x)$ blows up at infinity. Applying Lemma 1 for $T_m = \infty$ and $M = 4E(0)/(ka) > 0$ it follows that there exists $t_0 \geq 0$ such that $\psi(t_0) = \|u(t_0)\|^2 \geq 4E(0)/(ka) > 0$ and $\psi'(t_0) = (u(t_0), u_t(t_0)) > 0$. Thus, condition (15) holds for $b = t_0$. Statement (i_1) is proved.

(i_2) If (15) fails, then we have that for every $t \geq 0$ at least one of the following three conditions is satisfied:

$$\|u(t)\| = 0, \tag{23}$$

$$(u(t), u_t(t)) < 0 \tag{24}$$

or

$$E(0) \geq \frac{ka}{4}\|u(t)\|^2 + \frac{1}{2}\frac{(u(t), u_t(t))^2}{\|u(t)\|^2}. \tag{25}$$

To prove estimate (16), we consider the following two cases:

$$\text{Case 1:} \quad \|u_0\|^2 \leq \frac{4E(0)}{ka} \tag{26}$$

and

$$\text{Case 2:} \quad \|u_0\|^2 > \frac{4E(0)}{ka}. \tag{27}$$

Case 1. If $\psi(0) = \|u_0\|^2$ satisfies (26), then we will prove that

$$\psi(t) \leq \frac{4E(0)}{ka} \tag{28}$$

for every $t \geq 0$. Suppose by contradiction that estimate (28) fails for some $t_1 > 0$, i.e.,

$$\psi(t_1) > \frac{4E(0)}{ka}.$$

From this assumption it follows that there exist t_0 and t_* such that $0 \leq t_0 < t_* < t_1$,

$$\psi(t_0) = \frac{4E(0)}{ka},$$

$$(u(t_*), u_t(t_*)) = \frac{1}{2}\psi'(t_*) > 0, \text{ and } \|u(t_*)\|^2 = \psi(t_*) > \frac{4E(0)}{ka} > 0. \tag{29}$$

Inequalities (29) contradict our assumptions (23)–(25) at the time t_*. Therefore, in the hypothesis of Case 1, estimate (28) holds for every $t \geq 0$.

Case 2. In this case, we will show that

$$\|u(t)\|^2 = \psi(t) \leq \|u_0\|^2 \tag{30}$$

for every $t \geq 0$. Since (27) is fulfilled, then from (23)–(25) it follows that $\psi'(t) = (u(t), u_t(t)) < 0$ as long as $\psi(t) > \frac{4E(0)}{ka} > 0$, i.e., $\psi(t)$ is strictly decreasing.

If $\psi(t) > \frac{4E(0)}{ka} > 0$ for every $t \geq 0$, then (30) is satisfied for every $t \geq 0$.

Suppose there exists a time $t_2 > 0$ for which $\psi(t_2) = \frac{4E(0)}{ka}$. Then, applying Case 1 for t_2 instead of t_0 and $\psi(t_2)$ instead of $\|u_0\|^2$, we conclude that $\psi(t) \leq \frac{4E(0)}{ka} < \|u_0\|^2$ for every $t \in [t_2, \infty)$. Combining estimates (28) from Case 1 and (30) from Case 2, we obtain (16). The proof of (i_2) is completed.

(ii) Let $E(0) = 0$.

(ii_1) **Sufficiency.** Since (17) holds at some time $t = b$, then $\|u(b)\| > 0$ and consequently (15) is satisfied. The rest of the proof is identical to the proof of sufficiency of statement (i_1).

Necessity. As $u(t, x)$ blows up at infinity, then Lemma 1 for $T_m = \infty$ and $M = 0$ gives us that there exists $t_0 \geq 0$ such that $\psi(t_0) = \|u(t_0)\|^2 > 0$ and $\psi'(t_0) = (u(t_0), u_t(t_0)) > 0$. Thus, condition (17) holds for $b = t_0$ and (ii_1) is proved.

(ii_2) If (17) fails, then for every $t \geq 0$ we have

$$(u(t), u_t(t)) \leq 0.$$

Hence, the function $\psi(t)$ is decreasing for $t \in [0, \infty)$ and satisfies (18).

(iii) Let $E(0) < 0$.

The proof follows from Theorem 3 in [15] and we omit it. Thus, Theorem 2 is proved. □

Below, we give sufficient conditions on the initial data that guarantee infinite time blow up of the global solutions to problem (1)–(4).

Proposition 1 (Sufficient conditions). *Suppose $u(t, x)$ is a global weak solution to problem (1)–(4).*

(i) *If* $E(0) > 0$ *and*

$$\|u_0\| > 0, \quad (u_0, u_1) \geq 0, \quad E(0) < \frac{ka}{4}\|u_0\|^2 + \frac{1}{2}\frac{(u_0, u_1)^2}{\|u_0\|^2}, \tag{31}$$

then $u(t,x)$ *blows up at infinity.*

(ii) *If* $E(0) = 0$ *and* $(u_0, u_1) \geq 0$, *then every nontrivial solution* $u(t,x)$ *blows up at infinity.*

Proof. (i) The proof of this statement is a consequence of the sufficiency of Theorem $2(i_1)$ with $b = 0$.

(ii) If $E(0) = 0$ and $\|u_0\| > 0$, then from the sufficiency of Theorem $2(i_1)$ with $b = 0$, it follows that $u(t,x)$ blows up at infinity. If $\|u_0\| = 0$, since $u(t,x)$ is a nontrivial solution, there exists $b > 0$ such that $(u(b), u_t(b)) > 0$ and $u(t,x)$ blows up at infinity from the sufficiency of Theorem $2(ii_1)$. □

The following result is important for the comparison of the new sufficient conditions with the previous ones; see Section 5.

Proposition 2. *Under the conditions of Theorem 2 it follows that* $I(t) < 0$ *for every* $t \in [b, \infty)$.

The proof of this statement is given in **Step 1** and **Step 2** of the proof of sufficiency in Theorem $2(i_1)$ and it is valid for arbitrary energy $E(0)$.

Remark 1. *All results presented in Section 3 are valid for the case of pure logarithmic nonlinearity, i.e., when* $a_i(x) \equiv 0$ *for every* $i = 1, \ldots, r$.

4. Nonexistence of Global Solutions

Our aim in this section is to study the nonexistence of global solutions to problem (1)–(4) provided the additional Assumption (5); see Theorems 3 and 4. Later on, we show that (5) together with (41) changes the blow up type of the solutions: from infinite time blow up (under (3) and (4)) to finite time blow up (under (3)–(5) and (41)); see Theorems 5 and 6.

Theorem 3. *Suppose* (3)–(5) *hold and* $u(t,x)$ *is a weak solution of problem* (1), (2) *defined in the maximal existence time interval* $[0, T_m)$, $0 < T_m \leq \infty$.

(i) *Let* $E(0) > 0$, *then*

- (i_1) *If* (15) *is satisfied, then the maximal existence time of the solution* $u(t,x)$ *is finite, i.e.,* $T_m < \infty$.
- (i_2) *If* (15) *fails, then* $\|u(t)\|$ *is bounded for* $t \in [0, T_m)$ *and estimate* (16) *holds for* $t \in [0, T_m)$.

(ii) *Let* $E(0) = 0$, *then*

- (ii_1) *If* (17) *is satisfied, then the maximal existence time of* $u(t,x)$ *is finite, i.e.,* $T_m < \infty$.
- (ii_2) *If* (17) *fails, then* $\|u(t)\|$ *is bounded for every* $t \in [0, T_m)$ *and the estimate* (18) *holds for* $t \in [0, T_m)$.

(iii) *If* $E(0) < 0$, *then the maximal existence time of* $u(t,x)$ *is finite, i.e.,* $T_m < \infty$.

Proof. (i) Let $E(0) > 0$.

(i_1) Assume by contradiction that $T_m = \infty$. From Theorem $2(i_1)$ we obtain that $\psi(t) = \|u(t)\|^2$ blows up at infinity.

Now, we will prove that there exists $t_0 \geq b$ such that $\psi(t)$ satisfies, for every $t \geq t_0$, the following inequality:

$$\psi''(t)\psi(t) - \frac{(p+3)}{4}\psi'^2(t) \geq B_1(t), \tag{32}$$

where $p \in (1, p_1]$ and $B_1(t)$ is a suitable nonnegative function for $t_0 \geq b$, which will be defined later on.

Indeed, from (6)–(8) and (19), we have for every $p \in (1, p_1]$ the identity

$$\begin{aligned}\psi''(t) =&(p+3)\|u_t(t)\|^2 - 2(p+1)E(0) + (p-1)\|\nabla u(t)\|^2 + \frac{k(p+1)}{2}\int_\Omega a_0(x)|u|^2\,dx\\ &- k(p-1)\int_\Omega a_0(x)u^2\ln|u|\,dx + 2\sum_{i=1}^{r}\frac{p_i-p}{p_i+1}\int_\Omega a_i(x)|u|^{p_i+1}\,dx.\end{aligned} \tag{33}$$

Let us define the sets

$$\Omega_1(t) = \{x \in \Omega :\ |u(t,x)| < 1\},\quad \Omega_2(t) = \{x \in \Omega :\ |u(t,x)| \geq 1\}.$$

Hence, $\Omega(t) = \Omega_1(t) \cup \Omega_2(t)$.

For the last term on the right-hand side of (33) we obtain the estimate

$$\begin{aligned}2\sum_{i=1}^{r}\frac{p_i-p}{p_i+1}\int_\Omega a_i(x)|u|^{p_i+1}\,dx &\geq 2\sum_{i=1}^{r}\frac{p_i-p}{p_i+1}\int_{\Omega_2(t)} a_i(x)|u|^{p_i+1}\,dx\\ &\geq 2\sum_{i=1}^{r}\frac{p_i-p}{p_i+1}\int_{\Omega_2(t)} a_i(x)|u|^{p+1}\,dx\\ &\geq \frac{2(p_1-p)}{p_r+1}\int_{\Omega_2(t)}\left(\sum_{i=1}^{r}a_i(x)\right)|u|^{p+1}\,dx\\ &\geq K_1\int_{\Omega_2(t)}|u|^{p+1}\,dx,\end{aligned} \tag{34}$$

where

$$K_1 = \frac{2(p_1-p)}{p_r+1}K > 0.$$

To deal with the logarithmic term in (33) we use the trivial inequality

$$z^2\ln|z| \leq \frac{1}{\gamma}|z|^{2+\gamma} \quad \text{for } |z| \geq 1 \text{ and every } \gamma > 0.$$

Since

$$\int_{\Omega_1(t)} a_0(x)u^2\ln|u|\,dx \leq 0 \tag{35}$$

and

$$0 \leq \int_{\Omega_2(t)} a_0(x)u^2\ln|u|\,dx \leq \frac{A}{\gamma}\int_{\Omega_2(t)}|u|^{2+\gamma}\,dx, \tag{36}$$

we have the estimate

$$\begin{aligned}-k(p-1)&\int_\Omega a_0(x)u^2\ln|u|\,dx\\ &\geq -k(p-1)\int_{\Omega_1(t)} a_0(x)u^2\ln|u|\,dx - \frac{k(p-1)A}{\gamma}\int_{\Omega_2(t)}|u|^{2+\gamma}\,dx\\ &\geq -\frac{k(p-1)A}{\gamma}\int_{\Omega_2(t)}|u|^{2+\gamma}\,dx.\end{aligned} \tag{37}$$

If we choose

$$\gamma = \frac{p-1}{2},$$

then from (33), (34) and (37) it follows that

$$\begin{aligned}\psi''(t) \geq &(p+3)\|u_t(t)\|^2 - 2(p+1)E(0) + (p-1)\|\nabla u(t)\|^2 + \frac{k(p+1)a}{2}\|u(t)\|^2 \\ &+ K_1 \int_{\Omega_2(t)} |u|^{p+1}\,dx - 2kA \int_{\Omega_2(t)} |u|^{\frac{p+3}{2}}\,dx.\end{aligned} \tag{38}$$

Further, we represent $\Omega_2(t) = \Omega_3(t) \cup \Omega_4(t)$, where

$$\Omega_3(t) = \{x \in \Omega_2(t) : \ |u(t,x)| < M\}, \quad \Omega_4(t) = \{x \in \Omega_2(t) : \ |u(t,x)| \geq M\}$$

and

$$M = \left(\frac{2kA}{K_1}\right)^{\frac{2}{p-1}}.$$

Straightforward computations give us

$$\begin{aligned}&K_1 \int_{\Omega_2(t)} |u|^{p+1}\,dx - 2kA \int_{\Omega_2(t)} |u|^{\frac{p+3}{2}}\,dx \\ &\geq K_1 \int_{\Omega_4(t)} |u|^{p+1}\,dx - 2kA \int_{\Omega_3(t)} |u|^{\frac{p+3}{2}}\,dx - 2kA \int_{\Omega_4(t)} |u|^{\frac{p+3}{2}}\,dx \\ &\geq \int_{\Omega_4(t)} |u|^{\frac{p+3}{2}} (K_1 |u|^{\frac{p-1}{2}} - 2kA)\,dx - 2kAM^{\frac{p+3}{2}}|\Omega| \geq -2kAM^{\frac{p+3}{2}}|\Omega|.\end{aligned} \tag{39}$$

Finally, combining (38) and (39), we conclude that $\psi(t)$ satisfies (32) with

$$\begin{aligned}B_1(t) :=&(p-1)\|\nabla u(t)\|^2\|u(t)\|^2 + \frac{1}{2}k(p+1)a\|u(t)\|^4 \\ &+ (p+3)\Big(\|u_t(t)\|^2\|u(t)\|^2 - (u(t),u_t(t))^2\Big) \\ &- 2(p+1)E(0)\|u(t)\|^2 - 2kAM^{\frac{p+3}{2}}|\Omega|\|u(t)\|^2.\end{aligned}$$

Since $\psi(t) = \|u(t)\|^2$ blows up at infinity, there exists a sufficiently large time t_0 such that $B_1(t) \geq 0$ for every $t \geq t_0$. Then, according to Lemma 2, we obtain that $T_m < \infty$, which contradicts our assumption $T_m = \infty$. Thus, the solution $u(t,x)$ exists for a finite time and (i_1) is proved.

(i_2) The proof coincides with the proof of Theorem $2(i_2)$ and we omit it.

(ii) Let $E(0) = 0$.

(ii_1) Suppose that $T_m = \infty$. Then, from Theorem $2(ii_1)$ it follows that $u(t,x)$ blows up at infinity. The rest of the proof is similar to the proof of Theorem $3(i_1)$.

(ii_2) As in the proof of Theorem $2(ii_2)$, we obtain that function $\psi(t) = \|u(t)\|^2$ is a decreasing function for $t \in [0, T_m)$. Hence, the estimate (18) holds for $t \in [0, T_m)$.

(iii) Let $E(0) < 0$. If we suppose by contradiction that $T_m = \infty$, then it follows from Theorem $2(iii)$ that $u(t,x)$ blows up at infinity. The rest of the proof is similar to the proof of Theorem $3(i_1)$. □

Theorem 4. *Suppose* (3)–(5) *hold and* $u(t,x)$ *is a weak solution of problem* (1), (2) *defined in the maximal existence time interval* $[0, T_m)$, $0 < T_m \leq \infty$.

(i) If (11) *is satisfied, i.e.,*

$$\textit{there exists} \quad b \geq 0 \quad \textit{such that} \quad E(0) < \frac{ka}{4}\|u(b)\|^2 + \frac{\sqrt{ka}}{2}(u(b), u_t(b)),$$

then the maximal existence time is finite, i.e., $T_m < \infty$.

(ii) *If* (11) *fails, then* $\|u(t)\|$ *is bounded for* $t \in [0, T_m)$ *and estimate* (12) *holds for* $t \in [0, T_m)$, *i.e.,*

$$\|u(t)\|^2 \leq \left(\|u_0\|^2 - \frac{4E(0)}{ka}\right)e^{-\sqrt{ka}t} + \frac{4E(0)}{ka}.$$

holds for every $t \in [0, T_m)$.

Proof. (i) Assume by contradiction that $T_m = \infty$. Then, according to Theorem 1(i), we obtain that $\psi(t) = \|u(t)\|^2$ blows up at infinity. The rest of the proof is identical to the proof of Theorem 3(i_1).

(ii) If (11) fails, then for $t \in [0, T_m)$ we have

$$E(0) \geq \frac{ka}{4}\|u(t)\|^2 + \frac{\sqrt{ka}}{2}(u(t), u_t(t)) = \frac{ka}{4}\psi(t) + \frac{\sqrt{ka}}{4}\psi'(t). \tag{40}$$

Integrating (40), we obtain (12). The proof of Theorem 4 is completed. □

In Theorems 3(i_1) and 4(i), we prove that the maximal existence time of the weak solution to problem (1)–(5) is finite ($T_m < \infty$). However, we do not know whether $\|u(t)\| \to \infty$ as $t \to T_m$ or $\|\nabla u(t)\| \to \infty$ as $t \to T_m$. To exclude the case when $\|u(t)\|$ is bounded in $[0, T_m)$ but $\|\nabla u(t)\|$ blows up at T_m, we impose the following additional restrictions on the power exponents p_i, $i = 1, .., r$:

$$\begin{aligned} p_i &< \min\left\{\frac{n+4}{n}, \frac{n}{n-2}\right\} = \frac{n+4}{n}, \qquad \text{for } n = 1, 2, 3, 4; \\ p_i &\leq \min\left\{\frac{n+4}{n}, \frac{n}{n-2}\right\} = \frac{n}{n-2}, \qquad \text{for } n \geq 5. \end{aligned} \tag{41}$$

In the following two theorems, we formulate and prove our main assertions, namely, that under the additional restriction (41) the local weak solutions to (1)–(5) blow up at the maximal existence time $T_m < \infty$, i.e., $\limsup \|u(t)\| \to \infty$ as $t \to T_m$. Moreover, conditions (11), (15), and (17) become necessary and sufficient ones for finite time blow up of the solutions.

Theorem 5. *Suppose* (3)–(5) *and* (41) *hold. Let* $u(t, x)$ *be a local weak solution of problem* (1), (2), *defined in the maximal existence time interval* $[0, T_m)$, $0 < T_m \leq \infty$.

(i) *If* $E(0) > 0$, *then* $u(t, x)$ *blows up at finite time* T_m *if and only if assumption* (15) *is satisfied.*
(ii) *If* $E(0) = 0$, *then* $u(t, x)$ *blows up at finite time* T_m *if and only if assumption* (17) *is satisfied.*
(iii) *If* $E(0) < 0$, *then every local weak solution* $u(t, x)$ *blows up at finite time* T_m.

Proof. (i) Let $E(0) > 0$.

Sufficiency. Suppose (15) holds. Then, from Theorem 3(i_1) it follows that $T_m < \infty$. We assume by contradiction that $u(t, x)$ does not blow up at T_m, i.e., for every $t \in [0, T_m)$ and some constant C_1 we have

$$\limsup_{t \to T_m, t < T_m} \|u(t)\| < C_1 < \infty. \tag{42}$$

We will show that if (42) holds, then $\|\nabla u(t)\|$ and $\|u_t(t)\|$ are bounded in the maximal existence time interval $[0, T_m)$.

Now, the expression (7) and the conservation law (8) give us

$$\|\nabla u(t)\|^2 \leq 2E(0) + k\int_\Omega a_0(x)u^2(t, x)\ln|u(t, x)|\,dx + 2\sum_{i=1}^{r}\int_\Omega \frac{a_i(x)}{p_i + 1}|u(t, x)|^{p_i+1}\,dx. \tag{43}$$

We estimate the logarithmic term on the right-hand side of (43) by means of the technique proposed in the proof of Theorem 3*(i)*.

Using (35) and (36) with $\gamma = p_r - 1$, we obtain

$$\begin{aligned}\int_\Omega a_0(x)u^2 \ln|u(t,x)|\,dx &\le \int_{\Omega_1(t)} a_0(x)u^2 \ln|u(t,x)|\,dx + \int_{\Omega_2(t)} a_0(x)u^2 \ln|u(t,x)|\,dx \\ &\le \frac{A}{\gamma}\int_{\Omega_2(t)} |u(t,x)|^{2+\gamma}\,dx \le \frac{A}{\gamma}\|u(t)\|_{2+\gamma}^{2+\gamma} \\ &= \frac{A}{p_r-1}\|u(t)\|_{p_r+1}^{p_r+1}.\end{aligned} \tag{44}$$

Then, from (43) and (44), we obtain the estimate

$$\|\nabla u(t)\|^2 \le 2E(0) + \frac{kA}{p_r-1}\|u(t)\|_{p_r+1}^{p_r+1} + 2\sum_{i=1}^{r}\frac{A_i}{p_i+1}\|u(t)\|_{p_i+1}^{p_i+1}. \tag{45}$$

For every p_i, $i = 1,\dots,r$ satisfying (41), we apply the Gagliardo–Nirenberg inequality (14) with $j = 0$, $m = 1$, $r = q = 2$, and $\alpha = (p_i-1)n/(2(p_i+1))$. Thus, using (14) and (42), we have the estimate

$$\|u(t)\|_{p_i+1} \le C_{n,p_i}\|\nabla u(t)\|^{\frac{n(p_i-1)}{2(p_i+1)}}\|u(t)\|^{1-\frac{n(p_i-1)}{2(p_i+1)}} \le C_{n,p_i}C_1^{1-\frac{n(p_i-1)}{2(p_i+1)}}\|\nabla u(t)\|^{\frac{n(p_i-1)}{2(p_i+1)}}, \tag{46}$$

where the constant C_{n,p_i} depends only on n and p_i.

Applying (46) and the Young's inequality for

$$p_i^* = \frac{4}{n(p_i-1)}, \quad q_i^* = \frac{4}{4-n(p_i-1)}$$

and every $\epsilon > 0$, we obtain

$$\|u(t)\|_{p_i+1}^{p_i+1} \le \epsilon\|\nabla u(t)\|^2 + \frac{4+n-np_i}{4}(\epsilon p_i^*)^{-\frac{q_i^*}{p_i^*}}\left(C_{n,p_i}C_1^{1-\frac{n(p_i-1)}{2(p_i+1)}}\right)^{(p_i+1)q_i^*}. \tag{47}$$

If we choose

$$\epsilon = \left(4\sum_{i=1}^{r}\frac{A_i}{p_i+1} + \frac{2kA}{p_r-1}\right)^{-1},$$

we have from (45) and (47) the estimate

$$\|\nabla u(t)\|^2 \le 2E(0) + \frac{1}{2}\|\nabla u(t)\|^2 + C_2,$$

where the constant C_2 is given by the expression

$$\begin{aligned}C_2 =& 2\sum_{i=1}^{r}\frac{A_i}{p_i+1}\frac{4+n-np_i}{4}(\epsilon p_i^*)^{-\frac{q_i^*}{p_i^*}}\left(C_{n,p_i}C_1^{1-\frac{n(p_i-1)}{2(p_i+1)}}\right)^{(p_i+1)q_i^*} \\ &+ \frac{kA}{p_r-1}\frac{4+n-np_r}{4}(\epsilon p_r^*)^{-\frac{q_r^*}{p_r^*}}\left(C_{n,p_r}C_1^{1-\frac{n(p_r-1)}{2(p_r+1)}}\right)^{(p_r+1)q_r^*} < \infty.\end{aligned}$$

Hence,

$$\limsup_{t\to T_m, t<T_m}\|\nabla u(t)\| \le 4E(0) + 2C_2 < \infty. \tag{48}$$

Now, we will prove that $\|u_t(t)\|$ is bounded for every $t \in [0, T_m)$. Since (4) holds, it follows from the embedding theorem (13) and (48) that

$$\limsup_{t \to T_m, t < T_m} \|u(t)\|_{p_i+1} < \infty \text{ for } i = 1, \ldots r. \tag{49}$$

Then, using (7), the conservation law (8), the estimates (44), and (49), we obtain

$$\limsup_{t \to T_m, t < T_m} \|u_t(t)\| < \infty.$$

From the Continuation Principle, based on the local existence result, see [9,29], it follows that problem (1), (2) has a local weak solution in the interval $[0, T_m + \delta)$, $\delta > 0$. This contradicts that $[0, T_m)$ is the maximal existence time interval. Hence, $u(t, x)$ blows up for at $T_m < \infty$.

Necessity. The proof is identical to the proof of Necessity in Theorem $2(i_1)$.

(ii) Let $E(0) = 0$.

Sufficiency. If (17) holds, then Theorem $3(ii_1)$ gives us that $T_m < \infty$. Analogously to the proof of the sufficiency of Theorem $5(i)$, we conclude that $u(t, x)$ blows up at $T_m < \infty$.

The proof of **Necessity** is identical to the proof of necessity in Theorem $2(ii_1)$.

(iii) Let $E(0) < 0$. From Theorem $3(iii)$ it follows that $T_m < \infty$. The rest of the proof follows from the sufficiency of Theorem $5(i)$. Theorem 5 is proved. □

Theorem 6. *Suppose* (3)–(5) *and* (41) *hold. Then, every local weak solution* $u(t, x)$ *of problem* (1), (2), *defined in the maximal existence time interval* $[0, T_m)$, *blows up at finite time* T_m *if and only if* (11) *is satisfied.*

The proof follows the same ideas as the proof of Theorem 5. The only difference is that the conclusion $T_m < \infty$ is made by means of Theorem 4*(i)* instead of Theorem $3(i_1)$.

In the following proposition, we summarize our sufficient conditions for finite time blow up, obtained from Theorems 5 and 6.

Proposition 3 (Sufficient conditions)**.** *Suppose* (3)–(5) *and* (41) *hold. If* $u(t, x)$ *is a local weak solution of problem* (1), (2), *defined in the maximal existence time interval* $[0, T_m)$, $0 < T_m \leq \infty$, *then* $u(t, x)$ *blows up at finite time* T_m *if at least one of the following assumptions* (50)–(52) *or* (53) *holds:*

$$\|u_0\| > 0, \quad (u_0, u_1) \geq 0, \quad 0 < E(0) < \frac{ka}{4}\|u_0\|^2 + \frac{1}{2}\frac{(u_0, u_1)^2}{\|u_0\|^2}, \tag{50}$$

$$E(0) < \frac{ka}{4}\|u_0\|^2 + \frac{\sqrt{ka}}{2}(u_0, u_1), \tag{51}$$

$$E(0) = 0, \quad (u_0, u_1) \geq 0 \tag{52}$$

or

$$E(0) < 0. \tag{53}$$

5. Comparison and Discussion

Firstly, we comment on the results for blowing up at infinity of the global weak solutions to problem (1)–(4). We compare the result of Theorem 1, which is proved in [15], with the result of Theorem 2. Since the conditions of Theorems 1 and 2 are necessary and sufficient conditions for infinite time blow up, both results are equivalent. The only difference is the time $b \geq 0$, where these conditions hold. If the conditions in Theorem 1

are satisfied at some time $b_1 \geq 0$, then there exists time $b_2 \geq 0$ for which the conditions in Theorem 2 hold and vice versa.

As for the sufficient conditions given at $t = 0$, they are different. We compare the sufficient condition for arbitrary positive energy from Proposition 1 in [15], namely,

$$E(0) < \frac{ka}{4}\|u_0\|^2 + \frac{\sqrt{ka}}{2}(u_0, u_1), \tag{54}$$

with the sufficient condition (31) from Proposition 1. If

$$(u_0, u_1) > \sqrt{ka}\|u_0\|^2,$$

then condition (31) is better then (54). If the opposite inequality is satisfied, i.e., $(u_0, u_1) < \sqrt{ka}\|u_0\|^2$, then condition (54) is more general than (31). Finally, both conditions coincide when $(u_0, u_1) = \sqrt{ka}\|u_0\|^2$.

In order to compare our results with the results in [14], we consider problem (1), (2) for the following special case of the nonlinearity

$$f(x, u) = u \ln |u|^k, \quad k > 1. \tag{55}$$

Now, we show that for arbitrary positive energy, $E(0) > 0$, sufficient condition (31) in Proposition 1 is more general than condition (ii)(c) of Theorem 3.1 in [14], namely,

$$E(0) > 0, \ I(u_0) < 0, \ \|u_0\|^2 > \frac{4}{k}E(0) \ \text{ and } \ (u_0, u_1) > 0. \tag{56}$$

Indeed, if initial data u_0, u_1 satisfy (56), then they necessarily satisfy condition (31). Note that the assumption $I(u_0) < 0$ in (56) is superfluous. In fact, if $\|u_0\|^2 > \frac{4}{k}E(0)$, we conclude from (9) that $I(u_0) < 0$. Furthermore, the inequality $I(u_0) < 0$ results from the even more general assumptions (31); see Proposition 2.

Finally, let us mention that the problem (1), (2) with nonlinearity (55) is studied for different intervals of k. For example, the case $k > 1$ is considered in [12,14], the case $k \in (0, 1)$ is investigated in [10,19], while in [11] some additional restrictions on k are imposed. In contrast to the papers cited above, our results are valid for any $k > 0$.

6. Conclusions

- In this paper, we study the wave equation with combined logarithmic and power-type nonlinearities with nonnegative variable coefficients.
- New necessary and sufficient conditions for infinite time blow up of the global weak solutions to problem (1)–(4) are proved for every initial energy level; see Theorem 2.
- By introducing the additional assumptions (5) and (41), we establish novel necessary and sufficient conditions for finite time blow up of the local weak solutions to problem (1)–(5), (41); see Theorems 5 and 6. This demonstrates that the combination of conditions (5) and (41) significantly alters the blow up type of the weak solutions: from infinite time blow up (under (3) and (4)) to finite time blow up (under (3)–(5) and (41)).
- For arbitrary initial energy, sufficient conditions for infinite and finite time blow up are derived; see Propositions 1 and 3.
- All the results presented in the paper can be applied to other hyperbolic and parabolic equations with nonlinearity $f(x, u)$ considered in (1).

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References

1. Krolikowski, W.; Edmundson, D.; Bang, O. Unified model for partially coherent solitons in logarithmically nonlinear media. *Phys. Rev. E* **2000**, *61*, 3122–3126. [CrossRef]
2. Barrow, J.D.; Parsons, P. Inflationary models with logarithmic potentials. *Phys. Rev. D* **1995**, *52*, 5576–5587. [CrossRef] [PubMed]
3. Bialynicki-Birula, I.; Mycielski, J. Wave equations with logarithmic nonlinearities. *Bull. Acad. Polon. Sci.* **1975**, *3*, 461–466.
4. Enqvist, K.; McDonald, J. Q-balls and baryogenesis in the MSSM. *Phys. Lett. B* **1998**, *425*, 309–321. [CrossRef]
5. Rosen, G. Dilatation covariance and exact solutions in local relativistic field theories. *Phys. Rev.* **1969**, *183*, 1186–1188. [CrossRef]
6. Scott, T.C.; Zhang, X.; Mann, R.B.; Fee, G.J. Canonical reduction for dilatonic gravity in $3+1$ dimensions. *Phys. Rev. D* **2016**, *93*, 084017. [CrossRef]
7. De Martino, S.; Falanga, M.; Godano, C.; Lauro, G. Logarithmic Schrodinger-like equation as a model for magma transport. *Europhys. Lett.* **2003**, *63*, 472–475. [CrossRef]
8. Zloshchastiev, K. Applications of wave equations with logarithmic nonlinearity in fluid mechanics. *J. Phys. Conf. Ser.* **2018**, *1101*, 012051. [CrossRef]
9. Cazenave, T.; Haraux, A. Équations d'évolution avec non linéarité logarithmique. *Ann. Fac. Sci. Toulouse Math.* **1980**, *2*, 21–51. [CrossRef]
10. Górka, P. Logarithmic Klein–Gordon equation. *Acta Phys. Polon. B* **2009**, *40*, 59–66. Available online: https://www.actaphys.uj.edu.pl/R/40/1/59/pdf (accessed on 16 January 2025).
11. Ha, T. Sufficient condition for logarithmic nonlinearity in nonlinear evolution equations. *Math. Meth. Appl. Sci.* **2021**, *44*, 9611–9615. [CrossRef]
12. Lian, W.; Ahmed M.S.; Xu R. Global existence and blow up of solution for semilinear hyperbolic equation with logarithmic nonlinearity. *Nonlinear Anal.* **2019**, *184*, 239–257. [CrossRef]
13. Ye, Y.; Li, L. Global existence and blow-up of solutions for logarithmic Klein-Gordon equation. *AIMS Math.* **2021**, *6*, 6898–6914. [CrossRef]
14. Han, J.; Xu, R.; Yang C. Improved growth estimate of infinite time blowup solution for a semilinear hyperbolic equation with logarithmic nonlinearity. *Appl. Math. Lett.* **2023**, *143*, 108670. [CrossRef]
15. Dimova, M.; Kolkovska, N.; Kutev, N. Blow Up of Solutions to Wave Equations with Combined Logarithmic and Power-Type Nonlinearities. *Axioms* **2024**, *13*, 709. [CrossRef]
16. Rao, S. N.; Khuddush, M.; Singh, M., Meetei, M. Z. Infinite-time blowup and global solutions for a semilinear Klein–Gordan equation with logarithmic nonlinearity. *Appl. Math. Sci. Eng.* **2023**, *31*, 2270134. [CrossRef]
17. Hu, Q.; Zhang, H.; Liu, G. Asymptotic behavior for a class of logarithmic wave equations with linear damping. *Appl. Math. Optim.* **2019**, *79*, 131–144. [CrossRef]
18. Han, X. Global existence of weak solutions for a logarithmic wave equation arising from Q-ball dynamics. *Bull. Korean Math. Soc.* **2013**, *50*, 275–283. [CrossRef]
19. Ding, H.; Zhou, J. Infinite Time Blow-Up of Solutions for A Class of Logarithmic Wave Equations with Arbitrary High Initial Energy. *Appl. Math. Optim.* **2021**, *84*, 1331–1343. [CrossRef]
20. Lian, W.; Xu, R. Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term. *Adv. Nonlinear Anal.* **2020**, *9*, 613–632. [CrossRef]
21. Ding, H.; Zhou, J. Well-posedness of solutions for the dissipative Boussinesq equation with logarithmic nonlinearity. *Nonlinear Anal. Real World Appl.* **2022**, *67*, 103587. [CrossRef]
22. Han, Z.; Xu, R. Improved growth estimate for one-dimensional sixth-order Boussinesq equation with logarithmic nonlinearity. *Appl. Math. Lett.* **2025**, *160*, 109290. [CrossRef]
23. Shao, X.; Huang, N.; O'Regan, D. Infinite time blow-up of solutions for a plate equation with weak damping and logarithmic nonlinearity. *J. Math. Anal. Appl.* **2024**, *535*, 128144. [CrossRef]

24. Pişkin, E.; Ferreira, J.; Yuksekkaya, H.; Shahrouzi, M. Existence and asymptotic behavior for a logarithmic viscoelastic plate equation with distributed delay. *Int. J. Nonlinear Anal. Appl.* **2022**, *13*, 763–788. [CrossRef]
25. Chen, H.; Luo, P.; Liu, G. Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity. *J. Math. Anal. Appl.* **2015**, *422*, 84–98. [CrossRef]
26. Wang, X.; Wang, Y. Blowup for semilinear parabolic equation with logarithmic nonlinearity. *Discrete Contin. Dyn. Syst. Ser. S* **2024**, *17*, 2629–2639. [CrossRef]
27. Benramdane, A.; Mezouar, N.; Bensaber, F.; Boulaaras, S.; Jan, R. Blow-Up of Solution of Lamé Wave Equation with Fractional Damping and Logarithmic Nonlinearity Source Terms. *Mathematics* **2023**, *11*, 4591. [CrossRef]
28. Lian, W.; Ahmed, M.S.; Xu, R. Global existence and blow up of solution for semi-linear hyperbolic equation with the product of logarithmic and power-type nonlinearity. *Opuscula Math.* **2020**, *40*, 111–130. [CrossRef]
29. Dimova, M.; Kolkovska, N.; Kutev, N. Global behavior of the solutions to nonlinear wave equations with combined power-type nonlinearities with variable coefficients. *Nonlinear Anal.* **2024**, *242*, 113504. [CrossRef]
30. Dimova, M.; Kolkovska, N.; Kutev, N. Blow up of solutions to ordinary differential equations arising in nonlinear dispersive problems. *Electron. J. Differ. Equations* **2018**, *2018*, 1–16. Available online: https://ejde.math.txstate.edu/Volumes/2018/68/dimova.pdf (accessed on 16 January 2025).
31. Dimova, M.; Kolkovska, N.; Kutev, N. Global behavior of the solutions to nonlinear Klein–Gordon equation with critical initial energy. *Electron. Res. Arch.* **2020**, *28*, 671–689. [CrossRef]
32. Nirenberg, L. An extended interpolation inequality. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **1966**, *20*, 733–737.
33. Gagliardo, E. Ulteriori proprietà di alcune classi di funzioni in più variabili. *Ricerche Mat.* **1959**, *8*, 24–51.

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