mathematics

MDPI

Article

Uniqueness of Positive Solutions to Non-Local Problems of Brézis–Oswald Type Involving Hardy Potentials

Yun-Ho Kim

Department of Mathematics Education, Sangmyung University, Seoul 03016, Republic of Korea; kyh1213@smu.ac.kr

Abstract: The aim of this paper is to demonstrate the existence of a unique positive solution to non-local fractional p-Laplacian equations of the Brézis–Oswald type involving Hardy potentials. The main feature of this paper is solving the difficulty that arises in the presence of a singular coefficient and in the lack of the semicontinuity property of an energy functional associated with the relevant problem. The main tool for overcoming this difficulty is the concentration–compactness principle in fractional Sobolev spaces. Also, the uniqueness result of the Brézis–Oswald type is obtained by exploiting the discrete Picone inequality.

Keywords: fractional p-Laplacian; Hardy potential; concentration-compactness principle; discrete Picone inequality

MSC: 35B33, 35D30, 35J20, 35J60, 35J66

1. Introduction

In the last few decades, a great deal of work has been devoted to fractional Sobolev spaces and their corresponding non-local problems, arising in many fields such as fractional quantum mechanics, game theory, Lévy processes, image processing, thin obstacle problems, optimization, frame propagation, geophysical fluid dynamics, anomalous diffusion in plasma, and American options; see [1–5] and references therein for more details.

In this paper, we deal with a Brézis–Oswald-type problem driven by a non-local p-Laplacian fractional involving Hardy potential as follows:

$$\begin{cases} \mathcal{L}^{s,p}\phi(z) = \lambda \frac{|\phi(z)|^{p-2}\phi(z)}{|z|^{sp}} + g(z,\phi(z)) & \text{in} \quad \Omega, \\ \phi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where λ is a real parameter, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open set with Lipschitz boundary $\partial\Omega$, $0 \in \Omega$, $s \in (0,1)$, $p \in (1,+\infty)$, $sp < N$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is non-negative function, which will be specified later. Here, $\mathcal{L}^{s,p}$ is a non-local operator defined pointwise as

$$\mathcal{L}^{s,p}\phi(z) = 2\int_{\mathbb{R}^N} |\phi(z) - \phi(y)|^{p-2}(\phi(z) - \phi(y))\mathcal{K}(z,y)dy \quad \text{for all } z \in \mathbb{R}^N,$$

where the kernel $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \to (0,+\infty)$ is a measurable function with the following properties:

Academic Editor: Hovik Matevossian

Received: 14 December 2024
Revised: 9 January 2025
Accepted: 14 January 2025
Published: 18 January 2025

Citation: Kim, Y.-H. Uniqueness of Positive Solutions to Non-Local Problems of Brézis–Oswald Type Involving Hardy Potentials. *Mathematics* **2025**, *13*, 311. https://doi.org/10.3390/math13020311

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

($\mathcal{A}1$) There are positive constants γ_0 and γ_1 with $\gamma_0 \geq 1$ such that

$$\gamma_0|z-y|^{-(N+sp)} \leq \mathcal{K}(z,y) \leq \gamma_1|z-y|^{-(N+sp)}$$

for almost all $(z,y) \in \mathbb{R}^N \times \mathbb{R}^N$ and $z \neq y$;
($\mathcal{A}2$) $\mathcal{K}(z,y) = \mathcal{K}(y,z)$ for all $(z,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

It is obvious that $m\mathcal{K} \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$ by setting $m(z,y) = \min\{|z-y|^p, 1\}$. If $\mathcal{K}(z,y) = |z-y|^{-(N+sp)}$, $\mathcal{L}^{s,p}$ is the fractional p-Laplacian operator $(-\Delta)_p^s$ defined by

$$(-\Delta)_p^s \phi(z) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\rho(z)} \frac{|\phi(z)-\phi(y)|^{p-2}(\phi(z)-\phi(y))}{|z-y|^{N+sp}} dy, \quad z \in \mathbb{R}^N,$$

where $B_\rho(z) := \{z \in \mathbb{R}^N : |z-y| \leq \rho\}$.

The primary purpose of the present paper is dedicated to proving the existence and uniqueness of solutions to non-local fractional p-Laplacian problems of the Brézis–Oswald type involving Hardy potentials in the spirit of the celebrated paper by Brézis–Oswald on Laplacian problems. In [6], the authors delved into the existence and uniqueness of solutions to certain types of sublinear elliptic equations. In particular, they provided insights and results that have significant implications for the study of differential equations and their applications. Since this work, there has been a growing interest in supplying necessary and/or sufficient conditions for the solvability of more general problems, taking into account various operators and boundary conditions; see [7–16]. The authors in [16] examined the conditions that ensure the existence of positive solutions to p-Laplacian equations, and also provided criteria under which the uniqueness of these solutions is guaranteed. As an improvement on the classical Brézis–Oswald and Dìaz–Saa results in the Orlicz–Sobolev framework, the existence and uniqueness of solutions for a quasilinear elliptic problem can be found in [10]. As we know, the main tool for considering the Brézis–Oswald result for p-Laplacian problems or more general problems is the Dìaz–Saa-type inequalities, as given in [16,17]. In order to apply the Dìaz–Saa-type inequalities given in [16,17] to an elliptic problem, the classical Hopf boundary lemma is needed to ensure the fact that the quotient between solutions belongs to the $L^{₩\infty}$-space. However, the solutions of fractional-order equation are usually singular at the boundary, making it difficult to deal with their quotient between solutions because Hopf's boundary lemma is not preserved. Thus, in contrast with the references [10,12,13], the major difficulty is to obtain the result that a nonlinear problem of the Brézis–Oswald type driven by the fractional p-Laplacian possesses a unique positive weak solution. The authors in [7,11,15,18] overcame this difficulty by considering the discrete Picone inequality in [8,19]. By using this inequality, Mugnai–Pinamonti–Vecchi [15] discussed necessary and sufficient conditions that ensure the existence of a unique positive solution to the non-local Robin boundary value problem involving the fractional p-Laplacian. Motivated by this work, the author in [18] recently established the existence, uniqueness, and localization of positive solutions to non-local problems with discontinuous Kirchhoff-type functions via the global minimum principle of Ricceri [20]. Also, I refer to [11] for the case of the fractional (p,q)–Laplace equation with Dirichlet boundary conditions, and to [7] for some sublinear Dirichlet problems driven by a mixed local and a non-local operator.

Another interesting respect of the problem at hand is the emergence of Hardy potentials. In recent years, the stationary problem associated with singular coefficients has gained increasing attention as they can be demonstrated in models of various physical phenomena and in economic models. For more comprehensive details and examples, see [21–23]. In addition, by virtue of this great interest, these problems have been investigated more in recent years; see [24–27]. In particular, Ferrara and Bisci [24] revealed that a nonlinear

Dirichlet boundary problem with Hardy potential possesses at least one nontrivial solution. Inspired by this work, Khodabakhshi et al. [26] proved the existence of at least three distinct generalized solutions; see [27] for nonlinear elliptic problems involving p-Laplace-type operators. Here, we also cite the paper given in [25] for infinitely many solutions. These elliptic problems with singular coefficients pose some technical difficulties because it is not immediately obvious how to assure the compactness condition of the Palais–Smale type in the desired function space due to the presence of Hardy potential. For this reason, the authors of [24–27] discussed the multiplicity results of solutions by exploiting the various critical point theorems in [28,29] without substantiating the compactness condition of the Palais–Smale type.

In a different approach from [24–27], Fiscella in [30] recently demonstrated the existence of at least one non-trivial solution to a fractional p-Laplacian equation of the Schrödinger–Kirchhoff type with a Hardy term:

$$\left(\alpha+\beta[\phi]_{s,p}^{p(\theta-1)}\right)(-\Delta)_p^s\phi(z)+\mathcal{V}(z)|\phi|^{p-2}v=\lambda\frac{|\phi|^{p-2}\phi}{|z|^{sp}}+\mu g(z,\phi)\quad\text{in}\quad\mathbb{R}^N,$$

where $\alpha>0$, $\beta\geq 0$, $1<p<p^*$, $\theta\geq 1$, λ and μ are real numbers, $\mathcal{V}:\mathbb{R}^N\to(0,\infty)$ is a potential function, and g is a continuous function that fulfills the classical Ambrosetti–Rabinowitz condition. The primary tool for achieving this consequence is the mountain pass theorem. Motivated by this work, the existence of multiple solutions to equations of the Schrödinger-Hardy type driven by the non-local fractional p-Laplacian was obtained in [12]. In addition, the authors of [31] derived several multiplicity results and proved the regularity of nontrivial solutions for Kirchhoff–Schrödinger–Hardy-type p-Laplacian problems.

Unlike the aforementioned related studies, the main purpose of this paper is to establish the existence of a unique positive solution to non-local fractional p-Laplacian problems of the Brézis–Oswald type with Hardy potential without exploiting critical point theorems in [28,29] and the variational methods in [12,30,31]. However, this problem has some technical difficulties in showing the semicontinuity property of an energy functional corresponding to problem (1) because of the presence of the Hardy term. To overcome this difficulty, we utilize the concentration-compactness principle in fractional Sobolev spaces, which is given in [32,33]. As far as we are aware, this study is the first effort to develop the uniqueness result of positive solutions to non-local Brézis–Oswald-type elliptic problems with Hardy potentials.

The remainder of this paper is organized as follows. Section 2 presents some essential preliminary knowledge of function spaces to be utilized in the rest of the paper. In Section 3, we provide the variational framework related to problem (1); then, Section 4 derives the existence result of a unique positive solution under suitable assumptions. Finally, some useful comments related to the problems given in (1) are given in Section 5.

2. Preliminaries

In this section, we briefly give some useful definitions and elementary properties of fractional Sobolev spaces that will be utilized in the next sections. Let the fractional Sobolev space $W^{s,p}(\Omega)$ be defined as follows:

$$W^{s,p}(\Omega):=\left\{\phi\in L^p(\Omega):\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi(z)-\phi(y)|^p}{|z-y|^{N+ps}}\,dz\,dy<+\infty\right\},$$

endowed with the norm

$$\|\phi\|_{W^{s,p}(\Omega)}:=\left(\|\phi\|_{L^p(\Omega)}^p+|\phi|_{W^{s,p}(\mathbb{R}^N)}^p\right)^{\frac{1}{p}},$$

where

$$\|\phi\|^p_{L^p(\Omega)} := \int_\Omega |\phi(z)|^p \, dz \qquad \text{and} \qquad |\phi|^p_{W^{s,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\phi(z)-\phi(y)|^p}{|z-y|^{N+ps}} \, dz\, dy.$$

Then, $W^{s,p}(\Omega)$ is a separable and reflexive Banach space. Also, the space $C_0^\infty(\Omega)$ is dense in $W^{s,p}(\Omega)$—that is, $W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$ (see, e.g., [4,34]).

Lemma 1 ([4]). *Let $s \in (0,1)$ and $p \in (1,+\infty)$. Then, the following continuous embeddings hold:*

$$\begin{aligned} &W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) && \text{for all } \ q \in [1, p_s^*], && \text{if } \ sp < N;\\ &W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) && \text{for every } \ q \in [1,\infty), && \text{if } \ sp = N;\\ &W^{s,p}(\Omega) \hookrightarrow C_b^{0,\lambda}(\Omega) && \text{for all } \ \lambda < s - N/p, && \text{if } \ sp > N. \end{aligned}$$

In particular, the space $W^{s,p}(\Omega)$ is compactly embedded in $L^\ell(\Omega)$ for any $\ell \in [1, p_s^*)$, where p_s^* is the fractional critical Sobolev exponent, namely,

$$p_s^* := \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N,\\ +\infty & \text{if } sp \geq N. \end{cases}$$

Let the fractional Sobolev space $W^{s,p}_{\mathcal{K}}(\mathbb{R}^N)$ be defined as follows:

$$W^{s,p}_{\mathcal{K}}(\mathbb{R}^N) := \left\{ \phi \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi(y)-\phi(z)|^p \mathcal{K}(y,z)\, dy\, dz < +\infty \right\},$$

where the kernel function $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,+\infty)$ satisfies the conditions ($\mathcal{A}1$)–($\mathcal{A}2$). By the assumption ($\mathcal{A}1$), the function

$$(y,z) \mapsto (\phi(y)-\phi(z))\mathcal{K}^{\frac{1}{p}}(y,z) \in L^p(\mathbb{R}^N)$$

holds for any $\phi \in C_0^\infty(\mathbb{R}^N)$. Let us consider the problem (1) in the closed linear subspace defined as

$$X := \left\{ \phi \in W^{s,p}_{\mathcal{K}}(\mathbb{R}^N) : \phi(y) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}$$

with respect to the norm

$$\|\phi\|_X := \left(\|\phi\|^p_{L^p(\Omega)} + [\phi]^p_{s,p} \right)^{\frac{1}{p}},$$

where

$$[\phi]^p_{s,p} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi(y)-\phi(z)|^p \mathcal{K}(y,z)\, dy\, dz.$$

Throughout the sequel, let $s \in (0,1)$, with $p \in (1,+\infty)$ and $ps < N$, and let the kernel function $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ satisfy the conditions ($\mathcal{A}1$)–($\mathcal{A}2$).

Lemma 2 ([35]). *If $\phi \in X$, then $\phi \in W^{s,p}(\Omega)$. Additionally, one has*

$$\|\phi\|_{W^{s,p}(\Omega)} \leq \max\{1, \gamma_0^{-\frac{1}{p}}\} \|\phi\|_X,$$

where γ_0 is given in ($\mathcal{A}2$).

According to Lemmas 1 and 2, we can give the following assertion instantly:

Lemma 3 ([35]). *For any $\phi \in X$ and $1 \leq q \leq p_s^*$, there is a constant $\mathfrak{C}_0 = C_0(s,p,N) > 0$ such that*

$$\begin{aligned}\|\phi\|^p_{L^q(\Omega)} &\leq \mathfrak{C}_0 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\phi(z)-\phi(y)|^p}{|z-y|^{N+ps}} dz\,dy \\ &\leq \frac{\mathfrak{C}_0}{\gamma_0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi(z)-\phi(y)|^p \mathcal{K}(z,y)\,dz\,dy,\end{aligned}$$

where γ_0 is given in ($\mathcal{A}2$). As a result, the space X is continuously embedded in $L^\ell(\Omega)$ for any $\ell \in [1, p_s^*]$. Additionally, the embedding

$$X \hookrightarrow L^\ell(\Omega)$$

is compact for $\ell \in (1, p_s^*)$.

The following consequence is the fractional Hardy inequality given in [19].

Lemma 4. *For any $\phi \in X$, in case $sp < N$, and for all $\phi \in X\backslash\{0\}$, in case $sp > N$, we have*

$$\begin{aligned}\|\phi\|^p_{H_p} := \int_\Omega \frac{|\phi(z)|^p}{|z|^{sp}} dz &\leq c_H \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\phi(z)-\phi(y)|^p}{|z-y|^{N+sp}} dy\,dz, \\ &\leq \frac{c_H}{\gamma_0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi(z)-\phi(y)|^p \mathcal{K}(z,y) dy\,dz,\end{aligned} \tag{2}$$

where $c_H := c_H(N,s,p)$ is a positive constant.

3. Variational Setting and Useful Auxiliaries

In this section, we give the variational setting associated with problem (1). Also, we provide some useful auxiliary consequences, which are essential in obtaining our main result.

Let the functional $\Phi_\lambda : X \to \mathbb{R}$ be defined by

$$\Phi_\lambda(\phi) := \Phi(\phi) - \lambda \Phi_H(\phi),$$

where

$$\Phi(\phi) := \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi(z)-\phi(y)|^p \mathcal{K}(z,y)\,dy\,dz \quad \text{and} \quad \Phi_H(\phi) := \frac{\lambda}{p}\int_\Omega \frac{|\phi(z)|^p}{|z|^{sp}}\,dz.$$

Then, it is not difficult to show that Φ_λ is well defined on X, and the result below can be derived from following the lines of the proof of Lemma 2 in [36].

Lemma 5. *For each $\lambda \in \mathbb{R}$, the functional $\Phi_\lambda : X \to \mathbb{R}$ is of class $C^1(X,\mathbb{R})$ and its Fréchet derivative is*

$$\begin{aligned}\langle \Phi'_\lambda(\phi), \varphi\rangle = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi(z)-\phi(y)|^{p-2}(\phi(z)-\phi(y))(\varphi(z)-\varphi(y))\mathcal{K}(z,y) dy\,dz \\ &- \lambda\int_\Omega \frac{|\phi(z)|^{p-2}}{|z|^{sp}}\phi\varphi\,dz\end{aligned} \tag{3}$$

for any $\phi, \varphi \in X$, where $\langle\cdot,\cdot\rangle$ denotes the pairing of X and its dual X^*.

Proof. It is immediately provable that Φ_λ has a Fréchet derivative in X and (3) is satisfied for any $\phi, \varphi \in X$. Let $\{\phi_n\}_{n\in\mathbb{N}} \subset X$ be a sequence satisfying $\phi_n \to \phi$ strongly in X as $n \to \infty$. Without loss of generality, we assume that $\phi_n \to \phi$ a.e. in $\mathbb{R}^N$. Then, the sequence

$$\left\{ |\phi_n(z) - \phi_n(y)|^{p-2}(\phi_n(z) - \phi_n(y))\mathcal{K}(z,y)^{\frac{1}{p'}} \right\}_{n\in\mathbb{N}}$$

is bounded in $L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$, as well as a.e. in $\mathbb{R}^N \times \mathbb{R}^N$:

$$\begin{aligned} \mathcal{A}_n(z,y) &:= |\phi_n(z) - \phi_n(y)|^{p-2}(\phi_n(z) - \phi_n(y))\mathcal{K}(z,y)^{\frac{1}{p'}} \\ &\longrightarrow \mathcal{A}(z,y) := |\phi(z) - \phi(y)|^{p-2}(\phi(z) - \phi(y))\mathcal{K}(z,y)^{\frac{1}{p'}} \quad \text{as } n \to \infty. \end{aligned}$$

Thus, taking into account the Brézis–Lieb Lemma (see [37]), one has

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\mathcal{A}_n(z,y) - \mathcal{A}(z,y)|^{p'}\, dy\, dz \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (|\phi_n(z) - \phi_n(y)|^p \mathcal{K}(z,y) - |\phi(z) - \phi(y)|^p \mathcal{K}(z,y))\, dy\, dz. \end{aligned} \tag{4}$$

The fact that $\{\phi_n\}_{n\in\mathbb{N}}$ converges strongly to ϕ in X as $n \to \infty$ implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (|\phi_n(z) - \phi_n(y)|^p \mathcal{K}(z,y) - |\phi(z) - \phi(y)|^p \mathcal{K}(z,y))\, dy\, dz = 0.$$

Owing to (4), we infer that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\mathcal{A}_n(z,y) - \mathcal{A}(z,y)|^{p'}\, dy\, dz = 0. \tag{5}$$

On the other hand, the sequence

$$\left\{ \frac{|\phi_n(z)|^{p-2}\phi_n(z)}{|z|^{\frac{sp}{p'}}} \right\}_{n\in\mathbb{N}}$$

is bounded in $L^{p'}(\Omega)$, as well as a.e. in Ω:

$$\widetilde{\mathcal{A}}_n(z) := \frac{|\phi_n(z)|^{p-2}\phi_n(z)}{|z|^{\frac{sp}{p'}}} \longrightarrow \widetilde{\mathcal{A}}(z) := \frac{|\phi(z)|^{p-2}\phi(z)}{|z|^{\frac{sp}{p'}}} \quad \text{as } n \to \infty.$$

Thus, we have

$$\lim_{n\to\infty} \int_\Omega \left| \widetilde{\mathcal{A}}_n(z) - \widetilde{\mathcal{A}}(z) \right|^{p'} dz = \lim_{n\to\infty} \int_\Omega \left(\frac{|\phi_n(z)|^p}{|z|^{sp}} - \frac{|\phi(z)|^p}{|z|^{sp}} \right) dz. \tag{6}$$

Since $\{\phi_n\}_{n\in\mathbb{N}}$ converges strongly to ϕ in X as $n \to \infty$, it follows from Lemma 4 that

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega \left(\frac{|\phi_n(z)|^p}{|z|^{sp}} - \frac{|\phi(z)|^p}{|z|^{sp}} \right) dz \\ &= \lim_{n\to\infty} \int_\Omega \frac{|\phi_n(z) - \phi(z)|^p}{|z|^{sp}} dz \\ &\le \lim_{n\to\infty} \frac{c_H}{\gamma_0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |(\phi_n - \phi)(z) - (\phi_n - \phi)(y)|^p \mathcal{K}(z,y) dy\, dz \\ &\le \lim_{n\to\infty} \frac{c_H}{\gamma_0} \|\phi_n - \phi\|_X^p \\ &= 0. \end{aligned} \tag{7}$$

In accordance with (6) and (7), it follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left|\widetilde{\mathcal{A}}_n(z)-\widetilde{\mathcal{A}}(z)\right|^{p'}dz=0. \tag{8}$$

Combining (5) and (8) with the Hölder inequality, we obtain that

$$\|\Phi'_\lambda(\phi_n)-\Phi'_\lambda(\phi)\|_{X^*}=\sup_{\phi\in X,\|\phi\|_X=1}\left|\langle\Phi'_\lambda(\phi_n)-\Phi'_\lambda(\phi),\varphi\rangle\right|\longrightarrow 0$$

as $n\to\infty$. Therefore, we assert $\Phi_\lambda\in C^1(X,\mathbb{R})$. □

In view of Lemma 4, we have

$$\gamma_0 c_H^{-1}\le S_*:=\inf_{u\in X\setminus\{0\}}\frac{\|\phi\|_X}{\|\phi\|_{H_p}}. \tag{9}$$

Now, we present the following consequence, which provides the concentration-compactness principle in fractional Sobolev spaces; see [32,33]. This result is essential in showing the semicontinuity property of the functional Φ_λ.

Lemma 6. *Let $\mathcal{M}(\mathbb{R}^N)$ be the space of all signed finite Radon measures on $\mathbb{R}^N$ with the total variation norm. Let $\{\phi_n\}$ be a bounded sequence in X such that $\phi_n\rightharpoonup\phi$ in X. Let us assume that*

$$\begin{aligned}&\int_{\mathbb{R}^N}|\phi_n(z)-\phi_n(y)|^p\mathcal{K}(z,y)\,dy\ \overset{*}{\rightharpoonup}\ \mu\ \text{ in }\ \mathcal{M}(\mathbb{R}^N),\\&\frac{|\phi_n(z)|^p}{|z|^{sp}}\ \overset{*}{\rightharpoonup}\ \nu\ \text{ in }\ \mathcal{M}(\mathbb{R}^N).\end{aligned} \tag{10}$$

Then, there exist sets $\{\nu_i\}_{i\in I}\subset(0,\infty)$, $\{\mu_i\}_{i\in I}\subset(0,\infty)$, and $\{x_i\}_{i\in I}\subset\mathcal{C}$ where I is an at most countable index set, such that

$$\nu=\frac{|\phi(z)|^p}{|z|^{sp}}+\sum_{i\in I}\nu_i\delta_{x_i}, \tag{11}$$

$$\mu\ge\int_{\mathbb{R}^N}|\phi(z)-\phi(y)|^p\mathcal{K}(z,y)\,dy+\sum_{i\in I}\mu_i\delta_{x_i}, \tag{12}$$

$$\mu_i\ge S_*\nu_i\quad\text{for all}\quad i\in I, \tag{13}$$

where δ_{x_i} denotes the Dirac mass at x_i and

$$S_*:=\inf_{\phi\in X\setminus\{0\}}\frac{\|\phi\|_X}{\|\phi\|_{H_p}}>0. \tag{14}$$

Lemma 7. *There is a constant $\lambda^*>0$ such that the functional $\Phi_\lambda=\Phi-\lambda\Phi_H$ is sequentially weakly lower semicontinuous on X for any $\lambda\in(-\infty,\lambda^*)$; that is,*

$$\Phi_\lambda(\phi)\le\liminf_{n\to+\infty}\Phi_\lambda(\phi_n)\ \text{ if }\ \phi_n\to\phi\ \text{ weakly in }\ X.$$

Proof. Thanks to Theorem 6 in [38], we infer that $C_c^\infty(\Omega)$ is a dense subset of X. Hence, utilizing density arguments to show that Φ_λ is sequentially weakly lower semicontinuous on X, it suffices to prove that the functional

$$\Phi_\lambda\text{ is sequentially weakly lower semicontinuous on }C_c^\infty(\Omega) \tag{15}$$

for any $\lambda \in (-\infty, \lambda^*)$. So, let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence in $C_c^\infty(\Omega)$ such that

$$\phi_n \to \phi \text{ weakly in } X, \text{ as } n \to +\infty. \tag{16}$$

Thus, in accordance with Lemma 6, there exist two bounded measures ν and μ, positive real numbers $\{\nu_i\}_{i\in I}, \{\mu_i\}_{i\in I}$, and an at most countable set of indices I of distinct points $\{x_i\}_{i\in I} \subset \overline{\Omega}$ such that the following convergence holds weakly in the sense of measures:

$$\int_{\mathbb{R}^N} |\phi_n(z) - \phi_n(y)|^p \mathcal{K}(z,y)\, dy \rightharpoonup \mu \geq \int_{\mathbb{R}^N} |\phi(z) - \phi(y)|^p \mathcal{K}(z,y)\, dy + \sum_{i\in I} \mu_i \delta_{x_i}, \tag{17}$$

$$\frac{|\phi_n(z)|^p}{|z|^{sp}} \rightharpoonup \nu = \frac{|\phi(z)|^p}{|z|^{sp}} + \sum_{i\in I} \nu_i \delta_{x_i}. \tag{18}$$

Finally,

$$S_* \nu_i \leq \mu_i, \quad \text{for all} \quad i \in I. \tag{19}$$

By virtue of the continuity of the embedding $X \hookrightarrow L^p(\Omega)$, for every $p \in [1, p^*]$, we have that

$$\phi_n \to \phi \text{ strongly in } L^p(\Omega), \text{ as } n \to +\infty.$$

Set $\lambda^* = c_H^{-1}\gamma_0$, where γ_0 and c_H are given in ($\mathcal{A}2$) and Lemma 4. Let us consider $\lambda \in (0, \lambda^*)$ first. In view of Lemma 4, we know $\gamma_0 c_H^{-1} \leq S_*$. From this and (17)–(19), we obtain that

$$\begin{aligned}
&\liminf_{n\to+\infty} \Phi_\lambda(\phi_n) \\
&= \liminf_{n\to+\infty} \left[\frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\phi_n(z) - \phi_n(y)|^p \mathcal{K}(z,y)\, dy\, dz - \frac{\lambda}{p} \int_\Omega \frac{|\phi_n(z)|^p}{|z|^{sp}}\, dz \right] \\
&\geq \frac{1}{p} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\phi(z) - \phi(y)|^p \mathcal{K}(z,y)\, dy\, dz + \sum_{i=1}^k \mu_i \right) \\
&\qquad - \frac{\lambda}{p} \left(\int_\Omega \frac{|\phi(z)|^p}{|z|^{sp}}\, dz + \sum_{i=1}^k \nu_i \right) \\
&= \frac{1}{p} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\phi(z) - \phi(y)|^p \mathcal{K}(z,y)\, dy\, dz - \frac{\lambda}{p} \int_\Omega \frac{|\phi(z)|^p}{|z|^{sp}}\, dz \right) \\
&\qquad + \frac{1}{p} \sum_{i=1}^k \mu_i - \frac{\lambda}{p} \sum_{i=1}^k \nu_i \\
&\geq \Phi_\lambda(\phi) + \frac{1}{p} \sum_{i=1}^k \mu_i - \frac{\lambda}{p} \sum_{i=1}^k \frac{c_H \mu_i}{\gamma_0} \\
&= \Phi_\lambda(\phi) + \left(\frac{1 - \lambda c_H \gamma_0^{-1}}{p} \right) \sum_{i=1}^k \mu_i \\
&\geq \Phi_\lambda(\phi)
\end{aligned} \tag{20}$$

for any $\lambda \in (0, \lambda^*)$.

On the other hand, let $\lambda \in (-\infty, 0]$. Then, we have

$$\begin{aligned}
&\liminf_{n\to+\infty} \Phi_\lambda(\phi_n) \\
&= \liminf_{n\to+\infty} \left[\frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\phi_n(z) - \phi_n(y)|^p \mathcal{K}(z,y)\, dy\, dz - \frac{\lambda}{p} \int_\Omega \frac{|\phi_n(z)|^p}{|z|^{sp}}\, dz \right] \\
&\geq \frac{1}{p} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\phi(z) - \phi(y)|^p \mathcal{K}(z,y)\, dy\, dz + \sum_{i=1}^k \mu_i \right)
\end{aligned} \tag{21}$$

$$\geq \Phi_\lambda(\phi) + \frac{1}{p}\sum_{i=1}^{k} \mu_i$$
$$\geq \Phi_\lambda(\phi).$$

From relations (20) and (21), we deduce (15).

Now, let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence in X satisfying the same condition (16). Then, taking density arguments into account, we have for any $n \in \mathbb{N}$ that there exists $\{\phi_n^k\}_{k\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ such that

$$\phi_n^k \to \phi_n \text{ strongly in } X, \text{ as } k \to +\infty. \tag{22}$$

From (16) and (22), we have that

$$\left\langle \phi_n^k - \phi, \varphi \right\rangle = \left\langle \phi_n^k - \phi_n, \varphi \right\rangle + \langle \phi_n - \phi, \varphi \rangle \to 0, \text{ as } n, k \to +\infty$$

for any $\varphi \in X$. Then,

$$\phi_n^k \to \phi \text{ weakly in } X, \text{ as } n, k \to +\infty. \tag{23}$$

Since $\{\phi_n^k\}_{k\in\mathbb{N}}$ is contained in $C_c^\infty(\Omega)$ and statement (15) is satisfied, we deduce that

$$\liminf_{n,k\to+\infty} \Phi_\lambda(\phi_n^k) \geq \Phi_\lambda(\phi). \tag{24}$$

In addition, by (22), it is immediate to see that for any $n \in \mathbb{N}$, we have

$$\liminf_{k\to+\infty} \Phi_\lambda(\phi_n^k) = \Phi_\lambda(\phi_n).$$

So, passing to the limit inferior, we obtain

$$\liminf_{n,k\to+\infty} \Phi_\lambda(\phi_n^k) = \liminf_{n\to+\infty}\left(\lim_{k\to+\infty} \Phi_\lambda(\phi_n^k)\right) = \liminf_{n\to+\infty} \Phi_\lambda(\phi_n). \tag{25}$$

By (24) and (25), we obtain that

$$\liminf_{n\to+\infty} \Phi_\lambda(\phi_n) \geq \Phi_\lambda(\phi).$$

Therefore, Φ_λ is sequentially weakly lower semicontinuous on X. □

4. Main Result

In this section, we give the existence result of a unique nontrivial positive solution to problem (1), which is the main result of this paper. To accomplish this, suppose that the following conditions hold:

(F1) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function;

(F2) $0 \leq g(\cdot, \eta) \in L^\infty(\Omega)$ for every $\eta \geq 0$, and there exists a constant $\rho_1 > 0$ such that

$$g(z, \eta) \leq \rho_1\left(1 + |\eta|^{p-1}\right)$$

for all $\eta \geq 0$ and for almost all $z \in \Omega$;

(F3) $\lim_{\eta\to+\infty} \frac{g(z,\eta)}{\eta^{p-1}} = 0$ and $\lim_{\eta\to 0^+} \frac{g(z,\eta)}{\eta^{p-1}} = +\infty$ for almost all $z \in \Omega$;

(F4) The function $\eta \mapsto \frac{g(z,\eta)}{\eta^{p-1}}$ is decreasing in $(0, +\infty)$, uniformly in $z \in \Omega$.

Under Assumptions (F1) and (F2), let the functional $\mathcal{B} : X \to \mathbb{R}$ be defined by

$$\mathcal{B}(\phi) := \int_\Omega G(z, \phi(z))\, dz$$

for any $\phi \in X$, where $G(z,\eta) = \int_0^\eta g(z,t)dt$. Then, it is easy to show that $\mathcal{B} \in C^1(X,\mathbb{R})$, and its Fréchet derivative is

$$\langle \mathcal{B}'(\phi), \varphi \rangle = \int_\Omega g(z,\phi)\varphi \, dz$$

for any $\phi, \varphi \in X$. Next, the functional $\mathcal{J} : X \to \mathbb{R}$ is defined as

$$\mathcal{J}(\phi) = \Phi_\lambda(\phi) - \mathcal{B}(\phi).$$

Then, $\mathcal{J} \in C^1(X,\mathbb{R})$ and its Fréchet derivative is

$$\langle \mathcal{J}'(\phi), \varphi \rangle = \langle \Phi'_\lambda(\phi), \varphi \rangle - \langle \mathcal{B}'(\phi), \varphi \rangle \quad \text{for any } \phi, \varphi \in X.$$

We now introduce a discrete version of the celebrated Picone inequality that is given in Proposition 4.2 in [8]; see also Lemma 2.6 in [19].

Lemma 8 (Discrete Picone inequality)**.** *Let $1 < p < \infty$ and let $\alpha, \beta, \gamma, \delta \in [0, +\infty)$, with $\alpha, \beta > 0$. Then,*

$$\psi_p(\alpha - \beta)\left[\frac{\gamma^p}{\alpha^{p-1}} - \frac{\delta^p}{\beta^{p-1}}\right] \leq |\gamma - \delta|^p, \tag{26}$$

where $\psi_p(\eta) = |\eta|^{p-2}\eta$ for $\eta \in \mathbb{R}$. In addition, if equality holds in (26), then

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta}.$$

For any $\phi_i \in X$ and $\varepsilon > 0$, let us define the following truncation:

$$\phi_{i,\varepsilon} := \min\{\phi_i, \varepsilon^{-1}\}. \tag{27}$$

Now, we give a technical Lemma that will be very useful hereinafter. The proof of this assertion is completely the same as that of Lemma 2.3 in [15]; see also [18]. For the reader's convenience, we will give the proof.

Lemma 9. *Let $\phi_1, \phi_2 \in X$ with $\phi_1, \phi_2 \geq 0$ and set*

$$\varpi := \frac{\phi_{2,\varepsilon}^p}{(\phi_1 + \varepsilon)^{p-1}} - \phi_{1,\varepsilon},$$

where $\phi_{1,\varepsilon}, \phi_{2,\varepsilon}$ are as in (27). Then, we obtain that $\varpi \in X$.

Proof. Let $\varepsilon > 0$ be given. Since the function $\eta \mapsto \min\{|\eta|, \varepsilon^{-1}\}$ is 1-Lipschitz, we know

$$|\phi_{i,\varepsilon}(z) - \phi_{i,\varepsilon}(y)| \leq |\phi_i(z) - \phi_i(y)| \quad \text{for} \quad i = 1, 2, \tag{28}$$

which immediately implies that $\phi_{i,\varepsilon} \in X$. By virtue of the Lagrange theorem, we know that for any $\alpha, \beta \geq 0$ and for every $\gamma \geq 0$, we have that

$$|\alpha^\gamma - \beta^\gamma| \leq \gamma|\alpha - \beta|\max\{\alpha^{\gamma-1}, \beta^{\gamma-1}\}. \tag{29}$$

Then, since $\varepsilon^{p-1} \leq (\phi_{1,\varepsilon} + \varepsilon)^{p-1}$ and $\phi_{2,\varepsilon} \leq \frac{1}{\varepsilon}$, taking (28) and (29) into account, we have

$$\left| \frac{\phi_{2,\varepsilon}^p(z)}{(\phi_1(z) + \varepsilon)^{p-1}} - \frac{\phi_{2,\varepsilon}^p(y)}{(\phi_1(y) + \varepsilon)^{p-1}} \right|$$

$$
\begin{aligned}
&= \left| \frac{\phi_{2,\varepsilon}^p(z) - \phi_{2,\varepsilon}^p(y)}{(\phi_1(z)+\varepsilon)^{p-1}} + \phi_{2,\varepsilon}^p(y) \frac{(\phi_1(y)+\varepsilon)^{p-1} - (\phi_1(z)+\varepsilon)^{p-1}}{(\phi_1(z)+\varepsilon)^{p-1}(\phi_1(y)+\varepsilon)^{p-1}} \right| \\
&\leq \frac{p}{\varepsilon^{2p-2}} |\phi_{2,\varepsilon}(z) - \phi_{2,\varepsilon}(y)| + \frac{1}{\varepsilon^p} \left| \frac{(\phi_1(y)+\varepsilon)^{p-1} - (\phi_1(z)+\varepsilon)^{p-1}}{(\phi_1(z)+\varepsilon)^{p-1}(\phi_1(y)+\varepsilon)^{p-1}} \right| \\
&\leq \frac{p}{\varepsilon^{2p-2}} |\phi_{2,\varepsilon}(z) - \phi_{2,\varepsilon}(y)| \\
&\quad + \frac{p-1}{\varepsilon^p} \max\Big\{ (\phi_1(z)+\varepsilon)^{p-2}, (\phi_1(y)+\varepsilon)^{p-2} \Big\} \frac{|\phi_1(z) - \phi_1(y)|}{(\phi_1(z)+\varepsilon)^{p-1}(\phi_1(y)+\varepsilon)^{p-1}} \\
&\leq \frac{p}{\varepsilon^{2p-2}} |\phi_2(z) - \phi_2(y)| + \frac{p-1}{\varepsilon^{2p}} |\phi_1(z) - \phi_1(y)|
\end{aligned}
$$

for every $p > 1$. Thus, the Gagliardo seminorm of ϖ is finite. Further, one has

$$
\frac{\phi_{2,\varepsilon}^p}{(\phi_1+\varepsilon)^{p-1}} = \frac{\phi_{2,\varepsilon}^{p-1}}{(\phi_1+\varepsilon)^{p-1}} \phi_{2,\varepsilon} \leq \frac{1}{\varepsilon^{2p-2}} \phi_2,
$$

and thus,

$$
\begin{aligned}
\int_\Omega |\varpi|^p \, dz &\leq 2^{p-1} \left(\int_\Omega \left| \frac{\phi_{2,\varepsilon}^p}{(\phi_1+\varepsilon)^{p-1}} \right|^p dz + \int_\Omega |\phi_{1,\varepsilon}|^p \, dz \right) \\
&\leq C(p,\varepsilon) (\|\phi_2\|_{L^p(\Omega)} + \|\phi_1\|_{L^p(\Omega)}) < +\infty,
\end{aligned}
$$

where $C(p,\varepsilon) > 0$. This consequently implies that $\varpi \in X$. □

Definition 1. *We say that $\phi \in X$ is a weak solution of (1) if*

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\phi(z) - \phi(y)|^{p-2} (\phi(z) - \phi(y))(\varphi(z) - \varphi(y)) \mathcal{K}(z,y) \, dz dy \\
&\qquad = \lambda \int_\Omega \frac{|\phi(z)|^{p-2}\phi(z)}{|z|^{sp}} \varphi(z) \, dz + \int_\Omega g(z,\phi)\varphi \, dz
\end{aligned} \tag{30}
$$

for any $\varphi \in X$.

In accordance with Lemmas 7 and 9, we derive that problem (1) possesses at least one positive solution.

Lemma 10. *If* (F1)–(F3) *hold, then there is a positive constant λ^* such that problem* (1) *admits a positive solution for any $\lambda \in (-\infty, \lambda^*)$.*

Proof. Firstly, let us establish

$$
\lambda^* = \min\left\{ c_H^{-1}\gamma_0, \frac{(\mathfrak{C}_0 + \gamma_0)\gamma_0}{2\mathfrak{C}_0 c_H} \right\}, \tag{31}
$$

where $c_H = \left(\frac{p}{N-p}\right)^{-p}$, and γ_0 and $\mathfrak{C}_0$ are given in ($\mathcal{A}2$) and Lemma 3, respectively.

Moreover, since g has subcritical growth, $\mathcal{B}$ is a sequentially weakly continuous on X. Also, by virtue of Lemma 7, we obtain that $\Phi_\lambda - \mathcal{B}$ is sequentially weakly lower semicontinuous on X. Let us choose any positive real number ε satisfying either

$$
\varepsilon \in \left(0, \frac{(\mathfrak{C}_0 + \gamma_0) - 2\lambda\mathfrak{C}_0 c_H \gamma_0^{-1}}{2p\mathfrak{C}_0}\right) \quad \text{or} \quad \varepsilon \in \left(0, \frac{\mathfrak{C}_0 + \gamma_0}{2p\mathfrak{C}_0}\right).
$$

Since $\lim_{\eta\to+\infty}\frac{G(z,\eta)}{\eta^p}=0$, there exists a positive constant $C(\varepsilon)$ such that

$$G(z,\eta)\le\frac{\varepsilon}{p}|\eta|^p+\frac{C(\varepsilon)}{p} \tag{32}$$

for any $\eta\in\mathbb{R}$ and for almost all $z\in\Omega$. Thus, we have

$$\mathcal{B}(\phi)\le\varepsilon\int_\Omega|\phi(z)|^p dz+C(\varepsilon)|\Omega|,$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^N$. This, together with Lemmas 3 and 4, (32), and the definition of the X-norm, yields that for any $\phi\in X$ and for every $\lambda\in(0,\lambda^*)$,

$$\begin{aligned}\Phi_\lambda(\phi)-\mathcal{B}(\phi)&\ge\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\phi(z)-\phi(y)|^p\mathcal{K}(z,y)\,dz\,dy-\frac{\lambda}{p}\int_\Omega\frac{|\phi(z)|^p}{|z|^{sp}}\,dz\\&\quad-\varepsilon\int_\Omega|\phi(z)|^p dz-C(\varepsilon)|\Omega|\\&\ge\left(\frac{1}{2p}+\frac{\gamma_0}{2p\mathfrak{C}_0}\right)\|\phi\|_X^p-\frac{\lambda}{p}\int_\Omega\frac{|\phi(z)|^p}{|z|^{sp}}\,dz-\varepsilon\int_\Omega|\phi(z)|^p dz-C(\varepsilon)|\Omega|\\&\ge\left(\frac{\mathfrak{C}_0+\gamma_0}{2p\mathfrak{C}_0}-\frac{\lambda c_H\gamma_0^{-1}}{p}\right)\|\phi\|_X^p-\varepsilon\int_\Omega|\phi(z)|^p dz-C(\varepsilon)|\Omega|\\&\ge\left(\frac{(\mathfrak{C}_0+\gamma_0)-2\lambda\mathfrak{C}_0c_H\gamma_0^{-1}}{2p\mathfrak{C}_0}-\varepsilon\right)\|\phi\|_X^p-C(\varepsilon)|\Omega|.\end{aligned} \tag{33}$$

For any $\lambda\in(-\infty,0]$, it follows from the analogous argument (as in (33)) that

$$\begin{aligned}\Phi_\lambda(\phi)-\mathcal{B}(\phi)&\ge\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\phi(z)-\phi(y)|^p\mathcal{K}(z,y)\,dz\,dy\\&\quad-\varepsilon\int_\Omega|\phi(z)|^p dz-C(\varepsilon)|\Omega|\\&\ge\left(\frac{\mathfrak{C}_0+\gamma_0}{2p\mathfrak{C}_0}-\varepsilon\right)\|\phi\|_X^p-C(\varepsilon)|\Omega|.\end{aligned}$$

Hence, due to the choice of ε, we have

$$\lim_{\|\phi\|_X\to+\infty}(\Phi_\lambda(\phi)-\mathcal{B}(\phi))=+\infty.$$

Let us define the modified energy functional $\widetilde{\mathcal{J}}:X\to\mathbb{R}$ as

$$\begin{aligned}\widetilde{\mathcal{J}}(\phi):=&\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\phi(z)-\phi(y)|^p\mathcal{K}(z,y)\,dz\,dy\\&-\frac{\lambda}{p}\int_\Omega\frac{|\phi^+(z)|^p}{|z|^{sp}}\,dz-\int_{\mathbb{R}^N}G^+(z,\phi)\,dz,\quad\phi\in X,\end{aligned}$$

where $\phi^+=\max\{0,\phi\}$ and

$$G^+(z,\xi):=\int_0^\xi g^+(z,\eta)\,d\eta\ \text{ and }\ h^+(z,\xi):=\begin{cases}g(z,\xi), & \xi\ge0,\\ 0, & \xi<0\end{cases}$$

for all $\xi \in \mathbb{R}$ and for almost all $z \in \mathbb{R}^N$. Then, it is obvious by Lemma 7 and the above argument that $\widetilde{\mathcal{J}}$ is also sequentially weakly lower semicontinuous and coercive on X. Hence, it holds that there exists $\phi_0 \in X$ such that

$$\widetilde{\mathcal{J}}(\phi_0) = \inf\{\widetilde{\mathcal{J}}(\phi) : \phi \in X\}.$$

Now, we show that we can assume $\phi_0 \geq 0$. To this end, let us assume that ϕ_0 is sign-changing. By Lemma 9, we infer $\phi_0^+ \in X$ and so $\widetilde{\mathcal{J}}(\phi_0) \leq \widetilde{\mathcal{J}}(\phi_0^+)$. Since $\widetilde{\mathcal{J}}(\phi) = \mathcal{J}(\phi)$ when $\phi(z) \geq 0$ for almost all $z \in \Omega$, we have the following:

$$\begin{aligned}
\widetilde{\mathcal{J}}(\phi_0^+) &= \mathcal{J}(\phi_0^+) \\
&= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi_0^+(z) - \phi_0^+(y)|^p \mathcal{K}(z,y)\,dzdy - \frac{\lambda}{p}\int_\Omega \frac{|\phi_0^+(z)|^p}{|z|^{sp}}\,dz - \int_\Omega G(z,\phi_0^+)\,dz \\
&\leq \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi_0(z) - \phi_0(y)|^p \mathcal{K}(z,y)\,dzdy - \frac{\lambda}{p}\int_\Omega \frac{|\phi_0^+(z)|^p}{|z|^{sp}}\,dz - \int_\Omega G(z,\phi_0^+)\,dz \\
&= \widetilde{\mathcal{J}}(\phi_0).
\end{aligned}$$

Consequently, ϕ_0^+ is a non-negative solution of (1). For simplicity, let us write directly ϕ_0 in place of ϕ_0^+. Let us claim that $\phi_0 > 0$. Since $\phi_0(z) \geq 0$ for almost all $z \in \mathbb{R}^N$, it follows from arguments analogous to those in Theorem 3.7 of [18] that either $\phi_0(z) > 0$ or $\phi_0(z) = 0$ for almost all $z \in \mathbb{R}^N$. Hence, in order to show $\phi_0 > 0$, it is enough to show that $\widetilde{\mathcal{J}}(\phi_0) < 0$. Now, in the light of Theorem 4.1 in [39], fix any nonnegative function $\phi \in X$, with $\phi = 0$ on $\partial\Omega$, such that

$$\mu_1 \int_\Omega |\phi(z)|^p dz = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi(z) - \phi(y)|^p \mathcal{K}(z,y)\,dz\,dy,$$

where μ_1 is a positive eigenvalue that can be characterized as

$$\mu_1 = \min_{\{\phi \in X : \|\phi\|_{L^p(\Omega)} = 1\}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi(z) - \phi(y)|^p \mathcal{K}(z,y)\,dz\,dy.$$

By virtue of Theorem 3.2 in [39], we infer that $\phi \in L^\infty(\mathbb{R}^N)$. Let $\alpha_0 \in L^\infty(\Omega)$ with $\alpha_0 > 0$ and let $\kappa_0 \in (0, \|\alpha_0\|_{L^\infty(\Omega)})$ be fixed. Then the Lebesgue measure of the set $\Omega_{\kappa_0} := \{z \in \Omega : \alpha_0(z) \geq \kappa_0\}$ is positive. In addition, fix $\mathfrak{K}_1 > 0$ so that

$$\mathfrak{K}_1 > \frac{\mu_1 \int_\Omega |\phi(z)|^p\,dz}{\kappa_0 \int_{\Omega_{\kappa_0}} |\phi(z)|^p\,dz}.$$

According to the second condition in (F3), there is a constant $\eta_0 > 0$ such that

$$\frac{G(z,\eta)}{\eta^p} \geq \frac{\alpha_0(z)\mathfrak{K}_1}{p}$$

for all $\eta \in (0, \eta_0]$. Then, for $\varepsilon > 0$ small enough and for any $\lambda \in (0, \lambda^*)$, one has

$$\begin{aligned}
\int_\Omega \frac{G(z,\varepsilon\phi)}{\varepsilon^p}\,dz &\geq \frac{\mathfrak{K}_1}{p}\int_\Omega \alpha_0(z)|\phi(z)|^p\,dz \\
&\geq \frac{\mathfrak{K}_1\kappa_0}{p}\int_{\Omega_{\kappa_0}} |\phi(z)|^p\,dz \\
&> \frac{\mu_1}{p}\int_\Omega |\phi(z)|^p\,dz
\end{aligned} \tag{34}$$

$$
\begin{aligned}
&> \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\phi(z)-\phi(y)|^p\mathcal{K}(z,y)\,dz\,dy-\frac{\lambda}{p}\int_\Omega\frac{|\phi(z)|^p}{|z|^{sp}}\,dz\\
&=\frac{1}{p}[\phi]_{s,p}^p-\frac{\lambda}{p}\int_\Omega\frac{|\phi(z)|^p}{|z|^{sp}}\,dz.
\end{aligned}
$$

On the other hand, let us fix $\lambda\in(-\infty,0]$ and $\mathfrak{K}_2>0$ satisfying

$$
\mathfrak{K}_2>\frac{\mu_1(1-\lambda c_H\gamma_0^{-1})\int_\Omega|\phi(z)|^p\,dz}{\kappa_0\int_{\Omega_{\kappa_0}}|\phi(z)|^p\,dz}.
$$

Then, it follows from Lemma 4 and what is similarly given in (34) that

$$
\begin{aligned}
\int_\Omega\frac{G(z,\varepsilon\phi)}{\varepsilon^p}\,dz&\geq\frac{\mathfrak{K}_2}{p}\int_\Omega\alpha_0(z)|\phi(z)|^p\,dz\\
&\geq\frac{\mathfrak{K}_2\kappa_0}{p}\int_{\Omega_{\kappa_0}}|\phi(z)|^p\,dz\\
&>\frac{\mu_1}{p}\int_\Omega|\phi(z)|^p\,dz-\frac{\mu_1\lambda c_H\gamma_0^{-1}}{p}\int_\Omega|\phi(z)|^p\,dz\\
&=\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\phi(z)-\phi(y)|^p\mathcal{K}(z,y)\,dz\,dy\\
&\quad-\frac{\lambda c_H\gamma_0^{-1}}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\phi(z)-\phi(y)|^p\mathcal{K}(z,y)\,dz\,dy\\
&\geq\frac{1}{p}[\phi]_{s,p}^p-\frac{\lambda}{p}\int_\Omega\frac{|\phi(z)|^p}{|z|^{sp}}\,dz
\end{aligned}
\tag{35}
$$

for any $\lambda\in(-\infty,0]$.

Hence, using relations (34) and (35), we conclude that

$$
[\phi]_{s,p}^p-\lambda\int_\Omega\frac{|\phi(z)|^p}{|z|^{sp}}\,dz-p\int_{\mathbb{R}^N}\frac{G(z,\varepsilon\phi)}{\varepsilon^p}\,dz<0
$$

for any $\varepsilon>0$ sufficiently small, which is $\mathcal{J}(\varepsilon\phi)<0$, as claimed. Thus, problem (1) possesses a positive solution for any $\lambda\in(-\infty,\lambda^*)$. The proof is completed. □

We are in a position to provide our main assertion. The fundamental idea of the following consequence comes from [9,15].

Theorem 1. *If* (F1)–(F4) *hold, then for any* $\lambda\in(-\infty,\lambda^*)$, *problem* (1) *admits a unique positive solution, where* λ^* *is given in Lemma 10.*

Proof. In view of Lemma 10, we assume that problem (1) has two weak positive solutions ϕ_1 and ϕ_2. For any $\varepsilon>0$, let us define the truncations $\phi_{i,\varepsilon}$ as in (27) for $i=1,2$. We define the functions

$$
\varpi_{1,\varepsilon}:=\frac{\phi_{2,\varepsilon}^p}{(\phi_1+\varepsilon)^{p-1}}-\phi_{1,\varepsilon}
$$

and

$$
\varpi_{2,\varepsilon}:=\frac{\phi_{1,\varepsilon}^p}{(\phi_2+\varepsilon)^{p-1}}-\phi_{2,\varepsilon}.
$$

By Lemma 9, we deduce that $\varpi_{i,\varepsilon}\in X$ for $i=1,2$. Now, set

$$
\psi_p(\eta):=|\eta|^{p-2}\eta.
$$

Considering the weak formulation (30) of ϕ_i, by choosing $\varpi = \varpi_{i,\varepsilon}$ for $i = 1, 2$, we infer

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\psi_p(\phi_1(z)-\phi_1(y))(\varpi_{1,\varepsilon}(z)-\varpi_{1,\varepsilon}(y))\mathcal{K}(z,y)\,dz\,dy\\&\qquad=\lambda\int_\Omega\frac{|\phi_1(z)|^{p-2}\phi_1(z)}{|z|^{sp}}\varpi_{1,\varepsilon}(z)\,dz+\int_\Omega g(z,\phi_1)\varpi_{1,\varepsilon}(z)\,dz\end{aligned}\tag{36}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\psi_p(\phi_2(z)-\phi_2(y))(\varpi_{2,\varepsilon}(z)-\varpi_{2,\varepsilon}(y))\mathcal{K}(z,y)\,dz\,dy\\&\qquad=\lambda\int_\Omega\frac{|\phi_2(z)|^{p-2}\phi_2(z)}{|z|^{sp}}\varpi_{2,\varepsilon}(z)\,dz+\int_\Omega g(z,\phi_2)\varpi_{2,\varepsilon}(z)\,dz.\end{aligned}\tag{37}$$

Adding the above two equations, (36) and (37), and utilizing the fact that

$$\psi_p(\phi_i(z)-\phi_i(y))=\psi_p((\phi_i+\varepsilon)(z)-(\phi_i+\varepsilon)(y))\quad\text{for}\quad i=1,2,$$

we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\psi_p((\phi_1+\varepsilon)(z)-(\phi_1+\varepsilon)(y))\left(\frac{\phi_{2,\varepsilon}^p}{(\phi_1+\varepsilon)^{p-1}}(z)-\frac{\phi_{2,\varepsilon}^p}{(\phi_1+\varepsilon)^{p-1}}(y)\right)\mathcal{K}(z,y)\,dzdy\\&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\psi_p(\phi_1(z)-\phi_1(y))(\phi_{1,\varepsilon}(z)-\phi_{1,\varepsilon}(y))\mathcal{K}(z,y)\,dzdy\\&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\psi_p((\phi_2+\varepsilon)(z)-(\phi_2+\varepsilon)(y))\left(\frac{\phi_{1,\varepsilon}^p}{(\phi_2+\varepsilon)^{p-1}}(z)-\frac{\phi_{1,\varepsilon}^p}{(\phi_2+\varepsilon)^{p-1}}(y)\right)\mathcal{K}(z,y)\,dzdy\\&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\psi_p(\phi_2(z)-\phi_2(y))(\phi_{2,\varepsilon}(z)-\phi_{2,\varepsilon}(y))\mathcal{K}(z,y)\,dzdy\\&=\lambda\left(\int_\Omega\left[\frac{|\phi_1|^{p-2}\phi_1}{|z|^{sp}}\left(\frac{\phi_{2,\varepsilon}^p}{(\phi_1+\varepsilon)^{p-1}}-\phi_{1,\varepsilon}\right)+\frac{|\phi_2|^{p-2}\phi_2}{|z|^{sp}}\left(\frac{\phi_{1,\varepsilon}^p}{(\phi_2+\varepsilon)^{p-1}}-\phi_{2,\varepsilon}\right)\right]dz\right)\\&\quad+\int_\Omega\left[g(z,\phi_1)\left(\frac{\phi_{2,\varepsilon}^p}{(\phi_1+\varepsilon)^{p-1}}-\phi_{1,\varepsilon}\right)+g(z,\phi_2)\left(\frac{\phi_{1,\varepsilon}^p}{(\phi_2+\varepsilon)^{p-1}}-\phi_{2,\varepsilon}\right)\right]dz.\end{aligned}\tag{38}$$

Now, applying the discrete Picone inequality in Lemma 8 and the fact that $\eta \to \min\{|\eta|, \varepsilon^{-1}\}$ is 1-Lipschitz, we obtain

$$\psi_p((\phi_1+\varepsilon)(z)-(\phi_1+\varepsilon)(y))\left(\frac{\phi_{2,\varepsilon}^p}{(\phi_1+\varepsilon)^{p-1}}(z)-\frac{\phi_{2,\varepsilon}^p}{(\phi_1+\varepsilon)^{p-1}}(y)\right)\le|\phi_2(z)-\phi_2(y)|^p$$

and

$$\psi_p((\phi_2+\varepsilon)(z)-(\phi_2+\varepsilon)(y))\left(\frac{\phi_{1,\varepsilon}^p}{(\phi_2+\varepsilon)^{p-1}}(z)-\frac{\phi_{1,\varepsilon}^p}{(\phi_2+\varepsilon)^{p-1}}(y)\right)\le|\phi_1(z)-\phi_1(y)|^p.$$

As $\phi_{i,\varepsilon} \to \phi_i$ as $\varepsilon \to 0$ for $i = 1, 2$, by taking the limit in (38) and taking into account the Fatou Lemma in the first and third terms while applying the Dominated Convergence Theorem in all the other terms, we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\psi_p(\phi_1(z)-\phi_1(y))\left(\frac{\phi_2^p}{\phi_1^{p-1}}(z)-\frac{\phi_2^p}{\phi_1^{p-1}}(y)\right)\mathcal{K}(z,y)\,dzdy\\&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\phi_1(z)-\phi_1(y)|^p\mathcal{K}(z,y)\,dzdy\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\psi_p(\phi_2(z)-\phi_2(y))\left(\frac{\phi_1^p}{\phi_2^{p-1}}(z)-\frac{\phi_1^p}{\phi_2^{p-1}}(y)\right)\mathcal{K}(z,y)\,dzdy\\
&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\phi_2(z)-\phi_2(y)|^p\mathcal{K}(z,y)\,dzdy \qquad (39)\\
&\geq \lambda\left(\int_\Omega\left[\frac{|\phi_1|^{p-2}\phi_1}{|z|^{sp}}\left(\frac{\phi_2^p}{\phi_1^{p-1}}-\phi_1\right)+\frac{|\phi_2|^{p-2}\phi_2}{|z|^{sp}}\left(\frac{\phi_1^p}{\phi_2^{p-1}}-\phi_2\right)\right]dz\right)\\
&+\int_\Omega g(z,\phi_1)\left(\frac{\phi_2^p}{\phi_1^{p-1}}-\phi_1\right)+g(z,\phi_2)\left(\frac{\phi_1^p}{\phi_2^{p-1}}-\phi_2\right)dz\\
&=-\int_\Omega\left(\frac{g(z,\phi_1)}{\phi_1^{p-1}}-\frac{g(z,\phi_2)}{\phi_2^{p-1}}\right)(\phi_1^p-\phi_2^p)\,dz.
\end{aligned}
$$

Using Lemma 8 on the left-hand side of (39), we obtain

$$\int_\Omega\left(\frac{g(z,\phi_1)}{\phi_1^{p-1}}-\frac{g(z,\phi_2)}{\phi_2^{p-1}}\right)(\phi_1^p-\phi_2^p)\,dz\geq 0.$$

Hence, since the function $\eta\mapsto\frac{g(z,\eta)}{\eta^{p-1}}$ is decreasing in $(0,+\infty)$, we obtain that $\phi_1=\phi_2$. As a result, we conclude that problem (1) possesses a unique positive solution for any $\lambda\in(-\infty,\lambda^*)$. This completes the proof. □

5. Conclusions

The present paper is devoted to obtaining the existence of a unique positive solution to fractional p-Laplacian problems of the Brézis–Oswald type with Hardy potentials. The main difficulty in this paper is the lack of the semicontinuity property of the energy functional associated with our problem. To overcome this difficulty, we utilized the concentration–compactness principle in fractional Sobolev spaces. Also, we obtained the uniqueness result of the Brézis–Oswald type by exploiting the discrete Picone inequality given in [8,19]. However, Assumption (F3) can be regarded as a special case of that of [15,40], i.e., that the nonlinear term g satisfies conditions

$$\mathfrak{a}_0(z)=\lim_{\eta\to0^+}\frac{g(z,\eta)}{\eta^{p-1}}\qquad\text{and}\qquad\mathfrak{a}_\infty(z)=\lim_{\eta\to+\infty}\frac{g(z,\eta)}{\eta^{p-1}}$$

for almost all $z\in\Omega$. Let us define $\lambda_1(\mathcal{L}^{s,p}-\mathfrak{a}_0)$ and $\lambda_1(\mathcal{L}^{s,p}-\mathfrak{a}_\infty)$ as

$$\lambda_1(\mathcal{L}^{s,p}-\mathfrak{a}_0)=\inf_{\phi\in X}\left\{[\phi]_{s,p}^p-\int_\Omega\mathfrak{a}_0|\phi(z)|^p\,dz\,:\,\|\phi\|_{L^p(\Omega)}=1\right\}$$

and

$$\lambda_1(\mathcal{L}^{s,p}-\mathfrak{a}_0)=\inf_{\phi\in X}\left\{[\phi]_{s,p}^p-\int_\Omega\mathfrak{a}_\infty|\phi(z)|^p\,dz\,:\,\|\phi\|_{L^p(\Omega)}=1\right\}.$$

If $\lambda_1(\mathcal{L}^{s,p}-\mathfrak{a}_0)<0<\lambda_1(\mathcal{L}^{s,p}-\mathfrak{a}_0)$ instead of (F3) is satisfied, then it follows from obvious modifications of the proof of Theorem 1 and similar arguments as those in [15] that problem (1) admits a unique positive solution for any $\lambda\in(-\infty,\lambda^*)$.

Additionally, a new research direction would be to investigate mixed local and nonlocal problems of the Brézis–Oswald type involving Hardy potentials:

$$\begin{cases}-\Delta_p\,\phi+\mathcal{L}^{s,p}\phi(z)=\lambda\frac{|\phi|^{p-2}\phi}{|z|^p}+\mu g(z,\phi) & \text{in}\quad\Omega,\\ \phi>0 & \text{in}\quad\Omega,\\ \phi=0 & \text{on}\quad\mathbb{R}^N\setminus\Omega,\end{cases}\qquad(40)$$

where $1 < p < p_s^*$, $\lambda \in (-\infty, \lambda^*)$ for a positive constant λ^* and $\Delta_p \phi = \mathrm{div}(|\nabla \phi|^{p-2} \nabla \phi)$ is the classical p-Laplacian operator. Recently, the authors in [7] established necessary and sufficient conditions for the existence of a unique positive weak solution for problem (40) with $\lambda = 0$. However, to the best of my knowledge, there have been no results on the existence and uniqueness of a positive solution to problem (40) involving the Hardy potential. As a further study, one could therefore seek to obtain similar results as those given in [7] for problem (40).

Funding: This research was funded by a 2022 research Grant from Sangmyung University.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in this study are included in the article.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

References

1. Caffarelli, L. Non-local equations, drifts and games. *Nonlinear Partial Differ. Equ. Abel Symp.* **2012**, *7*, 37–52.
2. Gilboa, G.; Osher, S. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.* **2008**, *7*, 1005–1028. [CrossRef]
3. Laskin, N. Fractional quantum mechanics and Levy path integrals. *Phys. Lett. A.* **2000**, *268*, 298–305. [CrossRef]
4. Nezza, E.D.; Palatucci, G.; Valdinoci, E. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **2012**, *136*, 521–573. [CrossRef]
5. Servadei, R.; Valdinoci, E. Mountain Pass solutions for non-local elliptic operators. *J. Math. Anal. Appl.* **2012**, *389*, 887–898. [CrossRef]
6. Brezis, H.; Oswald, L. Remarks on sublinear elliptic equations. *Nonlinear Anal.* **1986**, *10*, 55–64. [CrossRef]
7. Biagi, S.; Mugnai, D.; Vecchi, E. A Brezis–Oswald approach for mixed local and nonlocal operators. *Commun. Contemp. Math.* **2024**, *26*, 2250057. [CrossRef]
8. Brasco, L.; Franzina, G. Convexity properties of Dirichlet integrals and Picone-type inequalities. *Kodai Math. J.* **2014**, *37*, 769–799. [CrossRef]
9. Brasco, L.; Squassina, M. Optimal solvability for a nonlocal at critical growth. *J. Differ. Equ.* **2018**, *264*, 2242–2269. [CrossRef]
10. Carvalho, M.L.; Goncalves, J.V.; Silva, E.D.; Santos, C.A.P. A Type of Brézis–Oswald Problem to the Φ-Laplacian Operator with Very Singular Term. *Milan J. Math. Vol.* **2018**, *86*, 53–80. [CrossRef]
11. Giacomoni, J.; Gouasmia, A.; Mokrane, A. Discrete Picone inequalities and applications to non local and non homogenenous operators. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **2022**, *116*, 100. [CrossRef]
12. Kim, Y.-H. Existence and uniqueness of solution to the p-Laplacian equations involving discontinuous Kirchhoff functions via a global minimum principle of Ricceri. *Minimax Theory Appl.* 2025, *in press*.
13. Kim, Y.-H. Existence and uniqueness of a positive solution to double phase problems involving discontinuous Kirchhoff type function. *Bull. Korean Math. Soc.* 2025, *in press*.
14. Kim, Y.-H. Existence and Uniqueness of Solutions to Non-Local Problems of Brézis–Oswald Type and Its Application. *Fractal Fract.* **2024**, *8*, 622. [CrossRef]
15. Mugnai, D.; Pinamonti, A.; Vecchi, E. Towards a Brezis-Oswald-type result for fractional problems with Robin boundary conditions. *Calc. Var. Partial Differ. Equ.* **2020**, *59*, 43. [CrossRef]
16. Díaz, J.I.; Saa, J.E. Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires. *Comptes Rendus Acad. Sci. Paris. Série I* **1987**, *305*, 521–524.
17. Faria, L.F.O.; Miyagaki, O.H.; Motreanu, D. Comparison and positive solutions for problems with the (p, q)-Laplacian and a convection term. *Proc. Edinb. Math. Soc.* **2014**, *57*, 687–698. [CrossRef]
18. Kim, I.H.; Kim, Y.-H. Existence, uniqueness, and localization of positive solutions to nonlocal problems of the Kirchhoff type via the global minimum principle of Ricceri. *AIMS Math. submitted*.
19. Frank, R.L.; Seiringer, R. Non-linear ground state representations and sharp Hardy inequalities. *J. Funct. Anal.* **2008**, *255*, 3407–3430. [CrossRef]

20. Ricceri, B. Existence, uniqueness, localization and minimization property of positive solutions for non-local problems involving discontinuous Kirchhoff functions. *Adv. Nonlinear Anal.* **2024**, *13*, 20230104. [CrossRef]
21. Nachman, A.; Callegari, A. A nonlinear singular boundary value problem in the theory of pseudoplastic fluids. *SIAM J. Appl. Math.* **1980**, *38*, 275–281. [CrossRef]
22. Diaz, J.I. Nonlinear Partial Differential Equations and Free Boundaries, V. I. Elliptic Equations. *Res. Notes Math.* **1985**, *106*, 182.
23. Diaz, J.I.; Morel, J.M.; Oswald, L. An elliptic equation with singular nonlinearity. *Commun. Partial Differ. Equ.* **1987**, *12*, 1333–1344. [CrossRef]
24. Ferrara, M.; Bisci, G.M. Existence results for elliptic problems with Hardy potential. *Bull. Sci. Math.* **2014**, *138*, 846–859. [CrossRef]
25. Khodabakhshi, M.; Aminpour, A.M.; Afrouzi, G.A.; Hadjian, A. Existence of two weak solutions for some singular elliptic problems. *RACSAM* **2016**, *110*, 385–393. [CrossRef]
26. Khodabakhshi, M.; Hadjian, A. Existence of three weak solutions for some singular elliptic problems. *Complex Var. Elliptic Equ.* **2018**, *63*, 68–75. [CrossRef]
27. Liu, J.; Zhao, Z. Existence of triple solutions for elliptic equations driven by p-Laplacian-like operators with Hardy potential under Dirichlet-Neumann boundary conditions. *Bound Value Probl.* **2023**, *2023*, 3. [CrossRef]
28. Ricceri, B. A general variational principle and some of its applications. *J. Comput. Appl. Math.* **2000**, *113*, 401–410. [CrossRef]
29. Ricceri, B. A further three critical points theorem. *Nonlinear Anal.* **2009**, *71*, 4151–4157. [CrossRef]
30. Fiscella, A. Schrödinger-Kirchhoff-Hardy p-fractional equations without the Ambrosetti-Rabinowitz condition. *Discret. Contin. Dyn. Syst.* **2020**, *13*, 1993–2007.
31. Kim, Y.-H.; Ahn, J.-H.; Lee, J.; Zeng, S. Multiplicity and a-priori bounds of solutions to Kirchhoff-Schrödinger-Hardy type equations involving the p-Laplacian. *Commun. Contemp. Math. submitted*.
32. Bonder, J.F.; Saintier, N.; Silva, A. The concentration-compactness principle for fractional order Sobolev spaces in unbounded-domains and applications to the generalized fractional Brezis-Nirenberg problem. *NoDEA Nonlinear Differ. Equ. Appl.* **2018**, *25*, 52. [CrossRef]
33. Ho, K.; Kim, Y.-H. The concentration-compactness principles for $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ and application. *Adv. Nonlinear Anal.* **2021**, *10*, 816–848. [CrossRef]
34. Adams, R.A.; Fournier, J.J.F. *Sobolev Spaces*, 2nd ed.; Academic Press: New York, NY, USA; London, UK, 2003.
35. Xiang, M.Q.; Zhang, B.L.; Ferrara, M. Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian. *J. Math. Anal. Appl.* **2015**, *424*, 1021–1041. [CrossRef]
36. Pucci, P.; Xiang, M.Q.; Zhang, B.L. Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$. *Calc. Var. Partial Differ. Equ.* **2015**, *54*, 2785–2806. [CrossRef]
37. Brézis, H.; Lieb, E. A Relation Between Pointwise Convergence of Functions and Convergence of Functionals. *Proc. Amer. Math. Soc.* **1983**, *88*, 486–490. [CrossRef]
38. Fiscella, A.; Servadei, R.; Valdinoci, E. Density properties for fractional Sobolev spaces. *Ann. Acad. Sci. Fenn. Math.* **2015**, *40*, 235–253. [CrossRef]
39. Franzina, G.; Palatucci, G. Fractional p-eigenvalues. *Riv. Mat. Univ. Parma* **2014**, *5*, 373–386.
40. Biagi, S.; Vecchi, E. On a Brezis-Oswald-type result for degenerate Kirchhoff problems. *Discrete Contin. Dyn. Syst.* **2024**, *44*, 702–717. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.