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Oscillation Solutions for Nonlinear Second-Order Neutral Differential Equations

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Abstract: This research investigates the oscillation criteria of nonlinear second-order neutral differential equations with multiple delays, focusing on their noncanonical forms. By leveraging an innovative iterative technique, new relationships are established to enhance the monotonic properties of positive solutions. These advancements lead to the derivation of novel oscillation criteria that significantly extend and refine the existing body of knowledge in this domain. The proposed criteria address gaps in the literature, providing a robust framework for analyzing such differential equations. To demonstrate their practical implications, three illustrative examples are presented, showcasing the applicability and effectiveness of the results in solving real-world problems involving delay differential equations.

Keywords: oscillation criteria; neutral differential equations; multiple delays; noncanonical forms

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1. Introduction

Recently, the incorporation of neutral equations into biological models has gained attention, as it allows for a more nuanced depiction of memory and hereditary properties inherent in biological systems. The time-fractional derivatives provide a flexible framework that can capture the long-term dependencies and anomalous diffusion processes observed in ecological contexts. As a result, delay equations and time-fractional models have proven to be more effective in describing real-world biological phenomena compared to traditional integer-order models [1].

In this paper, we investigate second-order neutral differential equations (NDEs) of the form:

$$\left(b(\mathrm{s})\left([\varkappa(\mathrm{s})+\mathrm{h}(\mathrm{s})\varkappa(\mathfrak{r}(\mathrm{s}))]'\right)^{\alpha}\right)'+\sum_{i=1}^{n}\mathrm{g}_i(\mathrm{s})\varkappa^{\beta}(\mathfrak{y}_i(\mathrm{s}))=0, \tag{1}$$

Throughout the paper we assume that:

(H_1) $0<\alpha\leq 1$, $\alpha>\beta$ are ratios of odd positive integers;

(H_2) $\mathrm{h}, \mathrm{g}\in C([\mathrm{s}_0,\infty),\mathbb{R}^+)$, $0\leq \mathrm{h}(\mathrm{s})<1$, $\mathrm{g}>0$, $\mathfrak{r}(\mathrm{s}), \mathfrak{y}(\mathrm{s})\in C^1([\mathrm{s}_0,\infty),\mathbb{R})$ satisfies $\mathfrak{y}(\mathrm{s})\leq \mathrm{s}$, $\mathfrak{r}(\mathrm{s})\leq \mathrm{s}$, $\mathfrak{y}'(\mathrm{s})>0$ and $\lim_{\mathrm{s}\to\infty}\mathfrak{r}(\mathrm{s})=\lim_{\mathrm{s}\to\infty}\mathfrak{y}(\mathrm{s})=\infty$;

(H$_3$) $b \in C([s_0, \infty), \mathbb{R}^+)$ satisfies $\mu(s_0) < \infty$, where:

$$\mu(s) := \int_s^\infty \frac{1}{b^{1/\alpha}(\varsigma)} d\varsigma;$$

(H$_4$) $h(s) < \mu(s)/\mu(\mathfrak{r}(s))$.

A function $\varkappa(s) \in C([s_\varkappa, \infty), \mathbb{R})$, $s_\varkappa \geqslant s_0$, is said to be a solution of (1) if which has the property $b(s)(\kappa'(s))^\alpha \in C^1[s_\varkappa, \infty)$, and

$$\sup\{|\varkappa(s)| : s \geqslant S\} > 0, \text{ for all } S \geq s_\varkappa.$$

A solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

Functional differential equations (FDEs) are a type of differential equation in which the derivative of an unknown function depends on the present and past values of the function, introducing the effect of memory into these systems. These equations are an essential tool for modeling dynamical systems whose behavior is influenced by the past and have wide applications in fields such as physics, biology, and engineering, where they explain phenomena such as population dynamics and delayed feedback systems. Among their types, NDEs stand out, in which the derivatives depend on the past values of the function and its past derivatives, distinguishing them by introducing a neutral term that affects the dynamic behavior of the system. NDEs play an important role in fields such as population dynamics, control theory, and electrical circuits, where time and memory effects play a crucial role. By understanding and solving these equations, deeper insights into the behavior of systems with memory can be gained, enhancing the ability to model complex phenomena more accurately, see [2–9].

Oscillation theorems form the basis for know the solutions to differential equations and their behavior and defining oscillation criteria. They play a vital role in fields such as engineering and physics, with practical applications in the design of control systems and signal processing, helping researchers build accurate models of periodic systems; see [10–17].

The study of neutral differential equations has witnessed significant development due to the contributions of many researchers. Hale [18] dealt with the basics of functional equations and the effect of time delays, while Agarwal et al. [19–21] contributed to added new oscillation criteria for neutral equations of the second order. Later, the authors in [22,23] provided precise oscillation criteria for equations with a delay term. The author in [24] focused on the oscillatory behavior of nonclassical equations. Finally, Bazighifan et al. [25,26] added new conditions sufficient for the oscillation of these complex mathematical systems.

Baculíková [27] explored the oscillations conditions of delay differential equations:

$$\left(b(s)\varkappa'(s)\right)' + g(s)\varkappa(\mathfrak{y}(s)) = 0. \tag{2}$$

in the noncanonical case:

$$\mu(s) := \int_s^\infty \frac{1}{b^{1/\alpha}(\varsigma)} d\varsigma.$$

Grammatikopoulos et al. [28] established conditions under which the NDE is oscillatory:

$$(\varkappa(s) + h(s)\varkappa(s - \mathfrak{r}))'' + g(s)\varkappa(s - \mathfrak{y}) = 0, \tag{3}$$

provided that:

$$0 < h(s) \leq 1, \ g(s) \geq 0,$$

and

$$\int_{s_0}^{\infty} g(\varsigma)(1 - h(\varsigma - \mathfrak{y}))d\varsigma = \infty,$$

then (3) is oscillatory. Subsequent studies by Xu and Xia [29] generalized this result to include:

$$(\varkappa(s) + h(s)\varkappa(s - \mathfrak{r}))'' + g(s)f(\varkappa(s - \mathfrak{y})) = 0,$$

under conditions

$$0 \leq h(s) < \infty,\ g(s) \geq M > 0.$$

Grace and Lalli [30] extended the analysis to the equation:

$$\Big(b(s)(\varkappa(s) + h(s)\varkappa(s - \mathfrak{r}))'\Big)' + g(s)f(\varkappa(s - \mathfrak{y})) = 0,$$

in the canonical case:

$$\int_{s_0}^{\infty} \frac{1}{b(\varsigma)}d\varsigma = \infty,$$

where

$$\frac{f(\varkappa)}{\varkappa} \geq k > 0.$$

Džurina and Stavroulakis [31] noted analogous properties between the nonlinear NDEs:

$$\Big(b(s)|\varkappa'(s)|^{\alpha-1}\varkappa'(s)\Big)' + g(s)|\varkappa(\mathfrak{y}(s))|^{\alpha-1}\varkappa(\mathfrak{y}(s)) = 0,$$

and its corresponding linear form:

$$\big(b(s)\varkappa'(s)\big)' + g(s)\varkappa(s) = 0.$$

Sun [32] established oscillation criteria for NDEs, utilizing a novel inequality of the form:

$$\Big(b(s)\Big|(\varkappa(s) + h(s)\varkappa(\mathfrak{r}(s)))'\Big|^{\alpha-1}(\varkappa(s) + h(s)\varkappa(\mathfrak{r}(s)))'\Big)' + g(s)f(s, \varkappa(\mathfrak{y}(s))) = 0,$$

Bohner et al. [33] and Grace et al. [34] investigated the oscillation of the second-order half-linear NDEs:

$$\Big(b(s)\Big[(\varkappa(s) + h(s)\varkappa(\mathfrak{r}(s)))'\Big]^{\alpha}\Big)' + g(s)\varkappa^{\alpha}(\mathfrak{y}(s)) = 0, \tag{4}$$

in the noncanonical case:

$$\int_{s_0}^{\infty} \frac{1}{b^{1/\alpha}(\varsigma)}d\varsigma < \infty.$$

Bohner et al. [35] expanded on their previous research into the oscillation of (4), offering new oscillation criteria that significantly refine and improve upon existing results in the literature.

Moaaz et al. [36] analyzed the oscillatory behavior of the neutral differential equation:

$$\Big(b(s)\Big[(\varkappa(s) + h(s)\varkappa(\mathfrak{r}(s)))'\Big]^{\alpha}\Big)' + \sum_{i=1}^{n} g_i(s)\varkappa^{\beta}(\mathfrak{y}_i(s)) = 0. \tag{5}$$

They proposed monotonic properties for the solutions and established oscillation conditions, guaranteeing that all solutions exhibit oscillatory behavior.

Despite the importance of these studies in the field of neutral differential equations, they largely focus on the study of linear and semilinear equations, leaving a clear gap in the

analysis of nonlinear equations that are characterized by their complexity and importance in developing the theoretical understanding of oscillatory behavior.

This paper seeks to address this gap in the scientific literature and to make a fundamental and distinctive contribution by expanding the scope of the study of the oscillatory behavior of second-order nonlinear neutral differential equations. Based on the pioneering work presented by Bakulikova [27], which dealt with the asymptotic and oscillatory behavior of second-order delay differential Equation (2), we show that the special case in which the values ($\alpha = \beta = 1$, $n = 1$, and $\mathrm{h}(\mathrm{s}) = 0$) match her results, making it a special case within the broader framework of our general equation. Furthermore, Moaz et al. [36] studied Equation (4), which represents a special case of Equation (5) when the values ($\alpha = \beta$). Based on those studies, this paper takes a broader step by extending the scope of analysis using similar techniques, while incorporating innovative improvements that enhance the analytical value of the results and contribute to deepening the theoretical understanding of nonlinear equations. This paper makes a qualitative contribution that enriches the current understanding of the oscillatory behavior of this class of equations, opening new horizons for theoretical development and enhancing the power of analytical tools used to study the complex dynamics governing them.

2. Preliminary Results

Let us define:

$$\kappa(\mathrm{s}) = \varkappa(\mathrm{s}) + \mathrm{h}(\mathrm{s})\varkappa(\mathfrak{r}(\mathrm{s}))$$

$$\widetilde{\mathfrak{y}}(\mathrm{s}) := \max\{\mathfrak{y}_i(\mathrm{s}),\ i = 1, 2, ..., n\}.$$

$$\widetilde{\mathrm{g}}(\mathrm{s}) := \sum_{i=1}^{n} \mathrm{g}_i(\mathrm{s}) \left(1 - \mathrm{h}(\mathfrak{y}_i(\mathrm{s})) \frac{\mu(\mathfrak{r}(\mathfrak{y}_i(\mathrm{s})))}{\mu(\mathfrak{y}_i(\mathrm{s}))}\right)^{\beta},$$

for $\mathrm{s}_1 \in [\mathrm{s}_0, \infty)$.

Lemma 1. *([37] Assume that $\varkappa(\mathrm{s})$ is an eventually positive solution of (1), then the corresponding function $\kappa(\mathrm{s})$ satisfies one of two cases eventually:*

$$\begin{aligned} \mathit{Case}(1) \quad &: \quad \kappa > 0,\ b\left(\kappa'\right)^{\alpha} > 0,\ \left(b\left(\kappa'\right)^{\alpha}\right)' < 0, \\ \mathit{Case}(2) \quad &: \quad \kappa > 0,\ b\left(\kappa'\right)^{\alpha} < 0,\ \left(b\left(\kappa'\right)^{\alpha}\right)' < 0, \end{aligned}$$

for $\mathrm{s} \geqslant \mathrm{s}_1 \geqslant \mathrm{s}_0$.

The following considerations aim to demonstrate that Case (2) is the fundamental one.

Lemma 2. *If*

$$\int_{\mathrm{s}_0}^{\infty} \left(\frac{1}{b(u)} \int_{\mathrm{s}_0}^{u} \widetilde{\mathrm{g}}(\varsigma)\mathrm{d}\varsigma\right)^{1/\alpha} \mathrm{d}u = \infty, \tag{6}$$

then, the positive solution $\varkappa(\mathrm{s})$ of (1) satisfies (C_2) in Lemma 1 and, moreover:
($\mathrm{B}_{1,1}$) $b^{1/\alpha}(\mathrm{s})\kappa'(\mathrm{s})\mu(\mathrm{s}) + \kappa(\mathrm{s}) \geqslant 0$;
($\mathrm{B}_{1,2}$) $\kappa(\mathrm{s})/\mu(\mathrm{s})$ *is increasing*;
($\mathrm{B}_{1,3}$) $\left(b(\mathrm{s})(\kappa'(\mathrm{s}))^{\alpha}\right)' \leq -\widetilde{\mathrm{g}}(\mathrm{s})\kappa^{\beta}(\mathfrak{y}(\mathrm{s}))$;
($\mathrm{B}_{1,4}$) $\lim_{\mathrm{s}\to\infty} \kappa(\mathrm{s}) = 0$.

Proof. Suppose on the contrary that $\varkappa$ is an eventually positive solution of (1) satisfying case(1) in Lemma 1 for $s \geq s_1 \geq s_0$. Then there exists a constant $c_0 > 0$ such that $\kappa(s) \geq c_0$ and $\kappa(\mathfrak{y}(s)) \geq c_0$ eventually. In view of the definition of κ, we have:

$$\varkappa(s) = \kappa(s) - h(s)\varkappa(\mathfrak{r}(s)) \geqslant \kappa(s) - h(s)\kappa(\mathfrak{r}(s)) \geqslant (1 - h(s))\kappa(s).$$

Then (1) become

$$\begin{aligned}\left(b(s)(\kappa'(s))^{\alpha}\right)' &= -\sum_{i=1}^{n} g_i(s)\varkappa^{\beta}(\mathfrak{y}_i(s)) \leq -\sum_{i=1}^{n} g_i(s)(1 - h(\mathfrak{y}_i(s)))^{\beta}\kappa^{\beta}(\mathfrak{y}_i(s)) \\ &\leq -\kappa^{\beta}(\widetilde{\mathfrak{y}}(s))\sum_{i=1}^{n} g_i(s)(1 - h(\mathfrak{y}_i(s)))^{\beta}. \end{aligned} \tag{7}$$

Since

$$\frac{\mu(\mathfrak{r}(\mathfrak{y}_i(s)))}{\mu(\mathfrak{y}_i(s))} \geq 1,$$

then

$$1 - h(\mathfrak{y}_i(s)) \geq 1 - \frac{\mu(\mathfrak{r}(\mathfrak{y}_i(s)))}{\mu(\mathfrak{y}_i(s))}h(\mathfrak{y}_i(s)). \tag{8}$$

Combining (7), (8), and integrating the resulting inequality from s_1 to s, we have

$$\begin{aligned} b(s_2)(\kappa'(s_2))^{\alpha} &\geq \int_{s_1}^{s} \kappa^{\beta}(\widetilde{\mathfrak{y}}(\varsigma))\sum_{i=1}^{n} g_i(\varsigma)\left(1 - h(\mathfrak{y}_i(\varsigma))\frac{\mu(\mathfrak{r}(\mathfrak{y}_i(\varsigma)))}{\mu(\mathfrak{y}_i(\varsigma))}\right)^{\beta} d\varsigma \\ &\geq c_0^{\beta}\int_{s_1}^{s}\sum_{i=1}^{n} g_i(\varsigma)\left(1 - h(\mathfrak{y}_i(\varsigma))\frac{\mu(\mathfrak{r}(\mathfrak{y}_i(\varsigma)))}{\mu(\mathfrak{y}_i(\varsigma))}\right)^{\beta} d\varsigma \\ &\geq c_0^{\beta}\int_{s_1}^{s} \widetilde{g}(\varsigma)d\varsigma, \end{aligned} \tag{9}$$

It follows from (6) and (H_3) that $\int_{s_1}^{s} \widetilde{g}(\varsigma)\mu^{\beta}(\widetilde{\mathfrak{y}}(\varsigma))d\varsigma$ must be unbounded. Furthermore, since $\mu'(s) < 0$, we deduce that

$$\int_{s_1}^{s} \widetilde{g}(\varsigma)d\varsigma \to \infty \text{ as } s \to \infty. \tag{10}$$

which with (9) gives a contradiction.

$(B_{1,1})$ From case(2) of Lemma 1, we note that $\kappa(s)$ is positive and decreasing for all $s \geq s_1 \geq s_0$. By the definition of $\kappa(s)$, we have $\kappa(s) \geq \varkappa(s)$ and

$$\varkappa(s) \geq \kappa(s) - h(s)\kappa(\mathfrak{r}(s)),\ s \geq s_1 \geq s_0. \tag{11}$$

Since $b(s)(\kappa'(s))^{\alpha}$ is decreasing, we get

$$b^{1/\alpha}(s)\kappa'(s) \geq b^{1/\alpha}(\varsigma)\kappa'(\varsigma) \text{ for } \varsigma \geq s.$$

Dividing the last inequality by $b^{1/\alpha}(\varsigma)$ and integrating the resulting inequality from s to ∞, we have

$$b^{1/\alpha}(s)\kappa'(s)\mu(s) + \kappa(s) \geqslant 0. \tag{12}$$

$(B_{1,2})$ From (12), we obtain

$$\left(\frac{\kappa(s)}{\mu(s)}\right)' = \frac{b^{1/\alpha}(s)\kappa'(s)\mu(s) + \kappa(s)}{b^{1/\alpha}(s)\mu^2(s)} \geqslant 0.$$

($B_{1,3}$) Since $\kappa(s)/\mu(s)$ is increasing, we get

$$\kappa(\mathfrak{r}(s)) \leq \frac{\mu(\mathfrak{r}(s))}{\mu(s)}\kappa(s).$$

In view of the definition of κ, we get

$$\varkappa(s) = \kappa(s) - h(s)\varkappa(\mathfrak{r}(s)) \geqslant \kappa(s) - h(s)\kappa(\mathfrak{r}(s)) \geqslant \kappa(s)\left(1 - h(s)\frac{\mu(\mathfrak{r}(s))}{\mu(s)}\right).$$

Thus, (1) becomes

$$\begin{aligned}\left(b(s)\left(\kappa'(s)\right)^{\alpha}\right)' &= -\sum_{i=1}^{n} g_i(s)\varkappa^{\beta}(\mathfrak{y}_i(s)) \\ &\leq -\sum_{i=1}^{n} g_i(s)\left(1 - h(\mathfrak{y}_i(s))\frac{\mu(\mathfrak{r}(\mathfrak{y}_i(s)))}{\mu(\mathfrak{y}_i(s))}\right)^{\beta}\kappa^{\beta}(\mathfrak{y}(s)) \\ &\leq -\kappa^{\beta}(\widetilde{\mathfrak{y}}(s))\sum_{i=1}^{n} g_i(s)\left(1 - h(\mathfrak{y}_i(s))\frac{\mu(\mathfrak{r}(\mathfrak{y}_i(s)))}{\mu(\mathfrak{y}_i(s))}\right)^{\beta} \\ &= -\widetilde{g}(s)\kappa^{\beta}(\widetilde{\mathfrak{y}}(s)).\end{aligned}$$

That is,

$$\left(b(s)\left(\kappa'(s)\right)^{\alpha}\right)' \leq -\widetilde{g}(s)\kappa^{\beta}(\widetilde{\mathfrak{y}}(s)). \tag{13}$$

($B_{1,4}$) Since $\kappa(s) > 0$, $\kappa'(s) < 0$, then $\lim_{s\to\infty}\kappa(s) = c \geq 0$. We claim that $c = 0$. If not, then $\kappa(s) \geq c > 0$ for $s \geq s_2 \geq s_1$. Then, integrating (1) from s_1 to s, we arrive at

$$\begin{aligned}b(s)\left(\kappa'(s)\right)^{\alpha} &\leq b(s_1)\left(\kappa'(s_1)\right)^{\alpha} - \int_{s1}^{s}\widetilde{g}(\varsigma)\kappa^{\beta}(\widetilde{\mathfrak{y}}(\varsigma))d\varsigma \\ &\leq -c^{\beta}\int_{s_1}^{s}\widetilde{g}(\varsigma)d\varsigma.\end{aligned}$$

By integrating the final inequality from s_1 to ∞, we deduce that

$$\kappa(s_1) \geq c^{\beta/\alpha}\int_{s_1}^{\infty}\left(\frac{1}{b(u)}\int_{s1}^{u}\widetilde{g}(\varsigma)d\varsigma\right)^{1/\alpha}du \to \infty \text{ as } s \to \infty,$$

which leads to a contradiction with (6). Therefore, $c = 0$. This concludes the proof of the lemma. □

3. Main Results

In this section, we shall show new monotonic properties for solutions of (1).

Lemma 3. *Suppose that $\varkappa(s)$ is a positive solution of (1). If $\delta_0 \in (0,1)$ with*

$$\frac{1}{\alpha}b^{1/\alpha}(s)\widetilde{g}(s)\mu^{\alpha+1}(s) \geqslant \delta_0^{\alpha}, \quad \rho_0 = c_1\delta_0, \tag{14}$$

then
($B_{2,1}$) *$\kappa(s)/\mu^{\rho_0}(s)$ is decreasing;*
($B_{2,2}$) $\lim_{s\to\infty}\kappa(s)/\mu^{\rho_0}(s) = 0$;
($B_{2,3}$) *$\kappa(s)/\mu^{1-\rho_0}(s)$ is increasing,*
where $c_1 := \kappa^{\beta/\alpha-1}(s) \geq l_1^{\beta/\alpha-1,\ if\ \alpha=\beta}l_1^{\beta/\alpha-1}$, if $\alpha > \beta$.

Proof. Assume that $\varkappa(\mathrm{s})$ is an eventually positive solution of (1). From (14) we find that

$$\begin{aligned}\int_{\mathrm{s}_0}^{\infty}\left(\frac{1}{b(u)}\int_{\mathrm{s}_1}^{u}\widetilde{\mathrm{g}}(\varsigma)\mathrm{d}\varsigma\right)^{1/\alpha}\mathrm{d}\varsigma &\geq \alpha^{1/\alpha}\delta_0\int_{\mathrm{s}_0}^{\infty}\left(\frac{1}{b(u)}\int_{\mathrm{s}_1}^{u}\frac{1}{b^{1/\alpha}(\varsigma)\mu^{\alpha+1}(\varsigma)}\mathrm{d}\varsigma\right)^{1/\alpha}\mathrm{d}u\\ &= \alpha^{1/\alpha}\delta_0\int_{\mathrm{s}_0}^{\infty}\frac{1}{b^{1/\alpha}(u)}\left(\int_{\mathrm{s}_1}^{u}\frac{1}{b^{1/\alpha}(\varsigma)\mu^{\alpha+1}(\varsigma)}\mathrm{d}\varsigma\right)^{1/\alpha}\mathrm{d}u\\ &= \delta_0\int_{\mathrm{s}_0}^{\infty}\frac{1}{b^{1/\alpha}(u)}\left(\mu^{-\alpha}(u)-\mu^{-\alpha}(\mathrm{s}_1)\right)^{1/\alpha}\mathrm{d}u.\end{aligned}$$

From the fact $\lim_{\mathrm{s}\to\infty}\kappa(\mathrm{s})=0$, there exists a $\mathrm{s}_1\geq\mathrm{s}_0$ such that $\mu^{-\alpha}(\mathrm{s})-\mu^{-\alpha}(\mathrm{s}_1)\geq\epsilon\mu^{-\alpha}(\mathrm{s})$ for $\epsilon\in(0,1)$. Thus

$$\begin{aligned}\int_{\mathrm{s}_0}^{\infty}\left(\frac{1}{b(u)}\int_{\mathrm{s}_1}^{u}\widetilde{\mathrm{g}}(\varsigma)\mathrm{d}\varsigma\right)^{1/\alpha}\mathrm{d}u &\geq \epsilon^{1/\alpha}\delta_0\int_{\mathrm{s}_0}^{\infty}\frac{1}{b^{1/\alpha}(u)\mu(u)}\mathrm{d}u\\ &= \epsilon^{1/\alpha}\delta_0\lim_{\mathrm{s}\to\infty}\ln\frac{\mu(\mathrm{s}_0)}{\mu(\mathrm{s})}\to\infty.\end{aligned}$$

Hence, From Lemma 2, we have ($\mathrm{B}_{1,1}$)–($\mathrm{B}_{1,4}$) hold.
($\mathrm{B}_{2,1}$) Integrating (1) from s_1 to s, we obtain

$$\begin{aligned}-b(\mathrm{s})\left(\kappa'(\mathrm{s})\right)^{\alpha} &= -b(\mathrm{s}_1)\left(\kappa'(\mathrm{s}_1)\right)^{\alpha}+\sum_{i=1}^{n}\mathrm{g}_i(\mathrm{s})\varkappa^{\beta}(\mathfrak{y}_i(\mathrm{s})))\mathrm{d}\varsigma\\ &\geq -b(\mathrm{s}_1)\left(\kappa'(\mathrm{s}_1)\right)^{\alpha}+\int_{\mathrm{s}_1}^{\mathrm{s}}\widetilde{\mathrm{g}}(\varsigma)\kappa^{\beta}(\widetilde{\mathfrak{y}}(\varsigma))\mathrm{d}\varsigma\\ &\geq -b(\mathrm{s}_1)\left(\kappa'(\mathrm{s}_1)\right)^{\alpha}+\kappa^{\beta}(\widetilde{\mathfrak{y}}(\mathrm{s}))\int_{\mathrm{s}_1}^{\mathrm{s}}\widetilde{\mathrm{g}}(\varsigma)\mathrm{d}\varsigma.\end{aligned}$$

By using (14), we get

$$\begin{aligned}-b(\mathrm{s})\left(\kappa'(\mathrm{s})\right)^{\alpha} &\geqslant -b(\mathrm{s}_1)\left(\kappa'(\mathrm{s}_1)\right)^{\alpha}+\kappa^{\beta}(\mathrm{s})\int_{\mathrm{s}_1}^{\mathrm{s}}\frac{\alpha\delta_0^{\alpha}}{b^{1/\alpha}(\varsigma)\mu^{\alpha+1}(\varsigma)}\mathrm{d}\varsigma\\ &= -b(\mathrm{s}_1)\left(\kappa'(\mathrm{s}_1)\right)^{\alpha}+\delta_0^{\alpha}\frac{\kappa^{\beta}(\mathrm{s})}{\mu^{\alpha}(\mathrm{s})}-\delta_0^{\alpha}\frac{\kappa^{\beta}(\mathrm{s})}{\mu^{\alpha}(\mathrm{s}_1)}\end{aligned}\tag{15}$$

Since $\lim_{\mathrm{s}\to\infty}\kappa(\mathrm{s})\to\infty$, there is a $\mathrm{s}_2\in(\mathrm{s}_1,\infty)$ such that

$$-b(\mathrm{s}_1)\left(\kappa'(\mathrm{s}_1)\right)^{\alpha}-\delta_0^{\alpha}\frac{\kappa^{\beta}(\mathrm{s})}{\mu^{\alpha}(\mathrm{s}_1)}\geq 0,$$

and so, (15) becomes

$$-b^{1/\alpha}(\mathrm{s})\kappa'(\mathrm{s})\geqslant\delta_0\frac{\kappa^{\beta/\alpha}(\mathrm{s})}{\mu(\mathrm{s})},\ \mathrm{s}\geq\mathrm{s}_2,$$

and so,

$$b^{1/\alpha}(\mathrm{s})\mu(\mathrm{s})\kappa'(\mathrm{s})+\delta_0\kappa^{\beta/\alpha}(\mathrm{s})\leq 0.\tag{16}$$

If $\alpha=\beta$, the result is straightforward. Now, consider the case where $\alpha>\beta$. Since $\kappa(\mathrm{s})$ is a nonincreasing positive function, there exists $l_1>0$ such that $\kappa(\mathrm{s})\leq l_1$, which implies that

$$\kappa^{\beta-\alpha}(\mathrm{s})\geq l_1^{\beta-\alpha},$$

and

$$\kappa^{\beta/\alpha-1}(\mathrm{s})\geq l_1^{\beta/\alpha-1}=c_1,$$

then

$$\kappa^{\beta/\alpha}(s) \geq c_1\kappa(s). \tag{17}$$

Using (17), we find

$$b^{1/\alpha}(s)\mu(s)\kappa'(s) + c_1\delta_0\kappa(s) \leq b^{1/\alpha}(s)\mu(s)\kappa'(s) + \delta_0\kappa^{\beta/\alpha}(s) \leq 0,$$

which leads to

$$b^{1/\alpha}(s)\mu(s)\kappa'(s) + \rho_0\kappa(s) \leq 0. \tag{18}$$

Consequently,

$$\left(\frac{\kappa(s)}{\mu^{\rho_0}(s)}\right)' = \frac{b^{1/\alpha}(s)\mu(s)\kappa'(s) + \rho_0\kappa(s)}{b^{1/\alpha}(s)\mu^{1+\rho_0}(s)} \leq 0.$$

($B_{2,2}$) Since $\kappa(s)/\mu^{\rho_0}(s)$ is positive and decreasing, $\lim_{s\to\infty}\kappa(s)/\mu^{\rho_0}(s) = c_1 \geqslant 0$. We claim that $c_1 = 0$. If not, then $\kappa(s)/\mu^{\rho_0}(s) \geqslant c_1 > 0$ eventually. Now, we introduce the function

$$w(s) = \left(b^{1/\alpha}(s)\kappa'(s)\mu(s) + \kappa(s)\right)\mu^{-\rho_0}(s).$$

In view of ($B_{1,1}$) in Lemma 2, we note that $w(s) > 0$ and

$$\begin{aligned}
w'(\varkappa) &= \left(b^{1/\alpha}(s)\kappa'(s)\right)'\mu^{1-\rho_0}(s) - (1-\rho_0)\kappa'(s)\mu^{-\rho_0}(s) + \kappa'(s)\mu^{-\rho_0}(s) + \rho_0\kappa(s)\frac{\mu^{-1-\rho_0}(s)}{b^{1/\alpha}(s)}\\
&= \frac{1}{\alpha}\left(b(s)(\kappa'(s))^{\alpha}\right)'\left(b^{1/\alpha}(s)\kappa'(s)\right)^{1-\alpha}\mu^{1-\rho_0}(s) + \rho_0\kappa'(s)\mu^{-\rho_0}(s) + \rho_0\kappa(s)\frac{\mu^{-1-\rho_0}(s)}{b^{1/\alpha}(s)}\\
&= -\frac{1}{\alpha}\left(b^{1/\alpha}(s)\kappa'(s)\right)^{1-\alpha}\mu^{1-\rho_0}(s)g(s)\varkappa^{\beta}(\eta(s)) + \rho_0\kappa'(s)\mu^{-\rho}(s) + \rho_0\kappa(s)\frac{\mu^{-1-\rho_0}(s)}{b^{1/\alpha}(s)}\\
&\leq -\frac{1}{\alpha}\left(b^{1/\alpha}(s)\kappa'(s)\right)^{1-\alpha}\mu^{1-\rho_0}(s)\widetilde{g}(s)\kappa^{\beta}(\widetilde{\eta}(s)) + \rho_0\kappa'(s)\mu^{-\rho_0} + \rho_0\kappa(s)\frac{\mu^{-1-\rho_0}(s)}{b^{1/\alpha}(s)}.
\end{aligned}$$

By using (14), (16), (17), and (18), we find

$$\begin{aligned}
w'(s) &\leq -\left(\frac{\delta_0\kappa^{\beta/\alpha}(s)}{\mu(s)}\right)^{1-\alpha}\mu^{1-\rho_0}(s)\frac{\delta_0^{\alpha}}{b^{1/\alpha}(s)\mu^{\alpha+1}(s)}\kappa^{\beta}(s) + \rho_0\kappa'(s)\mu^{-\rho_0} + \rho_0\kappa(s)\frac{\mu^{-1-\rho_0}(s)}{b^{1/\alpha}(s)}\\
&\leq -\delta_0\kappa^{\beta/\alpha}(s)\frac{\mu^{-1-\rho_0}(s)}{b^{1/\alpha}(s)} + \rho_0\kappa'(s)\mu^{-\rho_0}(s) + \rho_0\kappa(s)\frac{\mu^{-1-\rho_0}(s)}{b^{1/\alpha}(s)}\\
&\leq -c_1\delta_0\kappa(s)\frac{\mu^{-1-\rho_0}(s)}{b^{1/\alpha}(s)} + \rho_0\kappa'(s)\mu^{-\rho_0}(s) + \rho_0\kappa(s)\frac{\mu^{-1-\rho_0}(s)}{b^{1/\alpha}(s)}\\
&\leq -\rho_0\kappa(s)\frac{\mu^{-1-\rho_0}(s)}{b^{1/\alpha}(s)} + \rho_0\kappa'(s)\mu^{-\rho_0}(s) + \rho_0\kappa(s)\frac{\mu^{-1-\rho_0}(s)}{b^{1/\alpha}(s)}\\
&\leq \rho_0\kappa'(s)\mu^{-\rho_0}(s) \leq -\rho_0\mu^{-\rho_0}(s)\frac{\rho_0\kappa(s)}{b^{1/\alpha}(s)\mu(s)} \leq -\frac{\rho_0^2}{b^{1/\alpha}(s)\mu(s)}\frac{\kappa(s)}{\mu^{\rho_0}(s)}.
\end{aligned}$$

Using the fact that $\kappa(s)/\mu^{\rho_0}(s) \geqslant l_2$, we get

$$w'(s) \leqslant -\frac{\rho_0^2 l_2}{b^{1/\alpha}(s)\mu(s)} < 0.$$

By integrating the last inequality from s_1 to s, we have

$$w(s_1) \geq \rho_0^2 l_2 \ln\frac{\mu(s_1)}{\mu(s)} \to \infty \text{ as } s \to \infty,$$

which is a contradiction. Thus, $l_2 = 0$.

$(B_{2,3})$ Finally, we have

$$\begin{aligned}
\left(b^{1/\alpha}(s)\kappa'(s)\mu(s)+\kappa(s)\right)' &= \left(b^{1/\alpha}(s)\kappa'(s)\right)'\mu(s)-\kappa'(s)+\kappa'(s)\\
&= \left(b^{1/\alpha}(s)\kappa'(s)\right)'\mu(s) = \frac{1}{\alpha}\left(b(s)(\kappa'(s))^{\alpha}\right)'\left(b^{1/\alpha}(s)\kappa'(s)\right)^{1-\alpha}\mu(s)\\
&\leq -\frac{1}{\alpha}\widetilde{g}(s)\kappa^{\beta}(s)\left(b^{1/\alpha}(s)\kappa'(s)\right)^{1-\alpha}\mu(s)\\
&\leq -\delta_0^{\alpha}\frac{1}{b^{1/\alpha}(s)\mu^{1+\alpha}(s)}\kappa^{\beta}(s)\left(-\delta_0\frac{\kappa^{\beta/\alpha}(s)}{\mu(s)}\right)^{1-\alpha}\mu(s)\\
&\leq -\delta_0^{\alpha}\frac{1}{b^{1/\alpha}(s)\mu^{\alpha}(s)}\kappa^{\beta}(s)\left(\delta_0\frac{\kappa^{\beta/\alpha}(s)}{\mu(s)}\right)^{1-\alpha} \leq \frac{-\delta_0}{b^{1/\alpha}(s)\mu(s)}\kappa^{\beta/\alpha}(s)\\
&\leq \frac{-c_1\delta_0}{b^{1/\alpha}(s)\mu(s)}\kappa(s) \leq \frac{-\rho_0}{b^{1/\alpha}(s)\mu(s)}\kappa(s).
\end{aligned}$$

By integrating the final inequality from s to ∞, we deduce that

$$b^{1/\alpha}(s)\kappa'(s)\mu(s)+\kappa(s) \geq \rho_0\int_s^{\infty}\frac{1}{b^{1/\alpha}(\varsigma)}\frac{\kappa(\varsigma)}{\mu(\varsigma)}d\varsigma \geq \rho_0\frac{\kappa(s)}{\mu(s)}\int_s^{\infty}\frac{1}{b^{1/\alpha}(\varsigma)}d\varsigma \geq \rho_0\kappa(s).$$

Thus

$$b^{1/\alpha}(s)\kappa'(s)\mu(s)+(1-\rho_0)\kappa(s) \geq 0,$$

and hence

$$\left(\frac{\kappa(s)}{\mu^{1-\rho_0}(s)}\right)' = \frac{b^{1/\alpha}(s)\mu(s)\kappa'(s)+(1-\rho_0)\kappa(s)}{b^{1/\alpha}(s)\mu^{2-\rho_0}(s)} \geqslant 0.$$

Hence, the proof is complete. □

Theorem 1. *Suppose that there is a $\rho_0 \in (0,1)$ such that (14) holds.*

$$\rho_0 > \frac{1}{2}, \tag{19}$$

then, (1) is oscillatory.

Proof. Assume, for the sake of contradiction, that κ is an eventually positive solution of (1). Using Lemma 3, we obtain that the functions $\kappa(s)/\mu^{\rho_0}(s)$ and $\kappa(s)/\mu^{1-\rho_0}(s)$ are decreasing and increasing for $s \geq s_1$, respectively. In other words, we obtain

$$b^{1/\alpha}(s)\kappa'(s)\mu(s)+\rho_0\kappa(s) \leq 0, \tag{20}$$

and

$$b^{1/\alpha}(s)\kappa'(s)\mu(s)+(1-\rho_0)\kappa(s) \geq 0. \tag{21}$$

Combining (20) and (21), we get

$$\begin{aligned}
0 &\leq b^{1/\alpha}(s)\kappa'(s)\mu(s)+(1-\rho_0)\kappa(s)\\
&= b^{1/\alpha}(s)\kappa'(s)\mu(s)+\rho_0\kappa(s)+(1-2\rho_0)\kappa(s) \leq (1-2\beta_0)\kappa(s).
\end{aligned}$$

Since $\kappa(s) > 0$, we have $1-2\beta_0 \geq 0$, which mean that

$$\beta_0 \leq 1/2,$$

which is a contradiction. The proof is complete. □

If $\rho_0 \leq \frac{1}{2}$, then we can improve the results given in Lemma 3. Since $\mu(s)$ is decreasing, there exists a constant $\lambda \geq 1$ such that

$$\frac{\mu(\widetilde{\eta}(s))}{\mu(s)} \geq \lambda. \tag{22}$$

We introduce the constant $\rho_1 > \rho_0$ as follows

$$\rho_1 = \rho_0 \sqrt[\alpha]{\frac{\lambda^{\beta\rho_0}}{1-\frac{\beta}{\alpha}\rho_0}}. \tag{23}$$

Lemma 4. *Assume $\varkappa(s)$ is a positive solution of (1) and (14) holds. If (22) holds, then*
$(B_{3,1})$ *$\kappa(s)/\mu^{\rho_1}(s)$ is decreasing;*
$(B_{3,2})$ $\lim_{s\to\infty} \kappa(s)/\mu^{\rho_1}(s) = 0$;
$(B_{3,3})$ *$\kappa(s)/\mu^{1-\rho_1}(s)$ is increasing.*

Proof. Suppose that $\varkappa$ is an eventually positive solution of (1) satisfying condition Case(2) in Lemma 1 for $s \geq s_1 \geq s_0$. From Lemma 2, we have $(B_{1,1})$–$(B_{1,4})$ hold. Furthermore, it follows from Lemma 3 that $(B_{2,1})$–$(B_{2,3})$ hold.

$(B_{3,1})$ Integrating (1) from s_1 to s, we get

$$\begin{aligned} -b(s)\big(\kappa'(s)\big)^{\alpha} &\geqslant -b(s_1)\big(\kappa'(s_1)\big)^{\alpha} + \int_{s_1}^{s} -\sum_{i=1}^{n} g_i(\varsigma)\varkappa^{\beta}(\eta_i(\varsigma))d\varsigma \\ &\geq -b(s_1)\big(\kappa'(s_1)\big)^{\alpha} + \int_{s_1}^{s} \widetilde{g}(\varsigma)\kappa^{\beta}(\widetilde{\eta}(\varsigma))d\varsigma, \end{aligned}$$

by using the fact $\kappa(s)/\mu^{\rho_0}(s)$ is decreasing, we have

$$\begin{aligned} -b(s)\big(\kappa'(s)\big)^{\alpha} &\geqslant -b(s_1)\big(\kappa'(s_1)\big)^{\alpha} + \int_{s_1}^{s} \left(\frac{\kappa(\varsigma)}{\mu^{\rho_0}(\varsigma)}\right)^{\beta} \mu^{\beta\rho_0}(\widetilde{\eta}(\varsigma))\widetilde{g}(\varsigma)d\varsigma \\ &\geqslant -b(s_1)\big(\kappa'(s_1)\big)^{\alpha} + \left(\frac{\kappa(s)}{\mu^{\rho_0}(s)}\right)^{\beta} \int_{s_1}^{s} \mu^{\beta\rho_0}(\widetilde{\eta}(\varsigma))\widetilde{g}(\varsigma)d\varsigma. \end{aligned}$$

By using (14) and (22), we get

$$\begin{aligned} -b(s)\big(\kappa'(s)\big)^{\alpha} &\geqslant -b(s_1)\big(\kappa'(s_1)\big)^{\alpha} + \left(\frac{\kappa(s)}{\mu^{\rho_0}(s)}\right)^{\beta} \int_{s_1}^{s} \frac{\alpha\delta_0^{\alpha}\lambda^{\beta\rho_0}}{b^{1/\alpha}(\varsigma)\mu^{\alpha+1}(\varsigma)} \mu^{\beta\rho_0}(\varsigma)d\varsigma \\ &\geqslant -b(s_1)\big(\kappa'(s_1)\big)^{\alpha} + \alpha\delta_0^{\alpha}\lambda^{\beta\rho_0}\left(\frac{\kappa(s)}{\mu^{\rho_0}(s)}\right)^{\beta} \int_{s_1}^{s} \frac{\mu^{-1-\alpha+\beta\rho_0}(\varsigma)}{b^{1/\alpha}(\varsigma)}d\varsigma \\ &\geqslant -b(s_1)\big(\kappa'(s_1)\big)^{\alpha} + \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\left(\frac{\kappa(s)}{\mu^{\rho_0}(s)}\right)^{\beta} \Big[\mu^{\beta\rho_0-\alpha}(s) - \mu^{\beta\rho_0-\alpha}(s_1)\Big] \\ &\geqslant -b(s_1)\big(\kappa'(s_1)\big)^{\alpha} - \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\mu^{\beta\rho_0-\alpha}(s_1)\left(\frac{\kappa(s)}{\mu^{\rho_0}(s)}\right)^{\beta} \\ &\quad + \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\frac{\kappa^{\beta}(s)}{\mu^{\alpha}(s)}. \end{aligned}$$

Using $(B_{2,2})$, there is $s_2 \in [s_1, \infty)$, such that

$$-b(s_1)\big(\kappa'(s_1)\big)^{\alpha} - \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\mu^{\beta\rho_0-\alpha}(s_1)\left(\frac{\kappa(s)}{\mu^{\rho_0}(s)}\right)^{\beta} \geq 0,$$

for $s \geq s_2$, and so

$$-b(s)\left(\kappa'(s)\right)^{\alpha} \geqslant \frac{\delta_0^{\alpha}\lambda^{\beta\rho_0}}{(1-\frac{\beta}{\alpha}\rho_0)}\frac{\kappa^{\beta}(s)}{\mu^{\alpha}(s)},$$

and so

$$\begin{aligned}\kappa'(s) &\geq \delta_0\left(\frac{\lambda^{\beta\rho_0}}{1-\frac{\beta}{\alpha}\rho_0}\right)^{1/\alpha}\frac{1}{\mu(s)b^{1/\alpha}(s)}\kappa^{\beta/\alpha}(s) \geq c_1\delta_0\left(\frac{\lambda^{\beta\rho_0}}{1-\frac{\beta}{\alpha}\rho_0}\right)^{1/\alpha}\frac{1}{\mu(s)b^{1/\alpha}(s)}\kappa(s)\\ &= \rho_0\left(\frac{\lambda^{\beta\rho_0}}{1-\frac{\beta}{\alpha}\rho_0}\right)^{1/\alpha}\frac{1}{\mu(s)b^{1/\alpha}(s)}\kappa(s) = \rho_1\frac{1}{\mu(s)b^{1/\alpha}(s)}\kappa(s),\end{aligned}$$

or equivalently

$$b^{1/\alpha}(s)\mu(s)\kappa'(s) + \rho_1\kappa(s) \leq 0. \tag{24}$$

Consequently,

$$\left(\frac{\kappa(s)}{\mu^{\rho_1}(s)}\right)' = \frac{b^{1/\alpha}(s)\mu(s)\kappa'(s) + \rho_1\kappa(s)}{b^{1/\alpha}(s)\mu^{1+\rho_1}(s)} \leqslant 0.$$

so $\kappa(s)/\mu^{\rho_1}(s)$ is decreasing.

Following the same steps as in the proof of Lemma 3, we can confirm that conditions $(B_{3,2})$ and $(B_{3,3})$ are satisfied. □

If $\rho_1 < 1/2$, we can apply the same procedure to show that $\delta_2 > \delta_1$, as demonstrated below:

$$\rho_2 = \rho_0\sqrt[\alpha]{\frac{\lambda^{\beta\rho_1}}{1-\frac{\beta}{\alpha}\rho_1}}.$$

In general, if $\rho_i < 1/2$ for $i = 1, 2, ..., m-1$, we can define

$$\rho_m = \rho_0\sqrt[\alpha]{\frac{\lambda^{\beta\rho_{m-1}}}{1-\frac{\beta}{\alpha}\rho_{m-1}}}, \tag{25}$$

Additionally, following the same approach as in the proof of Lemma 4, we can confirm that

$(B_{m,1})$ $\kappa(s)/\mu^{\rho_m}(s)$ is decreasing;
$(B_{m,2})$ $\lim_{s\to\infty}\kappa(s)/\mu^{\rho_m}(s) = 0$;
$(B_{m,3})$ $\kappa(s)/\mu^{1-\rho_m}(s)$ is increasing.

Theorem 2. *Suppose there exists a $\rho_0 \in (0,1)$ such that condition (14) is satisfied. If there is some $n \in \mathbb{N}$ such that*

$$\rho_m > \frac{1}{2}, \tag{26}$$

then (1) is oscillatory.

Theorem 3. *Assume that (6), (14), and (22) hold. If there exists $m \in \mathbb{N}$ such that*

$$\liminf_{s\to\infty}\int_{\widetilde{\eta}(s)}^{s}\frac{\mu(\varsigma)\widetilde{g}(\varsigma)}{\mu^{1-\alpha}(\widetilde{\eta}(\varsigma))}d\varsigma > \frac{\alpha c_1^{-\alpha}\rho_m^{\alpha-1}(1-\rho_m)}{e}, \tag{27}$$

then (1) is oscillatory.

Proof. Assume on the contrary that (1) possesses an eventually positive solution $\varkappa(s)$. Condition (6) guarantees that $\varkappa(s)$ satisfies (C_2). From Lemma 2, we have that $(B_{1,1})$–$(B_{1,4})$ hold. We construct sequence $\{\rho_m\}$ by (25).

Now, we define the function

$$u(s) = b^{1/\alpha}(s)\kappa'(s)\mu(s) + \kappa(s).$$

In view of $(B_{1,1})$ in Lemma 2, we note that $u(s) > 0$ and from $(b_{n,1})$ we can obtain

$$b^{1/\alpha}(s)\kappa'(s)\mu(s) + \rho_m\kappa(s) \leq 0.$$

Then, from the definition of $u(s)$, we have

$$\begin{aligned} u(s) &= b^{1/\alpha}(s)\kappa'(s)\mu(s) + \rho_m\kappa(s) - \rho_m\kappa(s) + \kappa(s) \\ &\leq (1-\rho_n)\kappa(s). \end{aligned} \tag{28}$$

From $(B_{1,3})$ and (18), we obtain

$$\begin{aligned} u'(s) &= \left(b^{1/\alpha}(s)\kappa(s)\right)'\mu(s) \leq \frac{1}{\alpha}\left(b(s)\left(\kappa'(s)\right)^{\alpha}\right)'\left(b^{1/\alpha}(s)\kappa'(s)\right)^{1-\alpha}\mu(s) \\ &\leq -\frac{1}{\alpha}\widetilde{g}(s)\kappa^{\beta}(\widetilde{\eta}(s))\left(b^{1/\alpha}(s)\kappa'(s)\right)^{1-\alpha}\mu(s) \leq -\frac{1}{\alpha}\widetilde{g}(s)\kappa^{\beta}(\widetilde{\eta}(s))\left(\rho_m\frac{\kappa(s)}{\mu(s)}\right)^{1-\alpha}\mu(s) \\ &\leq -\frac{1}{\alpha}\rho_m^{1-\alpha}\widetilde{g}(s)\mu(s)\kappa^{\beta}(\widetilde{\eta}(s))\left(\frac{\kappa(s)}{\mu(s)}\right)^{1-\alpha}. \end{aligned} \tag{29}$$

from $(B_{1,2})$ in Lemma 2, we note that $\kappa(s)/\mu(s)$ is increasing, then

$$\frac{\kappa(\widetilde{\eta}(s))}{\mu(\widetilde{\eta}(s))} \leq \frac{\kappa(s)}{\mu(s)}.$$

Since $0 < \alpha \leq 1$, then

$$\left(\frac{\kappa(\widetilde{\eta}(s))}{\mu(\widetilde{\eta}(s))}\right)^{1-\alpha} \leq \left(\frac{\kappa(s)}{\mu(s)}\right)^{1-\alpha}.$$

From this (29) becomes

$$\begin{aligned} u'(s) &\leq -\frac{1}{\alpha}\rho_m^{1-\alpha}\widetilde{g}(s)\mu(s)\kappa^{\beta}(\widetilde{\eta}(s))\left(\frac{\kappa(\widetilde{\eta}(s))}{\mu(\widetilde{\eta}(s))}\right)^{1-\alpha} \\ &\leq -\frac{1}{\alpha}\rho_m^{1-\alpha}\widetilde{g}(s)\frac{\mu(s)}{\mu^{1-\alpha}(\widetilde{\eta}(s))}\kappa^{\beta-\alpha}(\widetilde{\eta}(s))\kappa(\widetilde{\eta}(s)). \end{aligned}$$

From (17), we know that $\kappa^{\beta-\alpha}(\widetilde{\eta}(s)) \geq c_1^{\alpha}$. Therefore the above inequality leads to

$$u'(s) \leq -\frac{c_1^{\alpha}}{\alpha}\rho_m^{1-\alpha}\widetilde{g}(s)\frac{\mu(s)}{\mu^{1-\alpha}(\widetilde{\eta}(s))}\kappa(\widetilde{\eta}(s)).$$

By applying (28), we observe that $u(s)$ is a positive solution of the inequality

$$u'(s) + \frac{c_1^{\alpha}}{\alpha}\frac{\rho_m^{1-\alpha}}{(1-\rho_m)}\frac{\mu(s)\widetilde{g}(s)}{\mu^{1-\alpha}(\widetilde{\eta}(s))}u(\widetilde{\eta}(s)) \leq 0. \tag{30}$$

which leads to a contradiction, as Theorem 2.1.1 in [38] confirms that condition (27) ensures the nonexistence of any positive solution for (30) has no positive solution. Therefore, the proof of the theorem is complete. □

Remark 1. *Theorem 3.5 in [27] gave the same result in Theorem 3 when we let* $\alpha = \beta = 1$, $n = 1$, *and* $h(s) = 0$.

We use illustrative examples to show the importance of the obtained results.

Example 1. *Consider the NDE*

$$\left(s^{2\alpha}\left((\varkappa(s) + h_0\varkappa(\tau_0 s))'\right)^{\alpha}\right)' + \sum_{i=1}^{n} g_0 s^{\alpha-1}\varkappa^{\beta}(\eta_i s) = 0,\ s \geq 1, \tag{31}$$

where $0 \leq h_0 < 1$, $\tau_0, \eta_i \in (0,1)$ *and* $g_0 > 0$. *By comparing (1) and (31) we note that* $b(s) = s^{2\alpha}$, $g_i(s) = g_0 s^{\alpha-1}$, $h(s) = h_0$, $\eta_i(s) = \eta_i s$, $\widetilde{\eta}(s) = \widetilde{\eta} s = \max\{\eta_i s,\ i = 1, 2, ..., n\}$, *and* $\tau(s) = \tau_0 s$. *It is easy to find that*

$$\mu(s) = \frac{1}{s},\ \frac{\mu(\tau(\widetilde{\eta}(s)))}{\mu(\widetilde{\eta}(s))} = \frac{1}{\tau_0},$$

and

$$\widetilde{g}(s) = n g_0 s^{\alpha-1}\left(1 - \frac{1}{\tau_0}h_0\right)^{\beta}.$$

For (14), we set

$$\delta_0 = n\sqrt[\alpha]{\frac{g_0\left(1 - \frac{1}{\tau_0}h_0\right)^{\beta}}{\alpha}}.$$

From (22), we have $\lambda = \frac{1}{\widetilde{\eta}}$. *Now, we define the sequence* $\{\beta_b\}_{b=1}^{m}$ *as*

$$\rho_n = \rho_0\sqrt[\alpha]{\frac{1}{1 - \frac{\beta}{\alpha}\rho_{n-1}}\left(\frac{1}{\widetilde{\eta}}\right)^{\beta\rho_{n-1}}},$$

with

$$\rho_0 = n c_1\sqrt[\alpha]{\frac{g_0\left(1 - \frac{1}{\tau_0}h_0\right)^{\beta}}{\alpha}}.$$

Then, condition (19) reduces to

$$g_0 > \frac{\alpha}{(2nc_1)^{\alpha}\left(1 - \frac{1}{\tau_0}h_0\right)^{\beta}}. \tag{32}$$

Condition (27) leads to

$$\begin{aligned}
\liminf_{s\to\infty}\int_{\widetilde{\eta}s}^{s}\frac{1}{\varsigma}\widetilde{\eta}^{1-\alpha}\varsigma^{1-\alpha}n g_0\varsigma^{\alpha-1}\left(1 - \frac{1}{\tau_0}h_0\right)^{\beta}d\varsigma &= \liminf_{s\to\infty}\int_{\widetilde{\eta}s}^{s}\frac{n}{\varsigma}\widetilde{\eta}^{1-\alpha}g_0\left(1 - \frac{1}{\tau_0}h_0\right)^{\beta}d\varsigma \\
&= n\widetilde{\eta}^{1-\alpha}g_0\left(1 - \frac{1}{\tau_0}h_0\right)^{\beta}\liminf_{s\to\infty}\int_{\widetilde{\eta}s}^{s}\frac{1}{\varsigma}d\varsigma \\
&= n\widetilde{\eta}^{1-\alpha}g_0\left(1 - \frac{1}{\tau_0}h_0\right)^{\beta}\ln\frac{1}{\widetilde{\eta}},
\end{aligned}$$

which holds when

$$g_0 > \frac{\alpha c_1^{-\alpha}\rho_n^{\alpha-1}(1-\rho_n)}{n\widetilde{\eta}^{1-\alpha}\left(1 - \frac{1}{\tau_0}h_0\right)^{\beta}\ln\frac{1}{\widetilde{\eta}}}\frac{1}{e}. \tag{33}$$

Using Theorems 1 and 3, we note that the solution of (31) is oscillatory if either (32) or (33) holds.

Example 2. *Consider the NDE*

$$\left(\mathrm{s}^{2/3}\left(\left(\varkappa(\mathrm{s})+\frac{1}{4}\varkappa\left(\frac{1}{2}\mathrm{s}\right)\right)'\right)^{1/3}\right)'+\frac{\mathrm{g}_0}{\mathrm{s}^2}\left[\varkappa^{1/5}\left(\frac{1}{3}\mathrm{s}\right)+\varkappa^{1/5}\left(\frac{1}{4}\mathrm{s}\right)+\varkappa^{1/5}\left(\frac{1}{5}\mathrm{s}\right)\right]=0,\ \mathrm{s}\geq 1, \tag{34}$$

Clearly: $n=3$, $\alpha=1/3$, $\beta=1/5$, $b(\mathrm{s})=\mathrm{s}^{2/3}$, $\mathrm{g}(\mathrm{s})=\frac{\mathrm{g}_0}{\mathrm{s}^{2/3}}$, $\mathrm{h}(\mathrm{s})=\frac{1}{4}$, $\widetilde{\mathfrak{y}}(\mathrm{s})=\frac{1}{3}\mathrm{s}$, *and* $\mathfrak{r}(\mathrm{s})=\frac{1}{2}\mathrm{s}$. *It is easy to find that*

$$\mu(\mathrm{s})=\frac{1}{\mathrm{s}},\ \frac{\mu(\mathfrak{r}(\widetilde{\mathfrak{y}}(\mathrm{s})))}{\mu(\widetilde{\mathfrak{y}}(\mathrm{s}))}=2,$$

and

$$\widetilde{\mathrm{g}}(\mathrm{s})=\frac{3\mathrm{g}_0}{2^{1/5}\mathrm{s}^{2/3}}.$$

For (14), we set

$$\delta_0=53.44\mathrm{g}_0^3.$$

From (22), we have $\lambda=3$. *Now, we define the sequence* $\{\beta_b\}_{b=1}^{m}$ *as:*

$$\rho_n=\rho_0\frac{1}{\left(1-\frac{3}{5}\rho_{n-1}\right)^3}3^{\frac{3\rho_{n-1}}{5}},$$

with

$$\rho_0=17.8\mathrm{g}_0^3c_1,\ \ \gamma_1>0.$$

Then, condition (26) reduces to

$$\mathrm{g}_0>\frac{0.3}{\sqrt[3]{c_1}}, \tag{35}$$

and condition (27) leads to

$$\begin{aligned}\liminf_{\mathrm{s}\to\infty}\int_{1/3\mathrm{s}}^{\mathrm{s}}\frac{1}{\varsigma}\frac{1}{3^{2/3}}3\mathrm{g}_0\frac{1}{2^{1/5}}\mathrm{d}\varsigma &= \frac{3}{2^{1/5}}\frac{1}{3^{2/3}}\mathrm{g}_0\liminf_{\mathrm{s}\to\infty}\int_{1/3\mathrm{s}}^{\mathrm{s}}\frac{1}{\varsigma}\mathrm{d}\varsigma\\ &= \frac{3}{2^{1/5}}\frac{1}{3^{2/3}}\ln(3)\mathrm{g}_0=1.4\mathrm{g}_0,\end{aligned}$$

which holds when

$$\mathrm{g}_0>\frac{\rho_n^{-2/3}\gamma_1^{-1/3}(1-\rho_n)}{4.2\mathrm{e}},\ \ \gamma_1>0. \tag{36}$$

Using Theorems 1 and 3, we note that the solution of (34) is oscillatory if either (35) or (36) holds.

Notation 1. *From the previous example we get*

$$\rho_0=17.8\mathrm{g}_0^3c_1,\ c_1>0.$$

Then condition (19) in Theorem 1 is satisfied if

$$17.8\mathrm{g}_0^3c_1>\frac{1}{2},$$

or

$$\mathrm{g}_0^3c_1>0.028\,09.$$

If we choose $c_1=0.5$, *then*

$$\mathrm{g}_0>0.383.$$

Example 3. *Consider the NDE*

$$\left(\mathrm{s}^2\left(\varkappa(\mathrm{s})+\frac{1}{4}\varkappa\left(\frac{1}{3}\mathrm{s}\right)\right)'\right)'+\mathrm{g}_0\left[\varkappa\left(\frac{1}{2}\mathrm{s}\right)+\varkappa\left(\frac{1}{3}\mathrm{s}\right)+\varkappa\left(\frac{1}{4}\mathrm{s}\right)\right]=0,\ \mathrm{s}\geq 1, \tag{37}$$

with $\mathrm{g}_0>0$. *Clearly:*

$$\mu(\mathrm{s})=\frac{1}{\mathrm{s}},\ \frac{\mu(\mathfrak{r}(\widetilde{\mathfrak{y}}(\mathrm{s})))}{\mu(\widetilde{\mathfrak{y}}(\mathrm{s}))}=3,\ \lambda=2 \text{ and } \rho_0=\frac{3}{4}\mathrm{g}_0.$$

By setting $\mathrm{g}_0=0.7$, *we have*

$$\rho_0=0.525,$$

and it is verified that condition (19) is satisfied, which implies that (37) oscillates.

Furthermore, for $\mathrm{g}_0=0.5$, *we obtain*

$$\rho_1=0.7781,$$

and condition (26) holds for $m=1$, *ensuring the oscillatory behavior of (37).*

Finally, for $\mathrm{g}_0=0.27$, *we find*

$$\rho_6=0.56315,$$

and condition (26) is satisfied with $m=6$, *again guaranteeing the oscillation of (37).*

4. Conclusions

This research has comprehensively studied nonlinear neutral differential equations of the second order with multiple delays. New criteria were derived that ensure the oscillation of all solutions of the equation under study. The study relied on the effective application of the iterative technique, which highlighted its efficiency and accuracy in analyzing the oscillatory behavior of this class of equations. The results represent a qualitative expansion compared to previous studies, which focused mainly on linear and semi-linear equations, by expanding the scope of analysis to include nonlinear equations, representing important progress in this field. In addition, the research has strengthened the analytical framework for studying this class of equations, which contributed to enriching the scientific literature related to oscillatory behavior. With respect to future directions, it is recommended to explore the cases in which $\beta>\alpha$ as a step to expand the comprehensiveness of the current criteria. It is also suggested that the iterative technique be employed to analyze higher-order differential equations, with the aim of developing more general and comprehensive criteria. The results obtained in this research form a solid theoretical basis for future studies that contribute to deepening the theoretical understanding of the oscillatory behavior of nonlinear neutral differential equations, which enhances their academic value in the scientific literature.

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