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Ricci–Bourguignon Almost Solitons with Vertical Torse-Forming Potential on Almost Contact Complex Riemannian Manifolds

Mancho Manev [1,2]

[1] Department of Algebra and Geometry, Faculty of Mathematics and Informatics, University of Plovdiv Paisii Hilendarski, 24 Tzar Asen St., 4000 Plovdiv, Bulgaria; mmanev@uni-plovdiv.bg; Tel.: +359-889-521-244
[2] Department of Medical Physics and Biophysics, Faculty of Pharmacy, Medical University of Plovdiv, 15A Vasil Aprilov Blvd, 4002 Plovdiv, Bulgaria

Abstract: Almost contact complex Riemannian manifolds, also known as almost contact B-metric manifolds, are equipped with Ricci–Bourguignon-like almost solitons. These almost solitons are a generalization of the known Ricci–Bourguignon almost solitons, in which, in addition to the main metric, the associated metric of the manifold is also involved. In the present paper, the soliton potential is specialized to be pointwise collinear with the Reeb vector field of the manifold structure, as well as torse-forming with respect to the two Levi-Civita connections of the pair of B-metrics. The forms of the Ricci tensor and the scalar curvatures generated by the pair of B-metrics on the studied manifolds with the additional structures have been found. In the three-dimensional case, an explicit example is constructed and some of the properties obtained in the theoretical part are illustrated.

Keywords: Ricci–Bourguignon almost soliton; η-Ricci–Bourguignon almost soliton; almost contact B-metric manifold; almost contact complex Riemannian manifold; torse-forming vector field; vertical potential

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1. Introduction

In 1981, J. P. Bourguignon introduced the concept of Ricci–Bourguignon flow [1]. Suppose that $g(t)$ is a time-dependent family of (pseudo-)Riemannian metrics on a smooth manifold $\mathcal{M}$. This family is said to evolve under Ricci–Bourguignon flow if $g(t)$ satisfies the following evolution equation

$$\frac{\partial}{\partial t}g = -2(\rho - \ell\tau g), \qquad g(0) = g_0,$$

where ℓ is a real constant, $\rho(t)$ and $\tau(t)$ are the Ricci tensor and the scalar curvature with respect to $g(t)$, respectively.

As is known, the solitons of any intrinsic geometric flow on $\mathcal{M}$ are fixed points or self-similar solutions of its evolution equation. The Ricci–Bourguignon soliton (briefly RB soliton) is determined by the following Equation [2,3]

$$\rho + \frac{1}{2}\mathcal{L}_\vartheta g + (\lambda + \ell\tau)g = 0, \tag{1}$$

where $\mathcal{L}_\vartheta g$ stands for the Lie derivative of g along the vector field ϑ called the soliton potential, and λ is the soliton constant. The solution is called an RB almost soliton if λ is a differential function on $\mathcal{M}$ [3].

An RB (almost) soliton is called expanding, steady, or shrinking if λ is positive, zero, or negative, respectively. It is called trivial if the soliton potential ϑ is a Killing vector field, i.e., $\mathcal{L}_{\vartheta} g = 0$.

The considered family of geometric flows is a generalization of the famous Ricci flow for $\ell = 0$, the Einstein flow for $\ell = \frac{1}{2}$, the traceless Ricci flow for $\ell = \frac{1}{m}$, and the Schouten flow for $\ell = \frac{1}{2(m-1)}$, where m is the dimension of the manifold [4,5].

A slightly more general notion of an RB (almost) soliton is obtained by perturbing (1) using a multiple of a (0, 2)-tensor field $\eta \otimes \eta$ for a certain 1-form η on the manifold. Namely, this is an η-Ricci–Bourguignon (almost) soliton (e.g., [6]).

Some recent research by other authors on (η-)Ricci–Bourguignon (almost) solitons is reported in [7–11].

In [12], the author of the present paper begins a study of RB almost solitons on almost contact almost complex Riemannian manifolds (abbreviated accR manifolds), exploiting the presence of a pair of metrics related to each other through the structure of the studied manifolds.

The rest of this paper is organized as follows. In Section 2, after the present introduction to the topic, we recall some known facts about the studied manifolds. In Section 3, we equip the manifold with a Ricci–Bourguignon-like almost soliton whose potential is pointwise collinear with the Reeb vector field and is moreover torse-forming with respect to the two Levi-Civita connections. We find properties and expressions of the Ricci tensor and the two scalar curvatures in terms of the almost soliton parameters. Finally, in Section 4 we give an explicit example from the lowest dimension that confirms the results.

2. The accR Manifolds

The manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$ is said to be an almost contact B-metric manifold or an almost contact complex Riemannian manifold (abbreviated accR manifold) if $\mathcal{M}$ is a smooth manifold of dimension $(2n+1)$, (φ, ξ, η) is an almost contact structure, and g is a B-metric on $\mathcal{M}$. This means that φ is an endomorphism of the tangent bundle $T\mathcal{M}$, ξ is a Reeb vector field, η is its dual contact 1-form, and g is a pseudo-Riemannian metric of signature $(n+1, n)$ such that

$$\begin{aligned} &\varphi\xi = 0, \qquad \varphi^2 = -\iota + \eta \otimes \xi, \qquad \eta \circ \varphi = 0, \qquad \eta(\xi) = 1, \\ &\qquad\qquad g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y), \end{aligned} \tag{2}$$

where ι denotes the identity on $\Gamma(T\mathcal{M})$ [13].

In the last equality and further on, by x, y, z we denote arbitrary elements of $\Gamma(T\mathcal{M})$ or vectors in the tangent space $T_p\mathcal{M}$ of $\mathcal{M}$ at an arbitrary point p in $\mathcal{M}$.

The direct consequences of (2) are the following identities

$$g(\varphi x, y) = g(x, \varphi y), \qquad g(x, \xi) = \eta(x), \qquad g(\xi, \xi) = 1, \qquad \eta(\nabla_x \xi) = 0, \tag{3}$$

where ∇ denotes the Levi-Civita connection of g.

Each accR manifold has one more B-metric $\tilde{g}$, which is associated with g through the (φ, ξ, η) structure in the following way

$$\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y). \tag{4}$$

The studied manifolds are classified into eleven basic classes $\mathcal{F}_i$, $i \in \{1, 2, \ldots, 11\}$ in [13]. The Ganchev–Mihova–Gribachev classification is made in regard to conditions for the (0, 3)-tensor F defined by

$$F(x, y, z) = g\big((\nabla_x \varphi) y, z\big)$$

and having the following basic properties:

$$F(x,y,z) = F(x,z,y) = F(x,\varphi y,\varphi z) + \eta(y)F(x,\xi,z) + \eta(z)F(x,y,\xi),$$
$$F(x,\varphi y,\xi) = (\nabla_x\eta)y = g(\nabla_x\xi,y). \tag{5}$$

The intersection of basic classes is the special class $\mathcal{F}_0$ defined by condition for the vanishing of F. It is the class of cosymplectic accR manifolds, where the structures φ, ξ, η, g, $\tilde{g}$ are parallel with respect to ∇ and $\tilde{\nabla}$.

The classification conditions for F are also related to the Lee forms on $(\mathcal{M},\varphi,\xi,\eta,g)$, i.e., the 1-forms defined as follows:

$$\theta(z) = g^{ij}F(e_i,e_j,z), \quad \theta^*(z) = g^{ij}F(e_i,\varphi e_j,z), \quad \omega(z) = F(\xi,\xi,z), \tag{6}$$

where (g^{ij}) is the inverse of the matrix (g_{ij}) of g with respect to a basis $\{e_i;\xi\}$ $(i = 1,2,\ldots,2n)$ of T_pM. Obviously, $\omega(\xi) = 0$ and $\theta^*\circ\varphi = -\theta\circ\varphi^2$ are valid.

Later in this work, we focus on the accR manifolds from $\mathcal{F}_5$, which is the counterpart of the class of well-known β-Kenmotsu manifolds among almost contact metric manifolds. The definition condition of $\mathcal{F}_5$-manifolds is the following

$$F(x,y,z) = -\frac{\theta^*(\xi)}{2n}\{g(x,\varphi y)\eta(z) + g(x,\varphi z)\eta(y)\}.$$

Therefore, the Lee forms on any $\mathcal{F}_5$-manifold satisfy the properties

$$\theta = 0, \quad \theta^* = \theta^*(\xi)\eta, \quad \omega = 0.$$

3. RB-like Almost Solitons with Torse-Forming Potential

In the present paper, we study an accR manifold $(\mathcal{M},\varphi,\xi,\eta,g)$. It has a pair of B-metrics g and $\tilde{g}$ related to each other with respect to the (φ,ξ,η) structure on $\mathcal{M}$. This gives us a reason to introduce in [12] a more natural generalization of the known RB (almost) soliton and η-RB (almost) soliton for the contact 1-form η as follows.

Let us recall that ρ and τ are the Ricci tensor and the scalar curvature of $\mathcal{M}$ with respect to g, respectively. Similarly, $\tilde{\tau}$ is the scalar curvature of $\mathcal{M}$ with respect to $\tilde{g}$. An accR manifold is said to be equipped with a Ricci–Bourguignon-like almost soliton (briefly RB-like almost soliton) with potential vector field ϑ if the following condition is satisfied

$$\rho + \frac{1}{2}\mathcal{L}_\vartheta g + \frac{1}{2}\mathcal{L}_\vartheta\tilde{g} + (\lambda+\ell\tau)g + (\tilde{\lambda}+\ell\tilde{\tau})\tilde{g} = 0, \tag{7}$$

where λ and $\tilde{\lambda}$ are a pair of functions on $\mathcal{M}$ and ℓ is a constant. We denote this object by $(g,\tilde{g};\vartheta;\lambda,\tilde{\lambda},\ell)$. In particular, if $(\lambda,\tilde{\lambda})$ is a pair of constants satisfying (7), then $(g,\tilde{g};\vartheta;\lambda,\tilde{\lambda},\ell)$ is called a Ricci–Bourguignon-like soliton (briefly RB-like soliton) [12].

Note that $\eta\otimes\eta$, which is used together with the metric in the definition of the η-RB (almost) soliton, is in our case included in both the B-metrics g and $\tilde{g}$ as their restriction on the vertical distribution $\mathrm{span}(\xi)$.

The following concept was introduced for accR manifolds in [14] and then developed in several papers (e.g., [12]). An accR manifold $(\mathcal{M},\varphi,\xi,\eta,g)$ is called almost Einstein-like if its Ricci tensor ρ has the following form

$$\rho = a\,g + b\,\tilde{g} + c\,\eta\otimes\eta, \tag{8}$$

where (a, b, c) is some triplet of functions. If a, b, c in (8) are constants on $\mathcal{M}$, then the manifold is called *Einstein-like*. In particular, if $b = 0$ ($b = c = 0$, respectively) the manifold is said to be an η-Einstein manifold (Einstein manifold, respectively).

3.1. The Soliton Potential Is Torse-Forming

We consider in the present paper the case when the soliton potential is a torse-forming vector field. A vector field ϑ on a (pseudo-)Riemannian manifold $(\mathcal{M}, g)$ is called a torse-forming vector field if the following condition is satisfied:

$$\nabla_x \vartheta = f\, x + \gamma(x)\vartheta, \tag{9}$$

where f is a differentiable function on $\mathcal{M}$ (called the conformal scalar of ϑ) and γ is a 1-form on $\mathcal{M}$ (called the generating form of ϑ) [15–17].

Remark 1. *Some special types of torse-forming vector fields have been studied by various authors. Namely, a vector field ϑ determined by (9) is said to be of the following type if the corresponding specializing condition is satisfied:*

- torqued, *if* $\gamma(\vartheta) = 0$ *[18];*
- concircular, *if* $\gamma = 0$ *[19];*
- concurrent, *if* $f - 1 = \gamma = 0$ *[20];*
- recurrent, *if* $f = 0$ *[21];*
- parallel, *if* $f = \gamma = 0$ *(e.g., [22]).*

In (9), the Levi-Civita connection ∇ of the basic B-metric g is used. For a similar purpose, we can use the twin B-metric $\tilde{g}$ and its Levi-Civita connection $\widetilde{\nabla}$ on the studied accR manifold. Furthermore, we require that the same vector field ϑ be torse-forming with respect to $\widetilde{\nabla}$, i.e., that the following condition be satisfied:

$$\widetilde{\nabla}_x \vartheta = \tilde{f}\, x + \tilde{\gamma}(x)\vartheta, \tag{10}$$

where $\tilde{f}$ and $\tilde{\gamma}$ are also a differentiable function and a 1-form on $\mathcal{M}$, respectively. Then $\tilde{f}$ and $\tilde{\gamma}$ are called the conformal scalar and the generating form of ϑ with respect to $\widetilde{\nabla}$, respectively.

Theorem 1. *Let an accR manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$ be equipped with an RB-like almost soliton $(g, \tilde{g}; \vartheta; \lambda, \tilde{\lambda}, \ell)$, where ϑ is torse-forming with respect to both ∇ and $\widetilde{\nabla}$. Then, the Ricci tensor of this manifold with respect to g has the following form in the cases $\ell \neq 0$ and $\ell = 0$, respectively:*

$$\begin{aligned}\rho(x,y) = &-(f+\lambda+\ell\tau)g(x,y)\\ &+\{[1+(2n+1)\ell]\tau+(2n+1)(f+\lambda)\\ &\quad+\gamma(\vartheta)+\tilde{\gamma}(\varphi\vartheta)+\eta(\vartheta)\tilde{\gamma}(\xi)\}\{g(x,\varphi y)+\eta(x)\eta(y)\}\\ &-\frac{1}{2}\{\gamma(x)g(\vartheta,y)+\gamma(y)g(\vartheta,x)+\tilde{\gamma}(x)g(\vartheta,\varphi y)+\tilde{\gamma}(y)g(\vartheta,\varphi x)\\ &\quad+\eta(\vartheta)\{\tilde{\gamma}(x)\eta(y)+\tilde{\gamma}(y)\eta(x)\}\},\end{aligned} \tag{11}$$

$$\begin{aligned}\rho(x,y) = &-(f+\lambda)g(x,y)-(\tilde{f}+\tilde{\lambda})\{g(x,\varphi y)+\eta(x)\eta(y)\}\\ &-\frac{1}{2}\{\gamma(x)g(\vartheta,y)+\gamma(y)g(\vartheta,x)+\tilde{\gamma}(x)g(\vartheta,\varphi y)+\tilde{\gamma}(y)g(\vartheta,\varphi x)\\ &\quad+\eta(\vartheta)\{\tilde{\gamma}(x)\eta(y)+\tilde{\gamma}(y)\eta(x)\}\}.\end{aligned} \tag{12}$$

Proof. Applying (9) and (10), we obtain the following expressions for the Lie derivatives of g and $\tilde{g}$ along ϑ, which is a torse-forming vector field with respect to ∇ and $\widetilde{\nabla}$:

$$(\mathcal{L}_\vartheta g)(x,y) = g(\nabla_x\vartheta, y) + g(x, \nabla_y\vartheta) = 2fg(x,y) + \gamma(x)g(\vartheta,y) + \gamma(y)g(\vartheta,x), \tag{13}$$

$$\begin{aligned}(\mathcal{L}_\vartheta \tilde{g})(x,y) &= \tilde{g}(\widetilde{\nabla}_x\vartheta, y) + \tilde{g}(x, \widetilde{\nabla}_y\vartheta) = 2\tilde{f}\tilde{g}(x,y) + \tilde{\gamma}(x)\tilde{g}(\vartheta,y) + \tilde{\gamma}(y)\tilde{g}(\vartheta,x)\\ &= 2\tilde{f}g(x,\varphi y) + 2\tilde{f}\eta(x)\eta(y) + \tilde{\gamma}(x)\{g(\vartheta,\varphi y) + \eta(\vartheta)\eta(y)\}\\ &\quad + \tilde{\gamma}(y)\{g(\vartheta,\varphi x) + \eta(\vartheta)\eta(x)\}.\end{aligned} \tag{14}$$

As a result of substituting (13) and (14) into (7), we obtain the following condition for an RB-like almost soliton with torse-forming potential:

$$\begin{aligned}&\rho(x,y) + (f+\lambda+\ell\tau)g(x,y) + (\tilde{f}+\tilde{\lambda}+\ell\tilde{\tau})\{g(x,\varphi y) + \eta(x)\eta(y)\}\\ &\quad + \frac{1}{2}\{\gamma(x)g(\vartheta,y) + \gamma(y)g(\vartheta,x) + \tilde{\gamma}(x)g(\vartheta,\varphi y) + \tilde{\gamma}(y)g(\vartheta,\varphi x)\\ &\qquad + \eta(\vartheta)\{\tilde{\gamma}(x)\eta(y) + \tilde{\gamma}(y)\eta(x)\}\} = 0.\end{aligned} \tag{15}$$

Taking the trace of the last equality, we obtain the expression of $\tilde{\tau}$ in terms of τ and the other parameters of the RB-like soliton. In the case $\ell \neq 0$, we have

$$\begin{aligned}\tilde{\tau} = -\frac{1}{\ell}\Big\{&\big[1 + (2n+1)\ell\big]\tau + (2n+1)(f+\lambda) + \tilde{f} + \tilde{\lambda}\\ &+ \gamma(\vartheta) + \tilde{\gamma}(\varphi\vartheta) + \eta(\vartheta)\tilde{\gamma}(\xi)\Big\}.\end{aligned} \tag{16}$$

Otherwise, i.e., $\ell = 0$ is true, the expression of ρ in (15) is specialized and taking its trace with respect to g gives the following result

$$\tau = -(2n+1)(f+\lambda) - \tilde{f} - \tilde{\lambda} - \gamma(\vartheta) - \tilde{\gamma}(\varphi\vartheta) - \eta(\vartheta)\tilde{\gamma}(\xi). \tag{17}$$

By substituting (16) into (15), we express the Ricci tensor ρ without $\tilde{\tau}$ for the case $\ell \neq 0$ as given in (11).

In the case $\ell = 0$, the expression of ρ looks like in (12). □

3.2. The Torse-Forming Potential Is Vertical

It is known that these vector fields, which have a special positioning concerning the structure under consideration, are naturally distinguished. The almost contact structure on $\mathcal{M}$ gives rise to two mutually orthogonal distributions with respect to g and $\tilde{g}$, namely the contact (or horizontal) distribution $\mathcal{H} = \ker(\eta) = \mathrm{im}(\varphi)$ and the vertical distribution $\mathcal{H}^\perp = \mathrm{span}(\xi) = \ker(\varphi)$.

For this reason, we study the case where the torse-forming vector field ϑ is vertical, i.e., $\vartheta \in \mathcal{H}^\perp$. Therefore, ϑ is pointwise collinear with ξ, i.e., the following equality holds

$$\vartheta = k\,\xi, \tag{18}$$

where k is a nowhere-vanishing function on $\mathcal{M}$ and obviously $k = \eta(\vartheta)$ is true. This means that we exclude from consideration the case of ϑ being torqued according to Remark 1.

Taking into account (9) and (18), we obtain

$$\mathrm{d}k(x)\,\xi + k\nabla_x\xi = fx + k\gamma(x)\,\xi, \tag{19}$$

which, after applying η and considering the last property in (3), gives

$$\mathrm{d}k(x) = f\eta(x) + k\gamma(x). \tag{20}$$

Thus, (2), (19), and (20) imply

$$\nabla_x \xi = -\frac{f}{k}\,\varphi^2 x. \tag{21}$$

Similarly, by virtue of (10) and (18), we obtain successively

$$\mathrm{d}k(x) = \tilde{f}\eta(x) + k\tilde{\gamma}(x), \tag{22}$$

$$\widetilde{\nabla}_x \xi = -\frac{\tilde{f}}{k}\,\varphi^2 x. \tag{23}$$

Taking into account (21), for the curvature tensor of g, we obtain

$$\begin{aligned} R(x,y)\xi = &-\frac{1}{k^2}\Big\{k\mathrm{d}f(x) - f\mathrm{d}k(x) + f^2\eta(x)\Big\}\varphi^2 y \\ &+\frac{1}{k^2}\Big\{k\mathrm{d}f(y) - f\mathrm{d}k(y) + f^2\eta(y)\Big\}\varphi^2 x. \end{aligned} \tag{24}$$

As immediate consequence of (24), we obtain the following expression

$$\rho(\xi,\xi) = -\frac{2n}{k^2}\Big\{k\mathrm{d}f(\xi) - f\mathrm{d}k(\xi) + f^2\Big\}. \tag{25}$$

Lemma 1. *Let $(\mathcal{M},\varphi,\xi,\eta,g)$ be an accR manifold with a vertical vector field ϑ that is torse-forming with respect to both ∇ and $\widetilde{\nabla}$. Then, the manifold belongs to $\mathcal{F}_5$ or to a direct sum of $\mathcal{F}_5$ with $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_6$, and $\mathcal{F}_{10}$. Furthermore, ϑ is recurrent if and only if the component of F relating $(\mathcal{M},\varphi,\xi,\eta,g)$ to $\mathcal{F}_5$ vanishes, i.e., $\mathcal{F}_5$ is restricted to $\mathcal{F}_0$.*

Proof. From (21), taking into account (5) and (6), we obtain $F(x,y,\xi) = -\frac{f}{k}g(x,\varphi y)$, which implies the following values of the Lee forms of the manifold in the considered case

$$\theta(\xi) = 0, \quad \theta^*(\xi) = 2n\frac{f}{k}, \quad \omega = 0. \tag{26}$$

These results show that the manifold belongs to $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_{10}$. In addition, among the five basic classes in the direct sum above, only $\mathcal{F}_5$ can contain such manifolds.

In $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_6$, and $\mathcal{F}_{10}$, all values of $\theta(\xi)$, $\theta^*(\xi)$, and ω are zero, which due to (26) means $f = 0$ and, therefore, ϑ is recurrent according to Remark 1. □

Lemma 2. *Let us consider an accR manifold $(\mathcal{M},\varphi,\xi,\eta,g)$ and a vertical vector field ϑ on it. If ϑ is torse-forming with respect to both ∇ and $\widetilde{\nabla}$ with conformal scalars f, $\tilde{f}$ and generating forms γ, $\tilde{\gamma}$, respectively, then we have the following*

$$\tilde{f} = f, \qquad \tilde{\gamma} = \gamma, \qquad \widetilde{\nabla}\vartheta = \nabla\vartheta. \tag{27}$$

Proof. A characterization of all basic classes of accR manifolds in terms of the symmetric tensor $\Phi(x,y) = \widetilde{\nabla}_x y - \nabla_x y$ is given in [23]. There, the relation between F and Φ, known from [13], is used. Using (21) and (23), and the expressions of $\Phi(x,\xi)$ in the mentioned above five basic classes, we obtain the first equality in (27). Then, due to (20) and (22), we also obtain the second equality in (27). Therefore, we have the third equality in (27). □

As a result of (27) and the condition for verticality of ϑ in (18), the expression of ρ in (11) is specialized in the following form

$$\begin{aligned}\rho(x,y) = &-(f+\lambda+\ell\tau)g(x,y) - \{\mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x)\}\\ &+\big\{[1+(2n+1)\ell]\tau + (2n+1)(f+\lambda) + 2\mathrm{d}k(\xi) - 2f\big\}g(x,\varphi y)\\ &+\big\{[1+(2n+1)\ell]\tau + (2n+1)(f+\lambda) + 2\mathrm{d}k(\xi)\big\}\eta(x)\eta(y),\end{aligned} \tag{28}$$

where we have taken into account $k\gamma = \mathrm{d}k - f\eta$ due to (20). This expression for ρ does not imply any restrictions on τ.

Given (28), calculating the associated quantity τ^* of τ with respect to φ, which is defined by $\tau^* = g^{ij}\rho(e_i, \varphi e_j)$, gives

$$\tau^* = -2n\big\{[1+(2n+1)\ell]\tau + (2n-1)f + (2n+1)\lambda + 2\mathrm{d}k(\xi)\big\}.$$

We pay special attention to the more general case when the potential ϑ is not recurrent, i.e., $f \neq 0$. This means that the studied accR manifold belongs to $\mathcal{F}_5$ or its direct sum with $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_6$, and $\mathcal{F}_{10}$ according to Lemma 1. A subclass of $\mathcal{F}_5$ with remarkable curvature properties is $\mathcal{F}_5^0$, defined by the additional condition $\mathrm{d}\theta^* = 0$ [24].

Proposition 1. *Let an accR manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$ belong to $\mathcal{F}_5$, a vector field ϑ on $\mathcal{M}$ be vertical, and ϑ be torse-forming with respect to both ∇ and $\widetilde{\nabla}$. Then, $(\mathcal{M}, \varphi, \xi, \eta, g)$ belongs to $\mathcal{F}_5^0$.*

Proof. From (26) follows that $\theta^* = \theta^*(\xi)\eta$ is valid in $\mathcal{F}_5$, and vanishes in $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_6$, and $\mathcal{F}_{10}$. It is easy to see that θ^* is closed if and only if the following condition is satisfied

$$\mathrm{d}(\theta^*(\xi))(x)\eta(y) = \mathrm{d}(\theta^*(\xi))\eta(x). \tag{29}$$

The latter necessary and sufficient condition follows due to the fact that η is a closed 1-form in the considered classes.

In the case of a vertical torse-forming potential, the expression of $\theta^*(\xi)$ in (26) implies the following equalities

$$\mathrm{d}(\theta^*(\xi))(x) = \mathrm{d}(\theta^*(\xi))(\xi)\eta(x), \qquad \mathrm{d}(\theta^*(\xi))(\xi) = \frac{2n}{k^2}(k\,\mathrm{d}f(\xi) - f\,\mathrm{d}k(\xi)). \tag{30}$$

The first equality of (30) shows that (29) is satisfied and, therefore, the studied manifold belongs to $\mathcal{F}_5^0$. □

Theorem 2. *Let an accR manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$ belonging to $\mathcal{F}_5$ be equipped with an RB-like almost soliton $(g, \tilde{g}; \vartheta; \lambda, \tilde{\lambda}, \ell)$, where the soliton potential ϑ is vertical and torse-forming with respect to both ∇ and $\widetilde{\nabla}$. Then, the Ricci tensor of this manifold with respect to g has the following form:*

$$\begin{aligned}\rho(x,y) = &-(f+\lambda+\ell\tau)g(x,y) - (f+\tilde{\lambda}+\ell\tilde{\tau})\tilde{g}(x,y)\\ &+2f\eta(x)\eta(y) - \mathrm{d}k(x)\eta(y) - \mathrm{d}k(y)\eta(x),\end{aligned} \tag{31}$$

where the scalar curvatures with respect to g and $\tilde{g}$ are expressed in terms of the potential parameters as follows:

$$\tau = -\frac{2n}{(1+2n\ell)k^2}\Big\{(f+\lambda)k^2 + f^2 + k\mathrm{d}f(\xi) - f\mathrm{d}k(\xi)\Big\}, \qquad \ell \neq -\frac{1}{2n}, \tag{32}$$

$$\tilde{\tau} = \frac{1}{\ell}\Big\{\frac{1}{1+2n\ell}(2n\ell f - \lambda) - \tilde{\lambda} - \frac{2}{k^2}\Big\{k^2 + n[1+(2n+1)\ell]f\Big\}\mathrm{d}k(\xi) + \frac{2n[1+(2n+1)\ell]}{(1+2n\ell)k^2}\Big[f^2 + k\mathrm{d}f(\xi)\Big]\Big\}, \qquad \ell \neq -\frac{1}{2n}, \quad \ell \neq 0. \tag{33}$$

In the case $\ell = -\frac{1}{2n}$*, the following equalities are valid:*

$$\lambda = -f - \frac{1}{k^2}\Big\{f^2 + k\,\mathrm{d}f(\xi) - f\,\mathrm{d}k(\xi)\Big\}, \tag{34}$$

$$\tau + \tilde{\tau} = 2n\Big\{-f + \tilde{\lambda} - \frac{2n+1}{k^2}\Big[f^2 + k\,\mathrm{d}f(\xi)\Big] + \Big[2 + \frac{(2n+1)f}{k^2}\Big]\mathrm{d}k(\xi)\Big\}. \tag{35}$$

If ℓ *vanishes, the following expressions hold:*

$$\tau = -\frac{2n}{k^2}\Big\{(f+\lambda)k^2 + f^2 + k\,\mathrm{d}f(\xi) - f\mathrm{d}k(\xi)\Big\}, \tag{36}$$

$$\tilde{\lambda} = -\lambda + \frac{2n}{k^2}\Big[f^2 + k\mathrm{d}f(\xi)\Big] - \Big[2 + \frac{2nf}{k^2}\Big]\mathrm{d}k(\xi). \tag{37}$$

Proof. Let us apply condition (18) as well as the findings in (20) and (27) to (13) and (14). Then, we obtain the following expressions of the Lie derivatives of g and $\tilde{g}$ along a torse-forming vertical potential with respect to ∇ and $\widetilde{\nabla}$, bearing in mind (4) and the last equality in (2):

$$\begin{aligned}(\mathcal{L}_{\vartheta}g)(x,y) &= -2fg(\varphi x, \varphi y) + \mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x)\\ &= 2fg(x,y) - 2f\eta(x)\eta(y) + \mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x),\end{aligned} \tag{38}$$

$$\begin{aligned}(\mathcal{L}_{\vartheta}\tilde{g})(x,y) &= 2fg(x, \varphi y) + \mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x)\\ &= 2f\tilde{g}(x,y) - 2f\eta(x)\eta(y) + \mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x).\end{aligned} \tag{39}$$

Using (7), (38), and (39), we obtain a consequence of the expression for the Ricci tensor in the case of an RB-like almost soliton with a torse-forming vertical potential, given in (31).

Substituting $x = y = \xi$ into (31) and using (25), we obtain the following equation for τ and $\tilde{\tau}$:

$$\ell(\tau + \tilde{\tau}) + \lambda + \tilde{\lambda} + \frac{2}{k^2}\Big(nf + k^2\Big)\mathrm{d}k(\xi) - \frac{2n}{k^2}\Big(k\,\mathrm{d}f(\xi) + f^2\Big) = 0. \tag{40}$$

Conversely, taking the trace of (31) with respect to g, we obtain another relation between the two scalar curvatures in the following form

$$\big[1 + (2n+1)\ell\big]\tau + \ell\tilde{\tau} + 2nf + (2n+1)\lambda + \tilde{\lambda} + 2\mathrm{d}k(\xi) = 0. \tag{41}$$

The last equality follows also from (16), (18), (20), and Lemma 2.

We solve the system of Equations (40) and (41) regarding τ and $\tilde{\tau}$ and obtain the expression of the two scalar curvatures of g and $\tilde{g}$, given in (32) and (33), respectively.

In the case $\ell = -\frac{1}{2n}$, Equations (40) and (41) are specialized into the following two equalities, respectively:

$$\tau + \tilde{\tau} = 2n\Big\{\lambda + \tilde{\lambda} + \frac{2}{k^2}\Big(nf + k^2\Big)\mathrm{d}k(\xi) - \frac{2n}{k^2}\Big(k\,\mathrm{d}f(\xi) + f^2\Big)\Big\}, \tag{42}$$

$$\tau + \tilde{\tau} = 2n\big\{2nf + (2n+1)\lambda + \tilde{\lambda} + 2\mathrm{d}k(\xi)\big\}. \tag{43}$$

Comparing (42) and (43), we first obtain the expression in (34) of the soliton function λ in terms of the functions f and k of the potential ϑ; and second, the equality in (35) of the sum of the two scalar curvatures without λ.

In the case $\ell = 0$, equality (41) implies

$$\tau = -2nf - (2n+1)\lambda - \tilde{\lambda} - 2\mathrm{d}k(\xi). \tag{44}$$

Formula (44) follows also from (17) when the torse-forming potential ϑ is vertical, ℓ vanishes and the manifold belongs to class $\mathcal{F}_5$.

Furthermore, the expression in (32) implies (36) at $\ell = 0$, which we combine with (44) to obtain the expression of the soliton function $\tilde{\lambda}$ in (37). □

Corollary 1. *Let an accR manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$ belonging to $\mathcal{F}_5$ be equipped with an RB-like almost soliton $(g, \tilde{g}; \vartheta; \lambda, \tilde{\lambda}, \ell)$, where the soliton potential ϑ is vertical and torse-forming with respect to both ∇ and $\widetilde{\nabla}$. This manifold is almost Einstein-like if and only if $k = \eta(\vartheta)$ is a constant on $\mathcal{H}$, i.e., $\mathrm{d}k = \mathrm{d}k(\xi)\eta$ is valid.*

Proof. The statement follows immediately from (8) and (31) in Theorem 2, as we have $a = -f - \lambda - \ell\tau$, $b = -f - \tilde{\lambda} - \ell\tilde{\tau}$, $c = 2f - 2\mathrm{d}k(\xi)$. □

4. Example on the Cone Over a 2-Dimensional Complex Space Form with Norden Metric

Let us consider an example of an accR manifold constructed in [25].

First, a two-dimensional manifold $\mathcal{N}$ equipped with an almost complex structure J and a Norden metric g' is given, i.e., g' is a pseudo-Riemannian metric with neutral signature such that $g'(Jx', Jy') = -g'(x', y')$ for arbitrary $x', y' \in \Gamma(T\mathcal{N})$. It is well-known that $(\mathcal{N}, J, g')$ is a complex space form with constant sectional curvature, which we can denote, for example, by k'.

Second, we consider the cone $\mathcal{C}(\mathcal{N})$ over $(\mathcal{N}, J, g')$, i.e., $\mathcal{C}(\mathcal{N})$ is the warped product $\mathbb{R}^+ \times_t \mathcal{N}$ with metric g generated in the following way

$$g\left(\left(x', a\tfrac{\mathrm{d}}{\mathrm{d}t}\right), \left(y', b\tfrac{\mathrm{d}}{\mathrm{d}t}\right)\right) = t^2\, g'(x', y') + ab,$$

where t is the coordinate on the set of positive real numbers $\mathbb{R}^+$, and a, b are differentiable functions on $\mathcal{C}(\mathcal{N})$. Furthermore, an almost contact structure (φ, ξ, η) is introduced on $\mathcal{C}(\mathcal{N})$ as follows

$$\varphi|_{\ker\eta} = J, \quad \xi = \tfrac{\mathrm{d}}{\mathrm{d}t}, \quad \eta = \mathrm{d}t, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

It is shown in [25] that $(\mathcal{C}(\mathcal{N}), \varphi, \xi, \eta, g)$ is a three-dimensional accR manifold of the class $\mathcal{F}_1 \oplus \mathcal{F}_5$. This manifold can belong in particular to $\mathcal{F}_5$ if and only if J is parallel with respect to the Levi-Civita connection of g'. However, the constructed manifold cannot belong to $\mathcal{F}_1$, nor to $\mathcal{F}_0$.

If $(\mathcal{C}(\mathcal{N}), \varphi, \xi, \eta, g)$ is an $\mathcal{F}_5$-manifold, then it is calculated that $\theta^*(\xi) = \frac{2}{t}$. It is easy to check that this manifold belongs to $\mathcal{F}_5^0$, since (29) holds.

Let the following basis $\{e_1, e_2, e_3\}$ be given for each tangent space at an arbitrary point on the cone:

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad e_3 = \xi,$$
$$g(e_1, e_1) = -g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_i, e_j) = 0,\ i \neq j.$$

Using this basis, the following results were obtained in [25]. The nonzero components of the curvature tensor R are determined by $R(e_1, e_2, e_1, e_2) = \frac{1}{t^2}(k' - 1)$ and the well-known properties of R. Therefore, $(\mathcal{C}(\mathcal{N}), \varphi, \xi, \eta, g)$ is flat if and only if $k' = 1$ for $(\mathcal{N}, J, g')$. The nonzero components of the Ricci tensor and the values of the scalar curvatures in the general case are the following

$$\rho(e_1, e_1) = -\rho(e_2, e_2) = \frac{1}{t^2}(k' - 1), \qquad \tau = \frac{2}{t^2}(k' - 1), \qquad \tilde{\tau} = -\frac{2}{t^2}. \tag{45}$$

The results $\nabla_{e_1} e_3 = \frac{1}{t} e_1$, $\nabla_{e_2} e_3 = \frac{1}{t} e_2$, $\nabla_{e_3} e_3 = 0$ of [25] imply the following formula for any x on the cone

$$\nabla_x \xi = -\frac{1}{t} \varphi^2 x.$$

Therefore, due to (21), it follows that

$$\frac{f}{k} = \frac{1}{t}.$$

In order to satisfy the last condition, we choose the following functions

$$f = t, \qquad k = t^2. \tag{46}$$

Therefore, the following equalities are true, which we need further:

$$\mathrm{d}f(\xi) = 1, \qquad \mathrm{d}k(\xi) = 2t, \qquad \mathrm{d}k = 2t\,\mathrm{d}t. \tag{47}$$

Let us consider the vertical vector field ϑ with $k = t^2$, i.e., due to (18), we have $\vartheta = t^2 \frac{\mathrm{d}}{\mathrm{d}t}$. Moreover, let ϑ be torse-forming with respect to ∇ and $\widetilde{\nabla}$ with

$$f = \tilde{f} = t, \qquad \gamma = \tilde{\gamma} = \frac{1}{t}\mathrm{d}t, \tag{48}$$

i.e., (9), (10), and Lemma 2 hold.

Then, let $(g, \tilde{g}; \vartheta; \lambda, \tilde{\lambda}, \ell)$ be an RB-like almost soliton with the introduced potential ϑ and $\ell = -\frac{1}{2}$. We need to determine the functions λ and $\tilde{\lambda}$ in (7).

Since we have $n = 1$ and $\ell = -\frac{1}{2n}$, the function λ must be calculated by (34). By virtue of (46)–(48), we obtain

$$\lambda = -f = -t.$$

Moreover, we have to verify (35). Due to (45), we have $\tau + \tilde{\tau} = \frac{2}{t^2}(k' - 2)$, and from (35), (46), and (47), we obtain $\tau + \tilde{\tau} = 2(3t + \tilde{\lambda})$. Therefore, we can choose the following

$$\tilde{\lambda} = -3t, \qquad k' = 2.$$

Thus, we specialize (45) for the constructed manifold as follows

$$\rho(e_1, e_1) = -\rho(e_2, e_2) = \frac{1}{t^2}, \qquad \tau = \frac{2}{t^2}, \qquad \tilde{\tau} = -\frac{2}{t^2}. \tag{49}$$

Finally, we check (31) in Theorem 2 for $\ell = -\frac{1}{2n}$ and obtain

$$\rho(x, y) = \frac{1}{t^2} g(x, y) + \left(2t - \frac{1}{t^2}\right) \tilde{g}(x, y) - 2t\,\eta(x)\eta(y),$$

which is consistent with (49) and shows that the constructed manifold is Einstein-like with coefficients $a = \frac{1}{t^2}$, $b = 2t - \frac{1}{t^2}$, and $c = -2t$, which support Corollary 1.

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