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Peano Theorems for Pedjeu–Ladde-Type Multi-Time Scale Stochastic Differential Equations Driven by Fractional Noises

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Abstract: This paper examines fractional multi-time scale stochastic functional differential equations that, in addition, are driven by fractional noises. Based on a specially crafted fixed-point principle for the so-called "local operators", we prove a Peano-type theorem on the existence of weak solutions, that is, those defined on an extended stochastic basis. To encompass all commonly used particular classes of fractional multi-time scale stochastic models, including those with random delays and impulses at random times, we consider equations with nonlinear random Volterra operators rather than functions. Some crucial properties of the associated integral operators, needed for the proofs of the main results, are studied as well. To illustrate major findings, several existence theorems, generalizing those known in the literature, are offered, with the emphasis put on the most popular examples such as ordinary stochastic differential equations driven by fractional noises, fractional stochastic differential equations with variable delays and fractional stochastic neutral differential equations.

Keywords: fixed-point principle; fractional calculus; multi-time scales; Peano's theorem; stochastic differential equations; Volterra operators

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1. Introduction

Fractional differential equations (fDEs) arise in a quite natural way in many different problems and have gained considerable popularity and importance in diverse fields of science and engineering. In the past few decades, fractional stochastic differential equations (fSDEs), which generalize complex processes both in the stochastic and in the fractional senses, are used to model the noise driving option pricing, inflation modeling in economics, the optimal management of populations, plasma physics, etc. Three of the most popular stochastic processes are frequently used: the fractional Brownian motion (fBm), the Riemann–Liouville process (RLfBm) and the fractional motion based on the Lévy process (fLm), which, in fact, is a generalization of fBm.

Whereas stochastic dynamics involving fractional noises are pervasive in the literature, they do not usually possess the key concept behind time-fractional models, i.e., the systems' memory effects, and the latter narrows the fields of applications of time-fractional equations driven by fractional noise. Note that over the last few years, a series of studies have found that many financial market time series display long-range dependence and momentum. Long-range dependence has an important feature that explains the well-documented evidence of volatility. The time-dependent viscosity is analyzed from a time series perspective by computing the elements used to describe stochastic processes in mathematical finance.

In this context, the viscoelastic effects are an ideal superposition of "purely elastic" and "purely viscous" elements to describe the stochastic volatility of financial markets. Such a superposition is often conveniently modeled in terms of the time-fractional Volterra operators. Also, such hybrid models could be used for heavy-tailed, long-range-dependent fluctuations in physical systems.

Inspired by a variety of applications of fractional stochastic differential equations, we think that the time is ripe to start a unified study of these models. Our key idea is to use the notion of fractional integration in multiple (deterministic and stochastic) time scales $T_j(t)$, $j = 1, 2, \dots, m$ introduced in [1]

$$(I_j f)(t) = \int_0^t f(s) dT_j(s).$$

In the paper [1], several independent deterministic scales defined via the Jumarie-type fractional differentials [2] and one (non-fractional) stochastic component given by the Itô differential were studied. Recall that the Jumarie derivative is defined for $0 < \alpha \leq 1$ as

$$f^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} [f(s) - f(0)] ds,$$

and the corresponding integration is given by

$$\int_0^t f(s)(ds)^\alpha = \alpha \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

In our paper, we suggest to extend the multi-scale integration by adding several fractional stochastic scales via the independent fractional Brownian motions. If we, finally, place the Caputo differential on the left-hand side, then we arrive at the following general equation, which covers most of the popular models and which is the main target of the present paper:

$$d^q x(t) = \sum_{j=1}^m \Big[(f_j x)(t)(dt)^{G_j} + (g_j x)(t) dB_j^{H_j}(t) \Big], \tag{1}$$

where d^q is the Caputo fractional differential, $(dt)^{G_j}$ are alternative time scale Jumarie differentials [2], $B_j^{H_j}$ are RLfBm processes with the Hurst indices H_j, $0 < q, G_j \leq 1$, $0 < H_j < 1$ ($j = 1, \dots, m$) and f_j, g_j are path-dependent, nonlinear Volterra operators; see Section 2 for more details. Note that RLfBm in (1) can be replaced by fBm, as fBm $=$ const $\times$ ($\lambda +$ RLfBm), where λ is a process with absolutely continuous paths and can therefore be included in the first group of coefficients.

Equation (1) covers several specific conceptual models known in the literature:
(A) $q = 1$, $j = 1$, $G_1 = 1$, $0 < H_1 < 1$ (models driven by a fractional Brownian motion);
(B) $0 < q < 1$, $j = 1$, $G_1 = 1$, $H_1 = 0.5$ (time-fractional models driven by the standard Brownian motion);
(C) $0 < q < 1$, $j = 2$, $f_2 = 0$, $H_1 = 0.5$, $0 < H_2 < 1$ (combinations of (A) and (B));
(D) $q = 1$, $j = 2$, $G_1 = 1$, $0 < G_2 < 1$, $g_2 = 0$, $H_1 = 0.5$ (a multi-time scale model).

Model (A) serves as a milestone in financial mathematics in describing an fBm market environment. The monograph [3] could be used as a source of reference for the results on this model obtained before 2008. Some interesting results and applications were obtained in recent publications [4–14].

Model (B) is quite popular as well; see, e.g., [15–23].

Model (C) was examined in [24–26].

Multi-time scale stochastic differential equations (D), based on the Jumarie differentials [2,27–29], were first introduced and examined in the most cited paper [1]. Recent qualitative analysis of the Ladde–Pedjeu model, see, e.g., [30], where a non-Lyapunov stability algorithm was designed for this model, as well as its wide applications in finance, epidemiology, robotics and networks [31–36], drive us to consider more general models, i.e., multi-time scale stochastic differential equations driven by fractional noise. In our analysis, we assume that the fractional differentials on the right-hand side of Equation (1) are expressed via the standard Lebesgue and Itô differentials. In the integral form, it reads as

$$\int_{t_0}^{t} \xi(s)(ds)^{G_j} = G_j \int_{t_0}^{t} \xi(s)(t-s)^{G_j-1} ds \quad \text{and}$$

$$\int_{t_0}^{t} \xi(s) dB_j^{H_j}(s) = \Gamma(H_j + 1/2) \int_{t_0}^{t} \xi(s)(t-s)^{H_j-1/2} dB_j(s),$$

see [1,3], respectively.

The analysis of the combined model (1) may pose new features which differ from those described by specific models and provide more insight into the description and explanation of the underlying phenomena. Also, it is intriguing from the pure mathematical point of view, as it offers technical challenges to the classical methods, since many of the standard techniques simply do not work in this setting. For instance, the standard martingale techniques, used to prove the existence of solutions, is only applicable in some special, e.g., linear, cases [37].

For stochastic differential equations, several profound generalizations of the classical Peano existence theorem were suggested. The main technical difficulty was that no solutions may be available on the underlying probability space, which therefore needs to be changed, thus giving rise to the notion of a weak solution. We prove the existence of weak solutions of fSDEs using a convenient fixed-point theorem for the so-called "local operators" [38]. The theorem was first published in [39], and its detailed proof can now be found in [40]. This result, in a slightly weaker form, is formulated in Appendix A for the sake of convenience.

This paper is organized as follows. Section 2 contains some preliminaries on function spaces, lists our assumptions and definitions, along with our motivation and examples. It also states the main object of this paper. In Section 3, the compactness, tightness and continuity of all integral operators are examined. Section 4, consisting of four subsections, contains the main results of this paper. In Section 4.1, we prove Theorem 1 using a specially crafted fixed-point principle for local operators, which serves as the theoretical basis of our studies. In Theorem 1, we chose to define the right-hand sides to be abstract nonlinear and random Volterra operators, in order to put different specific classes of fSDEs under the same umbrella. Some of these classes are treated in Sections 4.2–4.4, which illustrate the major findings. In these subsections, the existence theorem is applied to ordinary fractional stochastic differential equations, fractional stochastic differential equations with delay and fractional stochastic neutral differential equations, respectively.

Let us remark that the chosen level of generality of the main equation allows for including a countable number of arbitrary impulses at random times into any considered model. In addition, delays are not supposed to be continuous, so that our results cover impulsive delay effects. Some ideas concerning further research are discussed in Section 5, while Appendix A contains an overview of some relevant results from [40], including the fixed-point principle formulated in Section 4.1.

The appended bibliography is not intended to be highly comprehensive, and there is no pretense that all significant aspects of the subject are represented therein. Undoubtedly, we have overlooked some very relevant papers of which we are cognizant, and unfortunately others of which we are not aware. To the authors of all such papers, our regrets are extended herewith.

2. Preliminaries

Let $\mathcal{T} = [t_0, t_0 + T]$ be an arbitrary closed subinterval of the real axis and

$$\mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, P) \tag{2}$$

be a fixed stochastic basis with a non-decreasing, right-continuous filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$, containing complete (with respect to the measure P) σ-subalgebras of the σ-algebra $\mathcal{F}$. The corresponding expectation will be denoted by E. A $Bor(\mathcal{T}) \otimes \mathcal{F}$-measurable stochastic process $\xi(t) = \xi(t, \omega)$, $t \in \mathcal{T}$, is called $\mathcal{F}_t$-adapted [41] if $\xi(t, \cdot)$ is $\mathcal{F}_t$-measurable for all $t \in \mathcal{T}$. Here $Bor(\mathcal{T})$ is the σ-algebra of all Borel subsets of $\mathcal{T}$.

The stopping time on $\mathcal{B}$ is the mapping $\tau : \Omega \to \mathcal{T}$ satisfying the property $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for any $t \in \mathcal{T}$. Any stopping time τ generates the σ-algebra

$$\mathcal{F}_\tau \equiv \{A \subset \Omega : \ A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \in \mathcal{T}\}.$$

- $|.|$ is the matrix norm on $R^{m\times n}$.
- $\mathbf{1}_A$ is the indicator of the set A.
- X is the separable normed space with the norm $\|.\|_X$.
- For any separable metric space S, the set $\mathcal{P}(S)$ of all (equivalence classes of) measurable functions $x : \Omega \to S$ (also referred to as random points in S) is again a metric space equipped with the topology of convergence in probability; note that if X is a separable normed space, then $\mathcal{P}(X)$ becomes a linear metric, but not locally convex, space; to simplify the notation, we always disregard the difference between equivalence classes and their representatives considering the latter as elements of the space $\mathcal{P}(S)$ as well.
- Bounded subsets $\mathcal{A} \subset \mathcal{P}(X)$ can be described as follows: for any $\varepsilon > 0$ there is a ball $B_r = \{x \in X : \ \|x\| \leq r\}$ such that

  $$P\{x \notin B_r\} < \varepsilon$$

  for any $x \in \mathcal{A}$.
- $C_\mathcal{T} = C(\mathcal{T}; R^n)$ is the space of all n-dimensional functions that are continuous on $\mathcal{T}$.
- $L^p_\mathcal{T} = L^p(\mathcal{T}; R^n)$ $(1 \leq p \leq \infty)$ is the space of n-dimensional functions that are p-integrable on $\mathcal{T}$ with respect to the Lebesgue measure.
- $\mathcal{C}_\mathcal{T}$ contains all $\mathcal{F}_t$-adapted n-dimensional stochastic processes on $\mathcal{T}$, whose trajectories almost surely belong to the space $C_\mathcal{T}$.
- $\mathcal{L}^p_\mathcal{T}$ contains all $\mathcal{F}_t$-adapted n-dimensional stochastic processes on $\mathcal{T}$, whose trajectories almost surely belong to the space $L^p_\mathcal{T}$.
- In case $\mathcal{B}$ is replaced by another stochastic basis $\mathcal{B}^*$, we will write $\mathcal{C}_\mathcal{T}(\mathcal{B}^*)$ and $\mathcal{L}^p_\mathcal{T}(\mathcal{B}^*)$ instead of $\mathcal{C}_\mathcal{T}$ and $\mathcal{L}^p_\mathcal{T}$, respectively, so that $\mathcal{C}_\mathcal{T} \equiv \mathcal{C}_\mathcal{T}(\mathcal{B})$ and $\mathcal{L}^p_\mathcal{T} \equiv \mathcal{L}^p_\mathcal{T}(\mathcal{B})$.

Remark 1. *The equivalence classes of stochastic processes from $\mathcal{C}_\mathcal{T}$ and $\mathcal{L}^p_\mathcal{T}$ can be identified with the equivalence classes of $\mathcal{F}$-measurable mappings from Ω to $C_\mathcal{T}$ and $L^p_\mathcal{T}$, respectively. This enables us to equip the former spaces with the topologies of convergence in probability in the Banach spaces consisting of the respective trajectories.*

The following classes of operators will be systematically used in this paper.

Definition 1. *A mapping $F : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}, \mathcal{Y}$ are two spaces of functions defined on $\mathcal{T}$, is called a Volterra operator if for any $t \in \mathcal{T}$ and almost all $s \in [t_0, t]$, the equality $x_1(s) = x_2(s)$ $(x_1, x_2 \in \mathcal{X})$ implies that $(Fx_1)(s) = (Fx_2)(s)$.*

The main equation to be studied in this paper is

$$\begin{aligned} &x(t) - x(t_0) = \\ &\sum_{j=1}^{m} \Big(\int_{t_0}^{t} (t-s)^{\alpha_j - 1} (b_j x)(s) ds + \int_{t_0}^{t} (t-s)^{\beta_j - 1} (\sigma_j x)(s) dW_j(s) \Big), \end{aligned} \tag{3}$$

where $x(t)$ $(t \in \mathcal{T})$ is an $\mathcal{F}_t$-adapted stochastic process. To extend areas of applications we assume that b_j and σ_j are nonlinear random Volterra operators rather than functions to cover the wide classes of fSDEs used in multiple applications. It is quite important to remark that integrations in Equation (3) are treated as operators with singular kernels and not as the integrals with respect to the fractional Brownian motion [3].

Note that by putting $b_j x = G_j \Gamma^{-1}(q) f_j x$, $\sigma_j x = \Gamma^{-1}(q) \Gamma^{-1}(H_j + \frac{1}{2}) g_j x$, $\alpha_j = G_j + q - 1$ and $\beta_j = H_j + q - \frac{1}{2}$, we obtain the integral counterpart of fSDE (1). This follows from the representation formulas for the involved differentials, see [1,3,42], respectively. In particular, by a solution of fSDE (1), we understand an adapted stochastic process $x(t)$ satisfying the integral Equation (3) with b_j and σ_j defined in the above way.

The initial condition for Equation (3) reads as

$$x(t_0) = x_0, \tag{4}$$

where x_0 is an $\mathcal{F}_{t_0}$-measurable random point in R^n.

3. Basic Properties of the Integral Operators with Singular Kernels

In this section, we study qualitative properties of the integral operators I^α $(\alpha > 0)$ and J^β $(\beta > \frac{1}{2})$. Some of the properties of the deterministic operators I^α can be found in the literature, see, e.g., [43], but we prove them for the sake of completeness and because a similar technique is later used to study the stochastic operators J^β.

We define the integral operator I^α by

$$(I^\alpha u)(t) = \int_{t_0}^{t} (t-s)^{\alpha - 1} u(s) ds \quad (\alpha > 0). \tag{5}$$

Here u is a function defined on $\mathcal{T}$ and taking values in R^n. It is well known (see, e.g., [42]) that the integral (5) exists if $u \in L^1_{\mathcal{T}}$.

Proposition 1.

1. *The operator I^α is compact as an operator from $L^1_{\mathcal{T}}$ to $L^1_{\mathcal{T}}$.*
2. *If $\alpha \geq 1$, then the operator I^α maps $L^p_{\mathcal{T}}$ to $C_{\mathcal{T}}$ for any $p \geq 1$.*
3. *If $0 < \alpha < 1$, then the operator I^α maps $L^p_{\mathcal{T}}$ to $C_{\mathcal{T}}$ if and only if $p > \frac{1}{\alpha}$.*

Proof. To simplify the notation, assume below that $t_0 = 0$ and $T = 1$.

Part 1. Put

$$g_\nu(t) = \sum_{k=0}^{\nu-2} \frac{k}{\nu} \mathbf{1}_{[\frac{k+1}{\nu}, \frac{k+2}{\nu})}(t). \tag{6}$$

Consider a linear integral operator

$$(I^\alpha_\nu u)(t) = \int_0^{g_\nu(t)} (t-s)^{\alpha-1} u(s) ds = \sum_{k=0}^{\nu-2} \mathbf{1}_{[\frac{k+1}{\nu}, \frac{k+2}{\nu})}(t) \int_0^{\frac{k}{\nu}} (t-s)^{\alpha-1} u(s) ds.$$

Each of the operators $\int_0^{\frac{k}{\nu}} (t-s)^{\alpha-1} u(s) ds$ is compact as an operator from $L^1_{\mathcal{T}}$ to $L^1([\frac{k+1}{\nu}, \frac{k+2}{\nu}]); R^n$, as their kernels are continuous on $[0, \frac{k}{\nu}] \times [\frac{k+1}{\nu}, \frac{k+2}{\nu}]$ [44] [Ch. XI.3]. Hence, $I^\alpha_\nu : L^1_{\mathcal{T}} \to L^1_{\mathcal{T}}$ is compact as well.

On the other hand, for an arbitrary $u \in L^1_{\mathcal{T}}$

$$\begin{aligned}
&\|(I^\alpha - I^\alpha_\nu)u\|_{L^1_{\mathcal{T}}} = \int_0^1 dt \left| \int_0^1 \mathbf{1}_{(g_\nu(t),t]}(s)(t-s)^{\alpha-1}u(s)ds \right| \\
&\le \int_0^1 |u(s)|ds \int_0^1 \mathbf{1}_{(g_\nu(t),t]}(s)(t-s)^{\alpha-1}dt \\
&\le \|u\|_{L^1_{\mathcal{T}}} \sup_{s\in[0,1]} \int_0^1 \mathbf{1}_{(g_\nu(t),t]}(s)(t-s)^{\alpha-1}dt \\
&\le \|u\|_{L^1_{\mathcal{T}}} \sup_{s\in[0,1]} \int_0^1 \mathbf{1}_{[s,s+\frac{2}{\nu}]}(t)(t-s)^{\alpha-1}dt = \|u\|_{L^1_{\mathcal{T}}} \tfrac{(t-s)^\alpha}{\alpha}\Big|_{t=s}^{t=s+\frac{2}{\nu}} = \tfrac{2^\alpha}{\alpha\nu^\alpha}\|u\|_{L^1_{\mathcal{T}}},
\end{aligned}$$

Hence, $\|I^\alpha - I^\alpha_\nu\|_{L^1_{\mathcal{T}}} \to 0$ as $\nu \to \infty$, so that $I^\alpha : L^1_{\mathcal{T}} \to L^1_{\mathcal{T}}$ is compact.

Part 2. We prove the property of continuity in t for parts (2) and (3) simultaneously, and this proof essentially mimics the one offered in [44] [Ch. XI.3], although the corresponding result there does not formally imply statements (2) and (3) of Proposition 1. We still assume the interval $\mathcal{T}$ to be $[0,1]$ for the sake of simplicity.

For $\alpha = 1$, the statement is the standard property of the Lebesgue integral. We assume, therefore, that $\alpha \neq 1$. For a given $p > \max\{1; \frac{1}{\alpha}\}$, let $r = \frac{p}{p-1}$. Thus, $(\alpha-1)r + 1 > 0$, so that we can choose $0 < \gamma < 1$, satisfying

$$(\alpha - \gamma - 1)r + 1 > 0. \tag{7}$$

For a fixed, sufficiently small $h > 0$, we prove that

$$I = \int_0^1 \left| (t+h-s)^{\alpha-1}\mathbf{1}_{[0,t+h]}(s) - (t-s)^{\alpha-1}\mathbf{1}_{[0,t]}(s) \right|^r ds \le Ch^{\gamma r} \tag{8}$$

for some constant C. This will follow from the following estimates:

$$I_1 = \int_{t-2h}^t (t-s)^{(\alpha-1)r} ds \le C_1 h^{\gamma r}, \tag{9}$$

$$I_2 = \int_{t-2h}^{t+h} |t+h-s|^{(\alpha-1)r} ds \le C_2 h^{\gamma r} \tag{10}$$

and

$$I_3 = \int_0^{t-2h} \left| (t+h-s)^{\alpha-1} - (t-s)^{\alpha-1} \right|^r ds \le C_3 h^{\gamma r}. \tag{11}$$

To check (9), we write

$$I_1 = \int_{t-2h}^t (t-s)^{(\alpha-\gamma-1)r}(t-s)^{\gamma r} ds \le (2h)^{\gamma r} \int_{t-2h}^t (t-s)^{(\alpha-\gamma-1)r} ds = C_1 h^{\gamma r}.$$

The last integral is finite, as $(\alpha-\gamma-1)r > -1$ due to (7). A similar argument applies to (10).

To verify (11), we put $\Phi(t,s) = (\alpha-1)(t-s)^{\alpha-2}$, and notice that

$$\int_0^1 |\Phi^r(t,s)|(t-s)^{(1-\gamma)r} ds = |\alpha-1|^r \int_0^1 (t-s)^{(\alpha-2)r+(1-\gamma)r} ds \le C_0, \tag{12}$$

as $(\alpha-2)r + (1-\gamma)r = (\alpha-\gamma-1)r > -1$.

Now,

$$\begin{aligned}
&I_3 = \int_0^{t-2h} \left| \int_t^{t+h} \Phi(\lambda,s) d\lambda \right|^r ds \\
&\le 2^{(1-\gamma)r} \int_0^{t-2h} \left| \int_t^{t+h} \Phi(\lambda,s)(\lambda-s)^{(1-\gamma)} d\lambda \right|^r \tfrac{ds}{(t-s)^{(1-\gamma)r}},
\end{aligned}$$

because $|\lambda - s| \geq |t - s| - |\lambda - t| \geq |t - s| - h \geq \frac{t-s}{2}$. Then, due to Hölder's inequality,

$$I_3 \leq \frac{2^{(1-\gamma)r}}{(2h)^{(1-\gamma)r}} \int_0^{t-2h} \int_t^{t+h} |\Phi^r(\lambda, s)|(\lambda - s)^{(1-\gamma)r} d\lambda \times (\int_t^{t+h} d\lambda)^{r/p}$$

$$\leq \frac{h^{r/p}}{h^{(1-\gamma)r}} \int_t^{t+h} \int_0^{t-2h} |\Phi^r(\lambda, s)|(\lambda - s)^{(1-\gamma)r} ds d\lambda \leq C_0 h^{r(\gamma-1)+r/p+1} = C_0 h^{\gamma r},$$

which gives (11) with $C_3 = C_0$ due to (12).

Combining estimate (8) with Hölder's inequality yields

$$|(I^\alpha u)(t+h) - (I^\alpha u)(t)|^r \leq I \left(\int_0^1 |u(s)|^p ds \right)^{r/p} \leq C h^{\gamma r} \|u\|^r_{L^p_{\mathcal{T}}}.$$

Replacing t by $t - h$ we obtain

$$|I^\alpha u(t-h) - I^\alpha u(t)|^r \leq C h^{\gamma r} \|u\|^r_{L^p_{\mathcal{T}}}.$$

Thus, the function $(I^\alpha u)(t)$ is continuous.

Part 3. The examples below show that the estimate $p > \frac{1}{\alpha}$ is sharp. We assume that $n = 1$.

For any $0 < \alpha < 1$ and $p < \frac{1}{\alpha}$, let $\alpha < \gamma < \frac{1}{p}$ $t_0 = -1$, $T = 2$, $u(s) = |s|^{-\gamma}$. Then, $u \in L^p_{\mathcal{T}}$ as $p\gamma < 1$, but the integral

$$(I^\alpha u)(0) = \int_{-1}^0 (-s)^{\alpha-1} |s|^{-\gamma} ds = \int_{-1}^0 |s|^{\alpha-\gamma-1)} ds$$

diverges, as $\alpha - \gamma - 1 < -1$.

For $p = \frac{1}{\alpha}$ $(0 < \alpha < 1)$, let $t_0 = -0.5$, $T = 1$ and $u(s) = \frac{1}{|s|^\alpha \ln |s|}$. Then, $u \in L^p_{\mathcal{T}}$, as the integral $\int_{-0.5}^{0.5} \frac{ds}{|s| \ln^p |s|}$ is convergent. On the other hand, the integral

$$(I^\alpha u)(0) = \int_{-0.5}^0 \frac{(-s)^{\alpha-1}}{|s|^\alpha \ln |s|} ds = \int_{-0.5}^0 \frac{ds}{|s| \ln |s|}$$

is divergent. □

Now we will examine the stochastic singular operators

$$J^\beta = (J^\beta u)(s) = \int_{t_0}^t (t-s)^{\beta-1} u(s) dW(s) \quad (\beta > \frac{1}{2}), \tag{13}$$

where W is the standard scalar Wiener process defined on $\mathcal{T}$ and $u \in \mathcal{L}^2_{\mathcal{T}}$.

To describe the desired properties of the operator (13), we need two definitions.

Definition 2. *Let X and Y be separable metric spaces and $\mathcal{A} \subset \mathcal{P}(X)$. A (nonlinear) operator $H : \mathcal{A} \to \mathcal{P}(Y)$ is called local if*

$$x(\omega) = y(\omega) \text{ for } \omega \in A \text{ a.s. implies } Hx(\omega) = Hy(\omega) \text{ for } \omega \in A \text{ a.s.}$$

for any $x, y \in \mathcal{A}$ and $A \in \mathcal{F}$.

Let X be a separable Banach space and $\mathcal{P}(X)$ be the (metrizable) space of all $\mathcal{F}$-measurable random points $x : \Omega \to X$. Recall (see, e.g., [41]) that a set $\mathcal{K} \subset \mathcal{P}(X)$ is called tight if for any $\epsilon > 0$ there exists a compact set $K \subset X$ such that $P\{\omega : x(\omega) \notin K\} < \epsilon$ whenever $x \in \mathcal{K}$.

Definition 3. *Let X and Y be two separable Banach spaces. We say that an operator $h:\ \mathcal{A}\to\mathcal{P}(Y)$ defined on a subset $\mathcal{A}\subset\mathcal{P}(X)$, is tight (resp., tight–range) [40] [Def. 3.8] if*

1. *h is uniformly continuous on any tight subset of $\mathcal{A}$.*
2. *h maps bounded subsets of $\mathcal{A}$ (resp., the entire $\mathcal{A}$) to tight subsets of $\mathcal{P}(X)$.*

The next proposition summarizes some basic properties of the operators J^β.

Proposition 2.

1. *The operator J^β is tight as an operator from $\mathcal{L}^2_{\mathcal{T}}$ to $\mathcal{L}^2_{\mathcal{T}}$.*
2. *If $\beta\geq 1$, then the operator J^β maps $\mathcal{L}^{2q}_{\mathcal{T}}$ to $\mathcal{C}_{\mathcal{T}}$ for any $q\geq 1$.*
3. *If $\beta<1$ and $q>\frac{1}{2\beta-1}$, then the operator J^β maps $\mathcal{L}^{2q}_{\mathcal{T}}$ to $\mathcal{C}_{\mathcal{T}}$.*

Proof. Without loss of generality, we assume $\mathcal{T}=[0,1]$.

Part 1. Using the functions $g^\nu_t(t)$ defined in (6), consider the linear integral operator

$$\begin{aligned}(J^\beta_\nu u)(t)&=\textstyle\int_0^{g_\nu(t)}(t-s)^{\beta-1}u(s)dW(s)\\&=\textstyle\sum_{k=0}^{\nu-2}\mathbf{1}_{[\frac{k+1}{\nu},\frac{k+2}{\nu})}(t)\int_0^{\frac{k}{\nu}}(t-s)^{\beta-1}u(s)dW(s).\end{aligned}$$

Each of the operators $\int_0^{\frac{k}{\nu}}(t-s)^{\beta-1}u(s)dW(s)$ is tight as an operator from $\mathcal{L}^2_{\mathcal{T}}$ to $\mathcal{L}^2([\frac{k+1}{\nu},\frac{k+2}{\nu}];R^n)$, as their kernels are continuous on $[0,\frac{k}{\nu}]\times[\frac{k+1}{\nu},\frac{k+2}{\nu}]$ [40] [Ex. D.16]. Hence, $J^\beta_\nu:\ \mathcal{L}^2_{\mathcal{T}}\to\mathcal{L}^2_{\mathcal{T}}$ is tight as well.

Below we use

$$E\left|\int_0^1 v(s)dW(s)\right|^2=\left(E\int_0^1|v(s)|^2ds\right)\quad(v\in\mathcal{L}^2_{\mathcal{T}})$$

For an arbitrary $u\in\mathcal{L}^2_{\mathcal{T}}$, we have

$$\begin{aligned}E\|(J^\beta-J^\beta_\nu)u\|^2_{L^2_{\mathcal{T}}}&=\textstyle\int_0^1 dtE\left|\int_0^1\mathbf{1}_{(g_\nu(t),t]}(s)(t-s)^{\beta-1}u(s)dW(s)\right|^2\\&=\textstyle\int_0^1 dtE\left(\int_0^1\mathbf{1}_{(g_\nu(t),t]}(s)(t-s)^{2(\beta-1)}|u|^2(s)ds\right)\\&\leq E\|u\|_{L^2_{\mathcal{T}}}\sup_{s\in[0,1]}\textstyle\int_0^1\mathbf{1}_{(g_\nu(t),t]}(s)(t-s)^{2\beta-2}dt\\\leq E\|u\|_{L^2_{\mathcal{T}}}\sup_{s\in[0,1]}\textstyle\int_0^1\mathbf{1}_{[s,s+\frac{2}{\nu}]}(t)(t-s)^{2\beta-2}dt&=\tfrac{2^{2\beta-1}}{(2\beta-1)\nu^{2\beta-1}}E\|u\|_{L^2_{\mathcal{T}}}.\end{aligned}$$

Hence, $E\|J^\beta-J^\beta_\nu\|_{L^2_{\mathcal{T}}}\to 0$ as $\nu\to\infty$. Thus, the sequence $\{I^\beta_\nu\}$ of local and tight operators converges to the operator I^β, and this convergence is uniform on the sets $\{u\in\mathcal{L}^2_{\mathcal{T}}:\ E\|u\|_{L^2_t}\leq r\}$, $r>0$. Therefore, this convergence is also uniform on the sets $\mathcal{L}^2_{\mathcal{T}}\cap\mathcal{P}(B_r)$, where $B_r\subset L^2_{\mathcal{T}}$ is a ball of radius r centered in 0. By Proposition A4, the operator $J^\beta:\ \mathcal{L}^2_{\mathcal{T}}\to\mathcal{L}^2_{\mathcal{T}}$ is tight as well. Due to the continuous embedding of the space $L^2_{\mathcal{T}}$ into the space $L^1_{\mathcal{T}}$, any tight subset of $\mathcal{L}^2_{\mathcal{T}}$ is tight in $\mathcal{L}^1_{\mathcal{T}}$. This proves the tightness of the operator $J^\beta:\ \mathcal{L}^2_{\mathcal{T}}\to\mathcal{L}^1_{\mathcal{T}}$.

Parts 2 and 3. For $\beta=1$ the sample continuity in t is the standard property of Itô's integral. We assume therefore that $\beta\neq 1$. For a given $q>\max\{1;\ \frac{1}{2\beta-1}\}$, let $r=\frac{q}{q-1}$. Let us, in addition, choose $0<\gamma<1$ satisfying $2r(\beta-\gamma-1)+1>0$ and $\delta>0$ satisfying $2\delta\gamma>1$. As in the case of the operator I^α, we may assume that the parameter h below is positive.

Using the standard estimate for Itô's integral and Hölder's inequality yield for some $M > 0$,

$$\begin{aligned}
&E\left|\int_0^{t+h}(t+h-s)^{\beta-1}u(s)dW(s)-\int_0^t(t-s)^{\beta-1}u(s)dW(s)\right|^\delta\\
&\le ME\left(\int_0^1\left[(t+h-s)^{\beta-1}\mathbf{1}_{[0,t+h]}(s)-(t-s)^{\beta-1}\mathbf{1}_{[0,t]}(s)\right]^2u^2(s)ds\right)^{\delta/2}\\
&\le M\left(\int_0^1\left[(t+h-s)^{\beta-1}\mathbf{1}_{[0,t+h]}(s)-(t-s)^{\beta-1}\mathbf{1}_{[0,t]}(s)\right]^{2r}ds\right)^{\delta/(2r)}\\
&\times E\left(\int_0^1u^{2q}(s)ds\right)^{\delta/(2q)}.
\end{aligned}$$

By (8),

$$\int_0^1\left[(t+h-s)^{\beta-1}\mathbf{1}_{[0,t+h]}(s)-(t-s)^{\beta-1}\mathbf{1}_{[0,t]}(s)\right]^{2r}ds\le Ch^{2\gamma r}.$$

Hence,

$$E\left|(J^\beta u)(t+h)-(J^\beta u)(t)\right|^\delta\le MCh^{2\gamma r\cdot\delta/(2r)}E\|u\|^\delta_{L^{2q}_{\mathcal{T}}}=MCh^{2\gamma\delta}E\|u\|^\delta_{L^{2q}_{\mathcal{T}}}.$$

As $2\gamma\delta > 1$, the stochastic process $(J^\beta u)(t)$ satisfies Kolmogorov's criterion, according to which it is sample continuous provided that $\|u\|_{L^{2q}_{\mathcal{T}}}$ is bounded almost surely (which guarantees that $E\|u\|^\delta_{L^{2q}_{\mathcal{T}}} < \infty$). For an arbitrary $u \in \mathcal{L}^{2q}_{\mathcal{T}}$, take $\nu \in N$ and put $u_\nu(t) = \pi_\nu(u(t))$, where π_ν is the projection of the space $L^{2q}_{\mathcal{T}}$ onto the ball in this space of radius ν centered on the origin. Then, $u_\nu \in \mathcal{L}^{2q}_{\mathcal{T}}$ and $\|u\|_{L^{2q}_{\mathcal{T}}} \le \nu$ almost surely for any ν. Therefore, $(J^\beta u_\nu)(t)$ is sample continuous for all $\nu \in N$. On the other hand, for an arbitrary $\varepsilon > 0$, there is $\nu \in N$ such that $P\Omega_\nu < \varepsilon$, where $\Omega_\nu = \{\omega \in \Omega : u_\nu(\cdot,\omega) \ne u(\cdot,\omega)\}$. As the operator J^β is local (see Appendix A), we obtain $(J^\beta u)(\cdot,\omega) = (J^\beta(u_\nu))(\cdot,\omega)$ is sample continuous for almost all $\omega \in \Omega - \Omega_\nu$. As $\varepsilon > 0$ is an arbitrary positive number, the stochastic process $J^\beta u$ has almost surely continuous paths, i.e., it is sample continuous. □

4. Main Results

4.1. Existence of Weak and Strong Solutions

In this section, we consider main Equation (3), taking for granted the following assumptions:

(A1) *$W_1(t),\dots,W_m(t)$ are the standard scalar Wiener processes, not necessarily independent, defined for $t \in \mathcal{T}$ on the stochastic basis (2).*

(A2) *For each $j = 1,\dots,m$, $p_j = 1$ if $\alpha_j \ge 1$ and $p_j > \frac{1}{\alpha_j}$ if $0 < \alpha_j < 1$, $q_j = 1$ if $\beta_j \ge 1$ and $q_j > \frac{1}{2\beta_j-1}$ if $\frac{1}{2} < \beta_j < 1$.*

(A3) *$b_j(\cdot,\omega) : L^\infty_{\mathcal{T}} \to L^{p_j}_{\mathcal{T}}$, $\sigma_j(\cdot,\omega) : L^\infty_{\mathcal{T}} \to L^{2q_j}_{\mathcal{T}}$ $(\omega \in \Omega)$ are random continuous Volterra operators $(j = 1,\dots,m)$, which for any $u \in L^\infty_{\mathcal{T}}$ are $\mathcal{F}_t$-adapted stochastic processes on $\mathcal{T}$.*

Equation (3) is endowed with the initial condition (4).

It is well known (see, e.g., [45]) that even ordinary scalar stochastic differential equations with continuous coefficients may require different probability spaces to host their solutions. One of the constructions was suggested in the seminal paper [46], where the underlying probability space was replaced by its product with the space of trajectories, e.g., with the space of continuous functions. In our paper, we utilize this approach using more general extra factors. This explains the following terminology:

Definition 4. *The expansion $\mathcal{B}^* = (\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^*)_{t\in\mathcal{T}}, P^*)$ of the stochastic basis (2) will be called acceptable if it satisfies the following properties:*

1. *Ω^* is defined by $\Omega^* = \Omega \times \bar{Z}$, where $\bar{Z} = \prod_{m=1}^{\nu} Z^{(m)}$ equipped with the product topology, and $Z^{(m)}$ are all copies of either the space $L_{\mathcal{T}}^p$ or the space $C_{\mathcal{T}}$ and ν may be any natural value or ∞;*
2. *The marginal measure P_ω^* is the limit point (in the narrow topology) of a sequence of random Dirac measures $\delta_{\alpha_k(\omega)}$, where $\alpha_k \in \bar{Z}$ $(k \in N)$;*
3. *$\mathcal{F}^*$ and $\mathcal{F}_t^*$ are the P^*-completions of the σ-algebras $\mathcal{F} \otimes Bor(\bar{Z})$ and $\mathcal{F}_t \otimes (q^t)^{-1}(Bor(\bar{Z}_t))$ $(\bar{Z}_t = \prod_{m=1}^{\nu} Z_t^{(m)},\ (t \in \mathcal{T}))$, respectively, where $Z_t^{(m)}$ are copies of either the space $L_{[t_0,t]}^p$ or the space $C_{[t_0,t]}$.*

It is important to remark that this is a particular case of the classical definition of a weak solution. Yet, we will only use acceptable expansions, i.e., those in the sense of Definition 4, of the underlying probability space to construct weak solutions of Equation (1). An abstract counterpart of this notion can be found in [40] [Sec. D.5].

Definition 5. *Let $\mathcal{A}$ denote either $\mathcal{L}_{\mathcal{T}}^p$ $(1 \le p < \infty)$ or $\mathcal{C}_{\mathcal{T}}$ and an operator $H : \mathcal{A} \to \mathcal{A}$ be local and uniformly continuous on tight subsets of its domain. If there exists an acceptable expansion $\mathcal{B}^* = (\Omega^*, \mathcal{F}^*, \mathcal{F}_t^*, P^*)$ of the stochastic basis (2), where $\Omega^* = \Omega \times Z$ and $Z = L_{\mathcal{T}}^p$ or $C_{\mathcal{T}}$, respectively, with the appropriately defined $\mathcal{F}^*, \mathcal{F}_t^*$, and a random point $x^* \in \mathcal{A}^*$ such that $H^*x^* = x^*$, where $H^* : \mathcal{A}^* \to \mathcal{A}^*$ is a unique local and continuous extension of the operator H (see Proposition A5 of Appendix A), then x^* is called a weak fixed point of the operator H.*

Summarizing this definition, we note that it is only a new probability measure P^* which may vary in all constructions of the extended stochastic basis, while the other components are given as described in the definition and remain the same. Only this algorithm will be used in the **Fixed-Point Principle** below and, later, in Theorem 1.

Definition 5 enables us to formulate a useful fixed-point principle that is used to prove the main results of this paper.

Fixed-Point Principle. *Let $\mathcal{A}$ denote either $\mathcal{L}_{\mathcal{T}}^p$ $(1 \le p < \infty)$ or $\mathcal{C}_{\mathcal{T}}$.*

1. *If an operator $H : \mathcal{A} \to \mathcal{A}$ is local and tight-ranged, then it has at least one weak fixed point in the sense of Definition 5.*
2. *Assume that an operator $H : \mathcal{A} \to \mathcal{A}$ is local and uniformly continuous on tight subsets of $\mathcal{A}$. If H has at most one weak fixed point for any acceptable expansion $\mathcal{B}^*$ of the stochastic basis $\mathcal{B}$, then any weak point of H will be equivalent to a strong fixed point of H, i.e., the one belonging to $\mathcal{A}$.*

The justification (rather technical) of this principle can be found in [40] [Th. 5.2], where its more general form is offered. Note that this result was first announced in [39].

Returning to fSDE, let us first notice that if η is a stopping time on the stochastic basis $\mathcal{B}$ and if $\mathcal{B}^*$ is an acceptable expansion of $\mathcal{B}$, then $\eta \circ \mathfrak{c}$ will be a stopping time on $\mathcal{B}^*$, where $\mathfrak{c} : \Omega^* \to \Omega$ is the projection onto Ω if Ω^* is defined as in Definition 4. In what follows, we will always write η instead of $\eta \circ \mathfrak{c}$, because it usually does not cause any misunderstandings.

Definition 6. *Equation (3) has a (local) weak solution if there exist an acceptable expansion $\mathcal{B}^*$ of the initial stochastic basis $\mathcal{B}$, a stopping time $\tau : \Omega \to \mathcal{T}$ on $\mathcal{B}^*$, $\tau > t_0$ P^*-a.s., and a sample continuous, n-dimensional and $\mathcal{F}_t^*$-adapted stochastic process $x(t, \omega^*)$, $t \in \mathcal{T}$, $\omega^* \in \Omega^*$ such that*

$$x(t) - x(t_0) = \sum_{j=1}^{m} \left(\int_{t_0}^{t} (t-s)^{\alpha_j - 1} (b_j x)(s) ds + \int_{t_0}^{t} (t-s)^{\beta_j - 1} (\sigma_j x)(s) dW_j^*(s) \right) \tag{14}$$

P^*-almost surely on the random interval $[t_0, \tau]$, where

$$W_j^*(t, \omega^*) \equiv W_j(t, \mathfrak{c}(\omega^*)) \quad (j = 1, \dots, n)$$

are the standard scalar Wiener processes defined on the extended stochastic basis $\mathcal{B}^*$.

The fact that W_j^* are the standard Wiener processes follows from [40] [Ex. D.22].

The main result of this paper is the following Peano-type theorem.

Theorem 1. *Let Assumptions* **(A1)**–**(A3)**, *hold.*

1. *Equation (3) has at least one weak solution $x(t)$ satisfying the initial condition (4).*
2. *If the* sup*-norm of all weak solutions of the problem (3) and (4) on the interval $\mathcal{T}$ is a priori bounded in probability, then these solutions are a.s. defined on this interval.*
3. *If for any acceptable expansion $\mathcal{B}^*$ of the stochastic basis $\mathcal{B}$ the initial value problems (3) and (4) have at most one weak solution on the interval $\mathcal{T}$, then any weak solution on this interval is strong, i.e., it is defined on the original stochastic basis (2) for all $t \in \mathcal{T}$.*

Proof. We shall utilize the **Fixed-Point Principle** formulated above. It will be applied to a local and tight-ranged operator acting in the space $\mathcal{L}_{\mathcal{T}}^1$. The outcome will be a weak fixed point of this operator, i.e., a random point defined on some acceptable expansion of the original probability space in the sense of Definition 4. We assume that $\nu = 1$ in the first part and $\nu = \infty$ in the second part of the proof.

Part 1. Consider the random continuous projection κ_ω^1 of the space R^n onto the ball B_1 of radius 1 centered at the random initial value $x_0(\omega)$, and define the truncated coefficients as

$$b_{j,1}(x, \omega) = b_j(\kappa_\omega^1(x, \omega), \omega), \quad \sigma_{j,1}(x, \omega) = \sigma_j(\kappa_\omega^1(x, \omega), \omega).$$

By construction, $b_{j,\nu}$, $\sigma_{j,\nu}$ are random continuous Volterra operators acting in $L_{\mathcal{T}}^1$ and in $L_{\mathcal{T}}^2$ ($j = 1, \dots, m$), respectively, and satisfying the same measurability conditions with respect to the filtration $\mathcal{F}_t$ as the operators b_j and σ_j.

Consider the operator

$$(H_1 x)(t) = \sum_{j=1}^{m} (U_{j,1} x)(t) + \sum_{j=1}^{m} (V_{j,1} x)(t), \quad t \in \mathcal{T}, \tag{15}$$

where

$$(U_{j,1} x)(t) = \int_{t_0}^{t} (t-s)^{\alpha_j - 1} (b_{j,1} x)(s) ds$$

and

$$(V_{j,1} x)(t) = \int_{t_0}^{t} (t-s)^{\beta_j - 1} (\sigma_{j,1} x)(s) dW_j(s).$$

To check that the operator $H_1 : \mathcal{L}^1_{\mathcal{T}} \to \mathcal{L}^1_{\mathcal{T}}$ is local and tight-ranged, we consider the following superposition operators:

$$x(\omega) \mapsto b_{j,1}(x(\omega), \omega) \tag{16a}$$

$$x(\omega) \mapsto \sigma_{j,1}(x(\omega), \omega) \tag{16b}$$

$$x(\omega) \mapsto I^{\alpha_j}(x(\omega), \omega) \tag{16c}$$

By Assumption **(A3)**, the operators (16a), (16b) and (16c) map the space $\mathcal{P}(L^1_{\mathcal{T}})$ to the spaces $\mathcal{P}(L^{p^j}_{\mathcal{T}})$, $\mathcal{P}(L^{2q^j}_{\mathcal{T}})$ and $\mathcal{P}(L^1_{\mathcal{T}})$, respectively. By Proposition A2, the operators (16a), (16b) and (16c) are local and uniformly continuous on tight subsets of the space $\mathcal{P}(L^1_{\mathcal{T}})$. Moreover, due to the property of adaptedness required by Assumption **(A3)**, they map adapted stochastic processes to adapted ones, or in other words, they map the space $\mathcal{L}^1_{\mathcal{T}}$ to the spaces $\mathcal{L}^{p^j}_{\mathcal{T}}$, $\mathcal{L}^{2q^j}_{\mathcal{T}}$ and $\mathcal{L}^1_{\mathcal{T}}$, respectively.

Assumption **(A2)** combined with Proposition A3 implies that the integral operators $J^{\beta_j} : \mathcal{L}^2_{\mathcal{T}} \to \mathcal{L}^1_{\mathcal{T}}$ are local and tight. Hence, by Propositions A1 and A2, the operator $H_1 : \mathcal{L}^1_{\mathcal{T}} \to \mathcal{L}^1_{\mathcal{T}}$ is local and uniformly continuous on tight subsets of its domain as well.

To prove that the operator H_1 is tight-ranged, it remains to show that it maps the space $\mathcal{L}^1_{\mathcal{T}}$ to its tight subset.

According to Proposition 1, the linear integral operator $I^{\alpha_j} : L^1_{\mathcal{T}} \to L^1_{\mathcal{T}}$ is compact so that the superposition operator (16c) is tight by Proposition A2. On the other hand, the superposition operator (16a) maps the entire space $\mathcal{L}^1_{\mathcal{T}}$ onto some bounded subset of $\mathcal{L}^{p_j}_{\mathcal{T}}$. Hence, the operator $U_{j,1} : \mathcal{L}^1_{\mathcal{T}} \to \mathcal{L}^1_{\mathcal{T}}$ is tight-ranged.

Similarly, the linear integral operator $J^{\beta_j} : \mathcal{L}^2_{\mathcal{T}} \to \mathcal{L}^1_{\mathcal{T}}$ is tight by Proposition 2, and the superposition operator (16b) maps the space $\mathcal{L}^2_{\mathcal{T}}$ onto some bounded subset of $\mathcal{L}^{2q_j}_{\mathcal{T}}$. Hence, the operator $V_{j,1} : \mathcal{L}^1_{\mathcal{T}} \to \mathcal{L}^1_{\mathcal{T}}$ is tight-ranged.

We have, therefore, proven that the operator $H_1 : \mathcal{L}^1_{\mathcal{T}} \to \mathcal{L}^1_{\mathcal{T}}$ in (15) is local and tight-ranged.

Now, by the **Fixed-Point Principle**, there exists an acceptable expansion

$$\mathcal{B}^1 = (\Omega^1, \mathcal{F}^1, \mathcal{F}^1_t, P^1),$$

of the stochastic basis $\mathcal{B}$, where $\Omega^1 = \Omega \times Z$ and $Z = L^1_{\mathcal{T}}$, and a weak fixed point $x_1 \in \mathcal{L}^1_{\mathcal{T}}(\mathcal{B}^1)$ of the operator H_1, where W_j is replaced by the Wiener processes on $\mathcal{B}^1$, as it is described in Definition 6. Notice that $|x_1 - x_0| \le 1$ P^1-a.s. by construction. Evidently, this solution is sample continuous. Hence the stopping time $\tau_1(t) = \inf\{t : |x_1(t) - x_0| > 1\}$ is well-defined and $\tau_1 > t_0$ a.s., so that the restriction of x_1 to the random interval $[t_0, \tau_1]$ solves the initial value problem (3) and (4) on this interval. For the sake of simplicity, we may still denote this solution by x_1. This proves the first part of the theorem.

Part 2. Here, we iterate the procedure from Part 1 by induction. If $\nu \ge 2$ and $x_{\nu-1}$ is an already constructed weak solution defined on an acceptable expansion

$$\mathcal{B}^{\nu-1} = (\Omega^{\nu-1}, \mathcal{F}^{\nu-1}, \mathcal{F}^{\nu-1}_t, P^{\nu-1})$$

constructed as described in Definition 4. The solution $x_{\nu-1}$ is assumed to satisfy (3) and (4) on the random interval $[a, \tau_{\nu-1}]$ for some stopping time $\tau_{\nu-1}$ on $\mathcal{B}^{\nu-1}$ and to obey the estimate $|x_{\nu-1} - x_0| \le (\nu - 1)$ $P^{\nu-1}$-a.s. Put

$$\tilde{x}(t) = \begin{cases} x(t) & (t \ge \tau_{\nu-1}) \\ x_{\nu-1}(t) & (t < \tau_{\nu-1}) \end{cases}$$

and define the operator

$$\begin{aligned}&(H_\nu x)(t)\\&= \tilde{x}_0+\textstyle\sum_{j=1}^m \Big(\int_{t_0}^t (t-s)^{\alpha_j-1}(b_{j,\nu}x)(s)ds + \int_{t_0}^t (t-s)^{\beta_j-1}(\sigma_{j,\nu}x)(s)dW_j(s)\Big)\end{aligned} \tag{17}$$

where $W_j^{\nu-1}$ are the standard Wiener processes on $\mathcal{B}^{\nu-1}$, constructed as explained in Definition 4, and $b_{j,\nu}$ and $\sigma_{j,\nu}$ are superposition operators generated by the following random Volterra operators:

$$(b_{j,\nu}(x,\omega))(t) = \begin{cases} (b_j(\tilde{x}(\cdot,\omega)\circ\kappa_\omega^\nu),\omega)(t) & (t \geq \tau_{\nu-1}(\omega)) \\ (b_j(x_{\nu-1}(\cdot,\omega),\omega))(t) & (t < \tau_{\nu-1}(\omega)) \end{cases}$$

$$(\sigma_{j,\nu}x(\cdot,\omega))(t) = \begin{cases} (\sigma_j(\tilde{x}(\cdot,\omega)\circ\kappa_\omega^\nu),\omega)(t) & (t \geq \tau_{\nu-1}(\omega)) \\ (\sigma_j(x_{\nu-1}(\cdot,\omega),\omega))(t) & (t < \tau_{\nu-1}(\omega)) \end{cases}$$

where κ_ω^ν are random continuous projections of the space R^n onto the ball B_ν of radius ν centered at $x_0(\omega)$. Evidently, the operators $b_{j,\nu}$ and $\sigma_{j,\nu}$ satisfy Assumption **(A3)** with respect to the filtration $(\mathcal{F}_t^{\nu-1})$. Therefore, as it shown in the first part of the proof, the operator $H_\nu : \mathcal{L}_{\mathcal{T}}^1(\mathcal{B}^{\nu-1}) \to \mathcal{L}_{\mathcal{T}}^1(\mathcal{B}^{\nu-1})$ is local and tight-ranged. Hence, it has a weak fixed point x_ν defined on an acceptable expansion $\mathcal{B}^\nu$ of the stochastic basis $\mathcal{B}^{\nu-1}$, constructed as explained in Definition 4 (i.e., $\Omega^\nu = \Omega^{\nu-1} \times L_{\mathcal{T}}^1$). As before, x_ν is sample continuous, so that $\tau_\nu = \inf\{|x_\nu - x_0| > \nu\}$ is well-defined and satisfies $\tau_\nu > \tau_{\nu-1}$ a.s. Therefore, the weak fixed point of the operator H_ν gives rise to a local solution defined on $\mathcal{B}^\nu$ for all $t \in [t_0, \tau_\nu]$. We denote this solution by x_ν as well. By construction, it a.s. coincides with $x_{\nu-1}$ on the random interval $[a, \tau_{\nu-1}]$ and satisfies $|x_\nu - x_0| \leq \nu$ P^ν-a.s. The induction argument is completed.

Now, by letting $\nu \to \infty$ we obtain the stochastic basis

$$\mathcal{B}^* = (\Omega^*, \mathcal{F}^*, \mathcal{F}_t^*, P^*)$$

to be the projective limit of the sequence of the stochastic bases $\mathcal{B}^\nu$ (in particular, $\Omega^* = \Omega \times \bar{Z}$, where $\bar{Z} = \prod_{m=1}^{\nu} Z^{(m)}$ and $Z^{(m)}$ are all copies of the space $L_{\mathcal{T}}^1$). According to Definition 4, see also [40] [Ex. D.21], the stochastic basis $\mathcal{B}^*$ is acceptable. Clearly, all τ_ν remain stopping times on $\mathcal{B}^*$. Let us, therefore, put $x^*(t) = x_\nu(t)$ if $t \in (\tau_{\nu-1}, \tau_\nu]$, $\nu \in N$. Evidently, this stochastic process is sample continuous on the random interval $[t_0, \tau)$, where $\tau = \sup_{\nu\in N} \tau_\nu$ is a stopping time on $\mathcal{B}^*$ and satisfies the initial value problem (3) and (4), where W_j are replaced by the standard Wiener processes W_j^* on $\mathcal{B}^*$. Moreover, by construction, $\tau(t) = T$ if and only if $|x^*(t)| < \infty$ P^*-a.s. on $[t_0, \tau]$. This means that if the sup-norm of all local weak solutions of (3) and (4) on the interval $\mathcal{T}$ is a priori known to be bounded in probability, then $x^*(t)$ is a.s. defined for all $t_0 \leq t \leq T$.

Part 3. Finally, let us assume that the initial value problems (3) and (4) admit at most one weak solution on the interval $[a, b]$ for any acceptable expansion $\mathcal{B}^*$ of the stochastic basis $\mathcal{B}$. This means that the integral operator on the right-hand side of Equation (3), which is local and uniformly continuous on tight subsets of the space $\mathcal{L}_{\mathcal{T}}^1$, has at most one weak fixed point in this space for any acceptable expansion $\mathcal{B}^*$ of the stochastic basis $\mathcal{B}$. Then, by the **Fixed-Point Principle** part 2, any weak fixed point of this operator must be defined on the stochastic basis $\mathcal{B}$. This fixed point will be a strong solution of the initial value problems (3) and (4) on the interval $\mathcal{T}$. □

Remark 2. *Theorem 1 does not formally cover stochastic equations with impulses. However, as the initial condition (4) is random, we can always include models with a countable number of random impulses at increasing stopping times in our scheme by simply dividing the interval $\mathcal{T}$ into subintervals, where we can successively apply Theorem 1. This remark is also valid for all equations considered in the next section.*

In the next subsections, we examine some particular cases of Equation (3), the main tool being Theorem 1. From now on, the Lebesgue measure on $\mathcal{T}$ is denoted by μ.

4.2. Ordinary Fractional Stochastic Differential Equations

Consider

$$\begin{aligned} &x(t) - x(t_0) \\ &= \sum_{j=1}^{m} \Big(\int_{t_0}^{t} (t-s)^{\alpha_j - 1} f_j(s, x(s)) ds + \int_{t_0}^{t} (t-s)^{\beta_j - 1} g_j(s, x(s)) dW_j(s) \Big) \end{aligned} \tag{18}$$

on the interval $t \in \mathcal{T}$ under the following assumption:

(B1) *The functions $f_j(\omega, t, x)$ and $g_j(\omega, t, x)$ are $\mathcal{F} \otimes Bor(\mathcal{T})$-measurable in (ω, t) for all $x \in R^n$ and $j = 1, .., n$ $\mathcal{F}_t$-adapted in ω for any $t \in \mathcal{T}$ and $x \in R^n$, continuous in $x \in R^n$ for $P \otimes \mu$-almost all $(\omega, t) \in \Omega \times \mathcal{T}$ and for all $r > 0$*

$$\int_{\mathcal{T}} (\sup_{|x| \le r} |f_j(\omega, t, x)|)^{p_j} dt < \infty, \quad \int_{\mathcal{T}} (\sup_{|x| \le r} |g_j(\omega, t, x)|)^{2q_j} dt < \infty \text{ a.s.}$$

Theorem 2. *Let Assumptions **(A1)**, **(A2)** and **(B1)** hold. Then, for any initial condition (4), Equation (18) has at least one (local) weak solution.*

Proof. According to Assumption **(B1)** and the standard properties of the superposition operator, the random operators

$$b_j(x(\cdot), \omega) \equiv f_j(\omega, \cdot, x(\cdot)) \text{ and } \sigma_j(x(\cdot), \omega) \equiv g_j(\omega, \cdot, x(\cdot))$$

satisfy Assumption **(A3)**, which enables us to apply Theorem 1. □

Some conditions for the uniqueness property can be found in, e.g., [47]. Together with Theorem 1, this yields the existence of strong solutions of Equation (18).

4.3. Fractional Stochastic Differential Equation with Random Delays

Consider

$$\begin{aligned} &x(t) - x(t_0) \\ &= \sum_{j=1}^{m} \Big(\int_{t_0}^{t} (t-s)^{\alpha_j - 1} f_j(s, x(c_{1j}(s)), \dots, x(c_{kj}(s))) ds \\ &+ \int_{t_0}^{t} (t-s)^{\beta_j - 1} g_j(s, x(d_{1j}(s)), \dots, x(d_{kj}(s))) dW_j(s) \Big) \quad (t \in \mathcal{T}) \end{aligned} \tag{19}$$

under

(B2) *The random delays $c_{ij} = c_{ij}(t, \omega) \le t$, $d_{ij} = d_{ij}(\omega, t) \le t$ $(\omega \in \Omega)$ are $Bor(\mathcal{T}) \otimes \mathcal{F}$-measurable stopping times for all $t_0 < t \le t_0 + T$, $i = 1, \dots, k$, $j = 1, \dots, m$.*

Equation (19) is equipped with the initial condition (4) and the prehistory condition

$$x(s) = \varphi(s) \quad (s < t_0), \tag{20}$$

where the function φ is a $Bor(-\infty, t_0) \otimes \mathcal{F}_{t_0}$-measurable and a.s. locally bounded on $\Omega \times (-\infty, t_0)$ stochastic process.

The deterministic fractional delay equations are discussed in the monograph [42] [Ch.2]. The stochastic fractional delay equations were introduced in [48].

Theorem 3. *Let Assumptions **(A1)**, **(A2)** and **(B2)** hold. Then Equation (18) has at least one (local) weak solution for any initial condition (4) and prehistory condition (20).*

Proof. To represent Equation (19), together with the prehistory condition (20), as in Equation (3), we exploit the reduction algorithm, which is described in the monograph [49] and which consists of moving the prehistory process φ to the right-hand side of the equation in question. For any $i = 1, 2$, $j = 1, \dots, m$ and $t \in \mathcal{T}$ put

$$(C_{ij}x)(t) = \begin{cases} x(c_{ij}(t)) & (c_{ij}(t) \geq t_0) \\ 0 & (c_{ij}(t) < t_0) \end{cases}, \quad (D_{ij}x)(t) = \begin{cases} x(d_{ij}(t)) & (d_{ij}(t) \geq t_0) \\ 0 & (d_{ij}(t) < t_0) \end{cases} \tag{21}$$

and

$$(\varphi_{ij})(t) = \begin{cases} 0 & (c_{ij}(t) \geq t_0) \\ \varphi(c_{ij}(t)) & (c_{ij}(t) < t_0) \end{cases}, \quad (\psi_{ij})(t) = \begin{cases} 0 & (d_{ij}(t) \geq t_0) \\ \varphi(d_{ij}(t)) & (d_{ij}(t) < t_0) \end{cases}. \tag{22}$$

Clearly, C_{ij} and D_{ij} are bounded linear operators in the space $L^{\infty}_{\mathcal{T}}$. Therefore, by Assumption **(B2)** and the properties of the superposition operators generated by the functions f_j and g_j, listed in the proof of Theorem 2, the random operators

$$\begin{aligned} (b_j x)(\omega) &= f_j(\omega, (C_{1j}x)(\omega) + \varphi_{1j}(\omega, \cdot), \dots, (C_{kj}x)(\omega)) + \varphi_{kj}(\omega, \cdot)), \\ (\sigma_j x)(\omega) &= g_j(\omega, (D_{1j}x)(\omega) + \psi_{1j}(\omega, \cdot), \dots, (D_{kj}x)(\omega)) + \psi_{kj}(\omega, \cdot)) \end{aligned} \tag{23}$$

satisfy **(B1)**. By this, the systems (19) and (20) are represented as Equation (3). According to Assumption **(B1)** and the standard properties of the superposition operator, the random operators b_j and σ_j satisfy Assumption **(A3)**, which enables us to apply Theorem 1, ensuring the existence of a weak local solution $\bar{x}(t)$ ($[t_0, \tau]$), where $\tau > t_0$ a.s. is a stopping time, that satisfies the initial condition (4). Defining $x(t) = \bar{x}(t) I_{[t_0,\tau]} + \varphi I_{(-\infty,t_0)}$ and using Formula (23), we see that $x(t)$, by construction, satisfies Equation (19), the initial condition (4) and the prehistory condition (20). □

Remark 3. *As the delays $c_{ij}(t, \omega)$ and $d_{ij}(t, \omega)$ are not supposed to be continuous in t, the fractional differential Equation (19) includes the case of impulsive delays. This remark also applies to the fractional neutral differential Equation (24).*

4.4. Fractional Stochastic Neutral Differential Equation

Stochastic neutral (non-fractional) differential equations are quite popular in the literature, see, e.g., [50,51], and the references therein. A deterministic fractional version of neutral equations was introduced in [42] [Ch. 2.2].

Below, $h > 0$ is a fixed constant, $\mathcal{I} = [t_0 - h, t_0)$ and $\mathcal{J} = \mathcal{I} \cup \mathcal{T}$. Consider a stochastic version of the fractional neutral differential equation on $\mathcal{T}$

$$\begin{aligned} &d^{\alpha}[x - z(t, x(p_1(t)), \dots, x(p_k(t)))] \\ &= f(t, x(c_1(t)), \dots, x(c_k(t)))dt + \textstyle\sum_{j=1}^{m} g_j(t, x(d_{1j}(t)), \dots, x(d_{kj}(t)))dW_j(t) \end{aligned} \tag{24}$$

equipped with the initial condition (4) and the prehistory condition

$$x(t) = \varphi(t) \quad (t \in \mathcal{I}), \tag{25}$$

where $\varphi = \varphi(t,\omega)$ is a $Bor(\mathcal{I}) \otimes \mathcal{F}_{t_0}$-measurable and a.s. essentially bounded on the $\mathcal{I} \times \Omega$ stochastic process.

Introduce the assumptions

(B3) *$p = 1$, $q_j = 2$ if $\alpha = 1$ and $p > \frac{1}{\alpha}$ and $q_j > \frac{1}{2\alpha-1}$ if $\frac{1}{2} < \alpha < 1$. $(j = 1, \dots, m)$.*
(B4) *The functions $f(\omega,t,x)$ and $g_j(\omega,t,x)$ are $\mathcal{F} \otimes Bor(\mathcal{T})$-measurable in (ω,t) for all $x \in R^n$, $\mathcal{F}_t$-adapted in ω for any $t \in \mathcal{T}$ and $x \in R^n$, continuous in $x \in R^n$ for $P \otimes \mu$-almost all $(\omega,t) \in \Omega \times \mathcal{T}$ and for all $r > 0$ and $j = 1,..,n$*

$$\int_{\mathcal{T}} (\sup_{|x| \le r} |f(\omega,t,x)|)^p dt < \infty \quad \text{and} \quad \int_{\mathcal{T}} (\sup_{|x| \le r} |g_j(\omega,t,x)|)^{2q_j} dt < \infty \text{ a.s.}$$

(B5) *The random delays $c_i = c_i(t,\omega)$ $d_{ij} = d_{ij}(t,\omega)$ are $Bor(\mathcal{T}) \otimes \mathcal{F}$-measurable stopping times a.s. satisfying $t_0 - h \le c_i(t) \le t$, $t_0 - h \le d_{ij}(t) \le t$ for all $t_0 < t \le t_0 + T$, $i = 1, \dots, k$, $j = 1, \dots, m$.*
(B6) *$t_0 - h \le p_i(t) \le t$ are scalar Borel-measurable functions defined on $\mathcal{T}$ and $z(t,x^1,\dots,x^k)$ $(t \in \mathcal{T}$, $x^i \in R^n)$ is a continuous function with the property*

$$|z(t,x^1,\dots,x^k) - z(t,\bar{x}^1,\dots,\bar{x}^k)| \le M \sum_{i=1}^{k} |x^i - \bar{x}^i| \quad \text{for all} \quad x^i, \bar{x}^i \in R^n$$

for some $M \in R$, $0 < M < 1$.

By a local weak solution $x(t)$ $(t \in \mathcal{J})$ of (24) we mean a stochastic process, which is a sample continuous on $\mathcal{T}$ and which satisfies

$$\begin{array}{l} x(t) - z(t,x(p_1(t)),\dots,x(p_k(t))) = x(t_0) - z(t_0,x(p_1(t_0)),\dots,x(p_k(t_0))) \\ +\Gamma^{-1}(\alpha)\int_{t_0}^{t}(t-s)^{\alpha-1} f(s,x(c_1(s)),\dots,x(c_k(s)))ds \\ +\sum_{j=1}^{m}\Gamma^{-1}(\alpha)\int_{t_0}^{s}(t-s)^{\alpha-1} g_j(s,x(d_{1j}(s)),\dots,x(d_{kj}(s)))dW_j^*(s) \end{array} \tag{26}$$

for all $t \in [t_0 - h, \tau]$. Here W_j^* are the standard scalar Wiener processes defined on some acceptable expansion of the stochastic basis (2), and $\tau > t_0$ a.s. is a stopping time on this stochastic basis.

Theorem 4. *Let Assumptions **(A1)** and **(B3)–(B6)** hold. Then, for any initial condition (4) and the prehistory condition (25), Equation (24) has at least one (local) weak solution.*

Proof. To represent Equation (26) as Equation (3), we first represent it in a form similar to (19) from the previous subsection. To this end, consider the deterministic functional equation

$$x = Z(x,y), \quad Z(x,y)(t) = z(t,x(p_1(t)),\dots,x(p_k(t))) - y(t) \quad (t \in \mathcal{J}) \tag{27}$$

in the space $L_{\mathcal{J}}^{\infty}$, where $y \in L_{\mathcal{J}}^{\infty}$ is an arbitrary function. Due to Assumption **(B6)**, the operator $Z(\cdot,y)$ is a contraction with the constant $M < 1$ for all $y \in L_{\mathcal{J}}^{\infty}$. In particular, Equation (27) has a unique solution $x = Ky \in L_{\mathcal{J}}^{\infty}$ for each $y \in L_{\mathcal{J}}^{\infty}$ and $\lim_{n\to\infty} \|Ky - Z^n(0,y)\|_{\infty} = 0$.

Moreover, $\|Z(x_1,y_1) - Z(x_2,y_2)\|_{\infty} \le M\|x_1 - x_2\|_{\infty} + \|y_1 - y_2\|_{\infty}$ so that

$$\begin{array}{l} \|Z^N(x_1,y_1) - Z^N(x_2,y_2)\|_{\infty} = \|Z(Z^{N-1}(x_1,y_1),y_1) - Z(Z^{N-1}(x_2,y_2),y_2)\|_{\infty} \\ \le M\|Z^{N-1}(x_1,y_1) - Z^{N-1}(x_2,y_2)\|_{\infty} + \|y_1 - y_2\|_{\infty} \end{array}$$

and iterating this estimate yields

$$\|Z^N(x_1,y_1)-Z^N(x_2,y_2)\|_\infty \le M^N\|x_1-x_2\|_\infty+\sum_{l=0}^{N-1}M^l\|y_1-y_2\|_\infty$$
$$\le M^N\|x_1-x_2\|_\infty+\tfrac{M}{1-M}\|y_1-y_2\|_\infty.$$

From this, we deduce the inequality

$$\|Ky_1-Ky_2\|_\infty=\lim_{N\to\infty}\|Z^N(0,y_1)-Z^N(0,y_2)\|_\infty\le\frac{M}{1-M}\|y_1-y_2\|_\infty,$$

which means that the operator $K:L^\infty_{\mathcal{J}}\to L^\infty_{\mathcal{J}}$ is uniformly continuous. Put $(\mathcal{K}u)(\omega)=K(u(\omega))$. By construction, $\mathcal{K}$ is a (pointwise) limit of the sequence $\{Z^n(0,u)\}$ containing Volterra operators (as $p_i(t)\le t$ by Assumption **(B6)**) which evidently maps $\mathcal{F}_t$-adapted processes to $\mathcal{F}_t$-adapted ones. Therefore, $\mathcal{K}$ has these two properties as well. Moreover, the uniform continuity of $K:L^\infty_{\mathcal{J}}\to L^\infty_{\mathcal{J}}$ implies uniform continuity of $\mathcal{K}:\mathcal{L}^\infty_{\mathcal{J}}\to\mathcal{L}^\infty_{\mathcal{J}}$; see Proposition A2.

A similar argument applies to the vector equation

$$x_0=Z_0(x_0,y_0),\quad Z_0(x_0,y_0)=z(t_0,x(p_1(t_0)),\dots,x(p_k(t_0)))-y(t_0)\quad(t\in\mathcal{J})$$

in the space R^n ensuring the existence of a unique solution $x_0=K_0y_0$ of the latter equation, continuously depending on $y_0\in R^n$. The corresponding operator $\mathcal{K}_0$, defined as $(\mathcal{K}_0v)(\omega)=K_0(v(\omega))$, is continuous as an operator in the space of $\mathcal{F}_{t_0}$-measurable random points equipped with the topology of convergence in probability.

Using the substitutions

$$y(t)=x(t)-z(t,x(p_1(t)),\dots,x(p_k(t))),\quad y(t_0)=x(t_0)-z(t_0,x(p_1(t_0)),\dots,x(p_k(t_0)))$$

in Equation (26) and minding that $x=\mathcal{K}y$ and $x_0=\mathcal{K}y_0$ yield the equation

$$\begin{aligned}&y(t)-y(0)\\&=\Gamma^{-1}(\alpha)\textstyle\int_{t_0}^t(t-s)^{\alpha-1}f(s,\mathcal{K}y(c_1(s)),\dots,\mathcal{K}y(c_k(s)))ds\\&+\textstyle\sum_{j=1}^m\Gamma^{-1}(\alpha)\int_{t_0}^s(t-s)^{\alpha-1}g_j(s,\mathcal{K}y(d_{1j}(s)),\dots,\mathcal{K}y(d_{kj}(s)))dW_j(s)\end{aligned}\tag{28}$$

where $y(0)=y_0$. This equation is similar to Equation (19) from the previous section, the only considerable difference is the the delay operators on the right-hand side are subject to an additional operation represented by the uniformly continuous Volterra operator $\mathcal{K}:\mathcal{L}^\infty_{\mathcal{I}}\to\mathcal{L}^\infty_{\mathcal{I}}$. Thus, we can proceed here in a similar manner to reduce Equation (28) to the main Equation (3) by introducing the operators (21) and the stochastic processes (22) with the notational adjustments: c_{ij}, C_{ij}, φ_{ij}, and x will be replaced by c_i, C_i, φ_i and $\mathcal{K}y$, respectively, and φ will be replaced by $\varphi(t)-z(t,\varphi(p_1(t)),\dots,\varphi(p_k(t)))$ $(t_0-h\le t<t_0)$.

Following the scheme from the previous section, we define the operators $b_1:\mathcal{L}^\infty_{\mathcal{T}}\to L^p_{\mathcal{T}}$ and $\sigma_j:\mathcal{L}^\infty_{\mathcal{T}}\to L^{2q_j}_{\mathcal{T}}$ by putting

$$\begin{aligned}&(b_1y)(\omega)=\Gamma^{-1}(\alpha)f(\omega,(C_1\mathcal{K}y)(\omega)+\varphi_1(\omega,\cdot),\dots,(C_k\mathcal{K}y)(\omega))+\varphi_k(\omega,\cdot)),\\&(\sigma_jy)(\omega)=\Gamma^{-1}(\alpha)g_j(\omega,(D_{1j}\mathcal{K}y)(\omega)+\psi_{1j}(\omega,\cdot),\dots,(D_{kj}\mathcal{K}y)(\omega))+\psi_{kj}(\omega,\cdot)).\end{aligned}\tag{29}$$

It is straightforward to check that the operators b_1 and σ_j satisfy Assumption **(A3)**. By Theorem 1, Equation (3) with b_1 and σ_j defined by (29) has a weak local solution $\bar{y}(t)$, which is defined on some interval $[t_0,\tau]$, $\tau>t_0$ a.s. is a stopping time, and which satisfies the initial condition $y(0)=y_0$. Putting $y(t)=\bar{y}(t)I_{[t_0,\tau]}+\varphi I_{(t_0-h,t_0)}$ we see, as in the previous subsection, that $y=\mathcal{K}(yI_{\mathcal{T}}+\varphi I_{\mathcal{I}})$ will be a local weak solution of Equation (28) satisfying

the conditions $y(t_0) = y_0$ and $y(t) = \varphi(t) - z(t, \varphi(p_1(t)), \dots, \varphi(p_k(t)))$ $(t_0 - h \le t < t_0)$. By construction, the stochastic process $x = \mathcal{K}y$ satisfies Equation (24) on the interval $[t_0, \tau]$, the initial condition (4) and the prehistory condition (25). □

5. Conclusions

There is abundant literature on ordinary stochastic models driven by fractional Wiener processes, whereas for the combined models, the literature is scarce. The major model in this report covers wide classes of stochastic fractional-time differential equations with the driving fractional Wiener process, including equations with random bounded, unbounded and impulsive delays, neutral equations and equations with random impulses at random times. Using a specially crafted fixed-point theorem for local operators, we proved new existence theorems. This method is general and helps to avoid martingale-based techniques, which are inapplicable for our models. The existence of the solutions were obtained under rather mild assumptions. The compactness and tightness of all integral operators were examined; these results can be used for qualitative analyses of other classes of fractional stochastic models. The techniques offered in this report could be applied to similar analyses for the following cases:

1. Time-fractional stochastic differential equations driven by the fractional Levy noise, e.g., the fractional Poisson-like noise.
2. The existence of strong solutions for fSDEs by applying Okinawa's uniqueness theorem or its generalizations.
3. fSDEs with other types of fractional differentials, e.g., those with the time-dependent Hurst parameters.

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Abbreviations

The following abbreviations are used in this manuscript:

fDE	Fractional differential equations
fSDE	Fractional stochastic differential equations
fBm	Fractional Brownian motion
RLfBm	Riemann–Liouville process, i.e., fractional Brownian motion of the Riemann–Liouville type
fLm	Fractional motion based on the Lévy process
e.g.	exempli gratia (for example)
i.e.	id est (that is)

Appendix A. Fixed-Point Theorem and Main Properties of Local and Tight Operators

To describe the properties of local and tight operators, in particular, to formulate the fixed-point theorem used to prove the main result of this paper (Theorem 1). Recall

that local, tight and tight-ranged operators are defined in Definitions 2 and 3, and that the continuity of operators in the spaces of random points is understood in the sense of convergence in the probability of the sequences of random points. The results of this section are proven in [40], unless the proofs are explicitly offered below.

Proposition A1. *The following statements are straightforward:*

1. *A finite linear combination of local operators is local.*
2. *A composition of local operators is local.*

Proposition A2. *Let $F : X \times \Omega \to Y$ be a Carathéodory function, where X, Y are separable metric spaces, and define the superposition operator H_F by $x(\omega) \mapsto F(x(\omega), \omega)$.*

1. *The operator $H_F : \mathcal{P}(X) \to \mathcal{P}(Y)$ is local and uniformly continuous on tight subsets of $\mathcal{P}(X)$; in particular, H_F is continuous.*
2. *The operator H_F maps tight subsets of $\mathcal{P}(X)$ to tight subsets of $\mathcal{P}(Y)$.*
3. *If, in addition, $F(\cdot, \omega) : X \to Y$ is a.s. compact (resp., compact-ranged), then the operator $H_F : \mathcal{P}(X) \to \mathcal{P}(Y)$ is tight (resp., tight-ranged).*

Proposition A3. *Consider the linear integral operators $I^\alpha : L^1_{\mathcal{T}} \to L^1_{\mathcal{T}}$ and $J^\beta : \mathcal{L}^2_{\mathcal{T}} \to \mathcal{L}^1_{\mathcal{T}}$ from **Propositions 1** and 2, respectively.*

1. *For any $\alpha > 0$, the superposition operators $x(\omega) \mapsto I^\alpha(x(\omega), \omega)$ are local and tight as operators acting in the space $\mathcal{L}^1_{\mathcal{T}}$.*
2. *The operators J^β are local and tight for any $\beta > \frac{1}{2}$.*

Proof. The first statement follows from Propositions 1 and A2. The second statement follows from Proposition 2 and the property of locality of the Itô integral [40] [Ex. D.7]. □

Recall that $B_r = \{x \in X : \|x\|_X \le r\}$. In the next proposition, $X = L^p_{\mathcal{T}}$.

Proposition A4. *If a sequence of local and tight operators $H_\nu : \mathcal{L}^p_{\mathcal{T}} \to \mathcal{P}(Y)$, where $1 \le p < \infty$ and Y is a separable Banach space, converges to an operator H, and the convergence is uniform on any subset $\mathcal{L}^p_{\mathcal{T}} \cap \mathcal{P}(B_r)$ $(r > 0)$, then H is also local and tight.*

The space $\mathcal{L}^p_{\mathcal{T}}$ can be replaced by more general spaces; see [40] [Prop. C.2].

Let $\mathcal{B}^* = (\Omega^*, \mathcal{F}^*, (\mathcal{F}^*_t)_{t\in\mathcal{T}}, P^*)$ be any acceptable expansion of the stochastic basis (2) and $\mathfrak{c} : \Omega^* \to \Omega$ be the corresponding measurable projection.

Proposition A5. *Let $\mathcal{A}$ denote either $\mathcal{L}^p_{\mathcal{T}}$ $(1 \le p < \infty)$ or $\mathcal{C}_{\mathcal{T}}$ and an operator $H : \mathcal{A} \to \mathcal{A}$ be local and uniformly continuous on tight subsets of $\mathcal{A}$. Then, for any acceptable expansion $\mathcal{B}^*$ of the stochastic basis $\mathcal{B}$ (see Definition 4), there exists a unique local and continuous operator $H^* : \mathcal{A}^* \to \mathcal{A}^*$ that extends the operator H. The extension H^* is uniformly continuous on tight subsets of the set $\mathcal{A}^*$, where $\mathcal{A}^*$ is either $\mathcal{L}^{p,*}_{\mathcal{T}}$ $(1 \le p < \infty)$ or $\mathcal{C}^*_{\mathcal{T}}$, respectively.*

Proposition A6.

1. *If $H_F : \mathcal{P}(X) \to \mathcal{P}(Y)$ is the superposition operator generated by a random Carathéodory map $F(\cdot, \cdot) : \Omega \times X \to Y$, and X and Y are separable metric spaces, then the unique local and continuous extension of H_F to the space of all random points $x : \Omega^* \to X$ is given by $(H^*_F u)(\omega^*) = F(c(\omega^*), u(\omega^*))$.*

2. *If $\mathcal{B}^*$ is an acceptable expansion, $\mathcal{A} = \mathcal{L}^2_T$ and $\beta > \frac{1}{2}$, then the stochastic operators $J^\beta : \mathcal{L}^2_T \to \mathcal{L}^2_T$ admit the unique local and continuous extension to the space $\mathcal{A}^*$ given by*

$$(J^{\beta,*}u)(s) = \int_{t_0}^{t} (t-s)^{\beta-1} u(s) dW^*(s),$$

where $W^*(t, \omega^*) = W(t, \mathfrak{c}(\omega^*))$ is the standard scalar Wiener process on the stochastic basis $\mathcal{B}^*$.

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