

Article

Results on Solution Set in Certain Interval-Valued Controlled Models

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Abstract: In this paper, a class of controlled variational control models is studied by considering the notion of $(q, w) - \pi$ -invexity. Our aim is to investigate a solution set in the considered interval-valued controlled models. To achieve this, we establish some characterization results of solutions in the controlled interval-valued variational models. More precisely, necessary and sufficient conditions of optimality are highlighted as part of a feasible solution. To prove that the optimality conditions are sufficient, we impose generalized invariant convexity hypotheses for the involved multiple integral functionals. Finally, a duality result is provided in order to better describe the problem under study. The methodology used in this paper is a combination of techniques from the Lagrange–Hamilton theory, calculus of variations, and control theory. This study could be immediately improved by including an analysis of this class of optimization problems by using curvilinear integrals instead of multiple integrals. The independence of path imposed to these functionals and their physical significance would increase the applicability and importance of the paper.

Keywords: controlled variational models; optimal pair; $(q, w) - \pi$ -invexity; dual problem; feasible solution

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1. Introduction

Considering interval-valued functions or functionals and the corresponding analysis may result in easier models which will provide satisfactory accuracy results in practical situations. In this regard, we mention the research works of Charnes et al. [1], Steur [2], and Urli and Nadeau [3]. Hanson and Mond [4] established the necessary and sufficient conditions of optimality in constrained optimization. Chanas and Kuchta [5] considered multiobjective programming with interval-valued objective functions. Stancu-Minasian [6,7] studied multiobjective programming models with inexactness in the objective functions and, also, various approaches in multiple-objective fractional programming governed by set coefficients in the objectives. Jayswal et al. [8] formulated sufficiency and duality theorems in some programming problems that had interval values. Along the same lines, Treanță [9] studied the efficiency of multi-dimensional variational control problems with uncertain data. Also, Treanță [10] obtained results on solutions in interval-valued extremization problems with equality and inequality constraints. For other connected ideas, the reader

can read the works of Peng et al. [11], Wang et al. [12], Xi et al. [13], Saeed and Treanță [14], Treanță and Saeed [15].

Convexity and its generalizations play an important role in various aspects of mathematical programming, including sufficient optimality conditions and duality results (see Hanson [16], Mond and Hanson [17], Arrow and Enthoven [18], Craven [19], Jeyakumar [20], Antczak [21]). Later, Wu [22,23] studied Wolfe-type duality theorems associated with an interval-valued programming model. Mandal and Nahak [24] established duality theorems (weak and strong) under invexity assumptions. Zhang et al. [25] analyzed optimality conditions of KKT type for generalized convex optimization with interval-valued objectives. For further information on the various different ideas explored in this research area, the reader is directed to [26,27].

In this paper, based on the works of Mandal and Nahak [24], Ahmad et al. [28], and Treanță and Ciontescu [29], we establish necessary and sufficient optimality conditions for a class of multi-dimensional interval-valued controlled models involving generalized invex multiple integral functionals. Moreover, a duality theorem is presented in order to connect the studied model with a new variational problem. The study of such classes of interval-valued control problems, by considering generalized invex multiple integral functionals, represents the main novelty element. The main theorems derived in the paper are new in the specialized literature. The limitations associated with the existing papers and the principal novelties of this study are (i) the appearance of mixed-type constraints formulated in terms of partial differential equations; (ii) the appearance of control variables in the objective and constraint-type functionals; (iii) the use of Lagrange–Hamilton techniques to investigate the considered variational control models.

The rest of the current article is divided into the following sections: Section 2 provides the notations and basic elements such as the notion of generalized invex multiple integral functionals and the formulation of the controlled model. Also, the associated necessary Karush–Kuhn–Tucker-type optimality conditions are formulated. In Section 3, we provide some characterization results of solutions for the considered controlled model, under generalized invexity assumptions of the involved functionals. In Section 4, we formulate the final conclusions.

2. Preliminaries

Here, we establish the notations and elements used to formulate the main results stated in the present study.

Let I be the family of closed bounded real intervals. If $V \in I$, then $V = [o^L, o^U]$, with o^L and o^U as the lower and upper bounds for V , respectively. For $o^L = o^U = o$, we obtain $V = [o, o] = o$ as a real scalar. If $V = [o^L, o^U]$, $W = [b^L, b^U] \in I$, we define

- (i) $V + W = \{o + b : o \in V \text{ and } b \in W\} = [o^L + b^L, o^U + b^U]$,
- (ii) $-V = \{-o : o \in V\} = [-o^U, -o^L]$.

We notice that $V - W = V + (-W) = [o^L - b^U, o^U - b^L]$. We also have

- (i) $r + V = \{r + o : o \in V\} = [r + o^L, r + o^U]$,
- (ii) $rV = \{ro : o \in V\} = \begin{cases} [ro^L, ro^U] & \text{if } r \geq 0 \\ [ro^U, ro^L] & \text{if } r < 0, \end{cases} \quad \text{where } r \in \mathbb{R}.$

Let $\mathbb{R}^n$ denote the real n -dimensional Euclidean space. The function $Q : \mathbb{R}^n \times \mathbb{R}^p \rightarrow I$ is an interval-valued continuously differentiable functional; that is, $Q(\mu, u) = Q(\mu_1, \mu_2, \dots, \mu_n, u_1, u_2, \dots, u_p)$ is a closed bounded real interval, for each piecewise smooth state function $\mu : \Theta \subset \mathbb{R}^y \rightarrow \mathbb{R}^n$, $\mu = \mu(t) \in \mathbb{R}^n$, and piecewise continuous control function $u : \Theta \rightarrow \mathbb{R}^p$, $u = u(t) \in \mathbb{R}^p$. The interval-valued functional Q can be formulated

as $Q(\mu, u) = [Q^L(\mu, u), Q^U(\mu, u)]$, with $Q^L(\mu, u), Q^U(\mu, u)$ two real-valued functionals defined on $\mathbb{R}^n \times \mathbb{R}^p$ and satisfy the condition $Q^L(\mu, u) \leq Q^U(\mu, u)$, for each $\mu = \mu(t) \in \mathbb{R}^n$ and $u = u(t) \in \mathbb{R}^p$, with $t = (t^1, \dots, t^y) \in \Theta$. Let $\mathcal{X}$ be the space of piecewise smooth state functions $\mu : \Theta \rightarrow \mathbb{R}^n$ such that $\mu|_{\partial\Theta} = \text{given}$ and consider it is equipped with the norm $\|\mu\| = \|\mu\|_\infty + \|\mu_\xi\|_\infty$, $\mu_\xi := \frac{\partial \mu}{\partial t^\xi}$, $\xi = \overline{1, y}$. Also, let $\mathcal{U}$ be the space of piecewise continuous control functions $u : \Theta \rightarrow \mathbb{R}^p$, endowed with the uniform norm, as well.

Next, for $V = [o^L, o^U]$ and $W = [b^L, b^U]$, we use the partial ordering $V \leq_{LU} W$ if and only if $o^L \leq b^L$ and $o^U \leq b^U$. Also, we write $V <_{LU} W$ if and only if $V \leq_{LU} W$ but $V \neq W$. This means that $V <_{LU} W$ if and only if

$$\begin{aligned} o^L < b^L, \quad o^U \leq b^U, \quad \text{or} \\ o^L \leq b^L, \quad o^U < b^U, \quad \text{or} \\ o^L < b^L, \quad o^U < b^U. \end{aligned}$$

Next, in accordance with Mandal and Nahak [24], we introduce the following definitions. We use the following notations: $f_\mu(t, \theta, k) := \frac{\partial f}{\partial \mu}(t, \theta, k)$, $f_u(t, \theta, k) := \frac{\partial f}{\partial u}(t, \theta, k)$, $\mathbf{1}_\mu := (1, 1, \dots, 1) \in \mathbb{R}^n$, $\mathbf{1}_u := (1, 1, \dots, 1) \in \mathbb{R}^p$, $Q(\mu, u) := \int_\Theta f(t, \mu(t), u(t)) dt^1 \dots dt^y$, with $f : \Theta \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$, $Q : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$.

The following definitions will be used in future work in order to formulate and prove the main results derived in the present paper.

Definition 1. Consider q, w are two real scalars and $f : \Theta \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ is a smooth function. If there exist the functions $\tau : (\mathbb{R}^n \times \mathbb{R}^p)^2 \rightarrow \mathbb{R}^n$, $v : (\mathbb{R}^n \times \mathbb{R}^p)^2 \rightarrow \mathbb{R}^p$, $\sigma : (\mathbb{R}^n \times \mathbb{R}^p)^2 \rightarrow \mathbb{R}^n$ and the real scalar $\pi \in \mathbb{R}$, with $\tau|_{\partial\Theta} = v|_{\partial\Theta} = 0$, such that the following inequalities

$$\begin{aligned} \frac{1}{w} \left(e^{w(Q(\mu, u) - Q(\theta, k))} - 1 \right) (>) &\geq \frac{1}{q} \int_\Theta f_\mu(t, \theta, k) \left(e^{q\tau(\mu, u, \theta, k)} - \mathbf{1}_\mu \right) dt^1 \dots dt^y \\ &+ \frac{1}{q} \int_\Theta f_u(t, \theta, k) \left(e^{qv(\mu, u, \theta, k)} - \mathbf{1}_u \right) dt^1 \dots dt^y + \pi \|\sigma(\mu, u, \theta, k)\|^2, \quad \text{for } q \neq 0, w \neq 0, \\ \frac{1}{w} \left(e^{w(Q(\mu, u) - Q(\theta, k))} - 1 \right) (>) &\geq \int_\Theta f_\mu(t, \theta, k) \tau(\mu, u, \theta, k) dt^1 \dots dt^y \\ &+ \int_\Theta f_u(t, \theta, k) v(\mu, u, \theta, k) dt^1 \dots dt^y + \pi \|\sigma(\mu, u, \theta, k)\|^2, \quad \text{for } q = 0, w \neq 0, \\ Q(\mu, u) - Q(\theta, k) (>) &\geq \frac{1}{q} \int_\Theta f_\mu(t, \theta, k) \left(e^{q\tau(\mu, u, \theta, k)} - \mathbf{1}_\mu \right) dt^1 \dots dt^y \\ &+ \frac{1}{q} \int_\Theta f_u(t, \theta, k) \left(e^{qv(\mu, u, \theta, k)} - \mathbf{1}_u \right) dt^1 \dots dt^y + \pi \|\sigma(\mu, u, \theta, k)\|^2, \quad \text{for } q \neq 0, w = 0, \\ Q(\mu, u) - Q(\theta, k) (>) &\geq \int_\Theta f_\mu(t, \theta, k) \tau(\mu, u, \theta, k) dt^1 \dots dt^y \\ &+ \int_\Theta f_u(t, \theta, k) v(\mu, u, \theta, k) dt^1 \dots dt^y \\ &+ \pi \|\sigma(\mu, u, \theta, k)\|^2, \quad \text{for } q = 0, w = 0 \end{aligned}$$

are satisfied, then the real-valued functional Q is named (strictly) $(q, w) - \pi$ -invex at (θ, k) on $\mathcal{X} \times \mathcal{U}$ with respect to τ, v and σ .

Definition 2. Consider q, w are two real scalars and $f : \Theta \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ is a smooth function. If there exist the functions $\tau : (\mathbb{R}^n \times \mathbb{R}^p)^2 \rightarrow \mathbb{R}^n$, $\nu : (\mathbb{R}^n \times \mathbb{R}^p)^2 \rightarrow \mathbb{R}^p$, $\sigma : (\mathbb{R}^n \times \mathbb{R}^p)^2 \rightarrow \mathbb{R}^n$ and the real scalar $\pi \in \mathbb{R}$, with $\tau|_{\partial\Theta} = \nu|_{\partial\Theta} = 0$, such that the following inequalities

$$\begin{aligned} & \frac{1}{q} \int_{\Theta} f_{\mu}(t, \theta, k) \left(e^{q\tau(\mu, u, \theta, k)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \int_{\Theta} f_u(t, \theta, k) \left(e^{qv(\mu, u, \theta, k)} - \mathbf{1}_u \right) dt^1 \dots dt^y + \pi \|\sigma(\mu, u, \theta, k)\|^2 \geq 0 \\ & \Rightarrow \frac{1}{w} \left(e^{w(\mathbf{Q}(\mu, u) - \mathbf{Q}(\theta, k))} - 1 \right) (>) \geq 0, \quad \text{for } q \neq 0, w \neq 0, \\ & \int_{\Theta} f_{\mu}(t, \theta, k) \tau(\mu, u, \theta, k) dt^1 \dots dt^y \\ & + \int_{\Theta} f_u(t, \theta, k) \nu(\mu, u, \theta, k) dt^1 \dots dt^y + \pi \|\sigma(\mu, u, \theta, k)\|^2 \geq 0 \\ & \Rightarrow \frac{1}{w} \left(e^{w(\mathbf{Q}(\mu, u) - \mathbf{Q}(\theta, k))} - 1 \right) (>) \geq 0, \quad \text{for } q = 0, w \neq 0, \\ & \frac{1}{q} \int_{\Theta} f_{\mu}(t, \theta, k) \left(e^{q\tau(\mu, u, \theta, k)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \int_{\Theta} f_u(t, \theta, k) \left(e^{qv(\mu, u, \theta, k)} - \mathbf{1}_u \right) dt^1 \dots dt^y + \pi \|\sigma(\mu, u, \theta, k)\|^2 \geq 0 \\ & \Rightarrow \mathbf{Q}(\mu, u) - \mathbf{Q}(\theta, k) (>) \geq 0, \quad \text{for } q \neq 0, w = 0, \\ & \int_{\Theta} f_{\mu}(t, \theta, k) \tau(\mu, u, \theta, k) dt^1 \dots dt^y + \int_{\Theta} f_u(t, \theta, k) \nu(\mu, u, \theta, k) dt^1 \dots dt^y \\ & + \pi \|\sigma(\mu, u, \theta, k)\|^2 \geq 0 \\ & \Rightarrow \mathbf{Q}(\mu, u) - \mathbf{Q}(\theta, k) (>) \geq 0, \quad \text{for } q = 0, w = 0 \end{aligned}$$

are satisfied, then the real-valued functional $\mathbf{Q}$ is named (strictly) $(q, w) - \pi$ -pseudoinvex at (θ, k) on $\mathcal{X} \times \mathcal{U}$ with respect to τ, ν and σ .

Definition 3. Consider q, w are two real scalars and $f : \Theta \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ is a smooth function. If there exist the functions $\tau : (\mathbb{R}^n \times \mathbb{R}^p)^2 \rightarrow \mathbb{R}^n$, $\nu : (\mathbb{R}^n \times \mathbb{R}^p)^2 \rightarrow \mathbb{R}^p$, $\sigma : (\mathbb{R}^n \times \mathbb{R}^p)^2 \rightarrow \mathbb{R}^n$ and the real scalar $\pi \in \mathbb{R}$, with $\tau|_{\partial\Theta} = \nu|_{\partial\Theta} = 0$, such that the following inequalities

$$\begin{aligned} & \frac{1}{w} \left(e^{w(\mathbf{Q}(\mu, u) - \mathbf{Q}(\theta, k))} - 1 \right) \leq 0 \Rightarrow \frac{1}{q} \int_{\Theta} f_{\mu}(t, \theta, k) \left(e^{q\tau(\mu, u, \theta, k)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \int_{\Theta} f_u(t, \theta, k) \left(e^{qv(\mu, u, \theta, k)} - \mathbf{1}_u \right) dt^1 \dots dt^y \leq -\pi \|\sigma(\mu, u, \theta, k)\|^2, \quad \text{for } q \neq 0, w \neq 0, \\ & \frac{1}{w} \left(e^{w(\mathbf{Q}(\mu, u) - \mathbf{Q}(\theta, k))} - 1 \right) \leq 0 \Rightarrow \int_{\Theta} f_{\mu}(t, \theta, k) \tau(\mu, u, \theta, k) dt^1 \dots dt^y \\ & + \int_{\Theta} f_u(t, \theta, k) \nu(\mu, u, \theta, k) dt^1 \dots dt^y \leq -\pi \|\sigma(\mu, u, \theta, k)\|^2, \quad \text{for } q = 0, w \neq 0, \\ & \mathbf{Q}(\mu, u) - \mathbf{Q}(\theta, k) \leq 0 \Rightarrow \frac{1}{q} \int_{\Theta} f_{\mu}(t, \theta, k) \left(e^{q\tau(\mu, u, \theta, k)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \int_{\Theta} f_u(t, \theta, k) \left(e^{qv(\mu, u, \theta, k)} - \mathbf{1}_u \right) dt^1 \dots dt^y \leq -\pi \|\sigma(\mu, u, \theta, k)\|^2, \quad \text{for } q \neq 0, w = 0, \\ & \mathbf{Q}(\mu, u) - \mathbf{Q}(\theta, k) \leq 0 \Rightarrow \int_{\Theta} f_{\mu}(t, \theta, k) \tau(\mu, u, \theta, k) dt^1 \dots dt^y \\ & + \int_{\Theta} f_u(t, \theta, k) \nu(\mu, u, \theta, k) dt^1 \dots dt^y \\ & \leq -\pi \|\sigma(\mu, u, \theta, k)\|^2, \quad \text{for } q = 0, w = 0 \end{aligned}$$

are satisfied, then the real-valued functional $\mathbf{Q}$ is called $(q, w) - \pi$ -quasi-invex at (θ, k) on $\mathcal{X} \times \mathcal{U}$ with respect to τ, ν and σ .

Remark 1. The exponentials used in the above-mentioned inequalities are considered component-wise. Also, without loss of generality, in the rest of this paper, we assume that $w > 0, q > 0$.

Example 1. Consider $\Theta = [0, 1], f(t, \mu, u) = \ln(\mu^2 + 1) + t\mu^2 + \mu, q = 0, w = 1, \tau(\mu, u, \theta, k) = \mu - \theta, v(\mu, u, \theta, k) = u - k, \sigma(\mu, u, \theta, k) = 0, \pi \in \mathbb{R}$. By direct computation, we obtain the real-valued functional $Q(\mu, u) := \int_{\Theta} f(t, \mu(t), u(t)) dt$ is $(q, w) - \pi$ -invex at $(\theta = 0, k = 0)$ with respect to τ, v and σ .

In this study, we investigate the primal optimization problem (see also Treanță [9]) driven by an interval-valued cost functional:

$$(P) \quad \min_{(\mu, u) \in \mathcal{X} \times \mathcal{U}} \left\{ Q(\mu, u) = [Q^L(\mu, u), Q^U(\mu, u)] \right\}$$

subject to

$$g_{\epsilon}(t, \mu, u) \leq 0, \quad t \in \Theta, \epsilon = 1, 2, \dots, m$$

$$(\mu_{\xi})_i = h_i^{\xi}(t, \mu, u), \quad t \in \Theta, i = 1, 2, \dots, n, \xi = 1, 2, \dots, y$$

$$\mu|_{\partial\Theta} = \text{given},$$

where $Q : \mathcal{X} \times \mathcal{U} \rightarrow I$ is a C^1 -class interval-valued functional,

$$Q^L(\mu, u) := \int_{\Theta} f^L(t, \mu(t), u(t)) dt^1 \dots dt^y,$$

$$Q^U(\mu, u) := \int_{\Theta} f^U(t, \mu(t), u(t)) dt^1 \dots dt^y,$$

with $f : \Theta \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow I, f = [f^L, f^U]$, a differentiable interval-valued function, $g_{\epsilon} : \Theta \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}, \epsilon = 1, 2, \dots, m, h_i^{\xi} : \Theta \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}, i = 1, 2, \dots, n, \xi = 1, 2, \dots, y$ some given differentiable real-valued functions.

Let Ω be the set of all feasible solutions of (P).

Definition 4 (Treanță [9]). The pair $(\mu^*, u^*) \in \Omega$ is an LU-optimal pair of problem (P) if there exists no $(\mu_0, u_0) \in \Omega$ satisfying $Q(\mu_0, u_0) <_{LU} Q(\mu^*, u^*)$.

The following result presents the Karush–Kuhn–Tucker-type necessary optimality conditions associated with (P).

Theorem 1. If $(\mu^*, u^*) \in \Omega$ is an LU-optimal pair of primal variational control problems (P) and the constraint functions satisfy constraint qualification at (μ^*, u^*) , then there exist multipliers (functions on Θ) $0 < \gamma^L, \gamma^U \in \mathbb{R}, 0 \leq \zeta_{\epsilon} \in \mathbb{R}, \epsilon = 1, 2, \dots, m$, and $\lambda_i^{\xi} \in \mathbb{R}, i = 1, 2, \dots, n, \xi = 1, 2, \dots, y$, such that the next relations

$$\begin{aligned} & \gamma^L \frac{\partial f^L}{\partial \mu_i}(t, \mu^*, u^*) + \gamma^U \frac{\partial f^U}{\partial \mu_i}(t, \mu^*, u^*) + \sum_{\epsilon=1}^m \zeta_{\epsilon} \frac{\partial g_{\epsilon}}{\partial \mu_i}(t, \mu^*, u^*) \\ & + \sum_{i=1}^n \sum_{\xi=1}^y \lambda_i^{\xi} \frac{\partial h_i^{\xi}}{\partial \mu_i}(t, \mu^*, u^*) + \frac{\partial \lambda_i^{\xi}}{\partial t^{\xi}}(t) = 0, \quad i = \overline{1, n} \end{aligned} \quad (1)$$

$$\begin{aligned} & \gamma^L \frac{\partial f^L}{\partial u_s}(t, \mu^*, u^*) + \gamma^U \frac{\partial f^U}{\partial u_s}(t, \mu^*, u^*) + \sum_{\epsilon=1}^m \zeta_{\epsilon} \frac{\partial g_{\epsilon}}{\partial u_s}(t, \mu^*, u^*) \\ & + \sum_{i=1}^n \sum_{\xi=1}^y \lambda_i^{\xi} \frac{\partial h_i^{\xi}}{\partial u_s}(t, \mu^*, u^*) = 0, \quad s = \overline{1, p} \end{aligned} \quad (2)$$

$$\zeta_{\epsilon} g_{\epsilon}(t, \mu^*, u^*) = 0, \quad \epsilon = \overline{1, m} \quad (3)$$

are fulfilled for all $t \in \Theta$, except at discontinuities.

Proof. The proof follows in the same manner as in Theorem 6.1 of Treanță [30]. $\square$

3. On Solution Set for (P)

In the present section, we formulate and prove some characterization results of solutions for (P).

Next, by considering only the $(q, w) - \pi$ -invexity assumptions of the involved functionals, we state sufficient conditions of optimality for (P).

Theorem 2. Let $(\mu^*, u^*) \in \Omega$, the constraint functions satisfy the suitable Kuhn–Tucker constraint qualification at (μ^*, u^*) , and the relations (1)–(3) from Theorem 1 are satisfied for $t \in \Theta$, except at discontinuities. Moreover, we assume that Q^L and Q^U are $(q, w) - \pi_1$ -invex and $(q, w) - \pi_2$ -invex at (μ^*, u^*) with respect to τ, ν and σ , respectively, and $G(\mu, u) := \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu, u) dt^1 \cdots dt^y$ and $H(\mu, u) := \int_{\Theta} \sum_{i=1}^n \sum_{\xi=1}^y \lambda_i^{\xi} [h_i^{\xi}(t, \mu, u) - (\mu_{\xi})_i] dt^1 \cdots dt^y$ are $(q, w) - \pi_3$ -invex and $(q, w) - \pi_4$ -invex at (μ^*, u^*) with respect to τ, ν and σ , respectively. If $\gamma^L \pi_1 + \gamma^U \pi_2 + \pi_3 + \pi_4 \geq 0$, then (μ^*, u^*) is an LU-optimal pair to the problem (P).

Proof. We assume that (μ^*, u^*) is not an LU-optimal pair of (P). This fact implies there exists $(\mu_0, u_0) \in \Omega$, such that

$$Q(\mu_0, u_0) <_{LU} Q(\mu^*, u^*),$$

that is,

$$\begin{aligned} & \begin{cases} Q^L(\mu_0, u_0) < Q^L(\mu^*, u^*) \\ Q^U(\mu_0, u_0) < Q^U(\mu^*, u^*), \end{cases} \quad \text{or} \quad \begin{cases} Q^L(\mu_0, u_0) \leq Q^L(\mu^*, u^*) \\ Q^U(\mu_0, u_0) < Q^U(\mu^*, u^*), \end{cases} \\ & \text{or} \quad \begin{cases} Q^L(\mu_0, u_0) < Q^L(\mu^*, u^*) \\ Q^U(\mu_0, u_0) \leq Q^U(\mu^*, u^*). \end{cases} \end{aligned}$$

Since we have $w > 0$, by using the properties of exponential functions, we get

$$\begin{aligned} & \begin{cases} \frac{1}{w} [e^{w\{Q^L(\mu_0, u_0) - Q^L(\mu^*, u^*)\}} - 1] < 0 \\ \frac{1}{w} [e^{w\{Q^U(\mu_0, u_0) - Q^U(\mu^*, u^*)\}} - 1] < 0, \end{cases} \quad \text{or} \\ & \begin{cases} \frac{1}{w} [e^{w\{Q^L(\mu_0, u_0) - Q^L(\mu^*, u^*)\}} - 1] \leq 0 \\ \frac{1}{w} [e^{w\{Q^U(\mu_0, u_0) - Q^U(\mu^*, u^*)\}} - 1] < 0, \end{cases} \quad \text{or} \\ & \begin{cases} \frac{1}{w} [e^{w\{Q^L(\mu_0, u_0) - Q^L(\mu^*, u^*)\}} - 1] < 0 \\ \frac{1}{w} [e^{w\{Q^U(\mu_0, u_0) - Q^U(\mu^*, u^*)\}} - 1] \leq 0. \end{cases} \end{aligned}$$

Using the $(q, w) - \pi_1$ -invexity property of Q^L and the $(q, w) - \pi_2$ -invexity property of Q^U with respect to τ, ν and σ at (μ^*, u^*) at (μ^*, u^*) , the above inequalities imply

$$\begin{aligned} & \frac{1}{q} \int_{\Theta} f_{\mu}^L(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \cdots dt^y \\ & + \frac{1}{q} \int_{\Theta} f_{\nu}^L(t, \mu^*, u^*) \left(e^{q\nu(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\nu} \right) dt^1 \cdots dt^y + \pi_1 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 < 0, \\ & \frac{1}{q} \int_{\Theta} f_{\mu}^U(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \cdots dt^y \end{aligned}$$

$$+\frac{1}{q} \int_{\Theta} f_u^U(t, \mu^*, u^*) \left(e^{q\nu(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y + \pi_2 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 < 0,$$

or

$$\begin{aligned} & \frac{1}{q} \int_{\Theta} f_{\mu}^L(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \int_{\Theta} f_u^L(t, \mu^*, u^*) \left(e^{q\nu(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y + \pi_1 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 \leq 0, \\ & \frac{1}{q} \int_{\Theta} f_{\mu}^U(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \int_{\Theta} f_u^U(t, \mu^*, u^*) \left(e^{q\nu(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y + \pi_2 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 < 0, \end{aligned}$$

or

$$\begin{aligned} & \frac{1}{q} \int_{\Theta} f_{\mu}^L(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \int_{\Theta} f_u^L(t, \mu^*, u^*) \left(e^{q\nu(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y + \pi_1 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 < 0, \\ & \frac{1}{q} \int_{\Theta} f_{\mu}^U(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \int_{\Theta} f_u^U(t, \mu^*, u^*) \left(e^{q\nu(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y + \pi_2 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 \leq 0. \end{aligned}$$

Since $\gamma^L > 0$ and $\gamma^U > 0$, from the above inequalities, we get

$$\begin{aligned} & \frac{1}{q} \gamma^L \int_{\Theta} f_{\mu}^L(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \gamma^L \int_{\Theta} f_u^L(t, \mu^*, u^*) \left(e^{q\nu(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y + \gamma^L \pi_1 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 \\ & + \frac{1}{q} \gamma^U \int_{\Theta} f_{\mu}^U(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \gamma^U \int_{\Theta} f_u^U(t, \mu^*, u^*) \left(e^{q\nu(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y \\ & + \gamma^U \pi_2 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 < 0. \end{aligned} \quad (4)$$

Next, by considering the feasibility property of (μ_0, u_0) to (P), we obtain

$$g_{\epsilon}(t, \mu_0, u_0) \leq 0, \quad t \in \Theta, \quad \epsilon = \overline{1, m}.$$

Since we have $\zeta_{\epsilon} \geq 0$, $\epsilon = \overline{1, m}$, the relation (3) and the above inequality involve

$$\sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu_0, u_0) \leq \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu^*, u^*), \quad t \in \Theta,$$

which implies

$$\int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu_0, u_0) dt^1 \dots dt^y \leq \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu^*, u^*) dt^1 \dots dt^y, \quad t \in \Theta,$$

Since we have $w > 0$, by using the properties of exponential functions, we obtain

$$\frac{1}{w} \left[e^{w \left\{ \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu_0, u_0) dt^1 \dots dt^y - \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu^*, u^*) dt^1 \dots dt^y \right\}} - 1 \right] \leq 0,$$

and by considering that $G(\mu, u)$ is $(q, w) - \pi_3$ -invex at (μ^*, u^*) , with respect to τ, v and σ , we get

$$\begin{aligned} & \frac{1}{q} \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} \frac{\partial g_{\epsilon}}{\partial \mu}(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} \frac{\partial g_{\epsilon}}{\partial u}(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y \\ & + \pi_3 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 \leq 0. \end{aligned} \quad (5)$$

In a similar way, by considering the assumption that $H(\mu, u)$ is $(q, w) - \pi_4$ -invex at (μ^*, u^*) , with respect to τ, v and σ , we get

$$\begin{aligned} & \frac{1}{q} \int_{\Theta} \sum_{i=1}^n \sum_{\xi=1}^y \lambda_i^{\xi} \frac{\partial (h_i^{\xi} - (\mu_{\xi})_i)}{\partial \mu}(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \int_{\Theta} \sum_{i=1}^n \sum_{\xi=1}^y \lambda_i^{\xi} \frac{\partial (h_i^{\xi} - (\mu_{\xi})_i)}{\partial u}(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y \\ & + \pi_4 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 \leq 0. \end{aligned} \quad (6)$$

On adding (4), (5) and (6), from the hypothesis that $(\gamma^L \pi_1 + \gamma^U \pi_2 + \pi_3 + \pi_4) \geq 0$, and by considering (1), we get a contradiction. This completes the proof. $\square$

The next result, by considering the $(q, w) - \pi$ -pseudoinvexity and quasi-invexity assumptions of the involved functionals, introduces sufficient conditions of optimality for (P).

Theorem 3. Let $(\mu^*, u^*) \in \Omega$, the constraint functions satisfy the suitable Kuhn–Tucker constraint qualification at (μ^*, u^*) , and the relations (1)–(3) from Theorem 1 are satisfied for $t \in \Theta$, except at discontinuities. Moreover, we consider $\gamma^L Q^L + \gamma^U Q^U$ is $(q, w) - \pi_1$ -pseudoinvex at (μ^*, u^*) with respect to τ, v and σ , and $G(\mu, u) := \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu, u) dt^1 \dots dt^y$ and $H(\mu, u) := \int_{\Theta} \sum_{i=1}^n \sum_{\xi=1}^y \lambda_i^{\xi} [h_i^{\xi}(t, \mu, u) - (\mu_{\xi})_i] dt^1 \dots dt^y$ are $(q, w) - \pi_2$ -quasi-invex and $(q, w) - \pi_3$ -quasi-invex at (μ^*, u^*) with respect to τ, v and σ , respectively. If $\pi_1 + \pi_2 + \pi_3 \geq 0$, then (μ^*, u^*) is an LU-optimal pair for the problem (P).

Proof. Consider (μ^*, u^*) is not an LU-optimal pair of (P) and, consequently, there exists $(\mu_0, u_0) \in \Omega$, such that

$$Q(\mu_0, u_0) <_{LU} Q(\mu^*, u^*),$$

that is,

$$\begin{aligned} & \left\{ \begin{array}{l} Q^L(\mu_0, u_0) < Q^L(\mu^*, u^*) \\ Q^U(\mu_0, u_0) < Q^U(\mu^*, u^*), \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{l} Q^L(\mu_0, u_0) \leq Q^L(\mu^*, u^*) \\ Q^U(\mu_0, u_0) < Q^U(\mu^*, u^*), \end{array} \right\} \\ & \quad \text{or} \quad \left\{ \begin{array}{l} Q^L(\mu_0, u_0) < Q^L(\mu^*, u^*) \\ Q^U(\mu_0, u_0) \leq Q^U(\mu^*, u^*). \end{array} \right\}. \end{aligned}$$

Since $\gamma^L > 0$ and $\gamma^U > 0$, from the above inequalities, we have

$$\gamma^L Q^L(\mu_0, u_0) + \gamma^U Q^U(\mu_0, u_0) < \gamma^L Q^L(\mu^*, u^*) + \gamma^U Q^U(\mu^*, u^*),$$

or, equivalently,

$$\begin{aligned} & \int_{\Theta} [\gamma^L f^L(t, \mu_0, u_0) + \gamma^U f^U(t, \mu_0, u_0)] dt^1 \dots dt^y \\ & < \int_{\Theta} [\gamma^L f^L(t, \mu^*, u^*) + \gamma^U f^U(t, \mu^*, u^*)] dt^1 \dots dt^y. \end{aligned}$$

Since we have $w > 0$, by using the properties of exponential functions, it follows

$$\frac{1}{w} \left[e^{w\{(\gamma^L Q^L(\mu_0, u_0) + \gamma^U Q^U(\mu_0, u_0)) - (\gamma^L Q^L(\mu^*, u^*) + \gamma^U Q^U(\mu^*, u^*))\}} - 1 \right] < 0,$$

which together with the assumption that $\gamma^L Q^L + \gamma^U Q^U$ is $(q, w) - \pi_1$ -pseudoinvex at (μ^*, u^*) , with respect to τ, ν and σ , gives

$$\begin{aligned} & \frac{1}{q} \gamma^L \int_{\Theta} f_{\mu}^L(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \gamma^L \int_{\Theta} f_u^L(t, \mu^*, u^*) \left(e^{q\nu(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y + \pi_1 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 \\ & + \frac{1}{q} \gamma^U \int_{\Theta} f_{\mu}^U(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \gamma^U \int_{\Theta} f_u^U(t, \mu^*, u^*) \left(e^{q\nu(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y < 0. \end{aligned} \quad (7)$$

Next, by considering the feasibility property of (μ_0, u_0) to (P), we get

$$g_{\epsilon}(t, \mu_0, u_0) \leq 0, \quad t \in \Theta, \quad \epsilon = 1, 2, \dots, m.$$

But we have $\zeta_{\epsilon} \geq 0, \epsilon = 1, 2, \dots, m$. The relation (3) and the above inequality yield

$$\sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu_0, u_0) \leq \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu^*, u^*), \quad t \in \Theta,$$

which involves

$$\int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu_0, u_0) dt^1 \dots dt^y \leq \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu^*, u^*) dt^1 \dots dt^y, \quad t \in \Theta.$$

Since we have $w > 0$, by using the properties associated with exponential functions, we get

$$\frac{1}{w} \left[e^{w\{\int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu_0, u_0) dt^1 \dots dt^y - \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu^*, u^*) dt^1 \dots dt^y\}} - 1 \right] \leq 0,$$

which, together with the assumption that $G(\mu, u)$ is $(q, w) - \pi_2$ -quasi-invex at (μ^*, u^*) , with respect to τ, ν and σ , involves

$$\begin{aligned} & \frac{1}{q} \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} \frac{\partial g_{\epsilon}}{\partial \mu}(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} \frac{\partial g_{\epsilon}}{\partial u}(t, \mu^*, u^*) \left(e^{q\nu(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y \\ & + \pi_2 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 \leq 0. \end{aligned} \quad (8)$$

In a similar way, by considering the assumption that $H(\mu, u)$ is $(q, w) - \pi_3$ -quasi-invex at (μ^*, u^*) , with respect to τ, ν and σ , we get

$$\begin{aligned} & \frac{1}{q} \int_{\Theta} \sum_{i=1}^n \sum_{\xi=1}^y \lambda_i^{\xi} \frac{\partial (h_i^{\xi} - (\mu_{\xi})_i)}{\partial \mu}(t, \mu^*, u^*) \left(e^{q\tau(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_{\mu} \right) dt^1 \dots dt^y \\ & + \frac{1}{q} \int_{\Theta} \sum_{i=1}^n \sum_{\xi=1}^y \lambda_i^{\xi} \frac{\partial (h_i^{\xi} - (\mu_{\xi})_i)}{\partial u}(t, \mu^*, u^*) \left(e^{q\nu(\mu_0, u_0, \mu^*, u^*)} - \mathbf{1}_u \right) dt^1 \dots dt^y \\ & + \pi_3 \|\sigma(\mu_0, u_0, \mu^*, u^*)\|^2 \leq 0. \end{aligned} \quad (9)$$

On adding (7), (8) and (9), from the hypothesis that $(\pi_1 + \pi_2 + \pi_3) \geq 0$, and by considering (1), we get a contradiction. This completes the proof. $\square$

Next, we consider the following dual problem associated with the primal optimization problem (P):

$$(D-W) \quad \max_{(\theta,k) \in \mathcal{X} \times \mathcal{U}} \left\{ Q(\theta,k) + G(\theta,k) + H(\theta,k) \right\}$$

subject to

$$\begin{aligned} & \gamma^L \frac{\partial f^L}{\partial \mu_i}(t, \theta, k) + \gamma^U \frac{\partial f^U}{\partial \mu_i}(t, \theta, k) + \sum_{\epsilon=1}^m \zeta_\epsilon \frac{\partial g_\epsilon}{\partial \mu_i}(t, \theta, k) \\ & + \sum_{i=1}^n \sum_{\xi=1}^y \lambda_i^\xi \frac{h_i^\xi}{\partial \mu_i}(t, \theta, k) + \frac{\partial \lambda_i^\xi}{\partial t^\xi}(t) = 0, \quad i = \overline{1, n} \end{aligned} \quad (10)$$

$$\begin{aligned} & \gamma^L \frac{\partial f^L}{\partial u_s}(t, \theta, k) + \gamma^U \frac{\partial f^U}{\partial u_s}(t, \theta, k) + \sum_{\epsilon=1}^m \zeta_\epsilon \frac{\partial g_\epsilon}{\partial u_s}(t, \theta, k) \\ & + \sum_{i=1}^n \sum_{\xi=1}^y \lambda_i^\xi \frac{h_i^\xi}{\partial u_s}(t, \theta, k) = 0, \quad s = \overline{1, p} \end{aligned} \quad (11)$$

$$\theta|_{\partial\Theta} = \text{given}, \quad \gamma^L + \gamma^U = 1, \quad (12)$$

where Q, G, H are the same functionals defined in the previous section.

Definition 5. Let $(\theta^*, k^*, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$ be a feasible solution to the dual problem (D-W). We say that $(\theta^*, k^*, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$ is an LU-optimal point of the dual problem (D-W) if there exists no $(\theta, k, \gamma^{*L}, \gamma^{*U}, \zeta^*, \lambda^*)$, a feasible solution of the dual problem (D-W), such that $Q(\theta^*, k^*) + G(\theta^*, k^*) + H(\theta^*, k^*) <_{LU} Q(\theta, k) + G(\theta, k) + H(\theta, k)$.

Theorem 4. Suppose that (μ, u) and $(\theta, k, \gamma^L, \gamma^U, \zeta, \lambda)$ are some feasible points for (P) and (D-W), respectively. Also, consider that $\gamma^L > 0, \gamma^U > 0$ and $\zeta_\epsilon \geq 0, \epsilon = \overline{1, m}, \lambda_i^\xi \in \mathbb{R}, i = \overline{1, n}$, such that the functional $\gamma^L Q^L + \gamma^U Q^U + G + H$ is $(q, w) - \pi$ -invex at (θ, k) with respect to τ, v and σ , with $\pi \geq 0$. Then,

$$Q(\mu, u) \geq_{LU} Q(\theta, k) + G(\theta, k) + H(\theta, k).$$

Proof. Suppose, contrary to the result, that

$$Q(\mu, u) <_{LU} Q(\theta, k) + G(\theta, k) + H(\theta, k),$$

that is,

$$\begin{cases} Q^L(\mu, u) < Q^L(\theta, k) + \int_{\Theta} \sum_{\epsilon=1}^m \zeta_\epsilon g_\epsilon(t, \theta, k) dt^1 \cdots dt^y + \int_{\Theta} \sum_{i=1}^n \lambda_i [h_i(t, \theta, k) - \dot{\theta}_i] dt^1 \cdots dt^y \\ Q^U(\mu, u) < Q^U(\theta, k) + \int_{\Theta} \sum_{\epsilon=1}^m \zeta_\epsilon g_\epsilon(t, \theta, k) dt^1 \cdots dt^y + \int_{\Theta} \sum_{i=1}^n \lambda_i [h_i(t, \theta, k) - \dot{\theta}_i] dt^1 \cdots dt^y, \end{cases}$$

or

$$\begin{cases} Q^L(\mu, u) < Q^L(\theta, k) + \int_{\Theta} \sum_{\epsilon=1}^m \zeta_\epsilon g_\epsilon(t, \theta, k) dt^1 \cdots dt^y + \int_{\Theta} \sum_{i=1}^n \lambda_i [h_i(t, \theta, k) - \dot{\theta}_i] dt^1 \cdots dt^y \\ Q^U(\mu, u) \leq Q^U(\theta, k) + \int_{\Theta} \sum_{\epsilon=1}^m \zeta_\epsilon g_\epsilon(t, \theta, k) dt^1 \cdots dt^y + \int_{\Theta} \sum_{i=1}^n \lambda_i [h_i(t, \theta, k) - \dot{\theta}_i] dt^1 \cdots dt^y, \end{cases}$$

or

$$\begin{cases} Q^L(\mu, u) \leq Q^L(\theta, k) + \int_{\Theta} \sum_{\epsilon=1}^m \zeta_\epsilon g_\epsilon(t, \theta, k) dt^1 \cdots dt^y + \int_{\Theta} \sum_{i=1}^n \lambda_i [h_i(t, \theta, k) - \dot{\theta}_i] dt^1 \cdots dt^y \\ Q^U(\mu, u) < Q^U(\theta, k) + \int_{\Theta} \sum_{\epsilon=1}^m \zeta_\epsilon g_\epsilon(t, \theta, k) dt^1 \cdots dt^y + \int_{\Theta} \sum_{i=1}^n \lambda_i [h_i(t, \theta, k) - \dot{\theta}_i] dt^1 \cdots dt^y. \end{cases}$$

Since we have $\gamma^L > 0, \gamma^U > 0$ and $\gamma^L + \gamma^U = 1$, the feasibility property of (μ, u) to (P) and the above inequalities imply

$$\begin{aligned} & \gamma^L Q^L(\mu, u) + \gamma^U Q^U(\mu, u) + \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \mu, u) dt^1 \cdots dt^y \\ & \quad + \int_{\Theta} \sum_{i=1}^n \lambda_i [h_i(t, \mu, u) - (\mu_{\xi})_i] dt^1 \cdots dt^y \\ & < \gamma^L Q^L(\theta, k) + \gamma^U Q^U(\theta, k) + \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} g_{\epsilon}(t, \theta, k) dt^1 \cdots dt^y \\ & \quad + \int_{\Theta} \sum_{i=1}^n \lambda_i [h_i(t, \theta, k) - \theta_i] dt^1 \cdots dt^y. \end{aligned} \quad (13)$$

From the assumption that $\gamma^L Q^L + \gamma^U Q^U + G + H$ is $(q, w) - \pi$ -invex at (θ, k) with respect to τ, ν and σ , we have

$$\begin{aligned} & \frac{1}{w} \left[e^{w \left\{ (\gamma^L Q^L(\mu, u) + \gamma^U Q^U(\mu, u) + G(\mu, u) + H(\mu, u)) - (\gamma^L Q^L(\theta, k) + \gamma^U Q^U(\theta, k) + G(\theta, k) + H(\theta, k)) \right\}} - 1 \right] \\ & \geq \frac{1}{q} \gamma^L \int_{\Theta} f_{\mu}^L(t, \theta, k) \left(e^{q\tau(\mu, u, \theta, k)} - \mathbf{1}_{\mu} \right) dt^1 \cdots dt^y \\ & \quad + \frac{1}{q} \gamma^L \int_{\Theta} f_u^L(t, \theta, k) \left(e^{q\nu(\mu, u, \theta, k)} - \mathbf{1}_u \right) dt^1 \cdots dt^y \\ & \quad + \frac{1}{q} \gamma^U \int_{\Theta} f_{\mu}^U(t, \theta, k) \left(e^{q\tau(\mu, u, \theta, k)} - \mathbf{1}_{\mu} \right) dt^1 \cdots dt^y \\ & \quad + \frac{1}{q} \gamma^U \int_{\Theta} f_u^U(t, \theta, k) \left(e^{q\nu(\mu, u, \theta, k)} - \mathbf{1}_u \right) dt^1 \cdots dt^y \\ & \quad + \frac{1}{q} \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} \frac{\partial g_{\epsilon}}{\partial \mu}(t, \theta, k) \left(e^{q\tau(\mu, u, \theta, k)} - \mathbf{1}_{\mu} \right) dt^1 \cdots dt^y \\ & \quad + \frac{1}{q} \int_{\Theta} \sum_{\epsilon=1}^m \zeta_{\epsilon} \frac{\partial g_{\epsilon}}{\partial u}(t, \theta, k) \left(e^{q\nu(\mu, u, \theta, k)} - \mathbf{1}_u \right) dt^1 \cdots dt^y \\ & \quad + \frac{1}{q} \int_{\Theta} \sum_{i=1}^n \sum_{\xi=1}^y \lambda_i^{\xi} \frac{\partial (h_i^{\xi} - (\mu_{\xi})_i)}{\partial \mu}(t, \theta, k) \left(e^{q\tau(\mu, u, \theta, k)} - \mathbf{1}_{\mu} \right) dt^1 \cdots dt^y \\ & \quad + \frac{1}{q} \int_{\Theta} \sum_{i=1}^n \sum_{\xi=1}^y \lambda_i^{\xi} \frac{\partial (h_i^{\xi} - (\mu_{\xi})_i)}{\partial u}(t, \theta, k) \left(e^{q\nu(\mu, u, \theta, k)} - \mathbf{1}_u \right) dt^1 \cdots dt^y \\ & \quad + \pi \|\sigma(\mu, u, \theta, k)\|^2. \end{aligned}$$

The relations (10), (11), $\pi \geq 0$, and the above inequality imply the relation

$$\begin{aligned} & \frac{1}{w} \left[e^{w \left\{ (\gamma^L Q^L(\mu, u) + \gamma^U Q^U(\mu, u) + G(\mu, u) + H(\mu, u)) - (\gamma^L Q^L(\theta, k) + \gamma^U Q^U(\theta, k) + G(\theta, k) + H(\theta, k)) \right\}} - 1 \right] \\ & \geq 0. \end{aligned}$$

Since we have $w > 0$, by using the properties associated with exponential functions, we get

$$\begin{aligned} & \gamma^L Q^L(\mu, u) + \gamma^U Q^U(\mu, u) + G(\mu, u) + H(\mu, u) \\ & \geq \gamma^L Q^L(\theta, k) + \gamma^U Q^U(\theta, k) + G(\theta, k) + H(\theta, k), \end{aligned}$$

which is a contradiction with (13). $\square$

Remark 2. We could mention, as potential extensions of the proposed method for various types of extremization models or domains, the study of posedness and efficiency criteria for similar families of extremization problems governed by path-independent curvilinear integral functionals (which are crucial in applications due to their physical meaning (mechanical work)). This is a specific research question or unresolved issue that could be addressed in future studies to build upon the current findings. Therefore, we note the applicability of the proposed approach to larger and more complex extremization problems.

4. Conclusions and Further Developments

A class of controlled variational control models has been studied by considering the notion of $(q, w) - \pi$ -invexity and its associated variants. In particular, we have established some characterization results of solutions in the considered controlled interval-valued variational models. In this regard, necessary and sufficient conditions of optimality have been stated for a feasible solution. The latter have been formulated and proved by imposing generalized invariant convexity hypotheses for the involved multiple integral functionals. In addition, in the end of the paper, a duality theorem was presented in order to connect the studied model with a new variational problem. Related to some future research directions of the current work, let us consider, for instance, a situation in which the partial derivatives of second-order are included in the studied model.

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