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DP-4-Colorability on Planar Graphs Excluding 7-Cycles Adjacent to 4- or 5-Cycles

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Abstract: In order to resolve Borodin's Conjecture, DP-coloring was introduced in 2017 to extend the concept of list coloring. In previous works, it is proved that every planar graph without 7-cycles and butterflies is DP-4-colorable. And any planar graph that does not have 5-cycle adjacent to 6-cycle is DP-4-colorable. The existing research mainly focus on the forbidden adjacent cycles that guarantee the DP-4-colorability for planar graph. In this paper, we demonstrate that any planar graph G that excludes 7-cycles adjacent to k-cycles (for each $k = 4, 5$), and does not feature a Near-bow-tie as an induced subgraph, is DP-4-colorable. This result extends the findings of the previous works mentioned above.

Keywords: planar graph; DP-4-colorable; list coloring

MSC: 05C15; 05C10

1. Introduction

The graphs discussed are simple, planar graphs. Vertex coloring stands out as a significant area of study within graph theory. A graph G with *list assignment* allocates a color set $L(v)$ to each $v \in V(G)$. Considering ψ to be a proper coloring of G, which ensures that $\psi(u)$ differs from $\psi(v)$ across every edge uv in G. If $\psi(v) \in L(v)$ in respect to every vertex v, then ψ is defined as a *list coloring* of G, which proposed by was Vizing et al. [1,2] independently. Considering each list assignment with a size more than k, if G has a list coloring, then we could define G as *k-choosable*. The least amount k is the *choosable number*, represented by $\chi_l(G)$. Thomassen [3] demonstrated the 5-choosability of planar graph, while not all planar graphs are 4-choosable, as proven by Voigt [4]. Thus, many experts focus on identifying sufficient conditions under which planar graphs have the property of 4-choosability.

In list coloring, vertex identification cannot be used due to the different lists on vertices. To address this limitation, Dvořák et al. [5] proposed DP-coloring, which enabled them to resolve a long-standing conjecture by Borodin [6], namely that every planar graph G without cycles of lengths 4 to 8 is 3-choosable. Let X, Y be two disjoint vertex sets of G, and $[X, Y]$ denote the collection of edges with one end-vertex in X and the other end-vertex in Y. The following definition of DP-coloring was subsequently provided by Bernshteyn et al. [7].

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Definition 1. *Consider that a graph denoted as G along with its respective list assignment L. $\mathcal{M}_{(G,L)}$ qualifies as a cover for (G, L) when it meets these criteria:*

- $V(\mathcal{M}_{(G,L)}) = \bigcup_{w \in V(G)} L_w$, *where* $L_w = \{w\} \times L(w) = \{(w, c) : w \in V(G)$ *and* $c \in L(w)\}$;
- *for each vertex w in G, the collection L_w makes up a clique;*
- *for every edge uv, $[L_u, L_v]$ is a matching set (which may be empty), denoted by M_{uv};*
- *when $uv \notin E(G)$, $[L_u, L_v]$ is an empty set.*

For simplicity, we denote $E_M = \cup_{u,v \in V(G)} M_{uv}$. Figure 1b,c are two distinct covers of (C_4, L) with $|L(v)| = 2$ for each vertex of C_4.

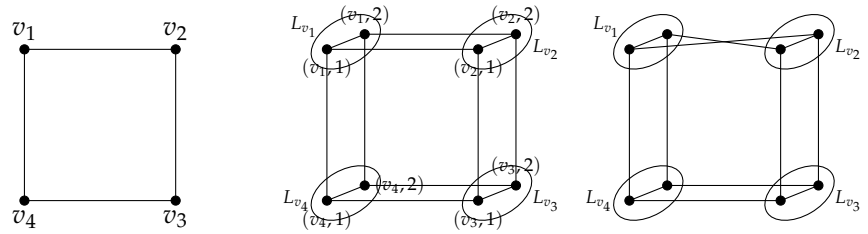
Definition 2. *Considering $\mathcal{M}_{(G,L)}$ denoted as a cover of (G, L). Then, the following two conditions are equivalent:*

(i) *$\mathcal{M}_{(G,L)}$ includes an independent subset I with size $|V(G)|$;*

(ii) *G has a coloring ψ on $V(G)$ satisfying $(v, \psi(v)) \in I$ across each vertex v. This coloring is said to $\mathcal{M}_{(G,L)}$-coloring.*

A graph is called DP-k-colorable when it has an $\mathcal{M}_{(G,L)}$-coloring for every L with $|L(u)| \geq k$ and every cover $\mathcal{M}_{(G,L)}$. The minimum such k is the DP-chromatic number, represented as $\chi_{DP}(G)$.

For every $uv \in E(G)$, distinct matchings between L_u and L_v correspond to different covers of G. If the matching between L_u and L_v is $\{(u, x)(v, x) |\ x \in L(u) \cap L(v)\}$, an $\mathcal{M}_{(G,L)}$-coloring of G becomes equivalent to a list coloring of G. It indicates that DP-coloring serves as a generalization of list coloring, leading to the conclusion that $\chi_l(G) \leq \chi_{DP}(G)$. From Figure 1, H_1 has an independent set $\{(v_1, 1), (v_2, 2), (v_3, 1), (v_4, 2)\}$; hence, C_4 has an H_1-coloring with vertices v_1, v_3 colored 1 and vertices v_2, v_4 colored 2, and this corresponding coloring is also a 2-list coloring of C_4. However, C_4 does not have an H_2-coloring for H_2 and has no 4-independent set. So, $\chi_{DP}(C_4) > 2$. The fact that $\chi_{DP}(C_4) = 3$ and $\chi_l(C_4) = 2$ marks an important difference between DP-coloring and list coloring.



(a) **(b)** **(c)**

Figure 1. H_1 and H_2 are two distinct covers of (C_4, L) with $|L(v_i)| = 2$ for each $i \in [4]$. **(a)** 4-cycle C_4. **(b)** H_1. **(c)** H_2.

Recently, DP-coloring has gained significant attention. Dvořák et al. [5] showed that χ_{DP} does not exceed 5, and it can be reduced to 3 for planar graphs with a girth of at least 5. Li et al. [8] proved that any planar graphs without intersecting 5-cycles is DP-4-colorable. Li et al. [9] discussed the DP-4-colorability of planar graphs with some restrictions on cycles. For more results about DP-colorability, the readers can see in [10–18]. Kim and Ozeki [12] improved results about 4-choosability, any planar graph that avoids a k-cycle (where k is in the set $\in \{3, 4, 5, 6\}$) is DP-4-colorable. Moreover, Farzad [19] confirmed the 4-choosability for every planar graph lacking 7-cycles. One naturally asks whether every planar graph lacking 7-cycles has the property of DP-4-colorability. For this purpose, Kim et al. [20] demonstrated the subsequent finding.

Theorem 1. *Any planar graph that lacks both butterflies (see Figure 2b) and 7-cycles is DP-4-colorable.*

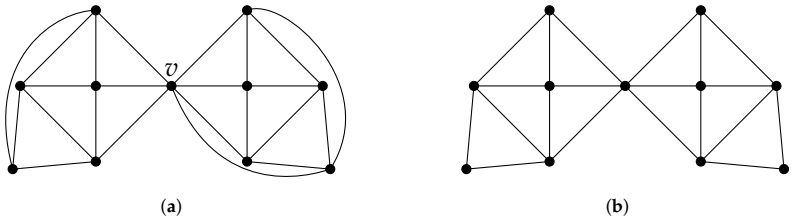


Figure 2. Near-bow-tie and butterfly. (**a**) Near-bow-tie, where v is a 7-vertex. (**b**) butterfly.

On the other hand, there is another way to study the sufficient conditions for DP-4-colorability, that is, forbidden adjacent cycles. For more details, the reader can refer to [11–13,16–18,21].

Theorem 2. *Any planar graph that does not have k_1-cycles adjacent to k_2-cycles is DP-4-colorable, where $(k_1, k_2) \in \{(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}$.*

In a planar graph G, a *cluster* refers to a subgraph formed by a maximal collection of 3-faces, ensuring that no other 3-face shares an edge with any face within this collection. When such a cluster contains k 3-faces, it is designated as *k-cluster* (see Figure 3). *Adjacent* cycles (faces) means that they share no less than one edge. A face f is said to be *adjacent* to one cluster H if there exists a face within H that shares adjacency with f. A *Near-bow-tie* is a graph isomorphic to the configuration depicted in Figure 2a where v is a 7-vertex. Inspired by Theorems 1 and 2, the subsequent result is formulated.

Theorem 3. *Consider a planar graph G without a Near-bow-tie as an induced subgraph. If G lacks 7-cycles that are adjacent to 4- or 5-cycles, then it is DP-4-colorable.*

If a graph G is free of 7-cycles that are adjacent to either 4-cycles or 5-cycles, then G may contain 7-cycles and a butterfly. Additionally, if the graph G lacks the Near-bow-tie as an induced subgraph, then it may still contain a butterfly. Therefore, Theorem 3 generalizes Theorem 1.

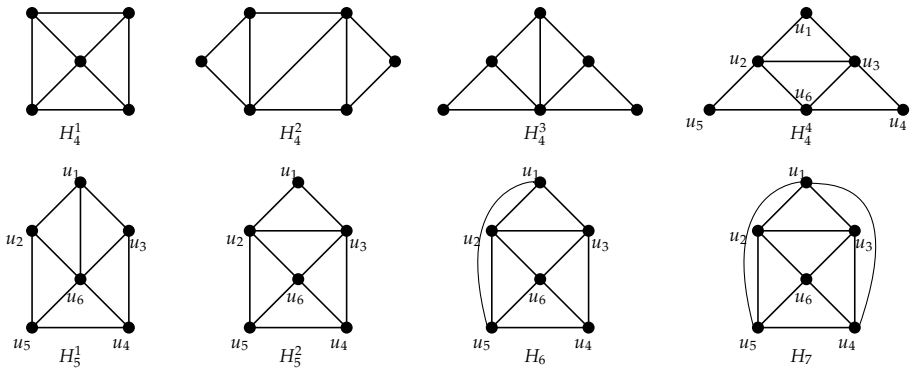


Figure 3. All non-isomorphic k-clusters for each $k \in \{4,5,6,7\}$.

A 3-cycle C is considered *bad* provided that it is isomorphic to the bordering of the outer face of H_7 in Figure 3; all other 3-cycles are *good*. Let $K \subseteq G$ denote a portion of graph G. Therefore, G is DP-k-colorable, provided that every DP-k-coloring restricted on K could be expanded to cover the whole G. To demonstrate Theorem 3, a stronger result is established as outlined below.

Theorem 4. *Consider a planar graph G without a Near-bow-tie as an induced subgraph. If G lacks 7-cycles that are adjacent to 4- or 5-cycles, then each DP-4-coloring restricted on one good 3-cycle within G can be extended to the entire graph.*

At the conclusion of this section, we will present some definitions and notations. The notation $[m]$ is defined as $\{1, 2, \ldots, m\}$, where m is a positive integer. A k-vertex represents a node whose degree is k, while a k^+ (or k^-)-vertex represents a node whose degree is no less (or no more) than k. A face labeled as $(d_1, d_2, \ldots, d_k)$-face corresponds to one k-face $[v_1 v_2 \ldots v_k]$ where v_i is d_i-vertex for $i \in [k]$. If $u \in V(f)$, then u is incident with f. The notation $d(C)$ denotes the number of edges associated with cycle (or face) C of G.

2. Preliminaries

This section focuses on the corresponding coloring on some small subgraph, which is crucial to the proof of Theorem 4.

Lemma 1. *Consider L as a list assignment on $K_3 = v_1 v_2 v_3$. If for each $i \in [3]$, the size of the list $L(v_i)$ is equal to 2, then K_3 has an $\mathcal{M}_{(K_3,L)}$-coloring except when the subgraph of $\mathcal{M}_{(K_3,L)}$ induced by E_M is a union of two 3-cycles.*

Moreover, if $L(v_i) \geq 2$ for each $i \in [3]$ and $|L(v_j)| > 2$ for some $j \in [3]$, then K_3 has an $\mathcal{M}_{(K_3,L)}$-coloring for any cover of (K_3, L).

Proof. Assume $L(v_i) = \{1, 2\}$ for every index $i \in [3]$. If the subgraph of $\mathcal{M}_{(K_3,L)}$ generated by E_M is a union of two 3-cycles, see Figure 4a, say $[(v_1, i)(v_2, i)(v_3, i)]$ for each $i = 1, 2$, then we cannot find a 3-independent set in this cover; hence, K_3 does not have an $\mathcal{M}_{(K_3,L)}$-coloring for this cover. Thus, we assume that the subgraph of $\mathcal{M}_{(K_3,L)}$ induced by E_M is not a union of two 3-cycles. Without loss of generality, suppose $(v_1, 1)(v_2, 1)(v_3, 1)$ forms a 2-path in this cover, but $(v_1, 1)(v_3, 1) \notin E_M$. Then, $\{(v_1, 1), (v_3, 1), (v_2, 2)\}$ is a 3-independent set of $\mathcal{M}_{(K_3,L)}$; see Figure 4b. Hence, we obtain an $\mathcal{M}_{(K_3,L)}$-coloring of K_3 for this cover.

If there is one $|L(v_i)| \geq 3$, for some $i \in [3]$, then we can identify a 3-independent set of $\mathcal{M}_{(K_3,L)}$ for any cover of (K_3, L). Consequently, an $\mathcal{M}_{(K_3,L)}$-coloring of K_3 can be achieved for any cover of (K_3, L). □

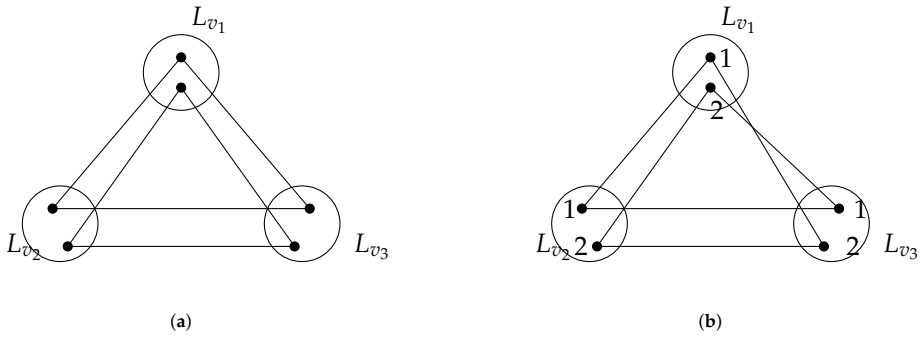


Figure 4. Different cover $\mathcal{M}_{(K_3,L)}$ of (K_3, L), where L_{v_i} is a clique for each $i \in [3]$. (**a**) E_M is a union of two 3-cycles. (**b**) E_M is not a union of two 3-cycles.

Lemma 2. *Consider L as a list assignment on K_4^-, where K_4^- comprises two 3-faces $[xyz_i]$ for $i \in \{1, 2\}$. If $|L(z_i)| \geq 2$ for each $i \in \{1, 2\}$ and $|L(x)|, |L(y)| \geq 3$, then we can assign colors to each z_i using at most one color from $L(z_i)$, so that K_4^- has an $\mathcal{M}_{(K_4^-,L)}$-coloring for any given cover of K_4^-.*

Proof. Assuming $L(x) = L(y) = [3]$ and $L(z_1) = [2]$ is valid for our analysis. We assume that there is a color, say $1 \in L(z_2)$, which cannot be chosen. This means if we choose $(z_2, 1)$ in $\mathcal{M}_{(K_4^-, L)}$, then by Lemma 1, x, y has two available colors and the corresponding vertices of z_1, x, y in $\mathcal{M}_{(K_4^-, L)}$ induce two triangles, say $[(x,i)(y,i)(z_1,i)]$ for every $i \in \{1,2\}$. In this case, $[(z_2,1)(x,3)(y,3)]$ is a 3-cycle of $\mathcal{M}_{(K_4^-, L)}$. Then, for each $d \in L(z_2) - \{1\}$ and for certain values i and j where $\{i,j\} = [2]$, the set $\{(z_2,d), (x,3), (z_1,i), (y,j)\}$ is an independent set in $\mathcal{M}_{(K_4^-, L)}$. Thus, each color in $L(z_2)$ other than 1 can be chosen so that K_4^- has an $\mathcal{M}_{(K_4^-, L)}$-coloring. For vertex z_1, we can discuss a similar method. □

Lemma 3. *Consider $P = v_1v_2v_3$ be a 2-path with a list assignment L on P, such that $|L(v_i)| \geq 1$ for every $i \in [3]$. If $|L(v_1)| + |L(v_3)| \geq |L(v_2)| + 1$, then we can assign colors to v_1, v_3, such that v_2 has at least $|L(v_2)| - 1$ available colors.*

Proof. If there is a vertex (v_1, a) in $\mathcal{M}_{(P,L)}$ such that (v_1, a) is not adjacent to any vertices in L_{v_2}, then we have completed this section. Thus, assume each vertex in L_{v_1} is adjacent to a vertex of L_{v_2}, and the same holds for L_{v_3}. Given that $|L(v_1)| + |L(v_3)| \geq |L(v_2)| + 1$, it follows that there exists a vertex in L_{v_2} that is adjacent to at least one vertex from both L_{v_1} and L_{v_3}. Without loss of generality, assume $(v_2, b)(v_1, a), (v_2, b)(v_3, c) \in E_M$, where $a \in L(v_1), c \in L(v_3)$. Consequently, v_1 can be colored with color a and v_3 with c, so that v_2 has at least $|L(v_2)| - 1$ available colors. □

Lemma 4. *Let H_6 be the graph depicted in Figure 3 with a list assignment L on H_6. If $|L(v)| \geq 2$ for every 3-vertex of H_6, $|L(u)| \geq 3$ for some $u \in \{u_3, u_5\}$ and $|L(v)| \geq 4$ for other three vertices in H_6, then we can color u with all colors but at most two of $L(u)$, ensuring that H_6 has an $\mathcal{M}_{(H_6,L)}$-coloring for any given cover of (H_6, L).*

Proof. Due to symmetry, we can assume that $|L(u_5)| \geq 3$. Thus, $|L(u_i)| \geq 4$ for each $i \in \{2,3,6\}$ and $|L(u_i)| \geq 2$ for each $i \in \{1,4\}$.

For 2-path $u_1u_5u_4$, if $|L(u_1)| + |L(u_4)| \geq |L(u_5)| + 1$, then by Lemma 3, we can color u_1, u_4 such that u_5 has $|L(u_5)| - 1 \geq 2$ available colors, denoted by $A(u_5)$. After such coloring, u_3 has at least two available colors and u_i has at least three available colors for each $i \in \{2,6\}$. For $\{u_5, u_2, u_3, u_6\}$, by Lemma 2, we can color u_5 with at least $|A(u_5)| - 1 = |L(u_5)| - 2$ colors such that $\{u_2, u_3, u_6\}$ can be colored. This means we can color u_5 with all colors in $L(u_5)$, except for at most two, ensuring that H_6 has an $\mathcal{M}_{(H_6,L)}$-coloring for any given cover of (H_6, L), which completes this section.

Thus, $|L(u_1)| + |L(u_4)| < |L(u_5)| + 1$. Without loss of generality, we assume $L(u_1) = L(u_4) = [2]$, $L(u_5) = [4]$. From the above discussion, we only need to prove the case that after coloring u_1 and u_4, u_5 has exactly two available colors. Without loss of generality, we assume $(u_1, i)(u_5, i), (u_4, i)(u_5, i+2) \in E_M$ for each $i \in [2]$. For $\{(u_1,1), (u_4,1)\}$, by Lemma 2, we can color u_5 with 2 or 4 such that $\{u_2, u_3, u_6\}$ can be colored. For the same reason, for $\{(u_1,2), (u_4,2)\}$, by Lemma 2, we can color u_5 with 1 or 3 such that $\{u_2, u_3, u_6\}$ can be colored. This means we can color u_5 with all colors in $L(u_5)$, except for at most two, ensuring that H_6 has an $\mathcal{M}_{(H_6,L)}$-coloring for any given cover of (H_6, L), which completes this section. □

Lemma 5. *Consider H_7 as the 7-cluster shown in Figure 3, along with a list assignment L. Let $S = \{u_1, u_4, u_5\}$. If there is at most one v in S for which $|L(v)| = 2$, while $|L(v)| \geq 3$ for remaining vertices in S, and $|L(v)| \geq 4$ for every vertex v in $V(H_7)$ except S, then H_7 admits an $\mathcal{M}_{(H_7,L)}$-coloring for any given cover of (H_7, L).*

Proof. By symmetry, for each $i \in \{4,5\}$, assume that $|L(u_1)| = 2$ and $|L(u_i)| = 3$. Furthermore, $|L(u_i)| = 4$ for each $i \in \{2,3,6\}$. Consequently, set $L(u_1) = \{1,2\}$, $L(u_4) = L(u_5) = \{1,2,3\}$, $L(u_2) = L(u_3) = L(u_6) = [4]$. Suppose otherwise that H_7 does not have an $\mathcal{M}_{(H_7,L)}$-coloring for some cover of (H_7, L).

Suppose the subgraph induced by E_M on vertices L_{u_1}, L_{u_4} and L_{u_5} contains two triangles in $\mathcal{M}_{(H_7,L)}$ since $|L(u_1)| = 2$; otherwise, by Lemma 1, we can color u_1 and u_5 such that u_4 has two available colors. Then, by Lemma 2, we can color $\{u_2, u_3, u_6, u_4\}$. Therefore, this results in an $\mathcal{M}_{(H_7,L)}$-coloring of H_7, leading to a contradiction. Let $[(u_1,i)(u_4,i)(u_5,i)]$ be the two triangles for $i = 1,2$ and $(u_4,3)(u_5,3) \in E_M$.

Since H_7 does not have an $\mathcal{M}_{(H_7,L)}$-coloring, we have the following proposition, Proposition 1, constructed easily using Lemma 1. □

Proposition 1. *If u_1, u_4, u_5 are colored, then the subgraph of $\mathcal{M}_{(H_7,L)}$ induced by edge E_M between $\{\{u_i\} \times A(u_i) | i = 2,3,6\}$ is two triangles, where $A(u_i)$ denotes the available colors in $L(u_i)$ after coloring $\{u_1, u_4, u_5\}$.*

By Proposition 1 and Lemma 1, there are four triangles between $\{u_i\} \times L(u_i)$ ($i \in \{2,3,6\}$) in $\mathcal{M}_{(H_7,L)}$. Let $[(u_2,i)(u_3,i)(u_6,i)]$ be the four triangles in $\mathcal{M}_{(H_7,L)}$ for each $i \in [4]$. We assume that $(u_1,1)(u_2,1) \in E_M$. Assume $(u_1,1)(u_3,i) \in E_M$ for some $i \in [4]$. By symmetry, we only need to discuss two cases: $i = 1$ or $i = 2$.

If $i = 2$, then by Proposition 1, $(u_4,3)(u_3,1) \in E_M$ for 3-independent set $\{(u_1,1),(u_4,3)\}$, $\{(u_5,2)\}$ and $(u_4,2)(u_3,1) \in E_M$ for 3-independent set $\{(u_1,1),(u_4,2),(u_5,3)\}$; contrary to this, $[L_{u_3}, L_{u_4}]$ is a matching set.

If $i = 1$, then by Proposition 1, $(u_5,2)(u_2,j), (u_4,3)(u_3,j) \in E_M$ for some $j \in \{2,3,4\}$, say $j = 2$. In this case, for 3-independent set $\{(u_1,2),(u_4,3),(u_5,1)\}$, $(u_1,2)(u_2,2)$, $(u_1,2)(u_3,k) \in E_M$ for some $k \in \{3,4\}$. The independent set $\{(u_1,2),(u_4,1),(u_5,3)\}$, $(u_4,1)(u_3,2) \in E_M$, contrary to $[L_{u_3}, L_{u_4}]$, is a matching set.

Lemma 6. *Consider H_7 as the 7-cluster shown in Figure 3, along with a list assignment L. If $|L(u_i)| \geq 3$ for $i = 4,5$ and $|L(v)| \geq 4$ for the other vertices of H_7, then we can color u_1 with all colors but at most one in $L(u_1)$, ensuring that H_7 maintains an $\mathcal{M}_{(H_7,L)}$-coloring for any given cover of (H_7, L).*

Proof. Without loss of generality, we assume that for each $i \in \{4,5\}$, $L(u_i) = [3]$ and $L(u_j) = [4]$ for each $j \in \{1,2,3,6\}$. For any given cover of (H_7, L), by contradiction, we assume that there are two colors that cannot be chosen in $L(u_1)$, say $\{1,2\}$. Since $[2] \subseteq L(u_1)$, by Lemma 5, there is an $\mathcal{M}_{(H_7,L)}$-coloring for any given cover of (H_7, L), a contradiction. This means we can color u_1 with all colors, but only use at most one in $L(u_1)$, ensuring that H_7 maintains an $\mathcal{M}_{(H_7,L)}$-coloring for any cover of (H_7, L). □

The following definition is due to Sittitrai and Nakprasit [22].

Definition 3. *Let $\mathcal{M}_{(G,L)}$ represent a cover of (G, L), with G' as an induced subgraph within G. The following definitions apply:*

(i) A list assignment $L|_{G'}$ is considered a restriction of L on G' if $L|_{G'}(u) = L(u)$ for every vertex of G'.

(ii) A graph $\mathcal{M}_{(G,L)}|_{G'}$ is termed a restriction of $\mathcal{M}_{(G,L)}$ on G' if $\mathcal{M}_{(G,L)}|_{G'} = \mathcal{M}_{(G,L)}[\{v \times L(v) : v \in V(G')\}]$. According to the cover definition, $\mathcal{M}_{(G,L)}|_{G'} \cong \mathcal{M}_{(G',L|_{G'})}$.

Assuming that G' has an $\mathcal{M}_{(G',L|_{G'})}$-coloring with an independent set I', the residual list assignment L^* for $G - G'$ is defined as follows:

$$L^*(y) = L(y) - \bigcup_{xy \in E(G)} \{c' \in L(y) : (x,c)(y,c') \in E(H) \text{ and } (x,c) \in I'\}$$

for every $y \in V(G - G')$. *The residual cover* $\mathcal{M}^*_{(G-G',L^*)}$ *is established by* $\mathcal{M}_{(G,L)}[\{w \times L^*(w) : w \in V(G - G')\}]$.

Lemma 7 ([22]). *Let* $\mathcal{M}_{(G,L)}$ *represent a cover of* (G, L) *with* G' *as an induced subgraph of* G. *If* G' *maintains an* $\mathcal{M}_{(G',L|_{G'})}$*- coloring, then the residual cover* $\mathcal{M}^*_{(G-G',L^*)}$ *serves as a cover for* $(G - G', L^*)$, *where* L^* *represents the residual list assignment. Additionally, if* $G - G'$ *is colorable with respect to* $\mathcal{M}^*_{(G-G',L^*)}$, *then* G *is also* $\mathcal{M}_{(G,L)}$*-colorable.*

For the remainder of this paper, G' is regarded as an induced subgraph of (G, L), where $|L(v)| \geq 4$ for every vertex v of G. By Lemma 7, to establish that G is $\mathcal{M}_{(G,L)}$-colorable, it suffices to demonstrate that G' maintains an $\mathcal{M}_{(G',L|_{G'})}$-coloring and G^* maintains an $\mathcal{M}^*_{(G-G',L^*)}$-coloring.

3. Reducible Configurations

Consider (G, C_0) as a counterexample to Theorem 4, where the number of vertices is minimized, and within that constraint, the number of edges is also minimized, with C_0 representing a good 3-cycle. The proof presented in [12] establishes that any planar graph devoid of 3-cycles is DP-4-colorable. Therefore, it is reasonable to assume that G includes a 3-cycle. Indeed, we can always identify a good 3-cycle within G by Lemma 5. Let L represent a 4-list assignment on the vertex set $V(G)$, and C_0 as the outer cycle of G. Let $\mathcal{M}_{(G,L)}$ be any cover of (G, L) and ϕ_0 the $\mathcal{M}_{(G,L)}$-coloring restricted on C_0. The following lemma holds true when G lacks any 7-cycle that is adjacent to 4- or 5-cycles.

Lemma 8. *G contains no k-cluster for $k \geq 8$; moreover, the graphs of Figure 3 are all k-clusters of G, where $k \in \{4, 5, 6, 7\}$.*

Consider C representing a cycle within G. The notation $int(C)$ (or $ext(C)$) is employed to indicate the collections of vertices lying inside (or outside) the cycle C. A cycle C is referred to as separating if it divides the vertex set such that both $int(C)$ and $ext(C)$ include vertices. A vertex u in G is classified as *internal* if $u \notin V(C_0)$. Additionally, a face f is classified as *internal* if all vertex of f is internal.

Lemma 9. *The statements below are all valid:*

(i) $V(G) \neq V(C_0)$;
(ii) *G does not have any good 3-cycle that is separating;*
(iii) *the degree of every internal vertex in G is at least 4.*

Proof. (i) Assume, for the purpose of contradiction, that $V(G) = V(C_0)$. Since C_0 is a good 3-cycle, the $\mathcal{M}_{(G,L)}$-coloring restricted on C_0 would also serve as an $\mathcal{M}_{(G,L)}$-coloring for G, which contradicts the initial assumption.

(ii) Suppose C is a separating good 3-cycle. Given the property of minimality in G, we can extend ϕ_0 to ext(C)∪C, say ϕ_1. This extension can then be restricted to C and further extended to create an $\mathcal{M}_{(G,L)}$-coloring ϕ_2 for $int(C)$. Consequently, by merging ϕ_1 and ϕ_2, we obtain an $\mathcal{M}_{(G,L)}$-coloring for G, contrary to the assumption.

(iii) Let v represent an internal 3^--vertex. Given that G is minimal, the $\mathcal{M}_{(G,L)}$-coloring restricted on C_0 can be applied to $G - v$. Since $d_G(v) \leq 3$, v has at least one available color for any cover of (G, L). This means that G has an $\mathcal{M}_{(G,L)}$-coloring, a contradiction. □

Lemma 10. *No two internal $(4, 4, 4)$-faces in G can share a common edge.*

Proof. Assume, for the sake of contradiction, that $f_i = [xyz_i]$ $(i = 1, 2)$ are two internal (4, 4, 4)-faces. Denote $G' = G - f_1 \cup f_2$. By the minimality of G, the graph G' maintains an $\mathcal{M}_{(G',L|_{G'})}$-coloring extended by ϕ_0 on C_0, and L^*, $\mathcal{M}^*_{(G-G',L^*)}$ are the residual list assignment and the residual cover of $(f_1 \cup f_2, L^*)$. Since each vertex of $f_1 \cup f_2$ is a 4-vertex, $|L^*(z_i)| \geq 2$ for every $i \in \{1, 2\}$ and $|L^*(x)|, |L^*(y)| \geq 3$. By Lemmas 2 and 7, we can obtain an $\mathcal{M}_{(G,L)}$-coloring for G, which contradicts the original assumption. □

A cluster is *special* if it is one of $\{H_4^4, H_5^2, H_6, H_7\}$ in Figure 3 with an internal (4, 4, 4)-face $[u_2u_3u_6]$ (or $[u_2u_5u_6]$ by symmetry if the cluster is H_6); otherwise non-special. An internal 4^+-vertex v located inside a cluster H is classified as *i-type* with respect to H if it is connected to precisely i edges of H. Moreover, if H is a special cluster, then this 4^+-vertex v is referred to as *i-type good*; otherwise, it is considered *i-type bad*. The following result is obvious from the fact that G contains no 7-cycle adjacent to a 4- or 5-cycle.

Lemma 11. *(i) A 1-cluster is adjacent to no more than a 4-face.*
(ii) The face of G adjacent to i-cluster should be 8^+-face for each $i \in \{4, 5, 6, 7\}$.
(iii) The face of G adjacent to 2- or 3-cluster should be 4-face or 8^+-face; moreover, s 2-cluster is adjacent to no more than two non-adjacent 4-faces, where these two 4-faces are adjacent to different 3-face of this 2-cluster. Similarly, a 3-cluster is adjacent to no more than one 4-face, which is incident to one 4-type 4-vertex to this 3-cluster.

Lemma 12. *No internal 5-vertex of G can belong to two clusters which are special.*

Proof. Let v represent an internal 5-vertex, and let its neighborhood be $N(v) = \{v_i : i \in [5]\}$. Assume, for the purpose of contradiction, that v is a vertex of two special clusters K_1, K_2. Without loss of generality, assume the internal (4, 4, 4)-faces of K_1 and K_2 are $[v_1v_2v_{12}]$ and $[v_3v_4v_{34}]$, respectively. Define H_1 and H_2 as the graphs induced by vertex set $\{v_1, v_2, v_{12}, v\}$ and $\{v_3, v_4, v_{34}, v\}$, respectively. Clearly, H_i is isomorphic to K_4^-. Let $G' = G - H_1 \cup H_2$ and L^*, $\mathcal{M}^*_{(G-G',L^*)}$ denote the residual list assignment and the residual cover of $G - G'$. Due to the minimality property of (G, C_0), G' maintains an $\mathcal{M}_{(G',L|_{G'})}$-coloring extended from ϕ_0. Notably, $|L^*(v)|$ and $|L^*(v_i)|$ are both at least 3 for each $i \in [4]$, while $|L^*(v_{12})|, |L^*(v_{34})| \geq 2$. Applying Lemma 2 to H_1, we can color v with at least two colors in $L^*(v)$, ensuring that H_1 maintains an $\mathcal{M}_{(H_1,L^*|_{H_1})}$-coloring for every cover of $(H_1, L^*|_{H_1})$. Applying Lemma 2 to H_2 subsequently yields an $\mathcal{M}^*_{(G-G',L^*)}$-coloring for $G - G'$. By Lemma 7, this implies that G maintains an $\mathcal{M}_{(G,L)}$-coloring, contrary to the assumption. □

Internal 6-vertex v is classified as *special* if it qualifies as a 4-type good vertex to an internal 6^+-cluster K_1 and as a 2-type good vertex in association with a k_2-cluster K_2, where $4 \leq k_2 \leq 5$ (see Figure 5a). Internal 6-vertex v is *poor* if it is a 3-type good vertex to an internal 6-cluster K_1 and a 3-type good vertex to a k_2-cluster K_2, where $5 \leq k_2 \leq 6$ (see Figure 5b).

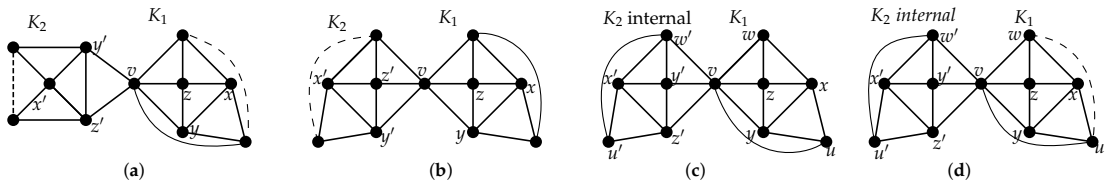


Figure 5. Special or poor vertex: all K_1 is special and internal and K_2 is special. (**a**) Special 6-vertex v. (**b**) Poor 6-vertex v. (**c**) Poor 7-vertex v when $u, w \in \mathcal{A}_1$. (**d**) Special 7-vertex v when $u' \in \mathcal{A}_1$ and $w' \in \mathcal{A}_2$.

Lemma 13. *Consider v as either a special or poor 6-vertex contained in K_1 and K_2 (see Figure 5a,b). Let $G' = G - K_1$, with L^*, $\mathcal{M}^*_{(G-G',L^*)}$ representing the residual list assignment and residual cover of $G - G'$. Then, $|L^*(v)| \geq 3$ if v is a special 6-vertex, and $|L^*(v)| \geq 2$ if v is a poor 6-vertex.*

Proof. Since v is either a special or poor 6-vertex, K_2 is a special cluster. Denote $[x'y'z']$ to represent the (4, 4, 4)-face of K_2. Denote $G'' = G' - \{x', y', z'\} = G - K_1 \cup \{x', y', z'\}$, with L^{**}, $\mathcal{M}^{**}_{(G-G'',L^{**})}$ as the residual list assignment and residual cover for $G - G''$. This implies that $|L^{**}(x')| \geq 2$ and $|L^{**}(u)| \geq 3$ for each $u \in \{y', z'\}$.

If v is a special 6-vertex, then $|L^{**}(v)| \geq 4$. If v is a poor 6-vertex, then $|L^{**}(v)| \geq 3$. In either case, by Lemma 2, we can precolor v using all colors in $L^{**}(v)$ except for at most one, allowing the vertices $\{x', y', z'\}$ to be colored as well. This means $|L^*(v)| \geq 3$ when the vertex v is classified as a special 6-vertex and $|L^*(v)| \geq 2$ when v is a poor 6-vertex. □

We use a recursive method to define the poor 7-vertex, the special 7-vertex, and the special 8-vertex.

Definition 4. *Consider v as a 7-vertex of G, which is 4-type good to one internal 6-cluster K_1 and 3-type good to one internal 6-cluster K_2.*

(i) *v is a **poor 7^0-vertex** if K_1 contains two 3-type vertices, each of which is in {3-type good 5-vertex, 3-type poor 6-vertex};*

(ii) *v is a **poor 7^m-vertex** if K_1 contains one 3-type poor 7^{m-1}-vertex and one 3-type vertices which is in {3-type good 5-vertex, 3-type poor 6-vertex, 3-type poor 7^k-vertex $(k \leq m-1)$}.*

*Then, v is called a **3-type poor 7-vertex** if v is a 3-type poor 7^m-vertex, where $m \in N$.*

Let $\mathcal{A}_1 =$ {3-type good 5-vertex, 3-type poor 6-vertex, 3-type poor 7-vertex}.

Definition 5. *Consider v as a 7-vertex of G which is 4-type good to one internal 6^+-cluster K_1 and 3-type good to one internal 6-cluster K_2.*

(i) *v is a **special 7^0-vertex** if K_2 contains one vertex of $\mathcal{A}_1$ and one vertex of {4-type good 5-vertex, 4-type special 6-vertex};*

(ii) *v is a **special 7^m-vertex** if K_2 contains one vertex of $\mathcal{A}_1$ and one special 7^{m-1}-vertex.*

*Then, v is called a **4-type special 7-vertex** if v is a 4-type special 7^m-vertex, where $m \in N$.*

Let $\mathcal{A}_2 =$ {4-type good 5-vertex, 4-type special 6-vertex, 4-type special 7-vertex}.

Definition 6. *Consider v as an 8-vertex of G which is 4-type good to one internal 7-cluster K_1 and 4-type good to one internal 6^+-cluster K_2.*

(i) *v is a **special 8^0-vertex** if K_1 contains two vertices which are in $\mathcal{A}_2$;*

(ii) *v is a **special 8^m-vertex** if K_1 contains one special 8^{m-1}-vertex and one vertex of $\mathcal{A}_2 \cup$ { 4-type special 8^l-vertex $(l \leq m-1)$}.*

*Then, v is called a **special 8-vertex** if v is a special 8^m-vertex, where $m \in N$.*

By Definition 6, in order to distinguish the above clusters K_1 and K_2, K_1 is called type I and K_2 is called type II.

Lemma 14. *Consider v as a 3-type vertex to an internal cluster K. Let $G' = G - K$, with L^* and $\mathcal{M}^*_{(G-G',L^*)}$ as the residual list assignment and residual cover on $G - G'$, respectively. If $v \in \mathcal{A}_1$, then $|L^*(v)| \geq 2$.*

Proof. If v is a 5-vertex, then the cardinality of $L^*(v)$ is at least 2. If v is a 3-type poor 6-vertex, then $|L^*(v)| \geq 2$ by Lemma 13. Then, v is a poor 7-vertex. By Definition 4, assuming v is a poor 7^m-vertex, $m \in N$. We use the symbol shown in Figure 5c, where $K_2 = K$ and K_1 are 6-clusters. Let $[xyz]$ and $[x'y'z']$ be (4, 4, 4)-faces in K_1 and K_2, respectively. Since v is poor, K_1 is also internal, and $u, w \in \mathcal{A}_1$. Let $G'' = G' - K_1 = G - (K_1 \cup K_2)$, and let L^{**} and $\mathcal{M}^{**}_{(G-G'',L^{**})}$ denote the residual list assignment and residual cover of $G - G''$. We use the induction hypothesis on m. If $m = 0$, then $\{u, w\}$ is a 3-type 5- or poor 6-vertex, by the above discussion, $|L^{**}(u)|$, $|L^{**}(w)| \geq 2$. Clearly, $|L^{**}(i)| = 4$ for each $i \in \{x, y, z, v\}$. Lemma 4 provides a minimum of two colors available in $L^{**}(v)$ to color K_1. This means $|L^*(v)| \geq 2$. By the induction hypothesis, it is assumed that the lemma is valid for $m \leq n - 1$.

Next, we prove that the lemma is valid if $m = n$.

By Definition 4, each of $\{u, w\}$ is a 5-, poor 6-, or 7^l-vertex ($l \leq n - 1$), using Lemma 13 and the induction hypothesis, $|L^{**}(u)| \geq 2$ and $|L^{**}(w)| \geq 2$. Then, using Lemma 4, we have completed this section. □

Lemma 15. *Consider v as a 4-type vertex to an internal cluster K. Let $G' = G - K$ with L^* and $\mathcal{M}^*_{(G-G',L^*)}$ representing the residual list assignment and residual cover of $G - G'$.*

(i) If $v \in \mathcal{A}_2$, then the cardinality of $L^*(v)$ is at least 3;
(ii) If v is a special 8-vertex and K is type II, then $|L^*(v)| \geq 3$.

Proof. (i) Clearly, $|L^*(v)| \geq 3$ if v is a 5-vertex. If v is a 4-type special 6-vertex, then $|L^*(v)| \geq 3$ by Lemma 13. Then, v is a special 7-vertex. By Definition 5, assuming v is a special 7^m-vertex, $m \in N$. We use the labeling shown in Figure 5d, where $K_1 = K$ and K_2 is an internal 6-cluster. Then, the (4, 4, 4)-face corresponding to the cluster K_2 is denoted by $[x'y'z']$ and $u' \in \mathcal{A}_1$. Let $G'' = G' - K_2 = G - (K_1 \cup K_2)$, with L^{**} and $\mathcal{M}^{**}_{(G-G'',L^{**})}$ representing the residual list assignment and residual cover of $G - G''$. Then, for each vertex $q \in \{v, x', y', z'\}$, $|L^{**}(q)| = 4$. Since $u' \in \mathcal{A}_1$, by Lemma 14, $|L^{**}(u')| \geq 2$.

Next, we use the induction hypothesis on m.

If $m = 0$, then w' is a good 5- or special 6-vertex. According to Lemma 13, we have $|L^{**}(w')| \geq 3$. Without loss of generality, we can take $L^{**}(w') = [3], L^{**}(u') = [2], L^{**}(q) = [4]$ for each $q \in \{x', y', z', v\}$. By Lemma 4, for $\{3, 4\} \subseteq L(v)$, there is an $\mathcal{M}_{(K_2,L^{**})}$-coloring, say ϕ_1, with $\phi_1(v) = 3$. Similarly, for $\{1, 2\} \subseteq L(v)$, there is an $\mathcal{M}_{(K_2,L^{**})}$-coloring, say ϕ_2, with $\phi_2(v) = 1$; for $\{2, 4\} \subseteq L(v)$, there is an $\mathcal{M}_{(K_2,L^{**})}$-coloring, say ϕ_3, with $\phi_3(v) = 2$. Hence, $|L^*(v)| \geq 3$.

By the induction hypothesis, it is assumed that the lemma is valid for $m \leq n - 1$.

Next we prove that the lemma is valid if $m = n$. By Definition 5, w' is a special 7^{n-1}-vertex. By the induction hypothesis, $|L^{**}(w')| \geq 3$. By Lemma 4 and the similar reasoning for the case $m = 0$, we have $|L^*(v)| \geq 3$.

(ii) Since v is a special 8-vertex, v is a special 8^m-vertex, $m \in N$. We use the induction hypothesis on m. If $m = 0$, then v is a special 8^0-vertex, K is an internal 6^+-cluster and v is also a vertex of internal 7-cluster K_1, and K_1 contains two vertices in $\mathcal{A}_2$, say u, w. Let $G'' = G' - K_1 = G - (K \cup K_1)$, with L^{**} and $\mathcal{M}^{**}_{(G-G'',L^{**})}$ representing the residual list assignment and residual cover of $G - G''$. Let $[xyz]$ be the (4, 4, 4)-face in K_1. Then, for each vertex $q \in \{v, x, y, z\}$, $|L^{**}(q)| = 4$. Since $u, w \in \mathcal{A}_2$, by Lemma 15(i), $|L^{**}(u)| \geq 3$, $|L^{**}(w)| \geq 3$. By Lemma 6, we can assign colors to v using all but at most one color, ensuring that K_1 maintains an $\mathcal{M}_{(K_1,L^{**})}$-coloring for any given cover of (K_1, L^{**}). This means $|L^*(v)| \geq 4 - 1 \geq 3$.

By the induction hypothesis, it is assumed that the lemma is valid for $m \leq n - 1$.

Next, we prove that the lemma is valid if $m = n$. By Definition 6, assuming that u is a special 8^{n-1}-vertex and w is a vertex from $\mathcal{A}_2 \cup \{\text{the 4-type special } 8^l\text{-vertex } (l \leq n-1)\}$. By the induction hypothesis, the cardinality of both $L^{**}(u)$ and $L^{**}(w)$ is at least 3. By Lemma 6, we have $|L^*(v)| \geq 3$. □

Lemma 16. *If v is a poor 7-vertex in G, then it cannot be a special 7-vertex of G.*

Proof. Since v is a poor 7-vertex, we will consider v as a good 4-type vertex in relation to the internal 6-cluster K_1 and as a good 3-type vertex to internal 6-cluster K_2. Clearly, $v \in \mathcal{A}_1$. Suppose, on the contrary, that v is also a special 7-vertex. This means $v \in \mathcal{A}_2$. Let $G' = G - K_1 \cup K_2$, with L^* and $\mathcal{M}^*_{(G-G',L^*)}$ representing the residual list assignment and residual cover for $G - G'$. Given the minimal property of G, G' has an $\mathcal{M}_{(G',L|_{G'})}$-coloring that extends ϕ_0. Since $v \in \mathcal{A}_2$, there are three available colors in $L^*(v)$ to choose, so that K_2 can be colored by Lemma 15. Since $v \in \mathcal{A}_1$, there are two available colors in $L^*(v)$ to choose, so that K_1 can be colored by Lemma 14. Since $|L^*(v)| = 4$, we can precolor v so that $K_1 \cup K_2$ can be colored. By Lemma 7, G is $(\mathcal{M}, L)$-coloring for any given cover of (G, L), leading to a contradiction. □

Lemma 17. *Consider v as a 7-vertex of G, where it is 4-type to a 7-cluster K_1 and 3-type to a 6^--cluster K_2. Denote $[x, y, z]$ represent a (4, 4, 4)-face within K_1 and $u, w \in V(K_1)$. If $u, w \in \mathcal{A}_2 \cup \{$the special 8-vertex and K_1 is type II$\}$, then K_2 cannot be special.*

Proof. Assume, to the contrary, that K_2 is special. In this case, let $[x'y'z']$ represent the internal (4, 4, 4)-face within K_2 with $y', z' \in N(v)$. Since $u, w \in \mathcal{A}_2 \cup \{$the special 8-vertex and K_1 is type II$\}$, K_1 is internal. Define $G' = G - K_1 - \{x', y', z'\}$, and let L^* and $\mathcal{M}^*_{(G-G',L^*)}$ be the residual list assignment and the residual cover of $G - G'$. In this setup, $|L^*(x')| \geq 2$ and $|L^*(y')|, |L^*(z')|, |L^*(v)| \geq 3$. By Lemma 15, $|L^*(u)|, |L^*(w)| \geq 3$. Clearly, $|L^*(x)|, |L^*(y)|$ and $|L^*(z)| \geq 4$. According to Lemma 2, v has a minimum of two colors that can be used from $L^*(v)$, allowing for the coloring of the set $\{x', y', z'\}$. Then, by Lemma 5, K_1 has an $\mathcal{M}^*_{(K_1,L^*|_{K_1})}$-coloring for any cover of (K_1, L^*). Therefore, $G - G'$ maintains an $\mathcal{M}^*_{(G-G',L^*)}$-coloring. By the minimality of G, G' also maintains an $\mathcal{M}_{(G',L|_{G'})}$-coloring. By Lemma 7, G maintains an $\mathcal{M}_{(G,L)}$-coloring, a contradiction. □

Lemma 18. *Consider $H_6 \subseteq G$ as an internal special 6-cluster with a (4, 4, 4)-face $[u_2u_3u_6]$, shown in Figure 3. If $u_i \in \mathcal{A}_1$ for each $i \in \{1, 4\}$, then $d(u_5) \geq 6$ and $u_5 \notin \mathcal{A}_2 \cup \{$the special 8-vertex, and H_6 is type II$\}$.*

Proof. By Lemma 10, $d(u_5) \geq 5$. Denote $G' = G - H_6$ with L^* and $\mathcal{M}^*_{(H_6,L^*)}$ as the residual list assignment and residual cover of $G - G'$. By the minimal property of G, G' maintains an $\mathcal{M}_{(G',L|_{G'})}$-coloring that extends ϕ_0. Suppose otherwise that $u_5 \in \mathcal{A}_2$ or u_5 is a special 8-vertex with H_6 type II. Then, by Lemma 15, $|L^*(u_5)| \geq 3$. Since $u_i \in \mathcal{A}_1$ for each $i \in \{1, 4\}$, by Lemma 14, $|L^*(u_i)| \geq 2$. Note that $|L^*(u_i)| \geq 4$ for each $i \in \{2, 3, 6\}$. By Lemma 4, H_6 has an $\mathcal{M}^*_{(H_6,L^*)}$-coloring. According to Lemma 7, G admits an $\mathcal{M}_{(G,L)}$-coloring, which is a contradiction. □

Lemma 19. *Consider $H_7 \subseteq G$ as an internal 7-cluster (see Figure 3) and $X = \{u_1, u_4, u_5\}$.*

(i) If X contains two vertices in $\mathcal{A}_2 \cup \{$the special 8-vertex, and H_7 is type II$\}$, then the third vertex of X is a 7^+-vertex;

(ii) X does not contain three vertices in $\mathcal{A}_2 \cup \{$the special 8-vertex, and H_7 is type II$\}$.

Proof. (i) By symmetry, assume $u_4, u_5 \in \mathcal{A}_2 \cup \{\text{the special 8-vertex, and } H_7 \text{ is type II}\}$. Suppose, to the contrary, that $d(u_1) \leq 6$.

Define $G' = G - H_7$ with L^* and $\mathcal{M}^*_{(H_7,L^*)}$ as the residual list assignment and the residual cover of $G - G'$. By the minimal property of G, G' maintains an $\mathcal{M}_{(G',L|_{G'})}$-coloring that extends ϕ_0, with $|L^*(u_1)| \geq 2$. Note that $|L^*(u_i)| \geq 4$ for each $i \in \{2,3,6\}$. Since $u_4, u_5 \in \mathcal{A}_2 \cup \{\text{the special 8-vertex, and } H_7 \text{ is type I}\}$, by Lemma 15, $|L^*(u_i)| \geq 3$, for each $i \in \{4,5\}$. By Lemma 5, an $\mathcal{M}^*_{(H_7,L^*)}$-coloring of H_7 can be achieved. Then, by Lemma 7, G admits an $\mathcal{M}_{(G,L)}$-coloring, a contradiction.

(ii) follows easily by Lemmas 5 and 15. □

4. Proof of Theorem 4

Consider V as the family of vertices of G and F as the family of faces. An initial charge is defined as $\mu(x) = d(x) - 4$ for every $x \in V \cup F \setminus \{C_0\}$, while $\mu(C_0) = d(C_0) + 4$, where C_0 is designated as the outer face of G. According to Euler's Formula, the sum $\sum_{x \in V \cup F} \mu(x) = 0$. Let $\mu^*(x)$ represent the charge of $x \in V \cup F$ after the discharging procedure. To reach a contradiction, the purpose of this is to demonstrate that $\mu^*(x) \geq 0$ for all $x \in V \cup F \setminus \{C_0\}$ and that $\mu^*(C_0) > 0$.

The discharging rules are as follows:

(R1) Every 5- or 8^+-face distributes $\frac{1}{2}$ to every neighboring 3-face, while a k-face ($k = 6, 7$) distributes $\frac{k-4}{k}$ to every neighboring 3-face;
Consider v as a vertex of k-face f and let t represent the number of edges of f that are incident to v, such that these edges are not in any 3-faces. In cases where $k \geq 8$, f distributes $\frac{t}{4}$ to v; f distributes $\frac{t(k-4)}{2k}$ to v when $k \in \{6,7\}$;

(R2) Every 2-cluster obtains $\frac{1}{4}$ from its 3-type 4^+-vertex and $\frac{1}{4}$ from its 2-type 4^+-vertex, which is on a 4-face adjacent to this cluster;

(R3) Every 3-cluster which is adjacent to 4-face obtains $\frac{1}{2}$ from its 3-type 4^+-vertex and $\frac{1}{4}$ from its 2-type 4^+-vertex; otherwise, 3-cluster obtains $\frac{1}{4}$ from its 3-type 4^+-vertex;

(R4) Every 4-cluster obtains the charge from its incident vertex v, defined by

$$\begin{cases} \frac{1}{2}, & \text{if } v \text{ is a 2-type good } 5^+\text{-vertex or a 3-type } 4^+\text{-vertex;} \\ 1, & \text{if } v \text{ is a 5-type } 5^+\text{-vertex;} \\ \frac{3}{2}, & \text{if } v \text{ is a 4-type } 5^+\text{-vertex.} \end{cases} \tag{1}$$

(R5) Every 5-cluster obtains the charge from its incident vertex v defined by

$$\begin{cases} \frac{1}{2}, & \text{if } v \text{ is a 2-type good } 5^+\text{-vertex or 3-type 4-vertex or 3-type bad 5-, 6-, 7-vertex;} \\ 1, & \text{if } v \text{ is a 3-type good 5-, 6-, 7-vertex or 3-type } 8^+\text{-vertex or 5-type 5-vertex;} \\ \frac{3}{2}, & \text{if } v \text{ is a 4-type } 5^+\text{-vertex.} \end{cases} \tag{2}$$

(R6) Every 6-cluster K obtains the charge from its incident vertex v defined by

$$\begin{cases} \frac{1}{2}, & \text{if } v \text{ is a 3-type 4-vertex or 3-type bad 5-, 6-, or 7-vertex;} \\ 1, & \text{if } v \in \mathcal{A}_1 \text{ or } v \text{ is a 3-type good, non-poor 6- or 7-vertex, while } K \text{ is not internal;} \\ 2, & \text{if } v \text{ is a 4-type good } 6^+\text{-vertex, while } K \text{ is internal and contains two vertices in } \mathcal{A}_1, \\ & \text{or a 4-type good } 9^+\text{-vertex;} \\ \frac{3}{2}, & \text{if } v \text{ is a 4-type } 5^+\text{-vertex except the above 4-type vertices, or 3-type } 8^+\text{-vertex, or} \\ & \text{3-type good, non-poor 6 or 7-vertex, while } K \text{ is internal.} \end{cases} \tag{3}$$

(R7) Every 7-cluster K obtains the charge from its incident internal vertex v defined by

$$\begin{cases} \frac{3}{2}, & \text{if } v \in \mathcal{A}_2 \cup \{\text{the special 8-vertex and } K \text{ is type II}\}, \\ & \text{or } v \text{ is a 7, 8-vertex or non-special 6-vertex while } K \text{ is not internal,} \\ 2, & \text{if } v \text{ is a non-special 6-vertex while } K \text{ is internal,} \\ & \text{or a non-special 7, 8-vertex while } K \text{ is internal and} \\ & \text{contains no two vertices in } \mathcal{A}_2 \cup \{\text{the special 8-vertex and } K \text{ is type II}\}, \\ \frac{5}{2}, & \text{if } v \text{ is a non-special 7, 8-vertex while } K \text{ is internal and contains two vertices (except } v) \\ & \text{in } \mathcal{A}_2 \cup \{\text{the special 8-vertex and } K \text{ is type II}\} \text{ or } 9^+\text{-vertex.} \end{cases} \tag{4}$$

(R8) C_0 receives $\mu(v)$ from every incident vertex, and distributes 1 to every non-internal 3-face;

(R9) At last, each face excepting C_0 transfers its present charge to C_0.

The following lemma can be easily obtained by discharging rules.

Lemma 20. *Assuming u is a k-type vertex within a cluster, if $d_G(u) \geq 4$, then u gives at most*

$$\begin{cases} \frac{1}{2}, & \text{if } k = 2, \\ \frac{3}{2}, & \text{if } k = 3, \\ \frac{5}{2}, & \text{if } k = 4, \\ 1, & \text{if } k = 5. \end{cases} \text{ to this cluster.} \tag{5}$$

We begin by examining the final charge assigned to each vertex of G. Consider v as a vertex of G. If $v \in V(C_0)$, then according to rule (R8), we have $\mu^*(v) \geq 0$. Now, consider the situation in which v is not in $V(C_0)$. By applying Lemma 9 (iii), the minimal degree of G is 4.

1. $d(v) = 4$. Suppose v is a k-type vertex of cluster K for some $k \in \{2, 3, 4\}$. If $k = 4$, then v does not participate in the discharge process, hence $\mu^*(v) = \mu(v) = 0$. If $k = 3$, then K is a 2^+-cluster and v must be contained in at least one 8^+-face. By (R2)–(R6), v gives $\frac{1}{4}$ or $\frac{1}{2}$ to K. By (R1), v obtains at least either $\frac{1}{4}$ if v is incident with a 4-face or $\frac{1}{2}$ otherwise. Therefore, if v is not incident with 4-face, then $\mu^*(v)$ is not negative. Assuming v is incident to a single 4-face. It can be inferred from Lemma 11 that K is a 2-cluster. By (R2) and (R1), $\mu^*(v) = \mu(v) + \frac{1}{4} - \frac{1}{4} = 0$. If $k = 2$, then K is a 1^+-cluster. By (R2)–(R5), v gives a charge to K when K is a 2- or 3-cluster. If K is a 2- or 3-cluster, then v gives $\frac{1}{4}$ to K when v is incident with a 4-face adjacent to K according to (R2)–(R3). Given that G lacks 7-cycles that are adjacent to 4- or 5-cycles, and according to Lemma 11, the three other faces incident with v in clockwise direction are 4-, 6^+-, and 8^+-faces, respectively. By applying (R1), we have $\mu^*(v) = \mu(v) - \frac{1}{4} + \frac{1}{6} + \frac{1}{4} > 0$.
2. $d(v) = 5$. Assume that v represents a k-type vertex of cluster K for some $k \in \{2, 3, 4, 5\}$. By applying Lemma 20, it can be inferred that $\mu^*(v)$ is at least $\mu(v) - 1$, which yields $\mu^*(v) \geq 0$ when $k = 5$. If $k = 4$, then K is a 3^+-cluster and v is contained in two 8^+-faces. By (R3)–(R7), v gives at most $\frac{3}{2}$ to K. By (R1), v receives $\frac{1}{4}$ from each incident 8^+-face, yielding a total of $\frac{1}{2}$. Therefore, we find that $\mu^*(v) \geq \mu(v) + \frac{1}{2} - \frac{3}{2} = 0$. If $k = 3$, then K is a 2^+-cluster. By rules (R2)–(R6), v contributes no more than 1 to K for $d(v) = 5$. If v is part of only a single cluster, then it follows that $\mu^*(v) = \mu(v) - 1 = 0$. Thus, let us consider the scenario where v is a 2-type vertex of other cluster, referred to as L. By Lemma 11, v is contained in two 8^+-faces. According to rules (R2)–(R5), v contributes $\frac{1}{2}$ to L when L is a special 4-, 5-cluster. By Lemma 12, K is not a special

cluster. According to rules (R2)–(R6), v contributes no more than $\frac{1}{2}$ to K. Then, $\mu^*(v) = \mu(v) - \frac{1}{2} \times 2 = 0$. The case of $k = 2$ can be proved similarly as $k = 3$.

3. $d(v) = 6$. Assume v is contained in no more than three clusters. By (R2)–(R7), v contributes no more than 2 to its incident cluster. Then, $\mu^*(v) \geq \mu(v) - 2 = 0$ when v is contained in only one cluster. If v is contained in three clusters, then it is a 2-type vertex for each cluster. According to Lemma 20, $\mu^*(v) = \mu(v) - \frac{1}{2} \times 3 > 0$. Therefore, we consider the case where v belongs to exactly two clusters, say K_1, K_2. Suppose firstly that v is a 2-type vertex for K_1. According to Lemma 20, assume v is a 4-type vertex to K_2 and v gives more than $\frac{3}{2}$ to K_2. By (R6)–(R7), v gives 2 to K_2, where K_2 is an internal 6- or 7-cluster. By (R6) and (R7), either v is a good but not special 6-vertex, or K_2 is a special 6-cluster containing two vertices in $\mathcal{A}_1$. In the latter situation, by Lemma 18, v is also good but not a special 6-vertex. In either case, v is a non-special 6-vertex, while K_2 is internal and special. By the definition of the special 6-vertex, K_1 is not a special 4-, 5-cluster. By (R2)–(R5), v gives no charge to K_1. In this case, $\mu^*(v) = \mu(v) - 2 = 0$. Now, let us consider the scenario where v acts as a 3-type vertex for K_1. Due to the symmetric properties, v is necessarily classified as a 3-type vertex in relation to K_2. According to rules (R2)–(R6), we can assume that K_i is 5^+-cluster for each $i \in \{1, 2\}$. By (R5)–(R6), assume v gives $\frac{3}{2}$ to K_1. Then, K_1 is an internal 6-cluster and v is a 3-type good 6-vertex which is not poor. By the definition of a poor 6-vertex, K_2 is not a special 5- or 6-cluster. Hence, v is a 3-type bad 6-vertex to K_2. By (R5)–(R6), v contributes at most $\frac{1}{2}$ to K_2. Thus, we have $\mu^*(v) = \mu(v) - \frac{3}{2} - \frac{1}{2} = 0$.
4. $d(v) = 7$. Then, v is contained in no more than three clusters and $\mu(v) = 3$. By Lemma 20, we only need to discuss the case that v is contained in two clusters, say K_1 and K_2, and v is classified as a 4-type vertex to K_1 and a 3-type vertex to K_2. Since v gives at most $\frac{3}{2}$ to K_2, we need only show the case that v gives at least 2 to K_1 cannot occurs by (R2)–(R7). Further, by rules (R2)–(R6), v contributes at most $\frac{5}{2}$ to K_1. If v contributes $\frac{5}{2}$ to K_1, then K_1 is an internal 7-cluster and contains two vertices in $\mathcal{A}_2 \cup \{\text{the special 8-vertex and } K_1 \text{ is type II}\}$ by (R7). By Lemma 17, K_2 is not special. According to (R2)–(R6), v contributes at most $\frac{1}{2}$ to K_2. Hence, we find that $\mu^*(v) \geq \mu(v) - \frac{5}{2} - \frac{1}{2} = 0$, and we complete this section. If v gives 2 to K_1, then one of the following applies: (i) v is a non-special 7-vertex, while K_1 is classified as an internal 7-cluster, lacking two vertices of $\mathcal{A}_2 \cup \{\text{the special 8-vertex and } K_1 \text{ is type II}\}$ by (R7); (ii) K_1 is an internal special 6-cluster that includes two vertices from $\mathcal{A}_1$ by (R6). When (i) occurs, K_2 is not a 6-cluster for G and contains no Near-bow-tie flaw. Following rules (R2)–(R5), v contributes no more than 1 to K_2. Therefore, we have $\mu^*(v) \geq \mu(v) - 2 - 1 = 0$, concluding our argument. When (ii) occurs, v is poor if K_2 is an internal and special 6-cluster. By the same rules (R2)–(R6), v contributes no more than 1 to K_2, again leading to $\mu^*(v) \geq \mu(v) - 2 - 1 = 0$, which completes the proof.
5. $d(v) = 8$. Hence, $\mu(v) = 4$. By applying Lemma 20, suppose v belongs to at least two clusters. If v is present in four clusters, then $\mu^*(v) \geq 4 - 4 \times \frac{1}{2} > 0$ according to Lemma 20. For the case where v belongs to three clusters, we have $\mu^*(v) \geq 4 - 2 \times \frac{3}{2} - \frac{1}{2} > 0$ or $\mu^*(v) \geq 4 - \frac{5}{2} - 2 \times \frac{1}{2} > 0$ by Lemma 20. Thus, suppose v is contained in just two clusters, say K_1 and K_2. By Lemma 20, suppose v gives $\frac{5}{2}$ to K_1 and v is a 4-type vertex of K_1. According to (R3)–(R7), K_1 is an internal 7-cluster and contains two vertices in $\mathcal{A}_2 \cup \{\text{the special 8-vertex and } K_1 \text{ is type II}\}$. We only need to prove cases that v gives 2 or $\frac{5}{2}$ to K_2 cannot occur. Suppose otherwise, by (R2)–(R7), one of the following applies: (i) K_2 is an internal and special 6-cluster and contains two vertices in $\mathcal{A}_1$; (ii) K_2 is an internal 7-cluster and v is a non-special 8-vertex. By Definition 6, v would indeed be a special 8-vertex if K_2 is an internal

7-cluster, contradicting the assumption that v is a non-special 8-vertex. This makes the second case (ii) impossible. Therefore, K_2 is an internal special 6-cluster containing two vertices from $\mathcal{A}_1$. Let $G' = G - K_1 \cup K_2$, where $L^*, \mathcal{M}^*_{(G-G',L^*)}$ be the residual list assignment and the residual cover of $G - G' = K_1 \cup K_2$. By the minimal property of G, G' admits $\mathcal{M}_{(G',L|_{G'})}$-coloring extending ϕ_0 with $|L^*(v)| \geq 4$. By Lemmas 4 and 14, v has at least two available colors to color K_2. Thus, by Lemmas 5 and 15, $K_1 \cup K_2$ has an $\mathcal{M}^*_{(K_1 \cup K_2, L^*)}$-coloring. By applying Lemma 7, an $\mathcal{M}_{(G,L)}$-coloring of G can be achieved, leading to a contradiction.

6. $d(v) \geq 9$. Then we have $\mu(v) \geq 5$. According to Lemma 20, $\mu^*(v) \geq 5 - 2 \times \frac{5}{2} = 0$, $\mu^*(v) \geq 5 - 3 \times \frac{3}{2} > 0$ or $\mu^*(v) \geq 5 - \frac{5}{2} - \frac{1}{2} - \frac{3}{2} > 0$.

Consider a face $f \neq C_0$ in G. By (R9), $\mu^*(f) = 0$. Now, we examine $\mu^*(C_0)$ in G. Consider that f_3 represents the number of 3-faces that have vertices in common with C_0 and let b denote the charge transferred to C_0 from other faces by (R9). Define s as the size of $[C_0, G - C_0]$. By Lemma 9 (i), $f_3 + s \geq 1$. If $f_3 \geq 1$, then $s \geq f_3 + 1$. If $f_3 = 0$, then $s \geq 1$, and so we also find $s \geq f_3 + 1$. Using rules (R8) and (R9), we can express $\mu^*(C_0)$ as follows:

$$\mu^*(C_0) = \mu(C_0) + \sum_{v \in V(C_0)} (d(v) - 4) - f_3 + b \tag{6}$$

$$= d(C_0) + 4 + \sum_{v \in V(C_0)} (d(v) - 2) - 2d(C_0) - f_3 + b$$

$$= 4 - d(C_0) + s - f_3 + b.$$

Since $s \geq f_3 + 1$, $\mu^*(C_0) \geq 4 - d(C_0) + 1 + b \geq 2 + b$ for $d(C_0) = 3$. If $b \geq 0$, then $\mu^*(C_0) > 0$, which completes this section.

Next, we prove that $b \geq 0$. Define $\mu'(f)$ as the charge assigned to face f following the application of rules (R1)–(R8). Clearly, $b = \sum_{f \in F \setminus \{C_0\}} \mu'(f)$. To begin with, let us examine the scenario where $d(f) = 4$. In this instance, f is not impacted by the discharging procedure. Thus, $\mu'(f) = \mu(f) = d(f) - 4 = 0$. If $d(f) = 5$, then f is adjacent to no more than a 3-face for G contains no 7-cycle adjacent to 4- or 5-cycles. By (R1), $\mu'(f) = \mu(f) - \frac{1}{2} > 0$. Now, assume f is a 6^+-face. Denote f to be adjacent to m 3-faces, meaning f has m edges shared with 3-faces. If f is a 6- or 7-face, then by (R1), $\mu'(f) = \mu(f) - \frac{\mu(f)}{d(f)} \times m - \frac{\mu(f)}{2d(f)} \times (d(f) - m) \times 2 = 0$. In the case of $d(f) \geq 8$, we find $\mu'(f) = \mu(f) - \frac{1}{2} \times m - \frac{1}{4} \times (d(f) - m) \times 2 = \frac{d(f)}{2} - 4 \geq 0$ by (R1). Finally, $d(f) = 3$. If f is a 3-face, then according to Lemma 8, f could be a subgraph of an i-cluster where $i \in [7]$.

Let K be an l-cluster. Note that C_0 is not a face of K. This means $|V(K) \cap V(C_0)| \leq 2$. We define the initial and the middle charges of K after applying (R1)–(R8), denoted by $\mu(K)$ and $\mu'(K)$, as the sum of the initial and middle charges of l 3-faces in it. Note that the final charge of every 3-face is 0 by (R9). We check $\mu'(K) \geq 0$ in the following.

1. $l = 1$. If K is not internal, then $\mu'(K) \geq \mu(K) + 1 = 0$ by (R8). Thus, assume that K is internal. Assume that f_i is the face adjacent to K for $i = 1, 2, 3$ with $d(f_1) \leq d(f_2) \leq d(f_3)$. If $d(f_1) = 4$, then $d(f_i) \geq 5$ and $d(f_i) \neq 6, 7$ for each $i \in \{2, 3\}$. By (R1), each f_i $(i = 2, 3)$ gives at least $\frac{1}{2}$ to K and $\mu'(K) \geq \mu(K) + \frac{1}{2} \times 2 \geq 0$. Thus, $d(f_1) \geq 5$. By (R1), $\mu'(K) \geq \mu(K) + 3 \times \frac{1}{3} = 0$.
2. $l = 2$. Assume that K is not internal. If $|V(C_0) \cap V(K)| = 2$, then by rule (R8), $\mu'(K) \geq \mu(K) + 1 \times 2 = 0$. So, suppose $|V(C_0) \cap V(K)| = 1$. According to Lemma 11(iii), K is adjacent to at least two 8^+-faces. Utilizing (R1) and (R8), $\mu'(K) \geq \mu(K) + 1 + \frac{1}{2} \times 2 = 0$.

   Thus, K is internal. By Lemma 11 (iii), the face adjacent to K is a 4-face or an 8^+-face and there are no more than two 4-faces. If all faces adjacent to K are 8^+-faces,

then by (R1), $\mu'(K) \geq \mu(K) + 4 \times \frac{1}{2} = 0$. Thus suppose one adjacent face is a 4-face. Utilizing (R2) and Lemma 9 (iii), each 3-type vertex of K gives $\frac{1}{4}$ to K. Then, according to (R1) and (R2), we find $\mu'(K) \geq \mu(K) + \frac{1}{2} \times 3 + \frac{1}{4} \times 2 = 0$ or $\mu'(K) \geq \mu(K) + \frac{1}{2} \times 2 + \frac{1}{4} \times 2 + \frac{1}{4} \times 2 = 0$.

3. $l = 3$. If $|V(C_0) \cap V(K)| = 2$, then K must be adjacent to at least two 8^+-faces. Utilizing (R1) and (R8), $\mu'(K) \geq \mu(K) + 2 + \frac{1}{2} \times 2 = 0$. If $|V(C_0) \cap V(K)| = 1$, then K is adjacent to a minimum of three 8^+-faces. By applying (R1), (R3) and (R8), it can be concluded that $\mu'(K) \geq \mu(K) + 1 + \frac{1}{2} \times 3 + \frac{1}{4} \times 2 \geq 0$ or $\mu'(K) \geq \mu(K) + 2 + \frac{1}{2} \times 3 + \frac{1}{4} \geq 0$. Thus $|V(C_0) \cap V(K)| = 0$. If the faces adjacent to K are all 8^+-faces, then $\mu'(K) \geq \mu(K) + \frac{1}{2} \times 5 + \frac{1}{4} \times 2 = 0$ by (R1) and (R3). Thus, according to Lemma 11(iii), there exists one adjacent 4-face which contains 4-type vertex of K. By applying (R1) and (R3), $\mu'(K) \geq \mu(K) + \frac{1}{2} \times 3 + \frac{1}{4} \times 2 + \frac{1}{2} \times 2 = 0$.
4. $l = 4$. By Lemma 8, K is a graph of $\{H_4^1, H_4^2, H_4^3, H_4^4\}$ shown in Figure 3. By Lemma 11 (ii), faces adjacent to K are 8^+-faces. If $|V(C_0) \cap V(K)| = t$, where $t \in \{1, 2\}$, then C_0 gives at least t to K when $K \in \{H_4^2, H_4^3, H_4^4\}$, $t + 1$ to K when $K \cong H_4^1$ by (R8). If $K \in \{H_4^2, H_4^3, H_4^4\}$, then there are at least $7 - t$ edges of K adjacent to 8^+-faces. By (R1) and (R8), $\mu'(K) \geq \mu(K) + (7 - t) \times \frac{1}{2} + t \geq 0$. If $K \cong H_4^1$, $\mu'(K) \geq \mu(K) + (5 - t) \times \frac{1}{2} + (t + 1) \geq 0$ by (R1) and (R8).
Now, suppose K is internal. Since H_4^1 has four 3-type 4^+-vertices and each of $\{H_4^2, H_4^3\}$ has two 3-type 4^+-vertices, $\mu'(K) \geq \mu(K) + 4 \times \frac{1}{2} + 4 \times \frac{1}{2} \geq 0$ or $\mu'(K) \geq \mu(K) + 6 \times \frac{1}{2} + 2 \times \frac{1}{2} \geq 0$ by (R1) and (R4). Suppose $K \cong H_4^4$. If K is special, then by (R1), (R4) and Lemma 10, $\mu'(K) \geq \mu(K) + 6 \times \frac{1}{2} + 3 \times \frac{1}{2} > 0$. Otherwise, $\mu'(K) \geq \mu(K) + 6 \times \frac{1}{2} + \frac{3}{2} > 0$.
5. $l = 5$. By Lemma 8, K is a graph of $\{H_5^1, H_5^2\}$ shown in Figure 3. By Lemma 11(ii), faces adjacent to K are 8^+-faces. If $|V(C_0) \cap V(K)| = 2$, then C_0 gives 3 to K by (R8). In this case, there are four edges of K adjacent to 8^+-faces. By (R1) and (R8), $\mu'(K) \geq \mu(K) + 3 + 4 \times \frac{1}{2} \geq 0$. If $|V(C_0) \cap V(K)| = 1$, then there are five edges of K adjacent to 8^+-faces. If K is H_5^1, then K has a 5-type 5-vertex and $\mu'(K) \geq \mu(K) + 2 + 5 \times \frac{1}{2} + 1 > 0$. Assume K is H_5^2. If $u_1 \in V(C_0)$, then $\mu'(K) \geq \mu(K) + 1 + 5 \times \frac{1}{2} + 2 \times 1 > 0$ when H_5^2 is special; otherwise, $\mu'(K) \geq \mu(K) + 1 + 5 \times \frac{1}{2} + \frac{1}{2} + \frac{3}{2} > 0$. If $u_1 \notin V(C_0)$, then $\mu'(K) \geq \mu(K) + 2 + 5 \times \frac{1}{2} + \frac{1}{2} \times 2 > 0$ by (R1), (R5) and (R8).
Thus, assume K is internal. If $K \cong H_5^1$, then K contains five 3-type bad 4^+-vertices and one 5-type 5-vertex. By (R1) and (R5), $\mu'(K) \geq \mu(K) + \frac{1}{2} \times 5 + \frac{1}{2} \times 5 + 1 > 0$. Assume K is H_5^2. If K is special, then K contains two 3-type good 5^+-vertices and one 2-type good 5^+-vertex. By (R1) and (R5), $\mu'(K) \geq \mu(K) + \frac{1}{2} \times 5 + 1 \times 2 + \frac{1}{2} = 0$. Otherwise, $\mu'(K) \geq \mu(K) + \frac{1}{2} \times 5 + \frac{3}{2} + \frac{1}{2} \times 2 = 0$.
6. $l = 6$. By Lemma 8, K is the graph H_6 shown in Figure 3. By Lemma 11(ii), the faces adjacent to K are 8^+-faces. If $|V(C_0) \cap V(K)| = 2$, then C_0 gives 4 to K by (R8). In this case, there are three edges of K adjacent to 8^+-faces. Due to symmetry, suppose that $u_1, u_3 \in V(C_0)$. Then u_4 is a 3-type 4^+-vertex. By (R6), u_4 gives at least $\frac{1}{2}$ to K. By (R1) and (R8), $\mu'(K) \geq \mu(K) + 3 \times \frac{1}{2} + 4 + \frac{1}{2} = 0$. If $|V(C_0) \cap V(K)| = 1$, then C_0 contributes at least 2 to K by (R8) and there are four edges of K adjacent to 8^+-faces. Suppose firstly, K is special. Then, K has at least two internal 5^+-vertices. Applying (R1), (R6), and (R8), we find that $\mu'(K) \geq \mu(K) + 4 \times \frac{1}{2} + 2 + 2 \times 1 \geq 0$. Suppose secondly, that K is not special. Then, K contains one 4-type 5^+-vertex and one 3-type bad 4^+-vertex. Utilizing (R1), (R6), and (R8), $\mu'(K) \geq \mu(K) + 4 \times \frac{1}{2} + 2 + \frac{3}{2} + \frac{1}{2} \geq 0$.
Thus, assume K is internal. If K is special, then K contains two 3-type good 5^+-vertices and one 4-type good 5^+-vertex by Lemma 10. If $u_1, u_4 \in \mathcal{A}_1$, then $d(u_5) \geq 6$ by Lemma 18. This means u_5 is a 4-type good 6^+-vertex and K contains two ver-

tices of $\mathcal{A}_1$. In this case, $\mu'(K) \geq \mu(K) + 4 \times \frac{1}{2} + 2 + 2 \times 1 = 0$ by (R1) and (R6). Then, assume $u_1 \notin \mathcal{A}_1$. By (R6), u_1 gives at least $\frac{3}{2}$ to K. By (R1) and (R6), $\mu'(K) \geq \mu(K) + 4 \times \frac{1}{2} + 1 + 2 \times \frac{3}{2} = 0$. Suppose K is not special. In this situation, u_i is classified as a 4-type 5^+-vertex for each $i \in \{3,5\}$. By (R1) and (R6), $\mu'(K) \geq \mu(K) + 4 \times \frac{1}{2} + 2 \times \frac{3}{2} + 2 \times \frac{1}{2} = 0$.

7. $l = 7$. By Lemma 8, K is the graph H_7 shown in Figure 3. By Lemma 11(ii), faces adjacent to K are 8^+-faces. If $|V(C_0) \cap V(K)| = 2$, then C_0 gives 5 to K by (R8) and according to Lemma 10, K contains one 4-type 5^+-vertex . If $|V(C_0) \cap V(K)| = 1$, then C_0 contributes 3 to K and K contains two 4-type 5^+-vertices similarly. By (R1), (R7) and (R8), $\mu'(K) \geq \mu(K) + 5 + 2 \times \frac{1}{2} + \frac{3}{2} > 0$ or $\mu'(K) \geq \mu(K) + 3 + 3 \times \frac{1}{2} + 2 \times \frac{3}{2} > 0$. Thus assume K is internal. By Lemma 10, $d(u_i) \geq 5$ for each $i \in \{1,4,5\}$. By Lemma 19(ii), there are at most two vertices of $\{u_1, u_4, u_5\}$ in $\mathcal{A}_2 \cup \{$the special 8-vertex and K is type II$\}$. If any vertex of $\{u_1, u_4, u_5\}$ is not in $\mathcal{A}_2 \cup \{$the special 8-vertex and K is type II$\}$, then $\mu'(K) \geq \mu(K) + 3 \times \frac{1}{2} + 3 \times 2 > 0$. If there is only one vertex in $\mathcal{A}_2 \cup \{$the special 8-vertex and K is type II$\}$, then $\mu'(K) \geq \mu(K) + 3 \times \frac{1}{2} + \frac{3}{2} + 2 \times 2 = 0$. Thus assume there are two vertices in $\mathcal{A}_2 \cup \{$the special 8-vertex and K is type II$\}$, say u_4, u_5. By Lemma 19(i), $d(u_1) \geq 7$. By (R1) and (R7), $\mu'(K) \geq \mu(K) + 3 \times \frac{1}{2} + 2 \times \frac{3}{2} + \frac{5}{2} = 0$.

5. Conclusions

DP-k-coloring has gained significant attention as a generalization of k-choosability. We demonstrate that any planar graph G, which does not have a Near-bow-tie as an induced subgraph and lacks 7-cycles adjacent to 4-cycles or 5-cycles, is DP-4-colorable. If the planar graph G does not have 7-cycles and a butterfly, then it still satisfies our condition. Thus, our results generalize the findings of Kim et al. [20] and promote research in the area of DP-4-colorability for planar graphs. Future research can focus on determining the DP-4-colorability on planar graphs such as $\chi_{DP}(G) = 4$ for any planar graph G without 7-cycles adjacent to 4-cycles.

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