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On Smooth Solution to Three-Dimensional Incompressible Navier–Stokes Equations Based on Numerical Solutions by Finite Element Approximation

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Abstract

In this paper, we develop a fully discrete finite element scheme, based on a second-order backward differentiation formula (BDF2), for numerically solving the three-dimensional incompressible Navier–Stokes equations. Under the assumption that the fully discrete solution remains bounded in a certain norm, we establish that any smooth initial data necessarily gives rise to a unique strong solution that remains smooth. Moreover, we demonstrate that the fully discrete numerical solution converges strongly to this exact solution as the temporal and spatial discretization parameters approach zero.

Keywords: incompressible Navier–Stokes equations; BDF2 scheme; finite element method; smooth solution

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1. Introduction

Consider the 3D incompressible Navier–Stokes equations

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla u - \mu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{0}, \\ \operatorname{div} \boldsymbol{u} = 0, \end{cases} \quad (x,t) \in \Omega \times (0,T], \tag{1}$$

where $\Omega \subset \mathbb{R}^3$ is a convex polyhedral domain, which satisfies compatibility conditions and has smooth boundary $\partial\Omega$, $\boldsymbol{u} = \boldsymbol{u}(x,t) \in \mathbb{R}^3$ and $p = p(x,t) \in \mathbb{R}$ are unknown velocity field and pressure, respectively, $\mu > 0$ represents the viscosity coefficient of the flow. We consider system (1) subject to the following Dirichlet boundary condition

$$\boldsymbol{u}(x,t)|_{\partial\Omega} = 0, \tag{2}$$

and the initial condition

$$\boldsymbol{u}(x,0) = \boldsymbol{u}^0(x), \quad \text{for } x \in \Omega. \tag{3}$$

As usual, one imposes the condition $\int_\Omega p(x,t)dx = 0$ for the uniqueness of pressure.

The three-dimensional incompressible Navier–Stokes equations serve as a cornerstone mathematical framework in fluid dynamics, with broad applicability across scientific and engineering disciplines such as meteorology, aerodynamics, and oceanic modeling. It is worth noting that the global existence of weak solutions to system (1) can be traced back to the work of Leray [1] and Hopf [2]. However, the global existence of strong solutions to system (1) with general smooth initial data remains an open challenge, primarily due to the strong nonlinearity and complex structure of the equations. A common approach to circumvent this difficulty has been to assume that the initial data is sufficiently small. Under such smallness conditions, several classical works have successfully established global well-posedness. For instance, Fujita and Kato [3] proved global existence for small initial data in $H^s(\mathbb{R}^3)$ with $s \geq \frac{1}{2}$. Subsequent improvements and extensions were obtained by Kato [4] in $L^3(\mathbb{R}^3)$, by Cannone [5] and Planchon [6] in Besov spaces, by Koch and Tataru [7] in the larger space BMO^{-1}, and by Lei and Lin [8] in the Lei–Lin space. In addition, significant progress has been made on regularity criteria for strong solutions to the 3D incompressible Navier–Stokes equations; we refer the reader to [9–12] and the references therein for further details.

Over the past several decades, a wide variety of numerical methods have been developed for approximating solutions to the incompressible Navier–Stokes equations. For instance, finite element methods are discussed in [13–17], finite difference methods in [18–20], Lagrange–Galerkin methods in [21–23], and spectral methods in [24–26]. It is important to note that previous convergence analyses of numerical schemes for the 3D Navier–Stokes equations have universally relied on the assumption that an exact smooth solution exists. This naturally leads to the question: if a numerical solution remains bounded in certain norms, what does this imply about the regularity of the true solution? To date, two relevant studies have addressed this issue: one by Li [27] and another by Cai and Zhang [28]. Both works show that for any $M > 0$, there exist positive constants τ_M and h_M such that, provided the time step $\tau \leq \tau_M$ and the mesh size $h \leq h_M$, boundedness of the numerical solution in specific norms implies both the existence of a unique smooth solution to the continuous problem (1)–(3) and the convergence of the numerical approximation to this true solution. The primary distinction between the two lies in the temporal discretization: Li employs a backward Euler scheme, whereas Cai and Zhang use a Crank–Nicolson approach.

In this paper, we study a numerical method based on a second-order backward differentiation formula (BDF2) for time discretization and finite elements for spatial discretization. As a widely adopted approach (see [29–34] and references therein), the BDF scheme provides higher-order temporal accuracy and improved stability over the backward Euler method. Furthermore, unlike the Crank–Nicolson scheme, the BDF2 method is a linear fully implicit scheme that is straightforward to implement and computationally efficient, as only linear systems need to be solved at each time step. Compared to the work of Cai and Zhang [28], our analysis requires weaker qualitative assumptions on the solution $\boldsymbol{u}$ of problem (1)–(3). Specifically, we only assume the same regularity conditions as in Li [27], yet still achieve higher-order convergence estimates comparable to those in [28].

Remark 1. *We note that the existence of weak solutions to the three-dimensional incompressible Navier–Stokes equations was established in seminal works by Leray [1] and Hopf [2]. Specifically, for an initial condition $\boldsymbol{u}_0 \in L^2(\Omega)$ with $\nabla \cdot \boldsymbol{u}_0 = 0$, where $\Omega \subseteq \mathbb{R}^3$ is either a bounded domain or the whole space, they proved the existence of a unique weak solution $\boldsymbol{u}$ in the space $L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\boldsymbol{H}^1(\Omega))$. However, due to the presence of the nonlinear convective term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$, standard PDE techniques are insufficient to derive higher-order a priori estimates. Consequently, it remains a major open problem and one of the seven Millennium Prize Problems to establish, without restrictions on the initial data, the existence of a strong solution $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}^k(\Omega)) \cap L^2(0,T;\boldsymbol{H}^{k+1}(\Omega))$ for any integer $k \geq 1$.*

In mathematical analysis, a common strategy to circumvent this difficulty is to assume the initial data are sufficiently small, which allows the conclusion of global strong solutions. In *numerical analysis, an alternative approach is taken: by assuming the numerical solution remains bounded in certain discrete norms, one can deduce the existence of a strong solution to the continuous problem. In this sense, the boundedness assumption on the numerical solution in our work plays a role analogous to the smallness assumption on the initial data in classical theoretical studies.*

The remainder of this paper is structured as follows. Section 2 introduces necessary notations and preliminary results, and states the main theorems. The detailed proof of the main result is then presented in Section 3.

2. Notations and Main Results

We employ standard notation for Lebesgue and Sobolev spaces. For any $p \in [1, \infty]$ and $K > 0$, let $P^p(\Omega)$ and $W^{k,p}(\Omega)$ be abbreviated as L^p and $W^{k,p}$, endowed with the norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$, respectively. When $p = 2$, we denote $H^k = W^{k,2}$ with norm $\|\cdot\|_{H^k}$. The norm and inner product in $L^2(\Omega)$ are written simply as $\|\cdot\|$ and $(\cdot, \cdot)$.

Define the space of zero-mean square-integrable functions as

$$\|L_0^2(\Omega) = \left\{ v \in L^2(\Omega) | \int_\Omega v dx = 0 \right\}.$$

Let $W_0^{k,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ and let $H_0^k(\Omega) = W_0^{k,2}(\Omega)$. Furthermore, the vector-valued Sobolev spaces are denoted by $\boldsymbol{W}^{k,p}(\Omega) = (W^{k,p}(\Omega))^3$, $\boldsymbol{H}_0^k(\Omega) = (H_0^k(\Omega))^3$ and $\boldsymbol{L}^p(\Omega) = (L^p(\Omega))^3$. The following vector-valued Sobolev spaces will be used frequently in the following:

$$\begin{aligned}
&\dot{\boldsymbol{D}}(\Omega) = \{\boldsymbol{v} \in C_0^\infty(\Omega)^3 : \mathrm{div}\boldsymbol{v} = 0\}, \\
&\dot{\boldsymbol{L}}(\Omega) = \text{The completion of } \dot{\boldsymbol{D}}(\Omega) \text{ in } \boldsymbol{L}^2(\Omega), \\
&\dot{\boldsymbol{H}}^1(\Omega) = \text{The completion of } \dot{\boldsymbol{D}}(\Omega) \text{ in } \boldsymbol{H}^1(\Omega), \\
&\dot{\boldsymbol{H}}^2(\Omega) = \dot{\boldsymbol{H}}^1(\Omega) \bigcap \boldsymbol{H}^2(\Omega), \\
&\boldsymbol{H}^{-1}(\Omega) = \text{The dual space of } \dot{\boldsymbol{H}}^1(\Omega).
\end{aligned}$$

In this paper, for convenience, we denote by $\|\cdot\|_{W^{k,p}}$ the norms of both $W^{k,p}(\Omega)$ and $\boldsymbol{W}^{k,p}(\Omega)$, denote by $\|\cdot\|_{H^k}$ the norms of both $H^k(\Omega)$ and $\boldsymbol{H}^k(\Omega)$, and denote by $\|\cdot\|_{L^p}$ the norms of both $L^p(\Omega)$ and $\boldsymbol{L}^p(\Omega)$. Denote the norm of $\dot{\boldsymbol{H}}^{-1}$ by $\|\cdot\|_{H^{-1}}$.

Let $\mathcal{J}_h$ be a quasi-uniform triangulation of the convex polyhedral domain Ω, consisting of tetrahedral elements $\mathcal{K}jj = 1^J$, where the mesh size is defined as $h = \max 1 \le j \le J\mathrm{diam}(\mathcal{K}_j)$. In order to discretize problem (1)–(3), we need to use a finite element space $\boldsymbol{X}_h \times V_h \subset \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ that satisfies

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{L^q} \le Ch^{l+1+\frac{3}{q}-\frac{3}{2}} \|\boldsymbol{v}\|_{H^{l+1}}, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \bigcap \boldsymbol{H}^{l+1}(\Omega), \tag{4}$$

$$\inf_{q_h \in V_h} \|q - q_h\|_{L^2} \le Ch\|q\|_{H^1}, \quad \forall q \in H^1(\Omega), \tag{5}$$

$$\|q_h\|_{L^2(\Omega)} \le C \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h, \boldsymbol{v}_h \neq \boldsymbol{0}} \frac{|(\nabla q_h, \boldsymbol{v}_h)|}{\|\nabla \boldsymbol{v}_h\|_{L^2(\Omega)}}, \quad \forall q_h \in V_h, \tag{6}$$

for $l = 0, 1$ and $q \in [2, 6]$, where C is a positive constant independent of h. We also need to assume that

$$\mathrm{div}\boldsymbol{v}_h \in V_h, \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \tag{7}$$

which ensures that the discrete divergence-free functions satisfy the divergence-free condition pointwise—a crucial property for the numerical approximation of problem (1)–(3).

Remark 2. *The condition specified in (7), namely that div,$\boldsymbol{v}_h \in V_h$ for all $\boldsymbol{v}_h \in \boldsymbol{X}_h$ and hence that discrete velocities are pointwise divergence-free, is only achievable with certain specialized finite elements, such as the Scott–Vogelius pair on barycentrically refined meshes. Standard MINI elements do not possess this property, and the original text does not specify any particular element choice or mesh condition to fulfill this requirement.*

Let $\{t_i\}_{i=0}^N$ be a uniform partition of the time interval $[0, T]$ with mesh size $\tau = \frac{T}{N}$. Then, for a sequence of functions g_n ($n = 1, 2, \cdots, N-1$), one defines

$$D_\tau g^{n+1} = \frac{3g^{n+1} - 4g^n + g^{n-1}}{2\tau}, \quad \bar{D}_\tau g^{n+1} = \frac{g^{n+1} - g^n}{\tau}, \quad \hat{g}^n = 2g^n - g^{n-1},$$

a BDF2 mixed finite element method of Navier–Stokes problem (1)–(3) is defined as: for given $\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n-1} \in \boldsymbol{X}_h$, $p_h^n \in V_h$, find $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}) \in \boldsymbol{X}_h \times V_h$ such that

$$(D_\tau \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + \mu(\nabla \boldsymbol{u}_h^{n+1}, \nabla \boldsymbol{v}_h) - (\boldsymbol{u}_h^{n+1}, \hat{\boldsymbol{u}}_h^n \cdot \nabla \boldsymbol{v}_h) - (p_h^{n+1}, \nabla \cdot \boldsymbol{v}_h) = 0, \tag{8}$$

and

$$(\nabla \cdot \boldsymbol{u}_h^{n+1}, q_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h,\ q_h \in V_h, \tag{9}$$

where $u_h^0 := \boldsymbol{R}_h \boldsymbol{u}^0(x)$ is the Stokes-Ritz projection of $\boldsymbol{u}^0$ onto $\boldsymbol{X}_h$, and $\boldsymbol{u}_h^1$ can be provided by a backward Euler finite element method:

$$(\bar{D}_\tau \boldsymbol{u}_h^1, \boldsymbol{v}_h) + \mu(\nabla \boldsymbol{u}_h^1, \nabla \boldsymbol{v}_h) - (\boldsymbol{u}_h^1, \hat{\boldsymbol{u}}_h^0 \cdot \nabla \boldsymbol{v}_h) - (p_h^1, \nabla \cdot \boldsymbol{v}_h) = 0, \tag{10}$$

$$(\nabla \cdot \boldsymbol{u}_h^1, q_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h,\ q_h \in V_h, \tag{11}$$

Then, for the solution $\boldsymbol{u}_h^n$ given by (8)–(11), one defines the piecewise constant numerical solution

$$\boldsymbol{u}_{h,\tau}(x,t) = \boldsymbol{u}_h^n(x), \quad x \in \Omega,\ t \in (t_{n-1}, t_n]. \tag{12}$$

By taking $(\boldsymbol{v}_h, q_h) = (\boldsymbol{u}_h^{n+1}, p_h^{n+1})$ in (8) and (9), note that

$$2(3a - 4b + c, a) = |a|^2 - |b|^2 + |2a - b|^2 - |2b - c|^2 + |a - 2b + c|^2,$$

we have

$$\frac{1}{4\tau}(\|\boldsymbol{u}_h^{n+1}\|_{L^2}^2 - \|\boldsymbol{u}_h^n\|_{L^2}^2 + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_{L^2}^2 - \|2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_{L^2}^2 + \|\|\boldsymbol{u}_h^{n+1} - 2\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}\|_{L^2}^2)$$
$$+ \mu\|\nabla \boldsymbol{u}_h^{n+1}\|_{L^2}^2 = 0.$$

Summing up from the time step t_1 to t_{n+1}, it yields that

$$\frac{1}{4}(\|\boldsymbol{u}_h^{n+1}\|_{L^2}^2 + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_{L^2}) + \mu\tau \sum_{n=1}^{N} \|\nabla \boldsymbol{u}_h^{n+1}\|_{L^2}^2 \leq \frac{1}{4}(\|\boldsymbol{u}_h^1\|_{L^2}^2 + \|2\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0\|_{L^2})$$
$$\leq \frac{1}{4}(5\|\boldsymbol{u}_h^1\|_{L^2}^2 + 2\|\boldsymbol{u}_h^0\|_{L^2}),$$

where $n = 1, 2, \cdots, N-1$. Taking $(\boldsymbol{v}_h, q_h) = (\boldsymbol{u}_h^1, p_h^1)$ in (10) and (11), it is easy to see that

$$\|\boldsymbol{u}_h^2\|_{L^2}^2 + 2\mu\tau\|\nabla \boldsymbol{u}_h^1\|_{L^2}^2 \leq \|\boldsymbol{u}_h^0\|_{L^2}^2.$$

Then, based on the above two inequalities, we easily derive that

$$\frac{1}{4}(\|\boldsymbol{u}_h^{n+1}\|_{L^2}^2 + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|_{L^2}) + \mu\tau \sum_{n=1}^{N} \|\nabla \boldsymbol{u}_h^{n+1}\|_{L^2}^2 \leq 2\|\boldsymbol{u}_h^0\|_{L^2}^2. \tag{13}$$

Next, we present our main result in the following theorem.

Theorem 1. *For any $M > 0$, there exist positive constants τ_M and h_M–both decreasing in M–that are independent of the solution $\boldsymbol{u}$, the initial data $\boldsymbol{u}^0$, and the time T, but may depend on the viscosity coefficient μ, such that when*

$$\tau < \tau_M, \quad h < h_M, \tag{14}$$

if a numerical solution $\boldsymbol{u}_{h,\tau}$ defined by (12) satisfies

$$\|\boldsymbol{u}_{h,\tau}\|_{L^\infty(0,T;L^4)} + \|\boldsymbol{u}^0\|_{H^2} + 1 \leq M, \tag{15}$$

then there exists a unqiue strong solution for problem (1)–(3) with regularity

$$\boldsymbol{u} \in L^\infty(0,T;\dot{H}^2), \quad \partial_t \boldsymbol{u} \in L^\infty(0,T;\dot{L}^2). \tag{16}$$

In the end of this section, we would like to provide a table of auxiliary functions and their properties (Table 1).

Table 1. Auxiliary functions and their properties.

Auxiliary Functions	Properties
α: decreasing function	depend on Ω and does not depend on $\boldsymbol{u}$ and T
Φ: increasing function	depend on Ω and does not depend on $\boldsymbol{u}$ and T, satisfies $\Phi(s) \geq s$

We remark that the constants C in the following are positive constants which are not only independent of h and τ, but also independent of the unknown solution u.

3. Proof of Theorem 1

Let $P : \boldsymbol{L}^2 \to \dot{\boldsymbol{L}}^2$ denote the L^2-orthogonal projection onto the space of divergence-free functions. Then, the H^2-regularity estimate of linear Stokes equations, as established in [35], implies that

$$\|\boldsymbol{v}\|_{H^2} \leq C\|P\Delta \boldsymbol{v}\|_{L^2} \leq C\|\Delta \boldsymbol{v}\|_{L^2}, \quad \forall \boldsymbol{v} \in \dot{\boldsymbol{H}}^2(\Omega). \tag{17}$$

Now, we give two Lemmas, which were proved in [27], and will be used in the proof of Theorem 1.

Lemma 1 ([27])**.** *There exists an increasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ such that if $\boldsymbol{u}^0 \in \dot{\boldsymbol{H}}^2$ and the Navier–Stokes problem (1)–(3) has a weak solution $\boldsymbol{u} \in L^\infty(0,T;\dot{\boldsymbol{H}}^2)$, then the solution has regularity (16) and satisfies the following quantitative estimate:*

$$\begin{aligned}
&\|\partial_{tt}\boldsymbol{u}\|_{L^2(0,T;\dot{H}^{-1})} + \|\partial_t \boldsymbol{u}\|_{L^\infty(0,T;L^2)} + \|\partial_t \boldsymbol{u}\|_{L^2(0,T;H^1)} \\
&+ \|\boldsymbol{u}\|_{L^2(0,T;H^2)} + \|\boldsymbol{u}\|_{L^\infty(0,T;H^2)} + \|p\|_{L^\infty(0,T;H^1)} \\
\leq &\Phi(\|\boldsymbol{u}\|_{L^\infty(0,T;L^4)} + \|\boldsymbol{u}^0\|_{\dot{H}^2}),
\end{aligned}$$

where the function Φ does not dependent on $\boldsymbol{u}$ and T.

Lemma 2 ([27])**.** *There exists a decreasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ such that if $\boldsymbol{u}^0 \in \dot{\boldsymbol{H}}^2$, then the strong solution exists on $[0, \alpha(\|\boldsymbol{u}^0\|_{H^2})]$ and satisfies*

$$\|\boldsymbol{u}\|_{L^\infty(0,\alpha(\boldsymbol{u}^0\|_{H^2});H^1)} \leq \|\boldsymbol{u}^0\|_{H^1} + 1,$$

where the function α does not depend on $\boldsymbol{u}$ and T.

To analyze the convergence behavior of the fully discrete scheme, we define the Stokes–Ritz projection operator $(\boldsymbol{R}_h, P_h) : H_0^1(\Omega) \times L^2(\Omega) \to \boldsymbol{X}_h \times V_h$:

$$\mu(\nabla(W - \boldsymbol{R}_h(W,p)), \nabla \boldsymbol{v}_h) - (p - P_h(W,p), \nabla \cdot \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \tag{18}$$

$$(\nabla \cdot \boldsymbol{R}_h(W,p), q_h) = 0, \quad \forall q_h \in V_h, \tag{19}$$

and the condition $\int_\Omega [p - P_h(\boldsymbol{u},p)]dx = 0$ is imposed to ensure uniqueness. This Stokes–Ritz projection exhibits the following approximation properties:

$$\|W - \boldsymbol{R}_h(W,p)\|_{L^q} \le Ch^{l-\frac{1}{2}+\frac{3}{q}}(\|W\|_{H^{l+1}} + \|p\|_{H^l}), \tag{20}$$

$$\|W - \boldsymbol{R}_h(W,p)\|_{H^1} + \|p - P_h(W,p)\|_{L^2} \le Ch^l(\|W\|_{H^{l+1}} + \|p\|_{H^l}), \tag{21}$$

for any $(W,p) \in \dot{H}^2(\Omega) \times H^1(\Omega)$, $2 \le q \le 6$ and $l = 0,1$. If we assume that $\bar{\boldsymbol{X}}_h$ is the divergence-free subspace of $\boldsymbol{X}_h$, then $\boldsymbol{R}_h(\boldsymbol{u},p) \in \bar{\boldsymbol{X}}_h$, $P_h(\boldsymbol{u},p) \in V_h$, and

$$\mu(\nabla(\boldsymbol{u} - \boldsymbol{R}_h(\boldsymbol{u},p)), \nabla \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \bar{\boldsymbol{X}}_h, \tag{22}$$

$$(p - P_h(\boldsymbol{u},p), \nabla \cdot \boldsymbol{v}_h) = (\nabla(\boldsymbol{u} - \boldsymbol{R}_h(\boldsymbol{u},p), \nabla v_h), \quad \forall \boldsymbol{v}_h \in X_h,\ v_h \in V_h. \tag{23}$$

Since the Stokes operator is independent of the pressure; hence, the term $\|p\|_{H^l}$ can be removed in (21). Therefore, we have [27,28]

$$h^{\frac{1}{2}-\frac{3}{q}}\|W - \boldsymbol{R}_h(W,p)\|_{L^q} + \|W - \boldsymbol{R}_h(W,p)\|_{H^1} \le Ch^l\|W\|_{H^{l+1}}, \tag{24}$$

$$\|p - P_h(W,p)\|_{L^2} \le Ch^l(\|W\|_{H^{l+1}} + \|p\|_{H^l}). \tag{25}$$

We also introduce the following inverse inequality [28]:

$$\|v_h\|_{W^{k,p}} \le Ch^{m-k+\frac{n}{p}-\frac{n}{q}}\|v_h\|_{W^{m,q}}, \quad 0 \le m \le k \le \infty,\ 1 \le q \le p \le \infty, \tag{26}$$

where $v_h \in \boldsymbol{X}_h$ or V_h and n is the spatial dimension.

Next, one proves the main results of Theorem 1. Define the positive constant M in Theorem 1 as

$$M := \|\boldsymbol{u}_{h,\tau}\|_{L^\infty(0,T;L^4)} + \|\boldsymbol{u}^0\|_{H^2} + 1. \tag{27}$$

Hence, the condition $\Phi(s) > s$ in Lemma 2 can be transformed as $\Phi(M) \ge M \ge 1$. In the following, to obtain the main results, we state a primary claim:

Claim 1 ([27,28])**.** *For each $k = 0,1,\cdots,N$, there exists a unique strong solution*

$$blue\boldsymbol{u} \in L^\infty(0,t_k;\dot{H}^2) \bigcap L^2(0,t_{k+1};\dot{H}^3),$$

of system (1)–(3) such that $\|\boldsymbol{u}_{h,\tau} - \boldsymbol{u}\|_{L^\infty(0,t_k;L^4)} \le 1$.

Note that $\boldsymbol{u}_h^0$ is the Stokes–Ritz projection of $\boldsymbol{u}^0$, we have $\|\boldsymbol{u}_h^0 - \boldsymbol{u}^0\|_{L^4} \le C_0 h^{\frac{5}{4}}\|\boldsymbol{u}^0\|_{H^2}$ for some positive constant C_0. Hence, if

$$h < (C_0 M)^{-\frac{4}{5}} \le (C_0\|\boldsymbol{u}^0\|_{H^2})^{-\frac{4}{5}},$$

then Claim 1 holds for $k = 0$. Further, according to ([27], Lemma 1), we easily see that Claim 1 also holds for $k = 1$. Then, one assumes that Claim 1 holds for $k \le n$, which implies

$$\begin{aligned} &\|\boldsymbol{u}\|_{L^\infty(0,t_k;L^4)} + \|\boldsymbol{u}^0\|_{H^2} \\ \le& \|\boldsymbol{u}_{h,\tau} - \boldsymbol{u}\|_{L^\infty(0,t_k;L^4)} + \|\boldsymbol{u}_{h,\tau}\|_{L^\infty(0,t_k;L^4)} + \|\boldsymbol{u}^0\|_{H^2} \\ \le& 1 + \|\boldsymbol{u}_{h,\tau}\|_{L^\infty(0,t_k;L^4)} + \|\boldsymbol{u}^0\|_{H^2} \le M. \end{aligned} \tag{28}$$

Employing Lemma 2 in conjunction with the inductive hypothesis and inequality (28), we conclude that there exists a unique strong solution $\boldsymbol{u} \in L^\infty(0, t_m; \dot{H}^2) \bigcap W^{1,\infty}(0, t_k; \dot{L}^2)$ defined in interval $(0, t_k)$ can be extended to $(0, t_{k+1})$, i.e.,

Problem (1)–(3) has a unique strong solution

$$\boldsymbol{u} \in L^\infty(0, t_{k+1}; \dot{H}^2) \bigcap L^2(0, t_{k+1}; \dot{H}^3), \tag{29}$$

satisfies

$$\begin{aligned} &\|\partial_{tt}\boldsymbol{u}\|_{L^2(0,t_{k+1};\dot{H}^{-1})} + \|\partial_t \boldsymbol{u}\|_{L^\infty(0,t_{k+1};\dot{L}^2)} + \|\partial_t \boldsymbol{u}\|_{L^2(0,t_{k+1};\dot{H}^1)} \\ &+ \|\boldsymbol{u}\|_{L^2(0,t_{k+1};\dot{H}^3)} + \|\boldsymbol{u}\|_{L^\infty(0,t_{k+1};\dot{H}^2)} + \|p\|_{L^\infty(0,t_{k+1};H^1)} \\ \le& \Phi(\|\boldsymbol{u}\|_{L^2(0,t_k;\dot{L}^4)} + \|\boldsymbol{u}^0\|_{H^2}) \\ \le& \Phi(M). \end{aligned} \tag{30}$$

Under the regularity (30), the solution $\boldsymbol{u}$ satisfies

$$(D_\tau \boldsymbol{u}^{n+1}, \boldsymbol{v}_h) - (\boldsymbol{u}^{n+1}, \hat{\boldsymbol{u}}^n \cdot \nabla \boldsymbol{v}) + \mu(\nabla \boldsymbol{u}, \boldsymbol{v}) - (p^{n+1}, \nabla \cdot \boldsymbol{v}_h) = (F_1^{n+1}, \boldsymbol{v}) + (F_2^{n+1}, \nabla \boldsymbol{v}), \tag{31}$$

$$(\nabla \cdot \boldsymbol{u}^{n+1}, q_h) = 0, \tag{32}$$

for $n = 1, 2, \cdots, k$, where

$$F_1^{n+1} = D_\tau \boldsymbol{u}^{n+1} - \partial_t \boldsymbol{u}^{n+1}, \tag{33}$$

$$F_2^{n+1} = -\hat{\boldsymbol{u}}^n \otimes \boldsymbol{u}^{n+1} + \boldsymbol{u}^{n+1} \otimes \boldsymbol{u}^{n+1}, \tag{34}$$

and the truncation errors of temporal discretization satisfies

$$\begin{aligned} &\sum_{n=1}^{k} \tau \|F_1^{n+1}\|^2_{\dot{H}^{-1}} + \sum_{n=1}^{k} \tau \|F_2^{n+1}\|^2_{L^2} \\ \le& C\|D_\tau \boldsymbol{u}^{n+1} - \partial_t \boldsymbol{u}^{n+1}\|^2_{L^2(0,t_{k+1};\dot{H}^{-1})} + C\|(\hat{\boldsymbol{u}}^n - \boldsymbol{u}^{n+1}) \otimes \boldsymbol{u}^{n+1}\|^2_{L^2(0,t_{k+1};L^2)} \\ \le& C\tau^4(\|\partial_{tt}\boldsymbol{u}\|^2_{L^2(0,t_{k+1};\dot{H}^{-1})} + \|\boldsymbol{u}\|^2_{L^\infty(0,t_{k+1};L^4)} \|\partial_{tt}\boldsymbol{u}\|^2_{L^2(0,t_{k+1};L^4)}) \\ \le& C\tau^4 \Phi^4(M), \end{aligned} \tag{35}$$

where the Taylor formula is used to get the second-to-last inequalities.

Assume that

$$e_{\boldsymbol{u}}^{n+1} := \boldsymbol{R}_h \boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1}, \quad e_p^{n+1} := P_h(\boldsymbol{u}^{n+1}, p^{n+1}) - p_h^{n+1}.$$

Subtracting (31) from (8), then we derive that

$$\begin{aligned} &(D_\tau e_{\boldsymbol{u}}^{n+1} \boldsymbol{v}_h) + \mu(\nabla e_{\boldsymbol{u}}^{n+1}, \nabla \boldsymbol{v}_h) - (e_p^{n+1}, \nabla \cdot \boldsymbol{v}_h) \\ =&(F_1^{n+1}, \boldsymbol{v}_h) + (F_2^{n+1}, \nabla \boldsymbol{v}_h) + (D_\tau(\boldsymbol{R}_h \boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1}), \boldsymbol{v}_h) \\ &+ (\boldsymbol{u}^{n+1} - \boldsymbol{R}_h \boldsymbol{u}^{n+1}, \hat{\boldsymbol{u}}^n \cdot \nabla \boldsymbol{v}_h) + (e_{\boldsymbol{u}}^{n+1}, \hat{\boldsymbol{u}}^n \cdot \nabla \boldsymbol{v}_h) \\ &+ (\boldsymbol{u}^{n+1}, (\hat{\boldsymbol{u}}^n - \boldsymbol{R}_h \hat{\boldsymbol{u}}^n) \cdot \nabla \boldsymbol{v}_h) + (\boldsymbol{u}^{n+1}, \hat{e}_{\boldsymbol{u}}^n \cdot \nabla \boldsymbol{v}_h) \\ =&\sum_{i=1}^{7} I_i(\boldsymbol{v}_h). \end{aligned} \tag{36}$$

Subtracting (32) from (9) yields

$$(\nabla \cdot e_{\boldsymbol{u}}^{n+1}, q_h) = 0. \tag{37}$$

Taking $\boldsymbol{v}_h = e_{\boldsymbol{u}}^{n+1}$ and $q_h = e_p^{n+1}$ in (36) and (37), respectively, summing them up, we deduce that

$$\begin{aligned} &\frac{1}{4\tau}(\|e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2 - \|e_{\boldsymbol{u}}^n\|_{L^2}^2 + \|2e_{\boldsymbol{u}}^{n+1} - e_{\boldsymbol{u}}^n\|_{L^2}^2 - \|2e_{\boldsymbol{u}}^n - e_{\boldsymbol{u}}^{n-1}\|_{L^2}^2 + \|e_{\boldsymbol{u}}^{n+1} + 2e_{\boldsymbol{u}}^n - e_{\boldsymbol{u}}^{n-1}\|_{L^2}^2) \\ &+ \mu\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2 = \sum_{i=1}^{6} I_i(e_{\boldsymbol{u}}^{n+1}). \end{aligned} \tag{38}$$

Based on (24) and (25), the terms on the right-hand side of (38) can be bounded as

$$I_1(e_{\boldsymbol{u}}^{n+1}) = (F_1^{n+1}, e\boldsymbol{u}^{n+1}) \le C\|F_1^{n+1}\|_{H^{-1}}^2 + \varepsilon\|\nabla e\boldsymbol{u}^{n+1}\|_{L^2}^2, \tag{39}$$

$$I_2(e_{\boldsymbol{u}}^{n+1}) = (F_2^{n+1}, \nabla e\boldsymbol{u}^{n+1}) \le C\|F_2^{n+1}\|_{L^2}^2 + \varepsilon\|\nabla e\boldsymbol{u}^{n+1}\|_{L^2}^2, \tag{40}$$

$$\begin{aligned} I_3(e_{\boldsymbol{u}}^{n+1}) =&(D_\tau(\boldsymbol{R}_h \boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1}), e_{\boldsymbol{u}}^{n+1}) \\ \le& Ch\|D_\tau \boldsymbol{u}^{n+1}\|_{H^1}\|e_{\boldsymbol{u}}^{n+1}\|_{L^2} \\ \le& Ch\|D_\tau \boldsymbol{u}^{n+1}\|_{H^1}\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2} \\ \le& \varepsilon\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2 + Ch^2\|D_\tau \boldsymbol{u}^{n+1}\|_{H^1}^2, \end{aligned} \tag{41}$$

$$\begin{aligned} I_4(e_{\boldsymbol{u}}^{n+1}) =&(\boldsymbol{u}^{n+1} - \boldsymbol{R}_h \boldsymbol{u}^{n+1}, \hat{\boldsymbol{u}}^n \cdot \nabla e_{\boldsymbol{u}}^{n+1}) \\ \le& \|\boldsymbol{u}^{n+1} - \boldsymbol{R}_h \boldsymbol{u}^{n+1}\|_{L^6}\|\hat{\boldsymbol{u}}^n\|_{L^3}\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2} \\ \le& \|\boldsymbol{u}^{n+1} - \boldsymbol{R}_h \boldsymbol{u}^{n+1}\|_{H^1}\|\hat{\boldsymbol{u}}^n\|_{L^3}\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2} \\ \le& Ch\|\boldsymbol{u}^{n+1}\|_{H^2}\|\nabla \hat{\boldsymbol{u}}^n\|_{L^2}\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2} \\ \le& \varepsilon\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2 + Ch^2\|D_\tau \boldsymbol{u}^{n+1}\|_{H^2}^2\|\|\nabla \hat{\boldsymbol{u}}^n\|_{L^2}^2, \end{aligned} \tag{42}$$

$$I_5(e_{\boldsymbol{u}}^{n+1}) = (e_{\boldsymbol{u}}^{n+1}, \hat{\boldsymbol{u}}^n \cdot \nabla e_{\boldsymbol{u}}^{n+1}) = 0, \tag{43}$$

$$\begin{aligned} I_6(e_{\boldsymbol{u}}^{n+1}) =&(\boldsymbol{u}^{n+1}, (\hat{\boldsymbol{u}}^n - \boldsymbol{R}_h \hat{\boldsymbol{u}}^n) \cdot \nabla e_{\boldsymbol{u}}^{n+1}) \\ \le& \|\boldsymbol{u}_h^{n+1}\|_{L^3}\|\hat{\boldsymbol{u}}^n - \boldsymbol{R}_h \hat{\boldsymbol{u}}^n\|_{L^6}\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2} \\ \le& C\|\boldsymbol{u}_h^{n+1}\|_{H^1}\|\nabla(\hat{\boldsymbol{u}}^n - \boldsymbol{R}_h \hat{\boldsymbol{u}}^n)\|_{L^2}\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2} \\ \le& C\|\nabla \boldsymbol{u}_h^{n+1}\|_{L^2} h\|\hat{\boldsymbol{u}}^n\|_{H^2}\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2} \\ \le& \varepsilon\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2 + Ch^2\|\nabla \boldsymbol{u}_h^{n+1}\|_{L^2}^2\|\hat{\boldsymbol{u}}^n\|_{H^2}^2, \end{aligned} \tag{44}$$

$$\begin{aligned} I_7(e_{\boldsymbol{u}}^{n+1}) =&(\boldsymbol{u}^{n+1}, \hat{e}_{\boldsymbol{u}}^n \cdot \nabla e_{\boldsymbol{u}}^{n+1}) \\ =&-(e_{\boldsymbol{u}}^{n+1}, \hat{e}_{\boldsymbol{u}}^n \cdot \nabla e_{\boldsymbol{u}}^{n+1}) + (\boldsymbol{R}_h\boldsymbol{u}^{n+1}, \hat{e}_{\boldsymbol{u}}^n \cdot \nabla e_{\boldsymbol{u}}^{n+1}) \\ =&(\boldsymbol{R}_h\boldsymbol{u}^{n+1}, \hat{e}_{\boldsymbol{u}}^n \cdot \nabla e_{\boldsymbol{u}}^{n+1}) \\ \le&\|\boldsymbol{R}_h\boldsymbol{u}^{n+1}\|_{L^\infty}\|\hat{e}_{\boldsymbol{u}}^n\|_{L^2}\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2} \\ \le&C\|\boldsymbol{R}_h\boldsymbol{u}^{n+1}\|_{H^2}\|\hat{e}_{\boldsymbol{u}}^n\|_L^2\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2} \\ \le&\varepsilon\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2 + C\|\boldsymbol{u}\|_{L^\infty(0,t_{k+1};H^2)}^2\|\hat{e}_{\boldsymbol{u}}^n\|_{L^2}^2, \end{aligned} \tag{45}$$

Supposing that $\varepsilon = \frac{\mu}{12}$, adding (38)–(45) together gives

$$\begin{aligned} &\frac{1}{2\tau}(\|e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2 - \|e_{\boldsymbol{u}}^n\|_{L^2}^2 + \|2e_{\boldsymbol{u}}^{n+1} - e_{\boldsymbol{u}}^n\|_{L^2}^2 - \|2e_{\boldsymbol{u}}^n - e_{\boldsymbol{u}}^{n-1}\|_{L^2}^2 + \|e_{\boldsymbol{u}}^{n+1} \\ &+ 2e_{\boldsymbol{u}}^n - e_{\boldsymbol{u}}^{n-1}\|_{L^2}^2) + \mu\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2 \\ \le&C(\|F_1^{n+1}\|_{H^{-1}}^2 + \|F_2^{n+1}\|_{L^2}^2) \\ &+ Ch^2(\|D_\tau\boldsymbol{u}^{n+1}\|_{H^1}^2 + \|\boldsymbol{u}^{n+1}\|_{H^2}^2\|\|\nabla\hat{\boldsymbol{u}}^n\|_{L^2}^2 + \|\nabla\boldsymbol{u}_h^{n+1}\|_{L^2}^2\|\hat{\boldsymbol{u}}^n\|_{H^2}^2) \\ &+ C\|\boldsymbol{u}\|_{L^\infty(0,t_{k+1};H^2)}^2\|\hat{e}_{\boldsymbol{u}}^n\|_{L^2}^2. \end{aligned} \tag{46}$$

Note that

$$\nabla\boldsymbol{u}^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\nabla\boldsymbol{u}(t)dt + \frac{1}{\tau}\int_{t_{n-1}}^{t_n}(s - t_{n-1})\partial_t\nabla\boldsymbol{u}(s)ds. \tag{47}$$

Using (30) and (35), we arrive at

$$\sum_{n=1}^{k}\tau(\|F_1^{n+1}\|_{H^{-1}}^2 + \|F_2^{n+1}\|_{L^2}^2) \le C\Phi^4(M)\tau^4, \tag{48}$$

$$\sum_{n=1}^{k}\tau\|D_\tau\boldsymbol{u}^{n+1}\|_{H^1}^2 \le C\|\partial_t\boldsymbol{u}\|_{L^2(0,t_{k+1};H^1)}^2 \le C\Phi^2(M), \tag{49}$$

and

$$\begin{aligned} &\sum_{n=1}^{k}\tau(\|\boldsymbol{u}^{n+1}\|_{H^2}^2\|\|\nabla\hat{\boldsymbol{u}}^n\|_{L^2}^2 + \|\nabla\boldsymbol{u}_h^{n+1}\|_{L^2}^2\|\hat{\boldsymbol{u}}^n\|_{H^2}^2) \\ \le&C\|\boldsymbol{u}\|_{L^\infty(0,t_{k+1};H^2)}^2\left(\sum_{n=1}^{k}\tau(\|\nabla\hat{\boldsymbol{u}}^n\|_{L^2}^2 + \|\nabla\boldsymbol{u}_h^{n+1}\|_{L^2}^2\right) \\ \le&C\Phi^2(M)\left(\|\nabla\boldsymbol{u}\|_{L^2(0,t_{k+1};L^2)}^2 + \tau^2\|\partial_t\nabla\boldsymbol{u}\|_{L^2(0,t_{k+1};L^2)}^2 + \tau\|\nabla\boldsymbol{u}^0\|_{L^2}^2 + \|\boldsymbol{u}^0\|_{L^2}^2\right) \\ \le&C\Phi^4(M). \end{aligned} \tag{50}$$

Plugging (48)–(50) into (46), one obtains

$$\begin{aligned} &\frac{1}{2}(\|e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2 + \|2e_{\boldsymbol{u}}^{n+1} - e_{\boldsymbol{u}}^n\|_{L^2}^2) + \mu\sum_{n=1}^{k}\tau\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2 \\ \le&\frac{1}{2}(\|e_{\boldsymbol{u}}^1\|_{L^2}^2 + \|2e_{\boldsymbol{u}}^1 - e_{\boldsymbol{u}}^0\|_{L^2}^2) + C\Phi^4(M)(\tau^4 + h^2) + C\tau\sum_{n=1}^{k}\|\boldsymbol{u}\|_{L^\infty(0,t_{k+1};H^2)}^2\|\hat{e}_{\boldsymbol{u}}^n\|_{L^2}^2. \end{aligned} \tag{51}$$

Cai and Zhang [28] proved that

$$\|e_{\boldsymbol{u}}^1\|_{L^2}^2 + \mu\tau\|\nabla e_{\boldsymbol{u}}^1\|_{L^2}^2 \le C\Phi^4(M)(\tau^4 + h^2), \tag{52}$$

as $\tau \leq \tau_1$, where τ_1 is a positive constant. We remark that M, which is only depends on the L^4-norm of numerical solution and the H^2-norm of the initial data, satisfies (27). Therefore, by (51) and (52) and Gronwall inequality, we find that

$$\begin{aligned}
&\max_{0\leq n\leq k}(\|e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2+\|2e_{\boldsymbol{u}}^{n+1}-2_{\boldsymbol{u}}^{n}\|_{L^2}^2)+2\mu\tau\sum_{n=1}^{k}\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2\\
\leq&\exp\left(C\sum_{n=1}^{k}\tau\|\boldsymbol{u}\|_{L^\infty(0,t_{n+1};H^2)}^2\right)\Phi^4(M)(\tau^4+h^2)\\
\leq&\exp\left(C\|\boldsymbol{u}\|_{L^\infty(0,t_{k+1};H^2)}^2\right)\Phi^4(M)(\tau^4+h^2)\\
\leq&\exp\left(C\Phi^2(M)\right)\Phi^4(M)(\tau^4+h^2)\\
\leq&\exp\left(C\Phi^4(M)\right)(\tau^4+h^2).
\end{aligned} \tag{53}$$

Next, applying the inverse inequality $\|e_{\boldsymbol{u}}^{n+1}\|_{L^4} \leq Ch^{-\frac{3}{4}}\|e_{\boldsymbol{u}}^{n+1}\|_{L^2}$ and the Sobolev embedding inequality $\|e_{\boldsymbol{u}}^{n+1}\|_{L^4} \leq C\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2}$, we have the following inequality:

$$\begin{aligned}
\max_{0\leq n\leq k}\|e_{\boldsymbol{u}}^{n+1}\|_{L^4}^2\leq&\min\left(Ch^{-\frac{3}{2}}\max_{0\leq n\leq k}\|e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2, C\max_{0\leq n\leq k}\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2}\right)\\
\leq&C\min(h^{-\frac{3}{2}},\tau^{-2})\left(\max_{0\leq n\leq k}\|e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2+\tau\sum_{n=0}^{k}\|\nabla e_{\boldsymbol{u}}^{n+1}\|_{L^2}^2+\tau\|\nabla e_{\boldsymbol{u}}^{1}\|_{L^2}^2\right)\\
\leq&\exp(C_1\Phi^4(M))(\tau^2+h^{\frac{1}{2}}),
\end{aligned} \tag{54}$$

where we have used (52) and (53) in the above estimates. Furthermore, for any $t \in (t_n, t_{n+1}]$ and $n = 0, 1, \cdots, k$, we have

$$\begin{aligned}
\max_{t\in(t_n,t_{n+1}]}\|\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}^{n+1}\|_{L^4}^2\leq&\max_{t\in(t_n,t_{n+1}]}(2\|\boldsymbol{u}(t)-\boldsymbol{u}^{n+1}\|_{L^4}^2+2\|\boldsymbol{u}^{n+1}-\boldsymbol{R}_h\boldsymbol{u}^{n+1}\|_{L^4}^2)\\
\leq&C\tau^2\|\partial_t\boldsymbol{u}\|_{L^\infty(t_n,t_{n+1};H^1)}^2+C\|\boldsymbol{u}^{n+1}-\boldsymbol{R}_h\boldsymbol{u}^{n+1}\|_{H^1}^2\\
\leq&C\tau^2\|\partial_t\boldsymbol{u}\|_{L^\infty(0,t_{n+1};H^1)}^2+Ch^2\|\boldsymbol{u}\|_{L^\infty(0,t_{n+1};H^2)}^2\\
\leq&C\Phi^2(M)(\tau^2+h^2)\\
\leq&\exp(C_2\Phi^4(M))(\tau^2+h^2).
\end{aligned} \tag{55}$$

Combining (54) and (55) together gives

$$\|\boldsymbol{u}_{h,\tau}-\boldsymbol{u}\|_{L^\infty(0,t_{k+1};L^4)}^2\leq\exp(C_3\Phi^4(M))(\tau^2+h^{\frac{1}{2}}), \tag{56}$$

that is

$$\|\boldsymbol{u}_{h,\tau}-\boldsymbol{u}\|_{L^\infty(0,t_{k+1};L^4)}^2\leq 1, \tag{57}$$

as $\tau^2 + h^{\frac{1}{2}} \leq \exp(-C_3\Phi^4(M))$.

Assume that

$$\begin{aligned}
\tau_0=&\min\left\{\phi(M),\tau_1,\frac{1}{2}\exp\left(-\frac{C_3}{2}\Phi^4(M)\right)\right\},\\
h_0=&\min\left\{(C_0M)^{-\frac{4}{5}},\frac{1}{2}\exp\left(-2C_3\Phi^4(M)\right)\right\}.
\end{aligned}$$

Then, on the basis of (29), (30) and (57), the mathematical induction of Claim 1 is complete. Consequently, the existence and uniqueness of the strong solution

$$\boldsymbol{u}\in L^\infty(0,T;\dot{H}^2)\bigcap W^{1,\infty}(0,T;\dot{L}^2)$$

is proved, which satisfies

$$\|\boldsymbol{u}\|_{L^\infty(0,T;L^4)} \le \|\boldsymbol{u}_{h,\tau}\|_{L^\infty(0,T;L^4)} + 1.$$

Then, we complete the proof of Theorem 1.

4. Conclusions

Existing studies on the convergence of numerical methods for the three-dimensional incompressible Navier–Stokes equations typically assume the existence of a sufficiently smooth exact solution. However, due to the nonlinear convective term and the intrinsic complexity of the equations, the global existence of strong solutions—in either bounded or unbounded domains—remains a major open problem. This raises a fundamental question: if a numerical solution remains bounded in certain norms, what can be deduced about the regularity of the true solution?

Partial answers were provided by Li [27] and Cai and Zhang [28], who showed for the backward Euler and Crank–Nicolson schemes, respectively, that boundedness of the numerical solution in specific norms implies the existence of a unique smooth solution to the continuous problem (1)–(3), and guarantees convergence of the numerical approximation to this strong solution.

In this work, we extend this analysis to the BDF2 scheme and prove that, under similar boundedness conditions on the discrete solution, there also exists a unique smooth solution to the continuous problem, and that the numerical solution converges to this strong solution. It should be noted that the primary aim of this paper is to establish the theoretical connection between the strong solution of the Navier–Stokes system and the numerical solution of the BDF2 discretization. Since numerical experiments and practical implementations of the BDF2 scheme have already been discussed in works such as [36,37], we do not focus on numerical examples here.

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