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# Cyclic, LCD, and Self-Dual Codes over the Non-Frobenius Ring $GR(p^2, m)[u]/\langle u^2, pu\rangle$

**Sami Alabiad * and Alhanouf Ali Alhomaidhi**

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; aalhomaidhi@ksu.edu.sa

* Correspondence: ssaif1@ksu.edu.sa

**Abstract**

Let $p$ be a prime number and $m$ be a positive integer. In this paper, we investigate cyclic codes of length $n$ over the local non-Frobenius ring $R = GR(p^2, m)[u]$, where $u^2 = 0$ and $pu = 0$. We first determine the algebraic structure of cyclic codes of arbitrary length $n$. For the case $\gcd(n, p) = 1$, we explicitly describe the generators of cyclic codes over $R$. Moreover, we establish necessary and sufficient conditions for the existence of self-dual and LCD codes, together with their enumeration. Several illustrative examples and tables are presented, highlighting the mass formula for cyclic self-orthogonal codes, cyclic LCD codes, and families of new cyclic codes that arise from our results.

**Keywords:** cyclic code; self-dual code; LCD code; non-frobenius ring

**MSC:** 11T71; 94B15; 94B05

## 1. Introduction

Cyclic codes constitute a basic category of linear codes in algebraic coding theory, owing to their intricate algebraic structure and effective encoding and decoding algorithms. Historically, these codes have been analyzed over finite fields, where they relate to ideals of the polynomial quotient ring $\frac{\mathbb{F}_{p^m}[x]}{\langle x^n-1\rangle}$. This connection facilitates the examination of cyclic codes via the factorization of $x^n - 1$ into irreducible polynomials [1]. When the code length $n$ is divisible by the characteristic of the underlying field, repeated-root cyclic codes emerge, which have been examined in relation to finite fields and chain rings [2–8].

Over the past three decades, research has expanded from fields to finite rings, with particular emphasis on cyclic codes over chain rings such as $\mathbb{Z}_{p^m}$ and Galois rings $GR(p^n, m)$. In these settings, cyclic codes have been systematically studied with respect to their generators, duals, and distance properties [9–12]. The Frobenius property of these rings plays a central role, guaranteeing the validity of key duality results such as MacWilliams' theorem and the identity $|C||C^{\perp}| = |R|^n$ [13]. The literature on cyclic and negacyclic codes over Frobenius rings is now extensive: Berman [1] initiated the study of repeated-root cyclic codes; Dinh [7,14] provided substantial contributions on constacyclic codes over Galois rings; Abualrub et al. [9] and Dougherty et al. [12] investigated cyclic codes over $\mathbb{Z}_4$ and modular rings; and further developments on cyclic codes over Galois rings of length $p^s$ were given by Alabiad et al. [15], Kiah et al. [10], and Jasbir et al. [11].

However, much less is known about cyclic codes over non-Frobenius rings, which do not admit these classical duality properties. Such rings arise naturally in algebra and coding theory, and recent studies have shown that they can produce interesting new

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families of codes. For instance, Samei and Alimoradi [16] examined cyclic codes over the non-Frobenius local ring $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2$, whereas Yildiz and Karadeniz [17] explored the structure of $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$. Research on self-dual and LCD codes encompasses the contributions of Massey [18], Shi et al. [19,20], and Sok et al. [21], among others. These studies illustrate both the challenges and opportunities of moving beyond the Frobenius framework.

Motivated by this direction, the present paper examines cyclic codes over the local non-Frobenius ring

$$R = \frac{GR(p^2, m)[u]}{\langle u^2, pu \rangle}, \qquad u^2 = pu = 0. \tag{1}$$

We provide a detailed description of cyclic codes of arbitrary length $n$ over this ring and, in the case where $\gcd(n, p) = 1$, we explicitly identify their generators. Furthermore, we establish the criteria for the existence of self-dual and linear complementary dual (LCD) codes and derive their enumerations. To support these theoretical results, we present worked examples and tables that clarify the mass formulae for cyclic self-orthogonal codes and enumerate families of cyclic LCD codes. In this way, this paper extends classical findings on cyclic codes over finite fields and chain rings to a broader class of local non-Frobenius rings. It also complements existing investigations of non-chain local rings such as those in [16,17], thereby advancing the understanding of duality and structural properties of codes in algebraic contexts where the Frobenius property is absent.

This paper is structured as outlined below. Section 2 presents the fundamentals of Galois rings, linear codes, cyclic codes, and duality, which furnish the requisite framework for our investigation. Section 3 examines the structure of cyclic codes over the Galois ring $GR(p^2, m)$, with particular emphasis on the case when $n = p^s$ to improve some previous related results (Cyclic Codes When $n = p^s$ (Corollary 2)). In Section 4, we focus on cyclic codes within the local non-Frobenius ring $R = \frac{GR(p^2,m)[u]}{\langle u^2,pu \rangle}$, characterized by the relations $u^2 = pu = 0$. First, we provide the algebraic structure of the ring $R$ and the lattice of its ideals. Section 4.1 generalizes these findings to cyclic codes over $R$ of any length $n$. In Section 4.2, we focus on the scenario where $\gcd(n, p) = 1$ and explicitly identify generators, dual codes, and the criteria for the presence of self-dual and LCD codes. Section 5 delineates mass formulae for cyclic codes over $R$ under the condition that $\gcd(n, p) = 1$, encompassing enumerative findings for self-orthogonal and LCD codes, alongside illustrative examples and tables that underscore newly derived quasi-cyclic codes. Throughout, illustrative examples and tables are included to enhance comprehension of the primary conclusions and constructs, including novel families of cyclic codes derived from our investigation.

## 2. Preliminaries

Let $p$ be prime and $m$ be a positive integer. Consider the Galois ring $GR(p^2, m)$ and define

$$R = \frac{GR(p^2, m)[u]}{\langle u^2, pu \rangle}, \tag{2}$$

with $u^2 = 0$ and $pu = 0$. Then, $R$ is a local non-chain ring whose unique maximal ideal is $J = \langle u, p \rangle$. Suppose $\xi$ generates the cyclic group $\mathbb{F}_{p^m}^*$ of order $p^m - 1$; then, each element of $GR(p^2, m)$ admits the unique $p$-adic expansion

$$\beta_0 + p\beta_1, \qquad \beta_i \in \Omega(m),$$

where $\Omega(m) = \langle \xi \rangle \cup \{0\} = \{0, 1, \xi, \xi^2, \ldots, \xi^{p^m-2}\}$. Moreover,

$$R = GR(p^2, m) + uGR(p^2, m),$$

so every $r \in R$ has a unique decomposition $r = r_0 + ur_1$ with $r_0, r_1 \in GR(p^2, m)$.

The unit group of $R$ has cardinality $(p^m - 1)p^{2m}$, while the set of non-units contains $p^{3m} - (p^m - 1)p^m$ elements. An element $r = r_0 + ur_1$ is a unit if and only if $r_0 \neq 0$ in $GR(p^2, m)$. Furthermore, any unit $v \in R$ can be written as

$$v = \omega(1 + p\beta_0 + u\beta_2), \tag{3}$$

where $\omega \in \Omega^*(m)$ and $\beta_i \in \Omega(m)$.

By a linear code of length $n$ over the ring $R$, we mean an $R$-submodule of $R^n$. Throughout this paper, all codes are assumed to be linear. An element of a code $C$ is called a *codeword*, and a codeword of length $n$ is represented as $(c_0, c_1, \ldots, c_{n-1})$ with entries $c_i \in R$ for $0 \leq i < n$.

Define the shift operator $\eta : R^n \to R^n$ by

$$\eta\big((c_0, c_1, \ldots, c_{n-1})\big) = (c_{n-1}, c_0, \ldots, c_{n-2}).$$

A code $C$ is said to be *cyclic* of length $n$ if $\eta(C) = C$. More generally, for integers $m, l \in \mathbb{N}$, a code of length $ml$ is called *quasi-cyclic of index $l$* provided that $\eta^{(l)}(C) = C$ and $l$ is the smallest such integer. Clearly, the case $l = 1$ corresponds exactly to cyclic codes.

Every codeword in $R^n$ can be associated with the polynomial in the residue class ring

$$\Re(R, n) = \frac{R[x]}{\langle x^n - 1 \rangle}. \tag{4}$$

This correspondence implies that a cyclic code $C$ of length $n$ over $R$ is naturally identified with an ideal $I$ in $\Re(R, n)$ (see [22]).

The dual of a code $C \subseteq R^n$ is defined based on the standard Euclidean inner product on $R^n$, and it is given by

$$C^\perp = \{d \in R^n : c \cdot d = 0 \text{ for all } c \in C\}. \tag{5}$$

A code $C$ is called *self-orthogonal* if $C \subseteq C^\perp$, *self-dual* if $C = C^\perp$, and *linear complementary dual (LCD)* if $C \cap C^\perp = \{0\}$.

For a polynomial $\lambda(x) \in \mathbb{F}_{p^m}[x]$, the reciprocal polynomial is defined by

$$\lambda^*(x) = x^{\deg(\lambda)}\lambda\Big(x^{-1}\Big). \tag{6}$$

It is easy to verify that $(\lambda(x)\pi(x))^* = \lambda^*(x)\pi^*(x)$ for all $\lambda(x), \pi(x) \in \mathbb{F}_{p^m}[x]$. The same identity holds in the quotient ring $\Re(\mathbb{F}_{p^m}, n) = \frac{\mathbb{F}_{p^m}[x]}{\langle x^n - 1 \rangle}$ whenever $\deg(\lambda\pi) < n$. Moreover, if $C$ is a cyclic code of length $n$ over $R$ with corresponding ideal $I \subseteq \Re(R, n)$, then the dual code $C^\perp$ corresponds to the ideal generated by $\{\lambda^* : \lambda \in \mathrm{Ann}(I)\}$.

We emphasize that there is a distinction between the notations used later to express ideals of $\Re(R, n)$. The single brackets $\langle \cdot \rangle$ denote the ideal (or code) generated by the listed elements in the usual sense. The double brackets $\langle\langle \cdot \rangle\rangle$, introduced here for convenience, indicate not only that the ideal is generated by the specified elements but also that this generating set is unique, and we call it *the representation*.

## 3. Various Structures of Cyclic Codes over $GR(p^2, m)$ of Length $n$

In this section, we investigate cyclic codes over $GR(p^2, m)$ of arbitrary length $n$. Cyclic codes of prime-power length over this ring have been studied in [10,15]. We first establish the representation for both cyclic codes without restrictions on $n$ and subsequently specialize our results to the case $n = p^s$ to improve the related existing results.

Assume $\hat{\lambda}(x) = \dfrac{x^n - 1}{\lambda(x)}$, where $\lambda(x) \in \Re(\mathbb{F}_{p^m}, n)$.

**Theorem 1.** *Suppose $C$ is a cyclic code of length $n$ over $GR(p^2, m)$. Then,*

$$C = \langle\langle \lambda_0(x) + p\pi_1(x), p\lambda_1(x) \rangle\rangle, \tag{7}$$

*where $\lambda_0(x), \pi_1(x), \lambda_1(x) \in \Re(\mathbb{F}_{p^m}, n)$, satisfying the following:*

$$\begin{cases} \lambda_1(x) \mid \lambda_0(x) \mid x^n - 1, \quad deg(\pi_1) < deg(\lambda_1) \text{ or } \pi_1(x) = 0, \\ \lambda_1(x) \mid \pi_1(x)\hat{\lambda}_0(x). \end{cases}$$

**Proof.** Let $\nu_p : \Re(GR(p^2, m), n)\rangle \to \Re(\mathbb{F}_{p^m}, n)$ be the map $\nu_p(r_0 + pr_1) = r_0$ for each $r_0, r_1 \in \Re(\mathbb{F}_{p^m}, n)$. Then, $\nu_p$ is a surjective ring homomorphism, and $\nu_p(C)$ is an ideal of $\Re(\mathbb{F}_{p^m}, n)$. Since $\Re(\mathbb{F}_{p^m}, n)$ is a principal ideal ring, there exists $\lambda_0(x) \in \Re(\mathbb{F}_{p^m}, n)$ such that $\lambda_0(x) \mid x^n - 1$ and $\nu_p(C)$ is the ideal of $\Re(\mathbb{F}_{p^m}, n)$ generated by $\lambda_0(x)$.

On the other hand, as $\ker(\nu_p) \cap C$ is an $\Re(\mathbb{F}_{p^m}, n)$-submodule of $p\Re(\mathbb{F}_{p^m}, n)$, there exists $\lambda_1(x) \in \Re(\mathbb{F}_{p^m}, n)$ such that $\lambda_1(x) \mid x^n - 1$ and $\ker(\nu_p) \cap C = \langle p\lambda_1(x) \rangle$. Therefore, we have

$$C = \langle \lambda_0(x) + p\pi_1(x), p\lambda_1(x) \rangle$$

for some $\pi_1(x) \in \mathbb{F}_{p^m}[x]$.

Additionally, $C$ can be constructed as

$$C = \langle \lambda_0(x) + p(\pi_1(x) - \lambda_1(x)h(x)), p\lambda_1(x) \rangle,$$

for each $h(x) \in \mathbb{F}_{p^m}[x]$. By the division algorithm, we can assume that $\deg(\pi_1) < \deg(\lambda_1)$ when $\deg(\lambda_1) \neq 0$ and $\pi_1(x) \neq 0$, while $\pi_1(x) = 0$ when $\deg(\lambda_1) = 0$.

It also follows that since $p\lambda_0(x) = p(\lambda_0(x) + p\pi_1(x)) \in C \cap \ker(\nu_p) = p\lambda_1(x)\Re(\mathbb{F}_{p^m}, n)$, we must have $\lambda_1(x) \mid \lambda_0(x)$. Similarly, since

$$p\pi_1(x)\hat{\lambda}_0(x) = (\lambda_0(x) + p\pi_1(x))\hat{\lambda}_0(x) \in C \cap \ker(\nu_p) = \langle p\lambda_1(x) \rangle,$$

we must have $\lambda_1(x) \mid \pi_1(x)\hat{\lambda}_0(x)$ in $\mathbb{F}_{p^m}[x]$. Thus, $C = \langle \lambda_0(x) + p\pi_1(x), p\lambda_1(x) \rangle$, and the uniqueness is forward. Therefore, $C$ has the representation

$$C = \langle\langle \lambda_0(x) + p\pi_1(x), p\lambda_1(x) \rangle\rangle. \tag{8}$$

*Cyclic Codes When $n = p^s$*

If we let $n = p^s$, then one can see that $x - 1$ is nilpotent in $\Re(GR(p^2, m), p^s)$,; in fact, $(x-1)^{2p^s} = 0$. Thus, the elements of $\Re(GR(p^2, m), p^s)$ are written in terms of $x - 1$ and coefficients of $\mathbb{F}_{p^m}$. In particular, the polynomials $\lambda_0(x)$, $\lambda_1(x)$ and $\pi_1(x)$ are used. This means the following

$$\lambda_0(x) = (x-1)^{\iota_0} p_1(x) \quad \text{and} \quad \lambda_1(x) = (x-1)^{\iota_1} p_2(x), \tag{9}$$

where $p_1(x)$ and $p_2(x)$ are either zeros or units in $\Re(\mathbb{F}_{p^m}, n)$. Hence, we have some results in this case. □

**Corollary 1.** *If $C$ is a cyclic code in $\Re(GR(p^2, m), p^s)$, then $C$ takes the form*

$$C = \langle\langle (x-1)^{\iota_0} + p(x-1)^e b(x), p(x-1)^{\iota_1} \rangle\rangle, \tag{10}$$

*where* $0 \leq \iota_0 < p^s, 0 \leq \iota_1 \leq min\{p^{s-1}, \iota_0\}$ *and* $e + deg(b) < \iota_1$.

**Proof.** Observe that $\pi_1(x) = (x-1)^e b(x)$, where $b(x)$ is a unit or $b(x) = 0$. By construction in Theorem 1, we have $\deg(\pi_1) = e + \deg(b) < \iota_1$. Now, it is enough to show that $\iota_1 \leq p^{s-1}$. Note that Theorem 1 implies that $\lambda_1(x) \mid [-\lambda_0(x) + \lambda_0(x) + p\pi_1(x)]\hat{\lambda}_0(x)$, which leads to $(x-1)^{\iota_1} \mid [-(x-1)^{\iota_0} + (x-1)^{\iota_0} + p\pi_1(x)](x-1)^{p^s-\iota_0}$. Because $\lambda_1(x) \mid \lambda_0(x)$, hence $(x-1)^{\iota_1} \mid [(x-1)^{\iota_0} + p\pi_1(x)](x-1)^{p^s-\iota_0}$. So, forward computations result in $(x-1)^{\iota_1} \mid [-p(x-1)p^{s-1}\vartheta_{p,s} + p(x-1)p^{s-\iota+th(x)}]$, and thus $(x-1)^{\iota_1} \mid p(x-1)p^{s-1}\left(-\vartheta_{p,s} + (x-1)p^{s-\iota_0-p^{s-1}+th(x)}\right)$, where $-\vartheta_{p,s} + (x-1)p^{s-\iota_0-p^{s-1}+th(x)}$ is unit, while $-\vartheta_{p,s}$ itself is nilpotent. Therefore, we conclude that, in any case, we have $\iota_1 \leq p^{s-1}$. □

In [10], Kiah et al. found that the dual of $C$ with representation in Theorem 1 is $C^\perp = \langle (x-1)^{p^s-\iota_1} - p_0(x), (x-1)^{p^s-\iota_0} \rangle$ for a certain $p_0(x)$ in $\Re(\mathbb{F}_{p^m}, n)$. However, we have $p^{s-1} \leq \frac{p^s}{2}$ and, by the representation of $C$, $\iota_1 \leq p^{s-1}$, and thus $\iota_1 \leq \frac{p^s}{2}$. This means that $\iota_1 \leq p^s - \iota_1$. Let $k = 0, 1$; we define the dual of a polynomial $h(x) = p^k(x-1)^t \sum_{i=0}^{\iota_k - 1} h_i(x-1)^i$ of $\Re(R, p^s)$ as

$$h_k^\perp(x) = \sum_{i=0}^{\iota_k-1} \left( \sum_{j=0}^{i} (-1)^{\iota_0+t+j+1} \binom{\iota_0 + \iota_1 - t - j}{i-j} h_j \right) (x-1)^i. \tag{11}$$

Thus, the construction of $C^\perp$ is given in the following corollary.

**Corollary 2.** *Suppose* $C$ *is a cyclic code in* $\Re(GR(p^2, m), p^s)$ *with representation*

$$\langle \langle (x-1)^{\iota_0} + p(x-1)^e \pi_1(x), p(x-1)^{\iota_1} \rangle \rangle. \tag{12}$$

*Then,* $C^\perp$ *has the representation*

$$C^\perp = \langle \langle (x-1)^{p^s-\iota_1} + p(x-1)^{p^s-\iota_0-\iota_1+e} \pi_1^\perp(x) + p\delta(x-1)^{p^{s-1}-\iota_1}, p(x-1)^{\iota_1} \rangle \rangle, \tag{13}$$

*where* $\delta = (-1)^{p^{s-1}-\iota_1}$ *if* $0 < p^{s-1} - \iota_1 < \iota_1$ *and* $\delta = 0$, *otherwise.*

**Proof.** It is not difficult to see that $\mathrm{Ann}(C)$ is an ideal in $\Re(GR(p^2, m), p^s)$ and that $\mathrm{Ann}(C) = \langle (x-1)^{p^s-\iota_1} - p(x-1)^{p^s-i_0-i_1+e} \sum_{j=0}^{\iota_1} h_j(x-1)^j + p(x-1)^{p^{s-1}-\iota_1}\vartheta_{p,s}, p(x-1)^{\iota_1} \rangle$. Since $\iota_1 \leq p^{s-1}$, then $\iota_1 < ip^{s-1} - \iota_1$ for $i \geq 2$. Thus, we have

$$\mathrm{Ann}(C) = \langle (x-1)^{p^s-\iota_1} - p(x-1)^{p^s-i_0-i_1+e} \sum_{j=0}^{\iota_1} h_j(x-1)^j + p(x-1)^{p^{s-1}-\iota_1}, p(x-1)^{\iota_1} \rangle.$$

Now, computing the reciprocal polynomials of $(x-1)^{p^s-\iota_1} - p(x-1)^{p^s-i_0-i_1+e} \sum_{j=0}^{\iota_1} h_j(x-1)^j + p(x-1)^{p^{s-1}-\iota_1}$ and $p(x-1)^{\iota_1}$, we found that the first one is $(x-1)^{p^s-\iota_1} + p(x-1)^{p^s-\iota_0-\iota_1+e}\pi_1^\perp(x) + p\delta(x-1)^{p^{s-1}-\iota_1}$, while the second polynomial is invariant under reciprocal operation. The value of $\delta$ depends on whether $0 < p^{s-1} - \iota_1 < \iota_1$ or not. Moreover, $\delta = (-1)^{p^{s-1}-\iota_1}$. Therefore,

$$C^\perp = \langle \langle (x-1)^{p^s-\iota_1} + p(x-1)^{p^s-\iota_0-\iota_1+e} \pi_1^\perp(x) + p\delta(x-1)^{p^{s-1}-\iota_1}, p(x-1)^{\iota_1} \rangle \rangle. \tag{14}$$

□

**Remark 1.** *Our results in Corollary 2 improves those related results presented in [10,15].*

We set

$$\delta = \begin{cases} (-1)^{p^{s-1}-\iota_1}, & \text{if} \quad p^{s-1}-\iota_1 < \iota_1, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we determine the self-dual codes when $p \neq 2$. Suppose that $C$ is a self-dual code over $GR(p^2, m)$ of length $p^s$. First, note that $\iota_0 + \iota_1 = p^s$. Furthermore, $(x-1)^{p^s} = p(x-1)^{p^{s-1}}\vartheta(x)$, where $\vartheta(x) \in \Re(GR(p^2, m), p^s)$. Then, $p(x-1)^{p^s} = 0$, and also

$$(x-1)^{2p^s-2\iota_1} = -p(x-1)^{p^s+p^{s-1}-2\iota_1}\vartheta(x) = 0,$$

when $\delta = 0$.

**Corollary 3.** *Assume $C$ is a cyclic code over $GR(p^2, m)$ of length $p^s$ and $\delta = 0$. Then, $C$ is self-dual if and only if $C$ has representation of*

$$C = \langle\langle (x-1)^{p^s-\iota_1}, p(x-1)^{\iota_1}\rangle\rangle. \tag{15}$$

On the other hand, when $\delta \neq 0$, the expression $(x-1)^{2p^s-2\iota_1}$ is not necessarily zero. However, $(x-1)^{2p^s-2\iota_1} = p(x-1)^{p^s+p^{s-1}-2\iota_1}$. So, if $p$ is odd, then $(x-1)^{2p^s-2\iota_1} + p(x-1)^{p^s+p^{s-1}-2\iota_1} = ((x-1)^{p^s-\iota_1} + p\gamma(x-1)^{p^{s-1}-\iota_1})^2 = 0$, where $\gamma = 2^{-1}$ in $\Re(GR(p^2, m), p^s)$.

**Corollary 4.** *Let $p$ be an odd prime number and $\delta \neq 0$. Moreover, suppose $C$ is a cyclic code over $GR(p^2, m)$; then, $C$ is self-dual if and only if $C$ is of the form*

$$C = \langle\langle (x-1)^{p^s-\iota_1} + p\gamma(x-1)^{p^{s-1}-\iota_1}, p(x-1)^{\iota_1}\rangle\rangle. \tag{16}$$

**Theorem 2.** *A cyclic code over $GR(p^2, m)$ of length $p^s$ is LCD if and only if $\iota_0 = p^s$ and $\iota_1 = min\{p^{s-1}, \iota_0\}$, i.e.,*

$$C = \langle\langle (x-1)^{p^s} + p(x-1)^e\pi_1(x), p(x-1)^{min\{p^{s-1},\iota_0\}}\rangle\rangle. \tag{17}$$

**Proof.** Assume $C$ is a cyclic code that is LCD. Then, by the structure of $C^\perp$ in Corollary 2, we have $p(x-1)^{\iota_1} \in C \cap C^\perp$. Thus, $C$ must have the second component zero, leaving the representation as $C = \langle\langle (x-1)^{\iota_0} + p(x-1)^e\pi_1(x), p(x-1)^{\min\{p^{s-1},\iota_0\}}\rangle\rangle = \langle\langle (x-1)^{\iota_0} + p(x-1)^e\pi_1(x)\rangle\rangle$. Assume that $C^\perp = \langle\langle (x-1)^{p^s-\min\{p^{s-1},\iota_0\}} - p(x-1)^{p^s-\iota_0-\min(p^{s-1},\iota_0)+e}\pi_1^\perp(x) + p\delta\pi(x)\rangle\rangle$, where $\pi(x) = (x-1)^{p^{s-1}-\min\{p^{s-1},\iota_0\}}$. Let $t = \max\{\iota_0, \min\{p^{s-1}, \iota_0\}\}$; then, $p(x-1)^t \in C \cap C^\perp$. Thus, if $t < p^s$, then $C \cap C^\perp \neq 0$. In order to have trivial intersection, we must have $t = p^s$, and hence $\iota_0 = p^s$. Therefore, the only $C$ is LCD if and only if $C = \langle\langle (x-1)^{p^s} + p(x-1)^e\pi_1(x), p(x-1)^{\min\{p^{s-1},\iota_0\}}\rangle\rangle$. □

**Theorem 3.** *There exists no proper self-orthogonal code over $GR(p^2, m)$ of length $p^s$.*

**Proof.** Suppose that $C$ is a proper self-orthogonal. In this case, $C \subset C^\perp$. Thus, we must have $\iota_0 + \iota_1 = p^s$, and then $\iota_0 = p^s - \iota_1$. This follows that $(x-1)^{p^s-\iota_1} + p(x-1)^e\pi_1(x) \in C^\perp$, and so there are $a(x)$ and $b(x)$ in $\Re(GR(p^2, m), p^s)$ such that $(x-1)^{p^s-\iota_1} + p(x-1)^e\pi_1(x) = a(x)\Big((x-1)^{p^s-\iota_1} + p(x-1)^{p^s-\iota_0-\iota_1+e}\pi_1^\perp(x) + p\delta(x-1)^{p^{s-1}-\iota_1}\Big) + b(x)(p(x-1)^{\iota_1})$. Then, comparing parts in two sides, we get $(x-1)^{p^s-\iota_1} = a(x)(x-1)^{p^s-\iota_1}$, and so $a(x) = 1$. So, $(x-1)^e\pi_1(x) = (x-1)^{p^s-\iota_0-\iota_1+e}\pi_1^\perp(x) + \delta(x-1)^{p^{s-1}-\iota_1} \pmod p$. This means that $(x-1)^e\pi_1(x) = (x-1)^{p^s-\iota_0-\iota_1+e}\pi_1^\perp(x) + \delta(x-1)^{p^{s-1}-\iota_1}$. Therefore, there are no proper self-orthogonal codes over $GR(p^2, m)$. □

**Example 1.** *Let $R = GR(9,1)$ of length 9. The purpose of this example is to list all self-dual and LCD codes. Table 1 provides the representations of all cyclic codes and their duals, while Table 2 classifies which of these codes are self-dual or LCD codes. After calculations, the polynomials $\pi_1^{\perp}(x)$ and $\pi(x) = p\delta(x-1)^{p^{s-1}-\iota_1}$ in Equation (13) take the forms*

$$\pi_1^{\perp}(x) = \begin{cases} 0, & \text{if } \iota_1 = 0, \\ h_0, & \text{if } \iota_1 = 1, \\ h_0 + (\iota_0 h_0 + h_1)(x-1), & \text{if } \iota_1 = 2, \\ h_0 + (\iota_0 h_0 + h_1)(x-1) + (\frac{\iota_0(\iota_0-1)}{2} h_0 + (\iota_0 - 1)h_1 + h_2)(x-1)^2 & \text{if } \iota_1 = 3, \end{cases}$$

$$\pi(x) = 3(x-1)^2 \quad \text{when} \quad 0 < 3 - \iota_1 < \iota_1 \quad \text{and} \quad \pi(x) = 0 \quad \text{otherwise.}$$

**Table 1.** Cyclic and dual codes over $GR(9,1)$.

| Cyclic Codes | Dual Cyclic Codes |
|---|---|
| $\langle\langle (x-1)^{\iota_0}, 3\rangle\rangle, 0 \le i_0 \le 8$ | $\langle\langle 3(x-1)^{9-\iota_0}\rangle\rangle$ |
| $\langle\langle (x-1)^8 + 3h_0, 3(x-1)\rangle\rangle$ | $\langle\langle (x-1)^8 - 3h_0, 3(x-1)\rangle\rangle$ |
| $\langle\langle (x-1)^7 + 3h_0, 3(x-1)\rangle\rangle$ | $\langle\langle (x-1)^8 - 3h_0, 3(x-1)^2\rangle\rangle$ |
| $\langle\langle (x-1)^{\iota_0} + 3h_0, 3(x-1)\rangle\rangle, 1 \le i_0 \le 6$ | $\langle\langle (x-1)^8 - 3h_0 + 3(x-1)^2, 3(x-1)^{9-\iota_0}\rangle\rangle$ |
| $\langle\langle (x-1)^8 + 3h_0 + 3h_1(x-1), 3(x-1)^2\rangle\rangle$ | $\langle\langle (x-1)^7 - 3h_0 - 3(-h_0+h_1)(x-1), 3(x-1)\rangle\rangle$ |
| $\langle\langle (x-1)^7 + 3h_0 + 3h_1(x-1), 3(x-1)^2\rangle\rangle$ | $\langle\langle (x-1)^7 - 3h_0 - 3(h_0+h_1)(x-1), 3(x-1)^2\rangle\rangle$ |
| $\langle\langle (x-1)^2 + 3h_0 + 3h_1(x-1), 3(x-1)^2\rangle\rangle$ | $\langle\langle (x-1)^7 - 3h_0 - 3(-h_0+h_1)(x-1) + 3(x-1)^2, 3(x-1)^7\rangle\rangle$ |
| $\langle\langle (x-1)^3 + 3h_0 + 3h_1(x-1), 3(x-1)^2\rangle\rangle$ | $\langle\langle (x-1)^7 - 3h_0 - 3(h_1)(x-1) + 3(x-1)^2, 3(x-1)^6\rangle\rangle$ |
| $\langle\langle (x-1)^4 + 3h_0 + 3h_1(x-1), 3(x-1)^2\rangle\rangle$ | $\langle\langle (x-1)^7 - 3h_0 - 3(h_0+h_1)(x-1) + 3(x-1)^2, 3(x-1)^5\rangle\rangle$ |
| $\langle\langle (x-1)^5 + 3h_0 + 3h_1(x-1), 3(x-1)^2\rangle\rangle$ | $\langle\langle (x-1)^7 - 3h_0 - 3(-h_0+h_1)(x-1) + 3(x-1)^2, 3(x-1)^4\rangle\rangle$ |
| $\langle\langle (x-1)^6 + 3h_0 + 3h_1(x-1), 3(x-1)^2\rangle\rangle$ | $\langle\langle (x-1)^7 - 3h_0 - 3(h_1)(x-1) + 3(x-1)^2, 3(x-1)^3\rangle\rangle$ |
| $\langle\langle (x-1)^8 + 3h_0 + 3h_1(x-1) + 3h_2(x-1)^2, 3(x-1)^3\rangle\rangle$ | $\langle\langle (x-1)^6 - 3h_0, 3(x-1)\rangle\rangle$ |
| $\langle\langle (x-1)^7 + 3h_0 + 3h_1(x-1) + 3h_2(x-1)^2, 3(x-1)^3\rangle\rangle$ | $\langle\langle (x-1)^6 - 3h_0 - 3(h_0+h_1)(x-1), 3(x-1)^2\rangle\rangle$ |
| $\langle\langle (x-1)^3 + 3h_0 + 3h_1(x-1) + 3h_2(x-1)^2, 3(x-1)^3\rangle\rangle$ | $\langle\langle (x-1)^6 - 3h_0 - 3h_1(x-1) - 3(-h_1+h_2-1)(x-1)^2, 3(x-1)^6\rangle\rangle$ |
| $\langle\langle (x-1)^4 + 3h_0 + 3h_1(x-1) + 3h_2(x-1)^2, 3(x-1)^3\rangle\rangle$ | $\langle\langle (x-1)^6 - 3h_0 - 3(h_0+h_1)(x-1) - 3(h_2-1)(x-1)^2, 3(x-1)^5\rangle\rangle$ |
| $\langle\langle (x-1)^5 + 3h_0 + 3h_1(x-1) + 3h_2(x-1)^2, 3(x-1)^3\rangle\rangle$ | $\langle\langle (x-1)^6 - 3h_0 - 3(-h_0+h_1)(x-1) - 3(h_0+h_1+h_2-1)(x-1)^2, 3(x-1)^4\rangle\rangle$ |
| $\langle\langle (x-1)^6 + 3h_0 + 3h_1(x-1) + 3h_2(x-1)^2, 3(x-1)^3\rangle\rangle$ | $\langle\langle (x-1)^6 - 3h_0 - 3(h_1)(x-1) - 3(h_1+h_2-1)(x-1)^2, 3(x-1)^3\rangle\rangle$ |

**Table 2.** Self-dual and LCD codes over $GR(9,1)$.

| Self-Dual Codes | LCD Codes |
|---|---|
| $\langle\langle (x-1)^8, 3(x-1)\rangle\rangle$ | $\langle\langle 3\rangle\rangle$ |
| $\langle\langle (x-1)^9, 3\rangle\rangle$ | $\langle\langle (x-1)^5, 3\rangle\rangle$ |
| $\langle\langle (x-1)^7 - 3(x-1), 3(x-1)^2\rangle\rangle$ | $\langle\langle (x-1)^6, 3\rangle\rangle$ |
| $\langle\langle (x-1)^6 - 3, 3(x-1)^3\rangle\rangle$ | $\langle\langle (x-1)^7, 3\rangle\rangle$ |
| | $\langle\langle (x-1)^8, 3\rangle\rangle$ |

## 4. Structures of Cyclic Codes over $R$ of Length $n$

This section delves into the fundamental ring-theoretic properties and the generators of ideals of $\Re(R,n)$. Recall that a prime ideal $P$ of a commutative ring $R$ is minimal if it

contains no prime ideals other than itself. Krull dimension 0 is assigned to a commutative ring in which each prime ideal is maximal, whereas Krull dimension 1 is assigned to a ring in which each prime ideal is either minimal or maximal but not all maximal. The Krull dimension of $R$ is 0, and $R[x]$ has $JR[x]$ as its maximal ideal. Consequently, $\dim(R[x]) = 1$. $\Re(R, n)$ is zero-dimensional, as $x^n - 1 \notin JR[x]$.

The ring $R$ is finite of order $p^{3m}$ with residue field $\mathbb{F}_{p^m}$, arising from the decomposition $R = GR(p^2, m) + uGR(p^2, m)$ and the relation $u^2 = pu = 0$. Its Jacobson radical is $J = \langle p, u \rangle$, and hence $\frac{R}{J} \cong \mathbb{F}_{p^m}$, so $R$ is local and non-chain. An element $a \in R$ is a zero-divisor if $ab = 0$ for some nonzero $b \in R$ and regular otherwise. A finite commutative ring is Frobenius if it is injective as a module over itself, equivalently if $\mathrm{Ann}(J) \cong \mathbb{F}_{p^m}$. In $R$, since $u^2 = pu = 0$, we have $pJ = uJ = 0$, showing that $p, u \in \mathrm{Ann}(J)$. Conversely, if $a \in \mathrm{Ann}(J) \setminus \{0\}$, then $pa = ua = 0$, forcing $a \in \langle p, u \rangle$. Hence, $\mathrm{Ann}(J) = \langle p, u \rangle$, and $R$ is non-Frobenius. Every unit of $R$ is uniquely expressed as

$$v = \beta_0 + p\beta_1 + u\beta_2, \qquad \beta_i \in \Omega(m),\ \beta_0 \neq 0,$$

and the ideals of $R$ form the lattice described below (Figure 1).

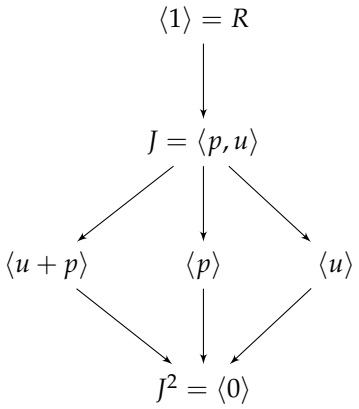



**Figure 1.** Lattice of ideals of $R$.

**Lemma 1.** *The group of units of $\Re(R, n)$ contains only elements of the form*

$$a_1(x) + pa_2(x) + ua_3(x),$$

*where $a_1(x) \neq 0$, $a_2(x)$, $a_3(x)$ are elements of $\Re(\mathbb{F}_{p^m}, n)$.*

**Proof.** Let $v(x)$ be a unit in $\Re(R, n)$. Then, we can express it as $v(x) = a_1(x) + pa_2(x) + ua_3(x)$, where $a_1(x)$, $a_2(x)$, and $a_3(x)$ are elements of $\Re(\mathbb{F}_{p^m}, n)$. It is important to note that $v(x)$ must be a regular element of $\Re(R, n)$, since it is a unit in that ring. Consequently, we have $a_1(x) \nmid x^n - 1$.

Indeed, if there exists some $a(x) \in \Re(\mathbb{F}_{p^m}, n)$ such that $a_1(x)a(x) = x^n - 1$, then it follows that $ua(x)v(x) = 0$. If $a(x) \neq 0$, this would imply that $v(x)$ is a zero-divisor, which is a contradiction. Thus, we conclude that $a(x) = 0$, leading us to determine that $\deg(a_1) = 0$ and $a_1(x) \neq 0$.

Conversely, suppose $v(x) = a_1(x) + a_2(x)p + ua_3(x)$, where $a_1(x)$, $a_2(x)$, $a_3(x)$ are elements of $\Re(\mathbb{F}_{p^m}, n)$, and assume that either $\deg(a_1(x)) = 0$ with $a_1(x) \neq 0$ or that $a_1(x) \nmid x^n - 1$. In this case, $a_1(x)$ is a regular element of $\Re(\mathbb{F}_{p^m}, n)$, and thereby, $v(x)$ is a regular element of $\Re(\mathbb{F}_{p^m}, n)$. Hence, $v(x)$ is a unit in $\Re(R, n)$. □

Let $\nu_u : R \to GR(p^2, m)$ be the natural projection defined by $\nu_u(r) = \bar{r} \pmod{u}$. Since $u^2 = 0$, the image of $\nu_u$ lies in $GR(p^2, m)$. We next determine generators of cyclic

codes over $GR(p^2, m)$ and, using this result, describe the explicit generators of the ideals of $\Re(R, n)$, which correspond to cyclic codes over $R$. Throughout, we first treat the general case without assuming $\gcd(n, p) = 1$ and subsequently impose $\gcd(n, p) = 1$ to obtain a more refined algebraic structure for these codes.

*4.1. Cyclic Codes of Arbitrary Length over R*

In this part, we describe the algebraic structure of cyclic codes of length $n$ over $R$, making use of the structural results on cyclic codes over $GR(p^2, m)$ established in Theorem 1.

**Theorem 4.** *Any cyclic code C of length n over R is represented by*

$$C = \langle\langle \lambda_0(x) + p\pi_1(x) + u\pi_2(x), p\lambda_1(x) + u\pi_3(x), u\lambda_2(x)\rangle\rangle, \tag{18}$$

*where* $\lambda_i(x), \pi_j(x) \in \Re(\mathbb{F}_{p^m}, n)$ *for* $i = 0, 1, 2$ *and* $j = 1, 2, 3$, *satisfying the following:*

$$\begin{cases} \lambda_1(x) \mid \lambda_0(x) \mid x^n - 1, & deg(\pi_1) < deg(\lambda_1) \text{ or } \pi_1(x) = 0, \\ \lambda_1(x) \mid \pi_1(x)\hat{\lambda}_0(x). \end{cases}$$

$$\begin{cases} \lambda_2(x) \mid \lambda_0(x), & deg(\pi_3) < deg(\lambda_3) \text{ or } \pi_3(x) = 0, \\ \lambda_2(x) \mid \pi_3(x)\hat{\lambda}_1(x), & deg(\pi_2) < deg(gcd(\pi_3, \lambda_3)) \text{ or } \pi_2(x) = 0, \\ \lambda_2(x) \mid \pi_2(x)\hat{\lambda}_0(x)\hat{\lambda}_1(x). \end{cases}$$

**Proof.** Consider the map $\nu_u : \Re(R, n)\rangle \to \Re(GR(p^2, m), n)$ defined by $\nu_u(r_0(x) + pr_1(x) + ur_2(x)) = r_0(x) + pr_1(x)$ for each $r_0(x), r_1(x), r_2(x) \in \Re(\mathbb{F}_{p^m}, n)$. Then, $\ker(\nu_u) \cap C$ is an $\Re(\mathbb{F}_{p^m}, n)$-submodule of $u\Re(\mathbb{F}_{p^m}, n)$, so $\ker(\nu_u) \cap C = \langle u\lambda_2(x)\rangle$ for some $\lambda_2(x) \in \Re(\mathbb{F}_{p^m}, n)$ with $\lambda_2(x) \mid x^n - 1$. Since $\nu_u$ is surjective, $\nu_u(C)$ is an ideal of $\Re(GR(p^2, m), n)$. By Theorem 1, $\nu_u(C) = \langle\langle \lambda_0(x) + p\pi_1(x), p\lambda_1(x)\rangle\rangle$ for some $\lambda_0(x), \lambda_1(x), \pi_1(x) \in \Re(\mathbb{F}_{p^m}, n)$, satisfying the conditions of Theorem 1. This means that $\lambda_1(x) \mid \lambda_0(x)$ and $\lambda_1(x) \mid \pi_1(x)\hat{\lambda_0}(x)$, where $\deg(\pi_1) < \deg(\lambda_1)$ or $\pi_1(x) = 0$. Thus,

$$C = \langle \lambda_0(x) + p\pi_1(x) + u\pi_2(x), p\lambda_1(x) + u\pi_3(x), u\lambda_2(x)\rangle,$$

for some $\pi_i(x) \in \Re(\mathbb{F}_{p^m}, n)$. Since $u\lambda_0(x) = u(\lambda_0(x) + p\pi_1(x)) \in \ker(\nu_u) \cap C = \langle u\lambda_2(x)\rangle$.

Note that the following reasoning is similar to that in the first paragraph of the proof of Theorem 1. In light of Theorem 1, $\lambda_1(x) \mid \lambda_0(x) \mid x^n - 1$. This also is extended to $\lambda_2(x) \mid \lambda_0(x)$. In addition, observe that $\lambda_1(x) \mid \pi_1(x)\hat{\lambda_0}(x)$. If we also consider the map $\nu_p : \Re(R, n) \to \Re(\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}, n)$, defined by $\nu_p(r_0(x) + pr_1(x) + ur_2(x)) = r_0(x) + ur_2(x)$ for each $r_0(x), r_1(x), r_2(x) \in \Re(\mathbb{F}_{p^m}, n)$. A similar argument leads to $\nu_p(C) = \langle \lambda_0(x) + u\pi_2(x), u\pi_2(x), u\lambda_2(x)\rangle$, and hence $\nu_p(C) = \langle \lambda_0(x) + u\pi_2(x), u\pi(x)\rangle$, where $\pi(x) = \gcd(\pi_3(x), \lambda_2(x))$. This gives $\deg(\pi_2) < \deg(\gcd(\pi_3(x), \lambda_2(x))$. Additionally, note that $u\pi_3(x)\hat{\lambda_0}(x)\hat{\lambda_1}(x) = \hat{\lambda_0}(x)\hat{\lambda_1}(x)\hat{\lambda_2}(x)[\lambda_0(x) + p\pi_1(x) + u\pi_2(x)]$ and $u\pi_3(x)\hat{\lambda_1}(x)\hat{\lambda_2}(x) = \hat{\lambda_0}(x)\hat{\lambda_1}(x)[p\lambda_1(x) + u\pi_3(x)]$. This implies that $\lambda_2(x) \mid \pi_2(x)\hat{\lambda_0}(x)\hat{\lambda_1}(x)$, and $\lambda_2(x) \mid \pi_3(x)\hat{\lambda_1}(x)$. Thus,

$$C = \langle\langle \lambda_0(x) + p\pi_1(x) + u\pi_2(x), p\lambda_1(x) + u\pi_3(x), u\lambda_2(x)\rangle\rangle.$$

□

*4.2. Cyclic Codes and Their Duals When $gcd(n, p) = 1$*

We demonstrate in this subsection that the generators of the corresponding ideal of a cyclic code $C$ of length $n$ over $R$ admit a simplified form when $\gcd(n, p) = 1$, as stated in Theorem 4. The dual code $C^{\perp}$ is described, the number of codewords in $C$ is determined, and the total number of self-orthogonal codes over $R_p$ of length $n$ is computed using these generators.

The polynomials $\lambda_0(x), \lambda_1(x), \lambda_2(x), \pi_1(x), \pi_2(x), \pi_3(x)$ are elements of $\Re(\mathbb{F}_{p^m}, n)$ that satisfy the conditions of Theorem 4. We assume that $\gcd(n, p) = 1$.

**Lemma 2.** *Let $C$ be a cyclic code over $R$ of length $n$.*

*(i) For each $a_1(x), a_3(x) \in \Re(\mathbb{F}_{p^m}, n)$, we have $a_1(x) \in \langle a_1(x) + pa_2(x) + ua_3(x)\rangle$. In particular, $\langle a_1(x), p\lambda_2(x) + ua_3(x)\rangle = \langle a_1(x) + p\lambda_2(x) + ua_3(x)\rangle$.*

*(ii) In $\Re(R, n)$, we have*

$$\begin{aligned}\big\langle \lambda_0(x) + p\pi_1(x) + u\pi_2(x),\, p\lambda_1(x) + u\pi_3(x),\, u\lambda_2(x)\big\rangle \\ = \langle \lambda_0(x) + p\lambda_1(x), u\lambda_2(x)\rangle.\end{aligned} \tag{19}$$

**Proof.** (i) Consider the factorization of $x^n - 1$ over $\mathbb{F}_{p^m}$ as

$$x^n - 1 = d_0(x)d_1(x)\cdots d_k(x),$$

where each $d_i(x)$ is irreducible. By Hensel's lemma, these lift to irreducible polynomials over $R$. Since $\gcd(n, p) = 1$, $x^n - 1$ has no repeated roots, ensuring that $d_i(x) \neq d_j(x)$ for $i \neq j$. Consequently, $d_0(x), \ldots, d_k(x)$ are distinct prime elements, and the Chinese Remainder Theorem gives

$$\Re(\mathbb{F}_{p^m}, n) \cong R_0 \times \cdots \times R_k, \quad R_i = \frac{\mathbb{F}_{p^m}[x]}{\langle d_i(x)\rangle} \text{ is a field.}$$

This decomposition extends to

$$\Re(R, n) \cong \prod_{i=0}^{k} (R_i + pR_i + uR_i).$$

Next, let $a_1(x), a_2(x), a_3(x) \in \Re(\mathbb{F}_{p^m}, n)$ and write $a_i(x) = (a_{i0}(x), \ldots, a_{ik}(x))$ componentwise with $a_{ij}(x) \in R_j$. Fix an index $j$. If $a_{1j}(x) \neq 0$, it is a unit in $R_j$, and we can write

$$a_{1j} = \big(a_{1j} + pa_{2j} + ua_{3j}\big)^{p^2} a_{1j}^{1-p^2} \in \langle a_{1j} + pa_{2j} + ua_{3j}\rangle.$$

If $a_{1j}(x) = 0$, the inclusion is immediate. Therefore, componentwise, we obtain

$$a_1(x) \in \langle a_1(x) + pa_2(x) + ua_3(x)\rangle,$$

which establishes part (i). The secondary claim follows directly.

(ii) Under the representation in Theorem 4, the distinct factorization of $x^n - 1$ implies that $\gcd(\lambda_1(x), \hat{\lambda}_0(x)) = 1$, hence $\lambda_1(x) \mid \pi_1(x)$. Applying degree arguments from Theorem 4, we deduce $\pi_1(x) = 0$. Repeating the same reasoning for the other polynomials shows that $\pi_i(x) = 0$ for $i = 2, 3$. Using part (i), we then have

$$C = \langle \lambda_0(x) + p\lambda_1(x),\, u\lambda_2(x)\rangle.$$

Thus, any non-principal ideal of $\Re(R,n)$ can be generated by three polynomials $\lambda_0(x), \lambda_1(x), \lambda_2(x) \in \Re(\mathbb{F}_{p^m}, n)$, with $\lambda_1(x) \mid \lambda_0(x)$, $\lambda_2(x) \mid \lambda_0(x)$, and $\lambda_0(x) \mid x^n - 1$.

□

**Corollary 5.** *Assuming* $\gcd(n,p) = 1$, *every ideal C of* $\Re(R,n)$ *can be expressed in the form*

$$C = \langle\langle \lambda_0(x) + p\lambda_1(x),\ u\lambda_2(x) \rangle\rangle. \tag{20}$$

The following theorem characterizes the structure of cyclic dual codes over $R$ of length $n$ under the assumption $\gcd(n,p) = 1$.

**Theorem 5.** *Let C be a cyclic code of length n over R, corresponding to the ideal*

$$I = \left\langle \lambda_0(x) + p\lambda_1(x),\ u\lambda_2(x) \right\rangle \subseteq \Re(R,n).$$

*Then, the annihilator of C is given by*

$$Ann(C) = \left\langle \frac{x^n - 1}{\gcd(\lambda_2(x), \lambda_1(x))} + p\hat{\lambda}_0(x),\ u\hat{\lambda}_0(x) \right\rangle. \tag{21}$$

*Moreover, the dual code* $C^{\perp}$ *is associated with the ideal*

$$\left\langle \frac{x^n - 1}{\gcd(\lambda_2(x), \lambda_1(x))^*} + p\hat{\lambda}_0^*(x),\ u\hat{\lambda}_0^*(x) \right\rangle. \tag{22}$$

**Proof.** First, note that $\lambda_1(x)$ and $\lambda_2(x)$ divide $x^n - 1$ since $x^n - 1$ factors into distinct irreducibles over $\mathbb{F}_{p^m}[x]$. By Lemma 2, the annihilator $\mathrm{Ann}(C)$ can be written as

$$\mathrm{Ann}(C) = \langle \lambda_0'(x) + p\lambda_1'(x),\ u\lambda_2'(x) \rangle = \langle \lambda_0'(x),\ p\lambda_1'(x),\ u\lambda_2'(x) \rangle,$$

for some $\lambda_0'(x), \lambda_1'(x), \lambda_2'(x) \in \Re(\mathbb{F}_{p^m}, n)$, satisfying Theorem 4. Since $(\lambda_0'(x) + p\lambda_1'(x))$ $u\lambda_2(x) = 0$, it follows that $x^n - 1$ divides $\lambda_0'(x)\lambda_2(x)$, implying

$$\frac{x^n - 1}{\lambda_2(x)} \mid \lambda_0'(x).$$

Similarly, using Lemma 2 (i), $p\lambda_1(x) \in C$ and $\lambda_0'(x) \in \mathrm{Ann}(C)$ yield

$$\lambda_0'(x) \cdot (p\lambda_1(x)) = 0 \quad \Rightarrow \quad \hat{\lambda}_1(x) \mid \lambda_0'(x).$$

Thus, we can write

$$\lambda_0'(x) = \frac{x^n - 1}{\gcd(\lambda_1(x), \lambda_2(x))}\, l(x)$$

for some $l(x) \in \Re(\mathbb{F}_{p^m}, n)$.

Analogously, the relations $\lambda_0(x) \cdot (u\lambda_1'(x)) = 0$ and $u\lambda_2'(x)\lambda_0(x) = 0$ imply

$$\lambda_1'(x) = \lambda_0(x) l_1(x), \quad \lambda_2'(x) = \hat{\lambda}_0(x) l_2(x)$$

for some $l_1(x), l_2(x) \in \Re(\mathbb{F}_{p^m}, n)$. Combining these observations and applying Lemma 2, we obtain

$$\mathrm{Ann}(C) = \left\langle \frac{x^n - 1}{\gcd(\lambda_2(x), \lambda_1(x))} + p\hat{\lambda}_0(x),\ u\hat{\lambda}_0(x) \right\rangle.$$

Finally, by the discussion preceding this theorem and Lemma 2, the corresponding dual code $C^{\perp}$ is generated by

$$\left\langle \frac{x^n-1}{\gcd(\lambda_2(x),\lambda_1(x))^*} + p\hat{\lambda}_0^*(x),\, u\hat{\lambda}_0^*(x) \right\rangle,$$

as claimed. □

**Lemma 3.** *Let $C$ and $C'$ be cyclic codes of length $n$ over $R$ with $\gcd(n,p)=1$, satisfying $C \subseteq C'$. Suppose that the polynomials*

$$\lambda_0(x), \lambda_1(x), \lambda_2(x), \lambda_0'(x), \lambda_1'(x), \lambda_2'(x) \in \Re(\mathbb{F}_{p^m}, n)$$

*satisfy the conclusion of Theorem 4 and generate the codes*

$$C = \langle \lambda_0(x) + p\lambda_1(x),\, u\lambda_2(x) \rangle, \quad C' = \langle \lambda_0'(x) + p\lambda_1'(x),\, u\lambda_2'(x) \rangle.$$

*Then, the divisibility relations*

$$\lambda_0'(x) \mid \lambda_0(x), \quad \lambda_1'(x) \mid \lambda_1(x), \quad \lambda_2'(x) \mid \lambda_2(x)$$

*hold.*

**Proof.** Since $C \subseteq C'$, any generator of $C$ can be expressed as a combination of the generators of $C'$. In particular, there exist polynomials $a_i(x), b_i(x) \in \Re(\mathbb{F}_{p^m}, n)$ such that

$$\lambda_0(x) = (\lambda_0'(x) + p\lambda_1'(x))(a_1(x) + pa_2(x) + ua_3(x)) + u\lambda_2'(x)(b_1(x) + ub_2(x)).$$

Comparing terms, it follows that

$$\lambda_0(x) = \lambda_0'(x)a_1(x),$$

establishing $\lambda_0'(x) \mid \lambda_0(x)$. Similarly, for the $p$-component,

$$\begin{aligned} p\lambda_1(x) = {} & (\lambda_0'(x) + p\lambda_1'(x))(c_1(x) + pc_2(x) + uc_3(x)) \\ & + u\lambda_2'(x)(c_5(x) + pc_6(x) + uc_7(x)) \end{aligned}$$

for some $c_i(x) \in \Re(\mathbb{F}_{p^m}, n)$. Extracting the coefficient of $p$ yields

$$\lambda_1(x) = \lambda_1'(x)c_1(x) + \lambda_0'(x)c_2(x),$$

from which the divisibility $\lambda_1'(x) \mid \lambda_1(x)$ follows. By the same argument, the divisibility $\lambda_2'(x) \mid \lambda_2(x)$ holds. Therefore, we conclude

$$\lambda_0'(x) \mid \lambda_0(x), \quad \lambda_1'(x) \mid \lambda_1(x), \quad \lambda_2'(x) \mid \lambda_2(x),$$

as required. □

Building on the preceding results, we present explicit generators for the ideals of $\Re(\mathbb{F}_{p^m}, n)$ that correspond to self-orthogonal and self-dual codes over $R$, respectively.

**Theorem 6.** *Let $C$ be a cyclic code of length $n$ over $R$ of the form*

$$C = \langle\langle \lambda_0(x) + p\lambda_1(x),\, u\lambda_2(x) \rangle\rangle. \tag{23}$$

*Then, the following holds:*

$(i)$ *$C$ is self-orthogonal if and only if*

$$\frac{x^n-1}{\gcd(\lambda_2(x),\lambda_1(x))^*} \mid \lambda_0(x),$$

$(ii)$ *No self-dual cyclic codes exist over $R$ under the condition* $\gcd(n,p)=1$.

**Proof.** $(i)$ Suppose first that $C$ is self-orthogonal. Then, by Theorem 5 and Lemma 3, we have

$$\frac{x^n-1}{\gcd(\lambda_2(x),\lambda_1(x))} \mid \lambda_0(x).$$

Conversely, assume that

$$\frac{x^n-1}{\gcd(\lambda_2(x),\lambda_1(x))} \mid \lambda_0(x).$$

Then,

$$x^n-1 = -(x^n-1)^* \mid \left(\lambda_0(x)\gcd(\lambda_2(x),\lambda_1(x))^*\right)^*,$$

which implies

$$x^n-1 \mid \lambda_0^*(x)\ \gcd(\lambda_2(x),\lambda_1(x)),$$

and thus

$$\frac{x^n-1}{\lambda_0^*(x)} \mid \gcd(\lambda_2(x),\lambda_1(x)).$$

By Lemma 3, this condition ensures that $C$ is self-orthogonal.

$(ii)$ Suppose, for the sake of contradiction, that $C$ is self-dual. Let $q(x) = \gcd(\lambda_2(x),\lambda_1(x))$. Then, by Lemma 2 and Theorems 4 and 5, there exist units $g_1(x), g_2(x), g_3(x) \in \Re(\mathbb{F}_{p^m}, n)$ such that

$$\lambda_0(x) = \frac{x^n-1}{q^*(x)}g_1(x), \quad \lambda_1(x) = \frac{x^n-1}{\lambda_0^*(x)}g_2(x), \quad \lambda_2(x) = \frac{x^n-1}{\lambda_0^*(x)}g_3(x).$$

It follows that

$$q(x) = \frac{x^n-1}{\lambda_0^*(x)}, \quad \text{and} \quad \lambda_0(x) = \frac{x^n-1}{q^*(x)}g_1(x) = \lambda_0^*(x)g_1(x).$$

Since $\lambda_1(x) \mid \lambda_0(x)$ by Theorem 6, we must have

$$\frac{x^n-1}{\lambda_0(x)} \mid \lambda_0(x),$$

which, due to the distinct factorization of $x^n-1$, forces $\lambda_0(x) = x^n-1$. Consequently, $C = \langle p, u\rangle$, which is impossible as $pC \neq 0$ and $uC \neq 0$. Therefore, no self-dual cyclic codes exist over $R$ when $\gcd(n,p)=1$. □

## 5. Mass Formulae of Cyclic Codes over $R$

In this section, we establish mass formulae that describe the total number of cyclic codes over $R$ of a given length. Such formulae provide a compact way to enumerate the codes up to equivalence, offering insight into their distribution and structure. In particular, we derive in Theorem 7 the number of self-orthogonal cyclic codes, and in Theorem 8 the number of LCD cyclic codes. Throughout, we continue to assume that $\gcd(n,p)=1$. For any two elements $r_1, r_2 \in R$, we write $r_1 \sim r_2$ if there exists a unit $v \in R$ such that $r_1 = vr_2$.

**Theorem 7.** *Let $d_1(x), \ldots, d_t(x)$ denote the irreducible factors of $x^n - 1$ in $\mathbb{F}_{p^m}[x]$, where $d_{2i-1}(x) \sim d_{2i}^*(x)$ for $1 \leq i \leq m$, and $d_j(x)$ are self-reciprocal for $2m+1 \leq j \leq t$. Define the set*

$$\mathcal{A} = \left\{ \lambda_0(x) \in \mathbb{F}_{p^m}[x] \;\middle|\; \begin{array}{l} d_{2m+1}(x) \cdots d_t(x) \mid \lambda_0(x) \mid x^n - 1, \\ \text{for each } i = 1, \ldots, m, \text{ at least one of } d_{2i-1}(x) \text{ or } d_{2i}(x) \text{ divides } \lambda_0(x) \end{array} \right\}.$$

*Then, the total number of cyclic self-orthogonal codes over $R$ of length $n$ is given by*

$$\sum_{\lambda_0(x) \in \mathcal{A}} \left( \sum_{\frac{x^n-1}{\lambda_0^*(x)} | \lambda_2(x) | \lambda_0(x)} \aleph_0 \right), \tag{24}$$

*where*

$$\aleph_0 = \sum_{\frac{x^n-1}{\lambda_0^*(x)} | \lambda_1(x) | \lambda_0(x)} p^{\deg(\lambda_2) - \deg(\gcd(\lambda_2(x), \lambda_1(x)))}.$$

**Proof.** Fix a polynomial $\lambda_0(x) \in \mathcal{A}$. By construction, for each $1 \leq i \leq m$, at least one of $d_{2i-1}(x)$ or $d_{2i}(x)$ divides $\lambda_0(x)$, and $d_j(x) \mid \lambda_0(x)$ for $j > 2m$.

Let $\lambda_2(x), \lambda_1(x) \in \Re(R, n)$ satisfy

$$\frac{x^n - 1}{\lambda_0^*(x)} \mid \lambda_2(x) \mid \lambda_0(x), \qquad \frac{x^n - 1}{\lambda_0^*(x)} \mid \lambda_1(x) \mid \lambda_0(x).$$

Using these polynomials, we form the ideal

$$I = \langle \langle \lambda_0(x) + p\lambda_1(x),\, u\lambda_2(x) \rangle \rangle.$$

The total number of distinct ideals constructed in this way is

$$\sum_{\lambda_0(x) \in \mathcal{A}} \left( \sum_{\frac{x^n-1}{\lambda_0^*(x)} | \lambda_2(x) | \lambda_0(x)} \aleph_0 \right),$$

where $\aleph_0$ accounts for the number of choices of $\lambda_1(x)$ relative to $\lambda_2(x)$, so

$$\aleph_0 = \sum_{\frac{x^n-1}{\lambda_0^*(x)} | \lambda_1(x) | \lambda_0(x)} p^{\deg(\lambda_2) - \deg(\gcd(\lambda_2, \lambda_1)}.$$

By Theorem 6, each ideal of the form $I$ corresponds uniquely to a cyclic self-orthogonal code over $R$ of length $n$. This completes the proof. □

We conclude this section by providing a characterization of cyclic LCD codes over $R$ of length $n$ and deriving a mass formula for their enumeration.

**Proposition 1.** *Let $C$ be a cyclic code over $R$ of length $n$ corresponding to the ideal*

$$C = \langle \langle \lambda_0(x) + p\lambda_1(x),\, u\lambda_2(x) \rangle \rangle. \tag{25}$$

*Then, $C$ is an LCD code if and only if*

$$\deg(\lambda_0(x)) = \deg(\lambda_1(x)) = \deg(\lambda_2(x)) \quad \textit{and} \quad \lambda_0^*(x) \mid \lambda_0(x).$$

**Proof.** By Theorem 4, the dual code $C^\perp$ corresponds to the ideal

$$\left\langle \frac{x^n-1}{\gcd(\lambda_2(x),\lambda_1(x))},\ p\frac{x^n-1}{\lambda_0^*(x)},\ u\frac{x^n-1}{\lambda_0^*(x)} \right\rangle.$$

Suppose $C$ is LCD, i.e., $C \cap C^\perp = \{0\}$. Then,

$$p\frac{x^n-1}{\lambda_0^*(x)}\lambda_0(x) \in C \cap C^\perp,$$

implying

$$\frac{x^n-1}{\lambda_0^*(x)}\lambda_0(x) = 0,$$

and hence $\lambda_0^*(x) \mid \lambda_0(x)$. Consequently, any irreducible factor $g(x)$ of $x^n-1$ dividing $\lambda_0(x)$ must also divide $\lambda_0^*(x)$, so $\lambda_0(x) = \beta\lambda_0^*(x)$ for some nonzero $\beta \in \mathbb{F}_{p^m}$.

Similarly, since

$$(p\lambda_1(x))\frac{x^n-1}{\lambda_0^*(x)} \in C \cap C^\perp,$$

it follows that $\lambda_0^*(x) \mid \lambda_1(x)$. By Theorem 3.4, $\lambda_1(x) \mid \lambda_0(x)$, so $\deg(\lambda_1) = \deg(\lambda_0)$. Applying the same argument to $u\lambda_2(x)$ shows that $\deg(\lambda_2) = \deg(\lambda_0)$.

Conversely, assume

$$\deg(\lambda_0) = \deg(\lambda_1) = \deg(\lambda_2) \quad \text{and} \quad \lambda_0^*(x) \mid \lambda_0(x).$$

Then, the corresponding ideals for $C$ and $C^\perp$ are

$$C = \langle \lambda_0(x),\ p\lambda_0(x),\ u\lambda_0(x)\rangle, \quad C^\perp = \left\langle \frac{x^n-1}{\lambda_0^*(x)},\ p\frac{x^n-1}{\lambda_0^*(x)},\ u\frac{x^n-1}{\lambda_0^*(x)} \right\rangle.$$

Let $l(x) \in C \cap C^\perp$. Then, there exist $\theta_i(x), \vartheta_i(x) \in \Re(R,n)$ for $i = 1,2,3$ such that

$$\begin{aligned} l(x) &= \lambda_0(x)\theta_1(x) + p\lambda_0(x)\theta_2(x) + u\lambda_0(x)\theta_3(x) \\ &= \frac{x^n-1}{\lambda_0^*(x)}\vartheta_1(x) + p\frac{x^n-1}{\lambda_0^*(x)}\vartheta_2(x) + u\frac{x^n-1}{\lambda_0^*(x)}\vartheta_3(x). \end{aligned}$$

Since $\lambda_0(x) = \beta\lambda_0^*(x)$ for some $\beta \in \mathbb{F}_{p^m}$, it follows that $\lambda_0(x) \mid \vartheta_i(x)$ for each $i$. Therefore,

$$\lambda_0(x)\theta_i(x) = \frac{x^n-1}{\lambda_0^*(x)}h_i(x) = 0 \quad \text{for all } i,$$

implying $l(x) = 0$. Hence, $C$ is LCD. □

**Theorem 8.** *Let $t$ (respectively, $e$) denote the number of irreducible factors $\lambda(x)$ of $x^n-1$ over $\mathbb{F}_{p^m}$, satisfying $\lambda(x) = \pm\lambda^*(x)$ (respectively, $\lambda(x) \neq \pm\lambda^*(x)$). Then, the total number $N$ of cyclic LCD codes over $R$ of length $n$ is*

$$N = 2^{t+\frac{e}{2}}. \tag{26}$$

**Proof.** By Proposition 1, a cyclic LCD code over $R$ of length $n$ is completely determined by a choice of polynomial $\lambda_0(x) \in \Re(R,n)$ such that whenever an irreducible factor of $x^n-1$ divides $\lambda_0(x)$, its reciprocal also divides $\lambda_0(x)$. Counting all possible combinations of self-reciprocal and non-self-reciprocal irreducible factors yields the formula $2^{t+\frac{e}{2}}$. □

Using Theorem 7, one can explicitly compute the number of cyclic LCD codes over $R$ of length $n$ with $\gcd(n,p) = 1$ in the following example.

**Example 2.** *In Table 3, we count the total number of LCD over R of length* $n = 9, 25, 26, 15$ *and* $n = 20$ *where* $p = 7, 3$ *and* $p = 11$.

**Table 3.** Number of LCD codes over $R$.

| $(n, p)$ | Factors of $x^n - 1$ | $t$ | $e$ | Number of LCD |
|---|---|---|---|---|
| $(9, 7)$ | $x^9 - 1 = (x-1)(x-2)(x-4)(x^3-2)(x^3-4)$ | 1 | 4 | 8 |
| $(25, 7)$ | $x^{25} - 1 = (x-1)(1+x+x^2+x^3+x^4)(1+2x+4x^2+2x^3+x^4)$ $(1+4x+4x^3+x^4)(1+4x+3x^2+4x^3+x^4)$ $(1+5x+5x^2+5x^3+x^4)(1+6x+5x^2+6x^3+x^4)$ | 3 | 4 | 32 |
| $(26, 3)$ | $x^{26} - 1 = (x-1)(x+1)(x^3-x+1)(x^3+x^2-1)$ $(x^3-x^2+1)(x^3-x-1)(x^3+x^2-x+1)$ $(x^3+x^2+x-1)(x^3-x^2-x-1)(x^3-x^2+x+1)$ | 2 | 8 | 64 |
| $(15, 11)$ | $x^{15} - 1 = (x-1)(x+6)(x+7)(x+8)(x+2)$ $(x^2+x+1)(x^2+3x+9)(x^2+4x+5)$ $(x^2+5x+3)(x^2+9x+4)$ | 2 | 8 | 64 |
| $(20, 3)$ | $x^{20} - 1 = (x-1)(x+1)(x^2+1)(x^4-x^3-x+1)$ $(x^4-x^3+x+1)(x^4-x^3+x^2-x+1)(x^4+x^3+x+1)$ | 3 | 4 | 32 |

## 6. Conclusions

This work examines cyclic codes of length $n$ over the local non-Frobenius ring $R = GR(p^2, m)[u]$, where $u^2 = 0$ and $pu = 0$. We established the algebraic structure of cyclic codes of arbitrary length $n$ and, in the scenario when $\gcd(n, p) = 1$, supplied explicit generators. Additionally, we determined the necessary and sufficient criteria for the existence of self-dual and LCD codes, together with their enumeration. Numerous examples and Tables 1–3 were provided to elucidate the mass formula for cyclic self-orthogonal codes, cyclic LCD codes, and novel families of quasi-cyclic codes.

The findings enhance the theory of codes over finite chain and non-chain rings and illustrate the complexity of cyclic structures in this context. Future research should focus on examining decoding algorithms, weight distributions, and the potential applications of these codes in communication systems and cryptographic frameworks.

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