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Weak Nearly $\mathcal{S}$- and Weak Nearly $\mathcal{C}$-Manifolds

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Abstract

The recent interest in geometers in the f-structures of K. Yano is motivated by the study of the dynamics of contact foliations, as well as their applications in theoretical physics. Weak metric f-structures on a smooth manifold, recently introduced by the author and R. Wolak, open a new perspective on the theory of classical structures. In this paper, we define structures of this kind, called weak nearly $\mathcal{S}$- and weak nearly $\mathcal{C}$-structures, study their geometry, e.g., their relations to Killing vector fields, and characterize weak nearly $\mathcal{S}$- and weak nearly $\mathcal{C}$-submanifolds in a weak nearly Kähler manifold.

Keywords: weak nearly $\mathcal{S}$-manifold; weak nearly $\mathcal{C}$-manifold; Killing vector field; submanifold; weak nearly Kähler manifold

MSC: 53C15; 53C25; 53D15

1. Introduction

The f-structure introduced by K. Yano [1] on a smooth manifold M^{2n+s} serves as a higher-dimensional analog of almost complex structures ($s=0$) and almost contact structures ($s=1$). This structure is defined by a (1,1)-tensor f of rank $2n$ such that $f^3+f=0$. The tangent bundle splits into two complementary subbundles: $TM=f(TM)\oplus\ker f$. The restriction of f to the $2n$-dimensional distribution $f(TM)$ defines a complex structure. The existence of the f-structure on M^{2n+s} is equivalent to a reduction of the structure group to $U(n)\times O(s)$; see [2]. A submanifold M of an almost complex manifold $(\bar M, J)$ that satisfies the condition $\dim(T_xM\cap J(T_xM)) = const > 0$ naturally possesses an f-structure; see [3]. An f-structure is a special case of an almost product structure, defined by two complementary orthogonal distributions of a Riemannian manifold (M,g). Foliations appear when one or both distributions are involutive. An interesting case occurs when the sub-bundle $\ker f$ is parallelizable, leading to a framed f-structure for which the reduced structure group is $U(n)\times \mathrm{Id}_s$. In this scenario, there exist vector fields $\{\xi_i\}_{1\le i\le s}$ (called Reeb vector fields) spanning $\ker f$ with dual 1-forms $\{\eta^i\}_{1\le i\le s}$, satisfying $f^2=-\mathrm{Id}+\sum_{i=1}^s\eta^i\otimes\xi_i$. Compatible Riemannian metrics, i.e.,

$$g(fX,fY)=g(X,Y)-\textstyle\sum_{i=1}^s\eta^i(X)\,\eta^i(Y),$$

exist on any framed f-manifold, and we obtain the metric f-structure; see [2,4–6].

To generalize concepts and results from almost contact geometry to metric f-manifolds, geometers have introduced and studied various broad classes of metric f-structures. A metric f-manifold is termed a $\mathcal{K}$-manifold if it is normal and $d\Phi=0$, where $\Phi(X,Y):= g(X,fY)$. Two important subclasses of $\mathcal{K}$-manifolds are $\mathcal{C}$-manifolds if $d\eta^i=0$ and $\mathcal{S}$-manifolds if $d\eta^i=\Phi$ for any i; see [2]. Omitting the normality condition, we obtain almost $\mathcal{K}$-manifolds, almost $\mathcal{S}$-manifolds and almost $\mathcal{C}$-manifolds, e.g., [7–9]. The distribution

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ker f of a $\mathcal{K}$-manifold is tangent to a $\mathfrak{g}$-foliation with flat totally geodesic leaves. An f-K-contact manifold is an almost $\mathcal{S}$-manifold, whose Reeb vector fields are Killing vector fields; the structure is intermediate between almost $\mathcal{S}$-structure and S-structure; see [6,10]. Nearly $\mathcal{S}$- and nearly $\mathcal{C}$-manifolds $(M^{2n+s}, f, \xi_i, \eta^i, g)$ are defined in the same spirit as the nearly Kähler manifolds of A. Gray [11] by a constraint only on the symmetric part of ∇f – starting from $\mathcal{S}$- and $\mathcal{C}$-manifolds (e.g., [12–15]):

$$(\nabla_X f)X = \begin{cases} g(fX, fX)\,\bar{\xi} + \bar{\eta}(X)f^2 X\,, & \text{nearly } \mathcal{S} - \text{manifolds.} \\ 0\,, & \text{nearly } \mathcal{C} - \text{manifolds.} \end{cases}$$

Here, $\bar{\eta} = \sum_{i=1}^{s} \eta^i$ and $\bar{\xi} = \sum_{i=1}^{s} \xi_i$. These counterparts of nearly Kähler manifolds play a key role in the classification of metric f-manifolds; see [2]. The Reeb vector fields ξ_i of nearly $\mathcal{S}$- and nearly $\mathcal{C}$-structures are unit Killing vector fields. The influence of constant-length Killing vector fields on Riemannian geometry has been studied by many authors, e.g., [16]. The interest of geometers in f-structures is also motivated by the study of the dynamics of contact foliations. Contact foliations generalize to higher dimensions the flow of the Reeb vector field on contact manifolds, and $\mathcal{K}$-structures are a particular case of uniform s-contact structures; see [17,18]. Dynamics and integration on s-cosymplectic manifolds are studied in [19]; they investigate the Lie integrability of s-evolution systems in this setting, and develop a Hamilton–Jacobi theory tailored to multi-time Hamiltonian systems, both via symplectification techniques.

In [20–22], we introduced and studied metric structures on a smooth manifold, see Definition 1, which generalize almost Hermitian, almost contact (e.g., Sasakian and cosymplectic) and f-structures. Such so-called "weak" structures (the complex structure on the contact distribution is replaced by a nonsingular skew-symmetric tensor) allow us a new look at the theory of classical structures and find new applications. A. Einstein worked on various variants of Unified Field Theory, more recently known as Non-symmetric Gravitational Theory (NGT), see [23]. In this theory, the symmetric part g of the basic tensor $G = g + F$ is associated with gravity, and the skew-symmetric one F is associated with electromagnetism. The theory of weak metric structures is fully consistent with the skew-symmetric part of G; thus, it provides new tools for studying NGT. S. Ivanov and M. Zlatanović developed NGT with linear connections of totally skew-symmetric torsion and gave examples with the skew-symmetric part F of the tensor G obtained using an almost contact metric structure; see [24]. In [25], the author and M. Zlatanović were the first to apply weak metric structures to NGT of totally skew-symmetric torsion with tensor $F(X,Y) = g(X, fY)$ of constant rank.

In this paper, we define and study new structures of this kind, generalizing nearly $\mathcal{S}$- and nearly $\mathcal{C}$-structures. Section 2, following the Introduction, recalls some results regarding weak nearly Kähler manifolds (generalizing nearly Kähler manifolds) and weak metric f-manifolds. Section 3 introduces weak nearly $\mathcal{S}$- and weak nearly $\mathcal{C}$-structures and studies their geometry. Section 4 characterizes weak nearly $\mathcal{C}$- and weak nearly $\mathcal{S}$-submanifolds in weak nearly Kähler manifolds and proves that a weak nearly $\mathcal{C}$-manifold with parallel Reeb vector fields is locally the Riemannian product of a Euclidean space and a weak nearly Kähler manifold. The proofs use the properties of new tensors, as well as classical constructions.

2. Preliminaries

Here, we review some results; see [20–22]. Nearly Kähler manifolds (M, J, g) were defined by A. Gray [11] using the condition that only the symmetric part of ∇J vanishes, where ∇ is the Levi-Civita connection, in contrast to the Kähler case, where $\nabla J = 0$. Several authors studied the problem of finding and classifying parallel skew-symmetric 2-tensors (other than almost-complex structures) on a Riemannian manifold, e.g., [26].

Definition 1. A Riemannian manifold (M, g) of even dimension equipped with a skew-symmetric (1,1)-tensor f such that the tensor f^2 is negative-definite is called a *weak Hermitian manifold*. Such (M, f, g) is called a *weak Kähler manifold* if $\nabla f = 0$. A weak Hermitian manifold is called a *weak nearly Kähler manifold* if

$$(\nabla_X f)Y + (\nabla_Y f)X = 0 \quad (X, Y \in \mathfrak{X}_M). \tag{1}$$

A *weak metric f-structure* on a smooth manifold M^{2n+s} $(n, s > 0)$ is a set (f, Q, ξ_i, η^i, g), where f is a skew-symmetric $(1,1)$-tensor of rank $2\,n$, Q is a self-adjoint nonsingular $(1,1)$-tensor, ξ_i $(1 \le i \le s)$ are orthonormal vector fields, η^i are dual 1-forms, and g is a Riemannian metric on M, satisfying

$$f^2 = -Q + \textstyle\sum_{i=1}^s \eta^i \otimes \xi_i, \quad \eta^i(\xi_j) = \delta^i_j, \quad Q\,\xi_i = \xi_i, \tag{2}$$

$$g(fX, fY) = g(X, Q\,Y) - \textstyle\sum_{i=1}^s \eta^i(X)\,\eta^i(Y) \quad (X, Y \in \mathfrak{X}_M). \tag{3}$$

In this case, $(M^{2n+s}, f, Q, \xi_i, \eta^i, g)$ is called a *weak metric f-manifold*.

The geometric meaning of (1) is the same as in the classical case: geodesics are f-planar curves. A curve γ is *f-planar* if the section $\dot\gamma \wedge f\dot\gamma$ is parallel along the curve. A framed weak f-manifold (i.e., only (2) holds) admits a compatible metric (i.e., also (3) holds) if f in (2) has a skew-symmetric representation, i.e., for any $x \in M$ there exists a frame $\{e_i\}$ on a neighborhood $U_x \subset M$, for which f has a skew-symmetric matrix.

Example 1. *Take $k > 1$ almost Hermitian manifolds (M_j, f_j, g_j). The Riemannian product $\prod_{j=1}^k (M_j, \lambda_j^{1/2} f_j, g_j)$, where $\lambda_j > 0$ are different constants, is a weak almost Hermitian manifold with $Q = \bigoplus_j \lambda_j \,\mathrm{Id}_j$. We call $\prod_j (M_j, \lambda_j^{1/2} f_j, g_j)$ a $(\lambda_1, \ldots, \lambda_k)$-weighed product of almost Hermitian manifolds (M_j, f_j, g_j); see [27]. The $(\lambda_1, \ldots, \lambda_k)$-weighed product of (nearly) Kähler manifolds is a weak (nearly) Kähler manifold. A nearly Kähler manifold of dimension ≤ 4 is a Kähler manifold; see [11]. The unit sphere S^6 in the set of purely imaginary Cayley numbers admits a strictly nearly Kähler structure. The classification of weak nearly Kähler manifolds in dimensions ≥ 4 is an open problem. The (λ_1, λ_2)-weighed products of 2-dimensional Kähler manifolds are 4-dimensional weak nearly Kähler manifolds. The $(\lambda_1, \lambda_2, \lambda_3)$-weighed products of 2-dimensional Kähler manifolds and (λ_1, λ_2)-weighed products of 2- and 4-dimensional Kähler manifolds are 6-dimensional weak nearly Kähler manifolds, and similarly for dimensions > 6.*

Putting $Y = \xi_j$ in (3), and using $\eta^i(\xi_j) = \delta^i_j$, we get

$$\eta^j(X) = g(X, \xi_j); \tag{4}$$

thus, ξ_j is orthogonal to the distribution $\mathcal{D} = \bigcap_{i=1}^s \ker \eta^i$. For a more intuitive understanding of the role of Q in the f-structure, we explain the following properties:

$$f\,\xi_i = 0, \quad \eta^i \circ f = 0, \quad \eta^i \circ Q = \eta^i, \quad [Q, f] = 0.$$

By (2), $f^2 \xi_i = 0$ is true. From this and (2), we get $f^3 + fQ = 0$. By this, $Q\xi_i = \xi_i$ and $f^2\xi_i = 0$ we get $0 = -f^3\xi_i = fQ\xi_i = f\xi_i$. By $f\xi_i = 0$, (4), and the skew-symmetry of f, we get $\eta^i(fX) = g(fX, \xi_i) = -g(X, f\xi_i) = 0$. From this and condition $\operatorname{rank} f = 2n$, we conclude that f the distribution $\mathcal{D}$ of a weak metric f-structure is f-invariant, $\mathcal{D} = f(TM)$

and $\dim \mathcal{D} = 2n$. By this and $f^3 + fQ = 0$, we get $f^3 X = f^2(fX) = -QfX$; hence, $f^3 + Qf = 0$. This and $f^3 + fQ = 0$ yield $fQ = Qf$. By symmetry of Q and $Q\xi_i = \xi_i$, we get $\eta^i(QX) = g(QX, \xi_i) = g(X, Q\xi_i) = g(X, \xi_i) = \eta^i(X)$.

Therefore, TM splits as complementary orthogonal sum of $\mathcal{D}$ and $\ker f$. A weak metric f-structure (f, Q, ξ_i, η^i, g) is said to be normal if the following tensor is zero:

$$\mathcal{N}^{(1)}(X,Y) = [f,f](X,Y) + 2\sum\nolimits_{i=1}^{s} d\eta^i(X,Y)\,\xi_i \quad (X,Y \in \mathfrak{X}_M).$$

The Nijenhuis torsion of a (1,1)-tensor S and the derivative of a 1-form ω are given by

$$[S,S](X,Y) = S^2[X,Y] + [SX,SY] - S[SX,Y] - S[X,SY] \quad (X,Y \in \mathfrak{X}_M),$$
$$d\omega(X,Y) = (1/2)\,\{X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])\} \quad (X,Y \in \mathfrak{X}_M).$$

Using the Levi-Civita connection ∇ of g, one can rewrite $[S,S]$ as

$$[S,S](X,Y) = (S\nabla_Y S - \nabla_{SY} S)X - (S\nabla_X S - \nabla_{SX} S)Y. \tag{5}$$

The fundamental 2-form Φ on $(M^{2n+s}, f, Q, \xi_i, \eta^i, g)$ is defined by

$$\Phi(X,Y) = g(X, fY) \quad (X,Y \in \mathfrak{X}_M).$$

Proposition 1. *A weak metric f-structure with condition $\mathcal{N}^{(1)} = 0$ satisfies*

$$\pounds_{\xi_i} f = d\eta^j(\xi_i, \cdot) = 0,$$
$$d\eta^i(fX,Y) - d\eta^i(fY,X) = \frac{1}{2}\,\eta^i([\widetilde{Q}X, fY]),$$
$$\nabla_{\xi_i} \xi_j \in \mathcal{D}, \quad [X, \xi_i] \in \mathcal{D} \quad (1 \le i,j \le s,\ X \in \mathcal{D}).$$

Moreover, $\nabla_{\xi_i} \xi_j + \nabla_{\xi_j} \xi_i = 0$, that is, $\ker f$ *defines a totally geodesic distribution.*

These tensors on a weak metric f-manifold are well known in the classical theory:

$$\mathcal{N}^{(2)}_i(X,Y) := (\pounds_{fX}\,\eta^i)(Y) - (\pounds_{fY}\,\eta^i)(X) = 2\,d\eta^i(fX,Y) - 2\,d\eta^i(fY,X),$$
$$\mathcal{N}^{(3)}_i(X) := (\pounds_{\xi_i} f)X = [\xi_i, fX] - f[\xi_i, X],$$
$$\mathcal{N}^{(4)}_{ij}(X) := (\pounds_{\xi_i}\,\eta^j)(X) = \xi_i(\eta^j(X)) - \eta^j([\xi_i, X]) = 2\,d\eta^j(\xi_i, X).$$

Example 2. *Let $M^{2n+s}(f, Q, \xi_i, \eta^i)$ be a weak framed f-manifold. Consider the product manifold $\bar{M} = M^{2n+s} \times \mathbb{R}^s$, where $\mathbb{R}^s$ is a Euclidean space with a basis $\partial_1, \ldots, \partial_s$, and define tensors J and $\bar{Q}$ on $\bar{M}$ putting $J(X, \sum_{i=1}^s a^i \partial_i) = (fX - \sum_{i=1}^s a^i \xi_i, \sum_j \eta^j(X)\partial_j)$ and $\bar{Q}(X, \sum_{i=1}^s a^i \partial_i) = (QX, \sum_{i=1}^s a^i \partial_i)$ for $a_i \in C^\infty(M)$. It can be shown that $J^2 = -\bar{Q}$. The tensors $\mathcal{N}^{(2)}_i, \mathcal{N}^{(3)}_i, \mathcal{N}^{(4)}_{ij}$ appear when we derive the integrability condition $[J, J] = 0$ and express the normality condition $\mathcal{N}^{(1)} = 0$ for (f, Q, ξ_i, η^i).*

Define a "small" (1, 1)-tensor $\widetilde{Q} := Q - \mathrm{Id}$ and note that $[\widetilde{Q}, f] = 0$ and $\eta^i \circ \widetilde{Q} = 0$. The following new tensor (vanishing at $\widetilde{Q} = 0$)

$$\mathcal{N}^{(5)}(X,Y,Z) := fZ\,(g(X, \widetilde{Q}Y)) - fY\,(g(X, \widetilde{Q}Z))$$
$$+ g([X, fZ], \widetilde{Q}Y) - g([X, fY], \widetilde{Q}Z) + g([Y, fZ] - [Z, fY] - f[Y,Z],\ \widetilde{Q}X),$$

which supplements the sequence $\mathcal{N}^{(1)}, \mathcal{N}_i^{(2)}, \mathcal{N}_i^{(3)}, \mathcal{N}_{ij}^{(4)}$, is needed to study the weak metric f-structure. We express the covariant derivative of f using a new tensor $\mathcal{N}^{(5)}$:

$$2\,g((\nabla_X f)Y, Z) = 3\,d\Phi(X, fY, fZ) - 3\,d\Phi(X, Y, Z) + g(\mathcal{N}^{(1)}(Y, Z), fX)$$
$$+ \sum\nolimits_{i=1}^{s} \big(\mathcal{N}_i^{(2)}(Y, Z)\,\eta^i(X) + 2\,d\eta^i(fY, X)\,\eta^i(Z) - 2\,d\eta^i(fZ, X)\,\eta^i(Y)\big) + \mathcal{N}^{(5)}(X, Y, Z),$$

where the derivative of a 2-form Φ is given by

$$3\,d\Phi(X, Y, Z) = X\,\Phi(Y, Z) + Y\,\Phi(Z, X) + Z\,\Phi(X, Y)$$
$$- \Phi([X, Y], Z) - \Phi([Z, X], Y) - \Phi([Y, Z], X).$$

Note that the above equality yields

$$3\,d\Phi(X, Y, Z) = (\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(Z, X) + (\nabla_Z \Phi)(X, Y). \tag{6}$$

For particular values of $\mathcal{N}^{(5)}$, we get $\mathcal{N}^{(5)}(\xi_i, \xi_j, Z) = \mathcal{N}^{(5)}(\xi_i, Y, \xi_j) = 0$ and

$$\mathcal{N}^{(5)}(X, \xi_i, Z) = -\mathcal{N}^{(5)}(X, Z, \xi_i) = g(\mathcal{N}_i^{(3)}(Z), \widetilde{Q}X),$$
$$\mathcal{N}^{(5)}(\xi_i, Y, Z) = g([\xi_i, fZ], \widetilde{Q}Y) - g([\xi_i, fY], \widetilde{Q}Z).$$

Definition 2. A weak metric f-structure is called a *weak almost $\mathcal{K}$-structure* if $d\Phi = 0$. We define its two subclasses as follows:

(i) A *weak almost $\mathcal{C}$-structure* if Φ and η^i $(1 \le i \le s)$ are closed forms;

(ii) A *weak almost $\mathcal{S}$-structure*

if the following is valid:

$$\Phi = d\eta^1 = \ldots = d\eta^s \quad (\text{hence, } d\Phi = 0). \tag{7}$$

Adding the normality condition, we get *weak $\mathcal{K}$-*, *weak $\mathcal{C}$-*, and *weak $\mathcal{S}$-structures*, respectively. A *weak f-K-contact structure* is a weak almost $\mathcal{S}$-structure, whose structure vector fields ξ_i are Killing, i.e., the tensor $(\pounds_{\xi_i} g)(X, Y) = g(\nabla_Y \xi_i, X) + g(\nabla_X \xi_i, Y)$ vanishes. For $s = 1$, weak (almost) $\mathcal{C}$- and weak (almost) $\mathcal{S}$-manifolds reduce to weak (almost) cosymplectic manifolds and weak (almost) Sasakian manifolds, respectively.

Remark 1. *The almost $\mathcal{S}$-structure is also called an f-contact structure, e.g., [21]; then, the $\mathcal{S}$-structure can be regarded as a normal f-contact structure.*

Example 3. *(i) To construct a weak metric f-structure (f, Q, ξ_i, η^i, g) on the Riemannian product $M = \bar{M} \times \mathbb{R}^s$ of a weak almost Hermitian manifold $(\bar{M}, \bar{f}, \bar{g})$ with $\Omega(X, Y) = \bar{g}(X, \bar{f}Y)$ and a Euclidean space $(\mathbb{R}^s, dy^2)$, we take any point (x, y) of M and set*

$$\xi_i = (0, \partial_{y^i}), \ \ \eta^i = (0, dy^i), \ \ f(X, \partial_{y^i}) = (\bar{f}X, 0), \ \ Q(X, \partial_{y^i}) = (-\bar{f}^2 X, \partial_{y^i}),$$

where $X \in T_x\bar{M}$. Note that $\nabla f = 0$ if and only if $\overline{\nabla}\bar{f} = 0$. On the other hand, $\overline{\nabla}\bar{f} = 0$ if and only if $d\Omega = 0$, see (6) with $\Phi = \Omega$, i.e., (M, Ω) is a symplectic manifold.

(ii) For a weak $\mathcal{C}$-structure, we obtain $g((\nabla_X f)Y, Z) = \frac{1}{2}\mathcal{N}^{(5)}(X, Y, Z)$. A weak metric f-structure with conditions $\nabla f = 0$ and $g([\xi_i, \xi_j], \xi_k) = 0$ is a weak $\mathcal{C}$-structure with the property $\mathcal{N}^{(5)} = 0$. For a weak $\mathcal{S}$-structure, we get

$$g((\nabla_X f)Y, Z) = g(fX, fY)\,\bar{\eta}(Z) - g(fX, fZ)\,\bar{\eta}(Y) + \frac{1}{2}\mathcal{N}^{(5)}(X, Y, Z);$$

ξ_i are Killing vector fields and $\ker f$ *defines a Riemannian totally geodesic foliation. In particular, for an $\mathcal{S}$-structure, we have*

$$(\nabla_X f)Y = g(fX, fY)\,\bar\xi + \bar\eta(Y) f^2 X\,. \tag{8}$$

For a weak almost $\mathcal{K}$-structure (and its special cases, a weak almost $\mathcal{S}$-structure and a weak almost $\mathcal{C}$-structure), the distribution $\ker f$ is involutive (tangent to a foliation). Moreover, weak almost $\mathcal{S}$- and weak almost $\mathcal{C}$-structures satisfy the following conditions (trivial for $s = 1$):

$$[\xi_i, \xi_j] = 0, \tag{9}$$

$$g(\nabla_X\, \xi_i,\ \xi_j) = 0 \quad (X \in \mathfrak{X}_M) \tag{10}$$

for $1 \le i, j \le s$. The following condition is a corollary of (10):

$$\eta^k(\nabla_{\xi_i}\, \xi_j) = 0 \quad (1 \le i, j, k \le s). \tag{11}$$

By (9), the distribution $\ker f$ of weak almost $\mathcal{S}$- and a weak almost $\mathcal{C}$-manifolds is tangent to a $\mathfrak{g}$-foliation with an abelian Lie algebra.

Remark 2 ([28])**.** *Let $\mathfrak{g}$ be a Lie algebra of dimension s. A foliation of dimension s on a smooth connected manifold M is called a $\mathfrak{g}$-foliation if there exist complete vector fields $\xi_1, \ldots, \xi_s$ on M which, when restricted to each leaf, form a parallelism of this submanifold with a Lie algebra isomorphic to $\mathfrak{g}$.*

3. Main Results

In this section, weak nearly $\mathcal{S}$- and weak nearly $\mathcal{C}$-structures are defined and studied; some of the statements generalize the results in [13–15].

The restriction on the symmetric part of (8) gives the following.

Definition 3. A weak metric f-manifold is called a *weak nearly $\mathcal{S}$-manifold* if

$$(\nabla_X f)Y + (\nabla_Y f)X = 2\, g(fX, fY)\,\bar\xi + \bar\eta(X) f^2 Y + \bar\eta(Y) f^2 X \tag{12}$$

for all $X, Y \in \mathfrak{X}_M$. A weak metric f-manifold is called a *weak nearly $\mathcal{C}$-manifold* if

$$(\nabla_X f)Y + (\nabla_Y f)X = 0. \tag{13}$$

Example 4. *Let a Riemannian manifold (M^{2n+s}, g) admit two nearly $\mathcal{S}$-structures (or, nearly $\mathcal{C}$-structures) $M^{2n+s}(f_k, Q, \xi_i, \eta^i, g)$ $(k = 1, 2)$ with common Reeb vector fields ξ_i and one-forms $\eta^i = g(\xi_i, \cdot)$. Suppose that $f_1 \ne f_2$ are such that $\psi := f_1 f_2 + f_2 f_1 \ne 0$. Then, $f := (\cos t) f_1 + (\sin t) f_2$ for small $t > 0$ satisfies (12) (and (13), respectively) and*

$$f^2 = -\mathrm{Id} + (\sin t \cos t)\,\psi + \textstyle\sum_{i=1}^s \eta^i \otimes \xi_i.$$

Thus, (f, Q, ξ_i, η^i, g) is a weak nearly $\mathcal{S}$-structure (and weak nearly $\mathcal{C}$-structure, respectively) on M^{2n+s} with $Q = \mathrm{Id} - (\sin t \cos t)\,\psi$.

The following condition is trivial when $Q = \mathrm{Id}_{\,TM}$:

$$(\nabla_X\, Q)Y = 0 \quad (X, Y \in \mathfrak{X}_M,\ Y \perp \ker f). \tag{14}$$

Using (14), we have

$$(\nabla_X Q)Y = \sum_{i=1}^s \eta^i(Y)(\nabla_X Q)\xi_i = -\sum_{i=1}^s \eta^i(Y)\,\widetilde{Q}\nabla_X \xi_i \quad (X, Y \in \mathfrak{X}_M).$$

Example 5. *To construct a weak (nearly) $\mathcal{C}$-structure (f, Q, ξ_i, η^i, g) on the Riemannian product $M = \bar{M} \times \mathbb{R}^s$ of a weak (nearly) Kähler manifold $(\bar{M}, \bar{f}, \bar{g})$ and a Euclidean space $(\mathbb{R}^s, dy^2)$, we take any point (x, y) of M and set*

$$\xi_i = (0, \partial_{y^i}), \ \ \eta^i = (0, dy^i), \ \ f(X, \partial_{y^i}) = (\bar{f}X, 0), \ \ Q(X, \partial_{y^i}) = (-\bar{f}^2 X, \partial_{y^i}),$$

as in Example 3(i). Note that if $\overline{\nabla}_X(\bar{f}^2) = 0$ $(X \in T\bar{M})$, then (14) holds.

The following result opens new applications to Killing vector fields.

Proposition 2. *Both on a weak nearly $\mathcal{S}$-manifold and a weak nearly $\mathcal{C}$-manifold satisfying (9) and (11), the distribution* $\ker f$ *defines a flat totally geodesic foliation; moreover, if conditions (10) and (14) hold, then the vector fields ξ_i are Killing.*

Proof. Putting $X = \xi_j$ and $Y = \xi_k$ in (12) or (13), we find $(\nabla_{\xi_j} f)\xi_k + (\nabla_{\xi_k} f)\xi_j = 0$; hence, $f(\nabla_{\xi_j}\xi_k + \nabla_{\xi_k}\xi_j) = 0$. Applying f to this and using (2), we obtain

$$0 = f^2(\nabla_{\xi_j}\xi_k + \nabla_{\xi_k}\xi_j) = -Q(\nabla_{\xi_j}\xi_k + \nabla_{\xi_k}\xi_j) + \sum_{i=1}^s \eta^i(\nabla_{\xi_j}\xi_k + \nabla_{\xi_k}\xi_j)\xi_i.$$

Since the (1,1)-tensor Q is nonsingular and (11) is true, we get $\nabla_{\xi_j}\xi_k + \nabla_{\xi_k}\xi_j = 0$. Combining this with $\nabla_{\xi_j}\xi_k - \nabla_{\xi_k}\xi_j = 0$, see (9), yields

$$\nabla_{\xi_j}\xi_k = 0 \quad (1 \le j, k \le s); \tag{15}$$

hence, $\ker f$ defines a flat totally geodesic foliation. Next, using (15) we calculate

$$\nabla_{\xi_i}\eta^j = g(\nabla_{\xi_i}\xi_j, \cdot) = 0. \tag{16}$$

Using (10) and (15), we obtain

$$(\pounds_{\xi_j} g)(\xi_k, \cdot) = g(\nabla_{\xi_j}\xi_k, \cdot) = 0.$$

Taking the ξ_j-derivative of (3) and using (14) and $\nabla_{\xi_j}\eta^i = 0$, we find (for $Y \perp \ker f$)

$$\begin{aligned} g((\nabla_{\xi_j} f)X, fY) + g(fX, (\nabla_{\xi_j} f)Y) = \nabla_{\xi_j}\, g(fX, fY) \\ = g(X, (\nabla_{\xi_j} Q)Y) + \textstyle\sum_{i=1}^s \{(\nabla_{\xi_j}\eta^i)(X)\,\eta^i(Y) + \eta^i(X)(\nabla_{\xi_j}\eta^i)(Y)\} = 0. \end{aligned}$$

For a weak nearly $\mathcal{S}$-manifold, using (12), (10), and $\eta \circ \widetilde{Q} = 0$ yields

$$\begin{aligned} & g((\nabla_{\xi_j} f)X, fY) + g(fX, (\nabla_{\xi_j} f)Y) \\ & = -g((\nabla_X f)\xi_j, fY) - g(fX, (\nabla_Y f)\xi_j) + g(f^2X, fY) + g(f^2Y, fX) \\ & = -g(\nabla_X \xi_j, f^2Y) - g(f^2X, \nabla_Y \xi_j) = g(\nabla_X \xi_j, QY) + g(QX, \nabla_Y \xi_j) \\ & = g(\nabla_X \xi_j, Y) + g(X, \nabla_Y \xi_j) + g(\nabla_X \xi_j, \widetilde{Q}Y) + g(\widetilde{Q}X, \nabla_Y \xi_j) \\ & = (\pounds_{\xi_j} g)(X, Y) - g(\xi_j, (\nabla_X \widetilde{Q})Y) - g((\nabla_Y \widetilde{Q})X, \xi_j) = (\pounds_{\xi_j} g)(X, Y). \end{aligned}$$

Here, we used $g(\xi_j, (\nabla_X \widetilde{Q})Y) = 0$. For a weak nearly $\mathcal{C}$-manifold, using (13) yields

$$(\pounds_{\xi_j} g)(X, Y) = g((\nabla_{\xi_j} f)X, fY) + g(fX, (\nabla_{\xi_j} f)Y) = 0. \tag{17}$$

From (17), for both cases we obtain $\pounds_{\xi_j} g = 0$, i.e., ξ_j is a Killing vector field. □

Remark 3. *Note that even for a nearly $\mathcal{S}$-manifold without conditions* (9) *and* (10)*, the vector fields ξ_i $(1 \le i \le s)$ are not Killing; see Corollary 1 in [13].*

Theorem 1. *There are no weak nearly $\mathcal{C}$-manifolds with conditions* (9)*,* (10)*, and* (14) *which satisfy $\Phi = d\eta^1 = \ldots = d\eta^s$; see* (7)*.*

Proof. Suppose that our weak nearly $\mathcal{C}$-manifold satisfies (7). Since also ξ_i are Killing vector fields (see Proposition 2), M is a weak f-K-contact manifold. By Theorem 1 in [22], the following holds:

$$\nabla \xi_i = -f \quad (1 \le i \le s). \tag{18}$$

By Proposition 6 in [22], the ξ-sectional curvature of a weak f-K-contact manifold is positive, i.e., $K(\xi_i, X) > 0$ $(X \perp \ker f)$. Thus, for any nonzero vector $X \perp \ker f$, using (13) and (18), we get

$$\begin{aligned} 0 < K(\xi_i, X) &= g(\nabla_{\xi_i} \nabla_X \xi_i - \nabla_X \nabla_{\xi_i} \xi_i - \nabla_{[\xi_i, X]} \xi_i, X) \\ &= g(-(\nabla_{\xi_i} f)X + f^2 X, X) = g((\nabla_X f)\xi_i, X) - g(fX, fX) \\ &= -g(f \nabla_X \xi_i, X) + g(f^2 X, X) = 2\, g(f^2 X, X). \end{aligned}$$

This contradicts the following equality: $g(f^2 X, X) = -g(fX, fX) \le 0$. □

Corollary 1. *There are no nearly $\mathcal{C}$-manifolds with conditions* (9) *and* (10) *which satisfy* (7)*.*

Theorem 2. *A weak nearly $\mathcal{C}$-manifold $(M^{2n+s}, f, Q, \xi_i, \eta^i, g)$ satisfies*

$$\nabla \xi_i = 0 \quad (1 \le i \le s) \tag{19}$$

if and only if the manifold is locally isometric to the Riemannian product of a Euclidean s-space and a weak nearly Kähler manifold.

Proof. For all vector fields X, Y orthogonal to $\ker f$, we have

$$2\, d\eta^i(X, Y) = g(\nabla_X \xi_i, Y) - g(\nabla_Y \xi_i, X). \tag{20}$$

Thus, if the condition $\nabla \xi_i = 0$ holds, then the contact distribution $\mathcal{D}$ is integrable. Moreover, any integral submanifold of $\mathcal{D}$ is a totally geodesic submanifold. Indeed, for $X, Y \perp \ker f$, we have $g(\nabla_X Y, \xi_i) = -g(Y, \nabla_X \xi_i) = 0$. Since $\nabla_{\xi_i} \xi_j = 0$, by de Rham Decomposition Theorem, the manifold is locally the Riemannian product $\bar{M} \times \mathbb{R}^s$. The metric weak f-structure induces on $\bar{M}$ a weak almost-Hermitian structure, which, by these conditions, is weak nearly Kähler.

Conversely, if a weak nearly $\mathcal{C}$-manifold is locally the Riemannian product $\bar{M} \times \mathbb{R}^s$, where $\bar{M}$ is a weak nearly Kähler manifold and $\xi_i = (0, \partial_{y^i})$ (see also Example 5), then $d\eta^j(X, Y) = 0$ $(X, Y \perp \ker f)$. By (20) and $\nabla_{\xi_i} \xi_j = 0$, we obtain $\nabla \xi_i = 0$. □

Corollary 2. *A nearly $\mathcal{C}$-manifold $(M^{2n+s}, f, \xi_i, \eta^i, g)$ satisfies (19) if and only if the manifold is locally isometric to the Riemannian product of $\mathbb{R}^s$ and a nearly Kähler manifold.*

Theorem 3. *Let a weak nearly $\mathcal{S}$-structure satisfy (9), (10), and (14); then, the following is true:*

(i) The condition $\eta^j \circ N^{(1)} = 0$ $(1 \le j \le s)$ yields $d\eta^j(X, Y) = \Phi(QX, Y)$ for all j.

(ii) The condition (7) yields $N^{(1)}(X, Y) = 2\,\Phi(\widetilde{Q}X, Y)\,\bar{\xi}$.

Proof. (i) We calculate, using (5), (12), and $\eta^j \circ f = 0$,

$$\begin{aligned}
\eta^j(N^{(1)}(X,Y)) - 2\,d\eta^j(X,Y) &= \eta^j([f,f](X,Y)) \overset{(5)}{=} \eta^j\big((\nabla_{fX} f)Y - (\nabla_{fY} f)X\big) \\
&\overset{(12)}{=} \eta^j\big((\nabla_X f)\,fY - (\nabla_Y f)\,fX\big) + 4\,g(f^2X, fY) \\
&= g\big((\nabla_X f^2)\,Y - (\nabla_Y f^2)\,X,\ \xi_j\big) - 4\,g(QX, fY) \\
&= (\nabla_X\,\eta^j)(Y) - (\nabla_Y\,\eta^j)(X) - 4\,g(QX, fY) \\
&= 2\,d\eta^j(X,Y) - 4\,g(QX, fY).
\end{aligned}$$

Here, we used the identity $2\,d\eta^j(X,Y) = (\nabla_X\,\eta^j)(Y) - (\nabla_Y\,\eta^j)(X)$.
Thus, if $\eta^i(N^{(1)}(X,Y)) = 0$, then $d\,\eta^j(X,Y) = g(QX, fY) = \Phi(QX, Y)$ for all j.

(ii) Using $d\,\Phi = 0$, (2), (6), and (12), where $\bar{\eta} = \sum_{i=1}^s \eta^i$ and $\bar{\xi} = \sum_{i=1}^s \xi_i$, we get

$$\begin{aligned}
&3\,d\,\Phi(X,Y,Z) = -g((\nabla_X f)Y, Z) + g((\nabla_Y f)X, Z) - g((\nabla_Z f)X, Y) \\
&= -g((\nabla_X f)Y, Z) + g\big(-(\nabla_X f)Y + 2\,g(fX, fY)\,\bar{\xi} + \bar{\eta}(X) f^2 Y + \bar{\eta}(Y) f^2 X,\ Z\big) \\
&\quad + g\big((\nabla_X f)Z - 2\,g(fX, fZ)\,\bar{\xi} - \bar{\eta}(X) f^2 Z - \bar{\eta}(Z) f^2 X,\ Y\big) \\
&= -3\,g((\nabla_X f)Y, Z) - 3\,g(f^2X, Y)\,\bar{\eta}(Z) + 3\,g(f^2X, Z)\,\bar{\eta}(Y).
\end{aligned}$$

Thus, (8) holds. Using (8) in (5) gives

$$[f,f] = 2\,g(f^2X, fY)\,\bar{\xi} = -2\,g(QX, fY)\,\bar{\xi} = -2\,\Phi(QX, Y)\,\bar{\xi},$$

hence, $N^{(1)}(X,Y) = 2\,\Phi(\widetilde{Q}X, Y)\,\bar{\xi}$. □

A consequence of Theorem 3 is a rigidity result for $\mathcal{S}$-manifolds; see Theorem 1 of [13].

Corollary 3. *A normal nearly $\mathcal{S}$-structure is an $\mathcal{S}$-structure.*

4. Submanifolds of Weak Nearly Kähler Manifolds

Here, we study weak nearly $\mathcal{S}$- and weak nearly $\mathcal{C}$- submanifolds in a weak nearly Kähler manifold. The second fundamental form h of a submanifold $M \subset (\bar{M}, \bar{g})$ is related with $\overline{\nabla}$ (the Levi-Civita connection of $\bar{g}$ restricted to M) and ∇ (the Levi-Civita connection of metric g induced on M via the Gauss equation) by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad (X, Y \in \mathfrak{X}_M). \tag{21}$$

A submanifold is said to be totally geodesic if $h = 0$. The shape operator $A_N : X \mapsto -\overline{\nabla}_X N$ with respect to a unit normal N is related with h via the equalities

$$h_N(X,Y) = \bar{g}(h(X,Y), N) = g(A_N(X), Y) \quad (X, Y \in \mathfrak{X}_M). \tag{22}$$

Lemma 1. *Let $(\bar M, \bar f, \bar g)$ be a weak Hermitian manifold and M^{2n+s} a submanifold of codimension s equipped with mutually orthogonal unit normals N_i $(i = 1, \dots, s)$ satisfying the condition*

$$\bar g(\bar f N_i, N_j) = 0 \quad (1 \le i, j \le s) \tag{23}$$

(trivial for $s = 1$). Then, M inherits a metric weak f-structure (f, Q, ξ_i, η^i, g) given by

$$\begin{aligned} &\xi_i = \bar f N_i, \quad \eta^i = \bar g(\bar f N_i, \cdot) \quad (i = 1, \dots, s), \quad g = \bar g|_M, \\ &f = \bar f + \textstyle\sum_{i=1}^s \bar g(\bar f N_i, \cdot)\, N_i, \quad Q = -\bar f^{\,2} + \textstyle\sum_{i=1}^s \bar g(\bar f^{\,2} N_i, \cdot)\, N_i. \end{aligned} \tag{24}$$

Moreover, (14) holds on M if $\bar f^{\,2} N_i \perp TM$ $(1 \le i \le s)$ and

$$((\overline\nabla_X \bar f^{\,2})Y)^\top = 0 \quad (X, Y \in \mathfrak{X}_M,\ Y \perp \ker f).$$

Proof. Using the skew-symmetry of $\bar f$ and (23), we verify (2):

$$\begin{aligned} &f^2 X = f(\bar f X - \textstyle\sum_{i=1}^s \bar g(\bar f X, N_i)\, N_i) \\ &= \bar f(\bar f X - \textstyle\sum_{i=1}^s \bar g(\bar f X, N_i)\, N_i) - \bar g(\bar f(\bar f X - \textstyle\sum_{i,j=1}^s \bar g(\bar f X, N_i)\, N_i), N_j)\, N_j \\ &= \bar f^{\,2} X - \textstyle\sum_j \bar g(\bar f^{\,2} N_j, X)\, N_j - \textstyle\sum_{i=1}^s \bar g(\bar f N_i, X)\, \bar f N_i + \textstyle\sum_{i,j=1}^s \bar g(\bar f X, N_i)\, \bar g(\bar f N_i, N_j)\, N_j \\ &= -QX + \textstyle\sum_{i=1}^s \eta^i(X)\, \xi_i \quad (X \in \mathfrak{X}_M). \end{aligned}$$

Since $\bar f^{\,2}$ is negative-definite, for nonzero $X \in \mathfrak{X}_M$ we obtain $\bar g(N_i, X) = 0$ and

$$g(QX, X) = \bar g(-\bar f^{\,2} X + \textstyle\sum_{i=1}^s \bar g(\bar f^{\,2} N_i, X)\, N_i,\ X) = -\bar g(\bar f^{\,2} X, X) > 0,$$

hence, the tensor Q is positive-definite on TM. Then, we calculate $(\nabla_X Q)Y$ for $X, Y \in \mathfrak{X}_M$ and $Y \perp \ker f$, using (21) and (24) and the condition $\bar f^2 N_i \perp TM$ $(1 \le i \le s)$:

$$\begin{aligned} &(\nabla_X Q)Y = \nabla_X(QY) - Q(\nabla_X Y) \\ &= \{\overline\nabla_X(-\bar f^2\, Y + \textstyle\sum_{i=1}^s g(\bar f^2 N_i, Y) N_i) - h(X, QY) + \bar f^2(\overline\nabla_X Y - h(X, Y)) \\ &- \textstyle\sum_{i=1}^s g(\bar f^2 N_i,\, \overline\nabla_X Y - h(X, Y)) N_i\}^\top \\ &= (-(\overline\nabla_X(\bar f^2 Y)) + \bar f^2(\overline\nabla_X Y))^\top - \textstyle\sum_{i=1}^s g(\bar f^2 N_i,\, Y)\, A_{N_i} X \\ &= -((\overline\nabla_X \bar f^2)Y)^\top, \end{aligned}$$

where $^\top$ is the TM-component of a vector. This completes the proof. □

The following theorem characterizes weak nearly $\mathcal{C}$- and weak nearly $\mathcal{S}$-submanifolds of a nearly Kähler manifold, using the property of the second fundamental form.

Theorem 4. *Let $(\bar M, \bar f, \bar g)$ be a weak nearly Kähler manifold and M^{2n+s} a submanifold of codimension s equipped with mutually orthogonal unit normals N_i $(i = 1, \dots, s)$ satisfying (23). If the second fundamental form of M and the induced metric weak f-structure (f, Q, ξ_i, η^i, g) on M, given by (24), satisfy*

$$\begin{aligned} &(i)\ h_{N_i}(X, Y) = g(QX,\, Y) + \textstyle\sum_{j,k=1}^s (h_{N_i}(\xi_j, \xi_k) - \delta_{j,k})\, \eta^j(X)\, \eta^k(Y), \\ &(ii)\ h_{N_i}(X, Y) = \textstyle\sum_{j,k=1}^s h_{N_i}(\xi_j, \xi_k)\, \eta^j(X)\, \eta^k(Y), \end{aligned} \tag{25}$$

and

$$h_{N_i}(\xi_j, \xi_k) = h_{N_j}(\xi_i, \xi_k) \quad (1 \le i, j \le s), \tag{26}$$

then (f, Q, ξ_i, η^i, g) *is*

$$(i)\ \text{a weak nearly } \mathcal{S}\text{-structure}; \quad (ii)\ \text{a weak nearly } \mathcal{C}\text{-structure}. \tag{27}$$

Proof. Substituting

$$\bar{f}\,Y = f\,Y - \sum\nolimits_{i=1}^{s} \bar{g}(\bar{f}N_i, Y)\,N_i = f\,Y - \sum\nolimits_{i=1}^{s} \eta^i(Y)\,N_i$$

in $(\overline{\nabla}_X \bar{f})Y$, where $X, Y \in \mathfrak{X}_M$, and using (21) and Lemma 1, we obtain

$$(\overline{\nabla}_X \bar{f})Y = \overline{\nabla}_X(\bar{f}\,Y) - \bar{f}(\overline{\nabla}_X Y) = (\nabla_X f)Y + \sum\nolimits_{i=1}^{s} \{\eta^i(Y) A_{N_i} X - h_{N_i}(X,Y)\,\xi_i\} \\ + \sum\nolimits_{i=1}^{s} \{X(\eta^i(Y)) - \eta^i(\nabla_X Y) + h_{N_i}(X, fY)\} N_i.$$

Thus, the TM-component of the weak nearly Kähler condition (1), using (21) and (22), takes the form

$$((\overline{\nabla}_X \bar{f})Y + (\overline{\nabla}_Y \bar{f})X)^{\top} = (\nabla_X f)Y + (\nabla_Y f)X \\ + \sum\nolimits_{i=1}^{s} \{\eta^i(X) A_{N_i} Y + \eta^i(Y) A_{N_i} X - 2\,h_{N_i}(X,Y)\,\xi_i\} = 0. \tag{28}$$

Using (22), one can show that (25) is equivalent to the following:

$$(i)\ A_{N_i} X = -f^2 X + \sum\nolimits_{j,k=1}^{s} h_{N_i}(\xi_j, \xi_k)\,\eta^j(X)\,\xi_k, \\ (ii)\ A_{N_i} X = \sum\nolimits_{j,k=1}^{s} h_{N_i}(\xi_j, \xi_k)\,\eta^j(X)\,\xi_k. \tag{29}$$

(i) If we have a weak nearly $\mathcal{S}$-structure, see (12), then from (28) we get

$$2\,g(fX, fY)\,\bar{\xi} + \bar{\eta}(Y) f^2 X + \bar{\eta}(X) f^2 Y \\ + \sum\nolimits_{i=1}^{s} \{\eta^i(X) A_{N_i} Y + \eta^i(Y) A_{N_i} X - 2\,h_{N_i}(X,Y)\,\xi_i\} = 0, \tag{30}$$

Substituting the expressions of $h_{N_i}(X,Y)$ and A_{N_i}, see (25)(i) and (29)(i), in (30) and using (26) gives identity; thus, we obtain a weak nearly $\mathcal{S}$-structure on M.

(ii) If we have a weak nearly $\mathcal{C}$-structure, see (13), then from (28) we get

$$\sum\nolimits_{i=1}^{s} \{\eta^i(X) A_{N_i} Y + \eta^i(Y) A_{N_i} X - 2\,h_{N_i}(X,Y)\,\xi_i\} = 0. \tag{31}$$

Substituting the expressions of $h_{N_i}(X,Y)$ and A_{N_i}, see (25)(ii) and (29)(ii), in (31) and using (26) gives identity; thus, we obtain a weak nearly $\mathcal{C}$-structure on M. □

For $Q = \mathrm{Id}$, the properties of (25) lead us to the following.

Definition 4. A codimension s submanifold M^{2n+s} of a Hermitian manifold $(\bar{M}, \bar{f}, \bar{g})$, equipped with mutually orthogonal unit normals N_i $(i = 1, \ldots, s)$ satisfying

$$h_{N_i}(X,Y) = a_i\,g(X,Y) + \sum\nolimits_{j,k=1}^{s} b_{i,j,k}\,\eta^j(X)\,\eta^k(Y), \tag{32}$$

where $a_i, b_{i,j,k} \in C^\infty(M)$ and η^i $(1 \le i \le s)$ are linear independent one-forms on M, will be called an *s-quasi-umbilical* submanifold. For $s = 1$, condition (32) reads as follows, see [15]:

$$h_N(X,Y) = a_1\,g(X,Y) + b_1\,\eta(X)\,\eta(Y).$$

The geometric meaning of (32) is that the restriction of h_{N_i} on the distribution $\bigcap_{i=1}^{s} \ker \eta^i$ looks similar to h for totally umbilical submanifolds: $h = (\mathrm{trace}_g\, h / \dim M)\, g$.

The following consequence of Theorem 4 extends the fact (see Theorem 4.1 in [14]) that a hypersurface of a nearly Kähler manifold is nearly Sasakian or nearly cosymplectic if and only if it is quasi-umbilical with respect to the almost contact form.

Corollary 4. *Let $(\bar M, \bar f, \bar g)$ be a nearly Kähler manifold and M^{2n+s} a submanifold of codimension s equipped with mutually orthogonal unit normals N_i $(i = 1, \dots, s)$ satisfying (23), and $(f, \xi_i, \eta^i, g = \bar g|_M)$ the induced metric f-structure on M, given by*

$$\xi_i = \bar f N_i, \quad \eta^i = \bar g(\bar f N_i, \cdot) \quad (i = 1, \dots, s), \quad f = \bar f + \sum\nolimits_{j=1}^{s} \bar g(\bar f N_j, \cdot)\, N_j.$$

If M^{2n+s} is an s-quasi-umbilical submanifold (with respect to the 1-forms η^i),

$$(i)\ h_{N_i}(X, Y) = g(X, Y) + \sum\nolimits_{j,k=1}^{s} \big(h_{N_i}(\xi_j, \xi_k) - \delta_{j,k}\big)\, \eta^j(X)\, \eta^k(Y),$$
$$(ii)\ h_{N_i}(X, Y) = \sum\nolimits_{j,k=1}^{s} h_{N_i}(\xi_j, \xi_k)\, \eta^j(X)\, \eta^k(Y),$$

and (26) are true, then (f, ξ_i, η^i, g) is (i) a nearly $\mathcal{S}$-structure; (ii) a nearly $\mathcal{C}$-structure.

5. Conclusions

We have shown that weak nearly $\mathcal{S}$- and weak nearly $\mathcal{C}$-structures are useful for studying metric f-structures, e.g., totally geodesic foliations, Killing vector fields, and s-quasi-umbilical submanifolds. Some classical results have been extended in this paper to weak nearly $\mathcal{S}$- and weak nearly $\mathcal{C}$-manifolds with additional conditions. Based on the numerous applications of nearly Kähler, nearly Sasakian, and nearly cosymplectic structures, we expect that weak nearly Kähler, $\mathcal{S}$- and $\mathcal{C}$-structures will be useful for geometry and theoretical physics, e.g., for NGT, the theory of s-cosymplectic structures and s-contact structures, multi-time Hamiltonian systems, and s-evolution systems.

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