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The Next Terracini Loci of Segre–Veronese Varieties and Their Maximal Weights

Edoardo Ballico †

Department of Mathematics, University of Trento, 38123 Trento, Italy; edoardo.ballico@unitn.it
† The author is a member of Gruppo Nazionale per le Strutture Algebriche e Geometriche e loro Applicazioni of Istituto di Alta Matematica, 00185 Rome, Italy.

Abstract

We describe all Terracini loci of Segre–Veronese varieties with at most roughly double the points of the minimal one. In this range we compute the maximum of all weights of the Terracini sets. To prove these results we use cohomological tools (residual exact sequences) applied to some critical schemes associated with a Terracini set and containing all of its points. We expect that these critical schemes will be a very useful tool for other related problems.

Keywords: Terracini locus; Segre–Veronese variety; zero-dimensional scheme; weight of a Terracini set; partially symmetric tensors

MSC: 14N07; 14M17; 14N05

1. Introduction

In recent years several papers have studied the Terracini loci of embedded varieties ([1–7]). For a long time they were studied only for general subsets of an embedded variety, because for general subsets they are related to the dimensions of the secant variety of the embedded varieties via the Terracini lemma ([8,9]). According to [3] they are considered for any finite union of smooth points of an embedded variety. We recall their definition.

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate n-dimensional complex projective variety. For any zero-dimensional subscheme $Z \subset \mathbb{P}^r$, let $\langle Z \rangle$ denote the linear span of Z, i.e., the intersection of all hyperplanes of $\mathbb{P}^r$ containing Z with the convention $\langle Z \rangle = \mathbb{P}^r$, if there is no such a hyperplane. Note that the zero-dimensional scheme Z is linearly independent if and only if $\dim\langle Z \rangle = \deg(Z) - 1$. Let X_{reg} denote the set of all smooth points of X. For all $p \in X_{\mathrm{reg}}$ let $2p$ be the degree $n+1$ of the zero-dimensional subscheme of X with $(\mathcal{I}_p)^2$ as its ideal sheaf. Note that $\deg(2p) = n+1$ and that $\langle 2p \rangle$ is the Zariski tangent space T_pX of p at X. Since $\dim T_pX = n$, $2p$ is linearly independent. For all positive integers x and all integral quasi-projective varieties W, let $S(W, x)$ denote the set of all subsets of W with cardinality x. For all $S \in S(X_{\mathrm{reg}}, x)$ set $2S := \cup_{p \in S} 2p$. We have $\deg(2S) = (n+1)x$, and hence, $\min\{r, (n+1)x - 1\}$ is the expected dimension of $\langle 2S \rangle$. We say that S is a Terracini set of X or of the embedding $X \subset \mathbb{P}^r$ and write $S \in \mathcal{T}(X, x)$ if $\dim\langle 2S \rangle \leq -1 + \min\{r, (n+1)x - 1\}$, i.e., if $\langle 2S \rangle \neq \mathbb{P}^r$ and $\dim\langle 2S \rangle \leq (n+1)x - 2$.

Outside the Terracini locus, a certain differential is injective by the Terracini Lemma in its general form ([8], Cor. 1.10). This observation is used to show that, outside the Terracini sets, a certain decomposition is locally unique. As an example we explain the case of the Segre variety X. Let T be a tensor with a format associated with X and with tensor rank x.

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A tensor decomposition of T with x addenda corresponds to a unique $S \in S(X,x)$, and if $S \notin \mathcal{T}(X,x)$ for each tensor T' near T (in the Euclidean or in the Zariski topology), there is a unique $S' \in S(X,x)$ near S such that T' has a tensor decomposition associated with S' and no other tensor decomposition near S. Hence, the tensor decomposition near T is well-behaved for small numerical errors. Our main results are for Segre–Veronese varieties, which are related to the additive decompositions of partially symmetric tensors.

Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a multiprojective space as in Theorems 1 and 2. For $i = 1, \dots, k$ let ϵ_i be the multi-index $(a_1, \dots, a_k) \in \mathbb{N}^k$ such that $a_i = 1$ and $a_j = 0$ for all $j \neq i$. A reduced curve $C \subset Y$ is said to have multidegree $(a_1, \dots, a_k)$ if $a_i = \deg(\mathcal{O}_C(\epsilon_i))$ for all i.

Theorem 1. *Fix an integer $k \geq 2$, and positive integers x, n_i, and d_i, $1 \leq i \leq k$. Set $n := n_1 + \cdots + n_k$, $t := \min\{d_1, \dots, d_k\}$, and assume that $x \leq t+1$. Set $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $\nu : Y \to \mathbb{P}^r$, where $r = -1 + \prod_{i=1}^{k} \binom{n_i+d_i}{n_i}$, be the Segre–Veronese embedding of Y of multidegree $(d_1, \dots, d_k)$. Fix a positive integer $x \leq t+1$ such that $(n+1)x \leq r+1$. Take $A \in S(Y,x)$. We have $\nu(A) \in \mathcal{T}(X,x)$ if and only if one of the following cases occur:*

(i) *$i \in \{1, \dots, k\}$ such that $\deg(A \cap L) \geq \lceil d_i/2 \rceil + 1$ for some curve L of multidegree ϵ_i.*

(ii) *$i, j \in \{1, \dots, k\}$ such that $i \neq j$, $d_i = d_j = t$, and $x = t+1$, and there is a reduced and connected curve $D \subset Y$ of multidegree $\epsilon_i + \epsilon_j$ such that $A \subset D$; if D is reducible, then x is even, $\mathrm{Sing}(D) \notin A$, and every irreducible component of D contains $1 + x/2$ points of A.*

(iii) *$x = t+1$, and there is a plane contained in one of the rulings of Y with $d_i = t$ such that A is contained in a conic C (perhaps reducible or a double line) of this plane; if $t = 1$, then C is a double line; if C is reducible and $t > 1$, then x is even, $\mathrm{Sing}(D) \notin A$, and every irreducible component of D contains $1 + x/2$ points of A.*

See Remark 7 for a discussion of case (iii) (if C is not smooth, it overlaps with case (i)). The statement in (ii) is chosen to avoid overlapping with case (i). Theorem 1 implies the well-known fact that $\mathcal{T}(X,x) = \emptyset$ if $2x \leq t+1$ ([7], Cor. 4.7). It is known that $\mathcal{T}(X, 1 + \lceil t/2 \rceil)$ is as described in case (i) with $d_i = t$, with the small assumption that $t = 1$ for at most one index i ([6], Proposition 7.4).

For each zero-dimensional scheme $Z \subset \mathbb{P}^r$, set $w(Z) := \deg(Z) - 1 - \dim\langle Z \rangle$ with the convention $\langle \emptyset \rangle = \emptyset$ and $\dim \emptyset = -1$. For all integral and non-degenerate varieties $X \subset \mathbb{P}^r$, let $w_{\max}(X,x)$ denote the maximum of all integers $\deg(2S) - 1 - \dim\langle 2S \rangle$ for all $S \in S(X_{\mathrm{reg}}, x)$. This is related to the weight considered in [1]. By definition of the Terracini locus, $S \in \mathcal{T}(X,x)$ if and only if $\langle 2S \rangle \neq \mathbb{P}^r$ and $w(2S) > 0$.

Theorem 2. *Take k, n_i, d_i, n, ν and t as in Theorem 1, with the only modification being that we allow the case $k = 1$. Assume that $x \leq t$.*

(a) *We have $w_{\max}(X,x) = \max\{0, 2x - t - 1\}$.*

(b) *Take $\nu(A) \in S(X,x)$ with $2x \geq t+2$. We have $w(2\nu(A)) = w_{\max}(X,x)$ if and only if there is $i \in \{1, \dots, k\}$ such that $d_i = t$ and a curve $L \subset Y$ of multidegree ϵ_i such that $A \subset L$.*

In Section 2 we give the preliminaries. A key concept is the generalization of the critical scheme of a Terracini set ([2,10]). It is a key notion for the proofs of Theorems 1 and 2 (see Lemma 1). We hope that it will be useful in other situations in which linearly dependent zero-dimensional schemes arise (as a tool or as the main subject).

In Section 3 we prove a cohomological result on the defectivity of zero-dimensional schemes of a multiprojective space (Proposition 1). This result is proven using several lemmas, each of them being a particular case of Proposition 1.

Section 4 contains the proofs of Theorems 1 and 2. Theorem 1 easily follows from Proposition 1. The proof of Theorem 2 is cohomological and use tools and ideas similar to the ones needed to prove Proposition 1.

Section 5 contains several examples on the weights of certain Terracini sets.

In the last section, Section 6, we discuss the previous literature and explain the main results of this paper and our hopes for the use of the critical schemes obtained in Lemma 1.

The interested reader may/should extend Theorem 1 to other varieties. The Veronese varieties (used for the additive decomposition of multivariate forms) are too easy and have been extensively studied (they are the case $k = 1$ of the variety arising in Theorem 1). The case of Grassmannians, i.e., the case of skew-symmetric tensors, seems promising and useful. A reasonable project is the extension of Theorem 1 to a large set of integers $x \geq t + 2$. In the statement of Theorem 1, several integers, k, $n_1, \dots, n_k$, and $d_1, \dots, d_k$, occur. Recall that $t := \min\{d_1, \dots, d_k\}$. We assume that $k \geq 2$. The key integers are the integers $d_1, \dots, d_k$. Assume, for instance, that $d_1 \leq d_2 \leq \cdots \leq d_k$, so that $t = d_1$. For $d_2 \gg t$ and, say, $x \leq 2t$, the list should be short. For $d_2 = d_1$ and $d_3 \gg d_2$ the list of cases should be longer, but still reasonable. The interested researcher should see how we handle the case $k = 2$. It would be useful to follow, in that order, the lemmas we used for the case $k = 2$ of Proposition 1 (Lemmas 4–8). Then we prove Theorem 1 through induction on the integer k. For $d_3 \gg d_2$ this inductive part should be easy to adapt (see the proof with induction of Proposition 1). Theorem 2 should be easier to adapt to other varieties $X \subset \mathbb{P}^r$ without having the equivalent of Theorem 1 for the embedded variety X.

2. Preliminaries

We work over an algebraically closed field with characteristic 0. The definitions and proofs do not depends on the characteristic, but even the usual definitions of Veronese varieties and Segre–Veronese varieties are a bit different for positive characteristics, say, for characteristics of at most $\max\{d_1, \dots, d_k\}$.

Remark 1. *The maximum in the definition of $w_{\max}(X, x)$ exists, because we obviously have $\deg(2S) = x(\dim X + 1)$ for all $S \in S(X_{\mathrm{reg}}, x)$.*

We need to compute double points or the double of a finite set S when the set S is contained in the smooth locus of different varieties. If $p \in C_{\mathrm{reg}}$ let $(2p, C)$ denote the zero-dimensional subscheme of C with $\mathcal{I}^2_{p,C}$ as its ideal sheaf. For all $S \in S(C_{\mathrm{reg}}, x)$ set $(2S, C) := \cup_{p \in S}(2p, C)$. If $C \subset X$ and $S \in C_{\mathrm{reg}} \cap X_{\mathrm{reg}}$, then $(2S, C) = C \cap (2S, X)$, where $\cap$ denotes the scheme-theoretic intersection.

Notation 1. *For each $i \in \{1, \dots, k\}$ let $\pi_i : Y \to \mathbb{P}^{n_i}$ be the projection on the i-th factor of Y, and let ϵ_i be the element $(a_1, \dots, a_k) \in \mathbb{N}^k$ such that $a_i = 1$ and $a_j = 0$ for all $j \neq i$. Now assume that $k \geq 2$. For each $i \in \{1, \dots, k\}$ let Y_i be the product of all $\mathbb{P}^{n_j}$, where $j \neq i$. Let $\eta_i : Y \to Y_i$ be a map which forgets the i-th component of each point of Y. For each irreducible curve $C \subset Y$, the multidegree of Y is the element $(b_1, \dots, b_k) \in \mathbb{N}^k$ such that $b_i = \deg(\mathcal{O}_C(\epsilon_i))$. For each reduced curve $T \subset Y$, the multidegree of Y is the sum of the multidegrees of the irreducible components of T.*

Remark 2. *Let $C \subset Y$ be a reduced curve with multidegree $(b_1, \dots, b_k)$. If C is irreducible, then $b_i = 0$ if and only if $\pi_i(C)$ is a point. If C is reducible, then $b_i = 0$ if and only if $\pi_i(C)$ is a finite set. If C is reducible, but connected, $\pi_i(C)$ is connected. Thus, for a connected curve we have $b_i = 0$ if and only if $\#\pi_i(C) = 1$.*

Take $S \in \mathcal{T}(X, x)$. Recall that we have $S \in S(X_{\mathrm{reg}}, x)$. The set S is said to be *minimal* if $S' \notin \mathcal{T}(X, \#S')$ for all $S' \subsetneq S$ ([2], Definition 2.3). Since $\langle 2S \rangle \neq \mathbb{P}^r$ according to the

definition of Terracini set, S is minimal if and only if $\dim\langle 2S'\rangle = (1+\dim X)\#S' - 1$ for all $S' \subsetneq S$. Every Terracini set $S \in \mathcal{T}(X,x)$ contains $S_1 \subseteq S$, $S_1 \neq \emptyset$, which is Terracini and minimal. The set S_1 may not be unique (Example 1). A zero-dimensional scheme $Z \subset X_{\mathrm{reg}}$ is said to be *critical* for S if $Z_{\mathrm{red}} \subseteq S$, each connected component of Z has at most degree 2, $\dim\langle Z\rangle \leq \deg(Z) - 2$, and $\dim\langle Z'\rangle = \deg(Z') - 1$ for all $Z' \subsetneq Z$. Obviously, $Z \subset X_{\mathrm{reg}}$ and $\deg(Z) \leq 2x$. Each critical set Z satisfies $\dim\langle Z\rangle = \deg(Z) - 2$ ([2], Lemma 2.11). Every Terracini set has at least one critical scheme ([2]). Easy examples show that critical schemes may not be unique (Example 1). If S is minimal, then each critical scheme Z of S satisfies $Z_{\mathrm{red}} = S$ ([2], Lemma 2.12). Hence, if S is not minimal and $S_1 \subsetneq S$ is minimal, a critical scheme W of S_1 (which is a critical scheme of S, too) satisfies $W_{\mathrm{red}} = S_1 \subsetneq S$. A non-minimal Terracini set S may have a critical scheme Z such that $Z_{\mathrm{red}} = S$ (Example 1).

Example 1. *Fix integers $d \geq 2$ and $x \geq 2 + \lceil d/2 \rceil$. Let $v_d : \mathbb{P}^2 \to \mathbb{P}^r$, where $r = \binom{d+2}{2} - 1$, be the order d Veronese embedding of $\mathbb{P}^2$. Set $X := v_d(\mathbb{P}^2)$. Take a line $L \subset \mathbb{P}^2$ and set $R := v_d(L)$. Take any $S \in S(R,x)$, say, $S = v_d(A)$, with $A \subset L$. Since $d \geq 2$, $h^0(\mathcal{I}_{2L}(d)) > 0$. Since $x \geq 1 + \lceil d/2 \rceil$, $h^0(\mathcal{I}_{2A}(d)) = h^0(\mathcal{I}_{2L}(d)) = \binom{d}{2}$. Hence, $\dim\langle 2S\rangle = \binom{d}{2} < 3(x-1)$. Hence, $S \in \mathcal{T}(X,x)$. Every $S_1 \subset S$ with $\#S_1 = 1 + \lceil d/2 \rceil$ is a minimal Terracini set. It is easy to check that no other subset of S is a minimal Terracini set. Now assume that $x = 2 + \lceil d/2 \rceil$. Note that $h^1(\mathcal{I}_A(d)) = 0$. With the notation of [2], we have $A \in \mathcal{T}(2,d;x)_1$. If we want to construct examples of type $v_d(B)$ with $B \in \mathcal{T}(2,d;z)$, it is sufficient to take $d \geq 4$ and set $B := A \cup \{p\}$ with $p \in \mathbb{P}^2 \setminus L$.*

Remark 3. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Take zero-dimensional subschemes W and Z of X such that $Z \subset W$. Set $z := \deg(Z)$ and $w := \deg(W)$. We have $\langle Z\rangle \subseteq \langle W\rangle$ and $\dim\langle W\rangle \leq \dim\langle Z\rangle + w - z$. Hence, if $\langle Z\rangle = \mathbb{P}^r$, then $\langle W\rangle = \mathbb{P}^r$, while if $\dim\langle W\rangle = w - 1$, then $\dim\langle Z\rangle = z - 1$.*

Remark 4. *We have $\dim\langle 2p\rangle = \deg(2p) - 1$ for all $p \in X_{\mathrm{reg}}$. Hence, according to Remark 3 we have $\dim\langle Z\rangle = \deg(Z) - 1$ for all $Z \subseteq 2p$.*

Remark 5. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety embedded by the complete linear system $|\mathcal{O}_X(1)|$. For each zero-dimensional scheme $Z \subset X$, we have $\deg(Z) - 1 - \dim\langle Z\rangle = h^1(\mathcal{I}_Z(1)) - h^1(\mathcal{O}_X(1))$. Since $\dim Z = 0$, we have $h^i(Z, \mathcal{O}_Z(1)) = 0$ for all integers $i \geq 1$. The long cohomology exact sequence of the exact sequence*

$$0 \to \mathcal{I}_Z(1) \to \mathcal{O}_X(1) \to \mathcal{O}_Z(1) \to 0$$

gives $h^2(\mathcal{I}_Z(1)) = h^2(\mathcal{O}_Z(1))$.

The following lemma is a key tool in proving Theorems 1 and 2. It is a generalization of the definition of the critical scheme of a Terracini set introduced by K. Chandler ([10]) and was used in [2] (Definition 2.3 and Lemma 2.9).

Lemma 1. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Let $W \subset X$ be a zero-dimensional scheme. Let $S \subset X_{\mathrm{reg}}$ be a finite set such that $S \cap W = \emptyset$ and $w(W \cup 2S) > w(W)$. Then there is a zero-dimensional scheme $E \subset X$ such that $E_{\mathrm{red}} \subseteq S$, each connected component of E has a degree of at most 2, $w(W \cup E') = w(W)$ for all $E' \subsetneq E$, and $w(W \cup E) = w(W) + 1$.*

Proof. Through induction on the integer $x := \#S$ (for all possible zero-dimensional schemes) we may assume that $w(2S' \cup W) = w(W)$ for all $S' \subsetneq S$. Fix $p \in S$ and set $S' := S \setminus \{p\}$. Since $w(2p \cup 2S' \cup W) > w(2S' \cup W)$, we have $T_pX \cap \langle 2S' \cup W\rangle \neq \emptyset$. Since the linear space T_pX is the union of all lines of T_pX containing p, there is a zero-

dimensional scheme $v \subset 2p$ such that $\deg(v) \leq 2$ and $\langle v \rangle \cap \langle 2S' \cup W \rangle \neq \emptyset$. Hence, $w(2S' \cup (v \cup W)) > w(W)$. If $w(v \cup W) > w(W)$ we take $E = v$. If $w(v \cup W) = w(W)$ we apply the inductive assumption to S' with respect to the scheme $v \cup W$. □

Lemma 2. *Take $S \in \mathcal{T}(X, x)$. There is a scheme $Z \subseteq 2S$ such that $Z_{\mathrm{red}} = S$, each connected component of S has at most a degree of 2, and either Z is a critical scheme of S or $\dim\langle S \rangle \leq x - 2$ and $Z = S$.*

Proof. If $\dim\langle S \rangle \leq x - 2$, by definition we take $Z = S$. Now assume that $\dim\langle S \rangle = x - 1$. According to Lemma 1 we get a zero-dimensional scheme Z such that $Z_{\mathrm{red}} = S$, each connected component of Z has a degree of at most 2, $Z_{\mathrm{red}} = S$, and $\dim\langle Z \rangle = \deg(Z) - 2$. □

3. A Cohomological Tool

This section is devoted to the proof of Proposition 1, which is essential for the proof of Theorem 1. The proof of Proposition 1 is a chain of lemmas, each of them proving a particular case of Proposition 1.

Fix an integral projective variety X, an effective Cartier divisor D of X, and a zero-dimensional scheme $Z \subset X$. The residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed subscheme of X, with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\mathrm{Res}_D(Z) \subseteq Z$ and $\deg(Z) = \deg(\mathrm{Res}_D(Z)) + \deg(Z \cap D)$, in which $Z \cap D$ is the scheme-theoretic intersection. For all line bundles $\mathcal{L}$ on X, we have an exact sequence,

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D) \to \mathcal{I}_Z \otimes \mathcal{L} \to \mathcal{I}_{Z \cap D, D} \otimes \mathcal{L}_{|D} \to 0,$$

often called the residual exact sequence of D.

The case $\deg(Z) \leq 2d + 1$ of the following lemma is as in [11] (Lemma 34).

Lemma 3. *Fix a positive integer d. Let $Z \subset \mathbb{P}^n$, where $n \geq 2$, be a zero-dimensional scheme such that $\deg(Z) \leq 2d + 2$. We have $h^1(\mathcal{I}_Z(d)) > 0$ if and only if either there is line $L \subset \mathbb{P}^n$ such that $\deg(Z \cap L) \geq d + 2$ or $\deg(Z) = 2d + 2$, and Z is contained in a plane conic $C \subset \mathbb{P}^n$ (we allow the case where C is a reducible conic or a double line).*

Proof. The "if" part follows from Remarks 3–5. Hence, it is sufficient to prove the "only if" part.

Set $z := \deg(Z)$. If $d = 1$, then $\deg(Z) \leq 4$. Since $h^1(\mathcal{I}_Z(1)) > 0$, Z is contained in a plane. Every degree 4 zero-dimensional scheme of $\mathbb{P}^2$ is contained in at least two conics, it may be reducible. Hence, the lemma is true if $d = 1$, and we may use induction on the integer d.

The lemma is easy and well-known if $n = 2$ ([12], Remarques at p. 116). Assume that $n > 2$ and that the lemma is true in lower-dimensional projective spaces. Let $H \subset \mathbb{P}^n$ be a hyperplane such that $w := \deg(Z \cap H)$ is maximal. If $w = z$, then we use the inductive assumption on n. Now assume that $Z \nsubseteq H$. If $h^1(H, \mathcal{I}_{Z \cap H, H}(d)) > 0$, then we use the inductive assumption on n to get the existence of a line $L \subset H$ such that $\deg(L \cap Z) \geq d + 2$.

Now assume that $h^1(H, \mathcal{I}_{Z \cap H, H}(d)) = 0$. The residual exact sequence of H gives $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$. We have $w \geq \dim|\mathcal{O}_{\mathbb{P}^n}(1)| = n$. Hence, $\deg(\mathrm{Res}_H(Z)) = z - w \leq 2d + 2 - 3 = 2(d-1) + 1$. The inductive assumption on d indicates the existence of a line R such that $\deg(R \cap \mathrm{Res}_H(Z)) \geq d + 1$. Taking a hyperplane containing R, we get $w \geq d + 1 + (n - 2)$, and hence, $\deg(\mathrm{Res}_H(Z)) \leq d$, a contradiction. □

Lemma 4. *Set $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Fix integers $b_1 \geq 0$ and $b_2 \geq 0$. Fix $D \in |\mathcal{O}_Y(1,1)|$ and a zero-dimensional scheme $Z \subset D$. We have $h^1(\mathcal{I}_Z(b_1, b_2)) = h^1(D, \mathcal{I}_{Z,D}(b_1, b_2))$. We have*

$h^1(\mathcal{I}_Z(b_1,b_2)) > 0$ *if and only if either* $\deg(Z) \geq b_1+b_2+2$ *or* $D = J \cup R$ *with* $J \in |\mathcal{O}_Y(1,0)|$, $R \in |\mathcal{O}_Y(0,1)|$, *and either* $\deg(Z \cap J) \geq b_2+2$ *or* $\deg(Z \cap R) \geq b_1+2$.

Proof. We have $h^1(\mathcal{I}_Z(b_1,b_2)) = h^1(D, \mathcal{I}_{Z,D}(b_1,b_2))$ because

$$h^1(\mathcal{O}_Y(b_1-1,b_2-1)) = h^2(\mathcal{O}_Y(b_1-1,b_2-1)) = 0$$

according to the Künneth formula. Since $h^1(\mathcal{O}_Y(b_1-1,b_2-1)) = 0$, we have $h^0(D,\mathcal{O}_D(b_1,b_2)) = b_1+b_2+1$. Assume that $\deg(Z) \leq b_1+b_2+1$. If D is irreducible, then $D \cong \mathbb{P}^1$ with $\deg(\mathcal{O}_D(b_1,b_2)) = b_1+b_2$, and hence, $h^1(D,\mathcal{I}_{Z,D}(b_1,b_2)) = 0$ according to the cohomology of line bundles on $\mathbb{P}^1$. Assume that $D = J \cup R$ with $J \in |\mathcal{O}_Y(1,0)|$, $R \in |\mathcal{O}_Y(0,1)|$, and $\deg(Z \cap R) \leq b_1+1$. Consider the residual exact sequence of R:

$$0 \to \mathcal{I}_{\mathrm{Res}_R(Z)}(b_1,b_2-1) \to \mathcal{I}_Z(b_1,b_2) \to \mathcal{I}_{R\cap Z,R}(b_1,b_2) \to 0 \tag{1}$$

Since $\deg(R \cap Z) \leq b_1+1$, we have $h^1(R,\mathcal{I}_{R\cap Z,R}(b_1,b_2)) = 0$. Hence, the long cohomology exact sequence of (1) gives $h^1(\mathcal{I}_{\mathrm{Res}_R(Z)}(b_1,b_2-1)) > 0$. Since $Z \subset J \cup R$, we have $\mathrm{Res}_R(Z) \subset J$. We have $h^1(\mathcal{I}_{\mathrm{Res}_R(Z)}(b_1,b_2-1)) = h^1(J,\mathcal{I}_{\mathrm{Res}_R(Z),J}(b_1,b_2-1))$, because $h^1(\mathcal{O}_Y(b_1-1,b_2-1)) = h^2(\mathcal{O}_Y(b_1-1,b_2-1)) = 0$. We get the inequality $\deg(\mathrm{Res}_R(Z)) \geq b_2+1$, and hence, $\deg(Z \cap J) \geq b_2+1$. Assume that $\deg(Z \cap J) \leq b_2+1$, and hence, $\deg(Z \cap J) = b_2+1$. Using the residual exact sequence of J, we get the inequality $\deg(\mathrm{Res}_J(Z)) \geq b_1+1$. Hence, $\deg(Z) \geq b_1+b_2+2$, contradicting one of our assumptions. □

Lemma 5. *Fix integers* $d_2 \geq d_1 \geq 1$. *Let* $Z \subset Y := \mathbb{P}^1 \times \mathbb{P}^1$ *be a zero-dimensional scheme such that* $\deg(Z) \leq 2d_1+2$. *We have* $h^1(\mathcal{I}_Z(d_1,d_2)) > 0$ *and* $h^1(\mathcal{I}_{Z'}(d_1,d_2)) = 0$ *for all* $Z' \subsetneq Z$ *if and only if one of the following cases occurs:*

1. $\deg(Z) = d_1+2$ *and there is* $L \in |\mathcal{O}_Y(0,1)|$ *such that* $Z \subset L$;
2. $\deg(Z) = d_2+2$ *and there is* $R \in |\mathcal{O}_Y(1,0)|$ *such that* $Z \subset R$;
3. $d_2 = d_1$, $\deg(Z) = d_1+d_2+2$ *and there is* $D \in |\mathcal{O}_Y(1,1)|$ *such that* $Z \subset D$.

Proof. The "if" part follows from Remarks 3–5. Hence, it is sufficient to prove the "only if" part.

Let ν be the Segre–Veronese embedding of Y. First assume that $d_1 = d_2 = 1$. Hence, we have $\deg(Z) \leq 4$. If $\deg(Z) \leq 3$, then $\nu(Z)$ shows that $\deg(Z) = 3$ and that Z is contained in a line of one of the 2 rulings of $\nu(Y)$. Now assume that $\deg(Z) = 4$. We determine that Z is contained in $D \in |\mathcal{O}_Y(1,1)|$. Since no degree 3 subscheme of $\nu(Z)$ is contained in a line, D is smooth and we are in the third case of the lemma.

The same proof works if $d_1 = 1$ and $d_2 > 1$.

Hence, we may assume that $d_1 \geq 2$ and use induction on the integer d_1.

Take $D \in |\mathcal{O}_Y(1,1)|$ such that $e_1 := \deg(Z \cap D)$ is maximal. Set $Z_1 := \mathrm{Res}_D(Z)$. If $h^1(\mathcal{I}_{Z\cap D}(d_1,d_2)) > 0$, then it is sufficient to use Lemma 4. Hence, we may assume that $h^1(\mathcal{I}_{Z\cap D}(d_1,d_2)) = 0$, i.e., $h^1(D,\mathcal{I}_{Z\cap D,D}(d_1,d_2)) = 0$. Hence the residual exact sequence of D gives $h^1(\mathcal{I}_{Z_1}(d_1-1,d_2-1)) > 0$. Let $Z' \subset Z_1$ be a minimal scheme such that $h^1(\mathcal{I}_{Z'}(d_1-1,d_2-1)) = 0$. Since $\dim|\mathcal{O}_Y(1,1)| = 3$, $e_1 \geq 3$. Hence, we get the inequality $\deg(Z') \leq \deg(Z)-e_1 \leq 2(d_1-1)+1$. Hence, the inductive assumption indicates that either $\deg(Z') = d_1+1$ and there is a line $L \in |\mathcal{O}_Y(0,1)|$ such that $Z \subset L$, or $\deg(Z') = d_2+1$ and there is $R \in |\mathcal{O}_Y(1,0)|$ such that $Z \subset R$. Since $\deg(Z') \geq d_1+1$, the definition of e_1 gives $e_1 \geq d_1+2$. Hence, $\deg(Z_1) \leq d_1$, a contradiction. □

Lemma 6. *Set $Y := \mathbb{P}^1 \times \mathbb{P}^2$. Let $Z \subset Y$ be a zero-dimensional scheme such that $\deg(Z) \leq 4$. We have $h^1(\mathcal{I}_Z(1,1)) > 0$ if and only if one of the following cases occurs:*

(i) There is a curve $L \subset Y$ of bidegree $(1,0)$ such that $\deg(Z \cap L) \geq 3$;
(ii) There is curve $R \subset Y$ of bidegree $(0,1)$ such that $\deg(Z \cap R) \geq 3$;
(iii) $d_1 = d_2$, $\deg(Z) = 2d_1 + 2$, and there is a reduced and connected curve $D \subset Y$ of bidegree $(1,1)$ such that $Z \subset D$;
(iv) $\deg(Z) = 4$, and Z is contained in a plane contained in the second ruling of Y.

Proof. The "if" part follows from Remarks 3–5. Hence, it is sufficient to prove the "only if" part.

The very ampleness of $\mathcal{O}_Y(1,1)$, the structure of the lines of the Segre variety, $\nu(Y)$ and the fact that $\nu(Y)$ is scheme-theoretically cut out by quadrics cover all cases with $\deg(Z) \leq 3$. Assume that $\deg(Z) = 4$. If neither (i) nor (ii) is true, then $\langle \nu(Z) \rangle$ is a plane. Take $H \in |\mathcal{O}_Y(0,1)|$ such that $w := \deg(H \cap Z)$ is maximal. We have $w \geq \dim |\mathcal{O}_Y(0,1)| = 2$, and hence, $\deg(\mathrm{Res}_H(Z)) = 4 - w \leq 2$. Consider the residual exact sequence of H:

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(1,0) \to \mathcal{I}_Z(1,1) \to \mathcal{I}_{H \cap Z,H}(1,1) \to 0 \tag{2}$$

If $h^1(H, \mathcal{I}_{Z \cap H,H}(1,1)) > 0$, then we use Lemma 5. Note that elements of the projective space $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a,b)|$ seen as curves have bidegree (b,a) and that each element $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)|$ is connected and with no multiple components. Hence, we may assume the equality $h^1(H, \mathcal{I}_{Z \cap H,H}(1,1)) = 0$. The long cohomology exact sequence of (2) gives $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(1,0)) > 0$. Hence, $w = 2$ and $\deg(\pi_1(\mathrm{Res}_H(Z))) = 1$. Hence, there is $M \in |\mathcal{O}_Y(1,0)|$ containing $\mathrm{Res}_H(Z)$. If $Z \subset M$, then we are in case (iv). If $h^1(\mathcal{I}_{Z \cap M}(1,1)) > 0$ and $\deg(Z \cap M) \leq 3$, then we are in case (i). Thus, we may assume the equality $h^1(\mathcal{I}_{Z \cap M}(1,1)) = 0$. The residual exact sequence of M gives $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(0,1)) > 0$. Hence, $\deg(\pi_2(\mathrm{Res}_M(Z))) = 1$. Since $\dim |\mathcal{O}_Y(0,1)| = 2$, we get $w \geq 3$, a contradiction. □

Lemma 7. *Set $Y := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ with $n_1 + n_2 \geq 3$. Let $Z \subset Y$ be a zero-dimensional scheme such that $\deg(Z) \leq 4$. We have $h^1(\mathcal{I}_Z(1,1)) > 0$ if and only if one of the following cases occurs:*

(i) There is a curve $L \subset Y$ of bidegree $(1,0)$ such that $\deg(Z \cap L) \geq 3$;
(ii) There is curve $R \subset Y$ of bidegree $(0,1)$ such that $\deg(Z \cap R) \geq 3$;
(iii) $\deg(Z) = 4$, and there is a reduced and connected curve $D \subset Y$ of bidegree $(1,1)$ such that $Z \subset D$;
(iv) $\deg(Z) = 4$, and Z is contained in a plane contained in one of the rulings of Y.

Proof. The "if" part follows from Remarks 3–5. Hence, it is sufficient to prove the "only if" part.

Based on the structure of lines in $\nu(Y)$ and the fact that $\nu(Y)$ is scheme-theoretically cut out by quadrics, we may assume that $\deg(Z) = 4$ and $h^1(\mathcal{I}_{Z'}(1,1)) = 0$ for all $Z' \subsetneq Z$. We use induction on the integer $n_1 + n_2$, with the case $n_1 + n_2 = 3$ being true according to Lemma 6. First assume $n_i \geq 3$ for some i, say, $n_2 \geq 3$. Take $H \in |\mathcal{O}_Y(0,1)|$ such that $w := \deg(H \cap Z)$ is maximal. We have $w \geq \dim |\mathcal{O}_Y(0,1)| = n_2 \geq 3$. Hence, $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(1,0)) = 0$. Hence, the long cohomology exact sequence of (2) gives $h^1(H, \mathcal{I}_{Z \cap H,H}(1,1)) > 0$. Hence, $Z \subset H$. Based on the inductive assumption, Z is contained in a plane contained in a ruling. Now assume that $n_1 = n_2 = 2$. Take $M \in |\mathcal{O}_Y(1,0)|$, where $c := \deg(Z \cap M)$ is maximal. If $Z \subset M$, then we use Lemma 6. If $Z \nsubseteq M$, then we get $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(0,1)) > 0$ with $\deg(\mathrm{Res}_M(Z)) = 4 - c \leq 2$. We get $\deg(\mathrm{Res}_M(Z)) = 2$ and $\deg(\pi_1(\mathrm{Res}_M(Z))) = 1$. Hence, $w \geq 3$, a contradiction. □

Lemma 8. *Set $Y := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ with $n_1 + n_2 \geq 3$. Let $Z \subset Y$ be a zero-dimensional scheme such that $\deg(Z) \leq 4$. We have $h^1(\mathcal{I}_Z(1,t)) > 0$ for some $t \geq 2$ if and only if one of the following cases occurs:*

(i) There is a curve $L \subset Y$ of bidegree $(1,0)$ such that $\deg(Z \cap L) \geq 3$;
(ii) $t = 2$, $\deg(Z) = 4$, and there is curve $R \subset Y$ of bidegree $(0,1)$ such that $\deg(Z \cap R) \geq 4$;
(iii) $\deg(Z) = 4$, and Z is contained in a plane contained in the first rulings of Y.

Proof. The "if" part follows from Remarks 3–5. Hence, it is sufficient to prove the "only if" part.

According to Lemma 5 we may use induction on the integer $n_1 + n_2$. Hence, we may assume that Z is contained in no lower-dimensional multiprojective spaces. Since $h^1(\mathcal{I}_Z(1,1)) > 0$ and we may apply Lemma 5 for the case $n_1 = n_2 = 1$, we are in case (i), (ii), (iii), or (iv) of Lemma 7. Case (i) or case (iv) for the first ruling concludes the proof. Assume the existence of a curve R of bidegree $(0,1)$ such that $\deg(R \cap Z) \geq 3$. Take $H \in |\mathcal{I}_H(1,0)|$, where $w := \deg(Z \cap H)$ is maximal. If $n_2 \geq 2$, we have $Z \subset H$ and we use the inductive assumption. Assume that $n_2 = 1$ and $w = 3$. Since $\mathcal{O}_Y(1,0)$ is globally generated, $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(1,0)) = 0$. The residual exact sequence of H gives $h^1(H, \mathcal{I}_{Z\cap H}(1,t)) > 0$. Since $h^1(\mathcal{I}_R(1,t)) = 0$, we get $h^1(R, \mathcal{I}_{Z\cap R,R}(1,t)) = 0$, contradicting the assumption $t \geq 2$. If $n_2 \geq 2$, and Z is contained in a plane G of the second ruling, we use the fact that $h^1(\mathcal{I}_G(1,t)) = 0$ and that $h^1(G, \mathcal{I}_{Z,G}(1,t)) = 0$, unless $t = 2$ and Z is contained in a line contained in G and, hence, is of bidegree $(0,1)$. Now assume that Z is contained in a reduced and connected curve $D \subset Y$ of bidegree $(1,1)$. Since $h^1(\mathcal{I}_D(1,t)) = 0$ and $t \geq 2$, we determine that D is reducible, say, $D = L_1 \cup R_1$, with L_1 of bidegree $(1,0)$ and R_1 of bidegree $(0,1)$. We determine that either $\deg(Z \cap L_1) \geq 3$ or $\deg(Z) = 4$, $t = 2$, and $Z \subset R_1$. □

Lemma 9. *Fix integers $n_1 \geq 1$, $n_2 \geq 1$, and $d_2 \geq d_1 \geq 1$. Set $Y := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$. Fix a zero-dimensional scheme $Z \subset Y$ such that $\deg(Z) \leq 2d_1 + 2$. We have $h^1(\mathcal{I}_Z(d_1,d_2)) > 0$ if and only if one of the following cases occurs:*

(i) There is a curve $L \subset Y$ of bidegree $(1,0)$ such that $\deg(Z \cap L) \geq d_1 + 2$;
(ii) There is curve $R \subset Y$ of bidegree $(0,1)$ such that $\deg(Z \cap R) \geq d_2 + 2$;
(iii) $d_1 = d_2$, $\deg(Z) = 2d_1 + 2$, and there is a reduced and connected curve $D \subset Y$ of bidegree $(1,1)$ such that $Z \subset D$;
(iv) $\deg(Z) = 2d_1 + 2$, and there is a plane contained in one of the rulings of Y such that Z is contained in a conic (perhaps reducible or a double line) of this plane; if the ruling is the latter, then $d_2 = d_1$.

Proof. The "if" part follows from Remarks 3–5. Hence, it is sufficient to prove the "only if" part.

The case $n_1 = n_2 = 1$ is true according to Lemma 5. Note that elements of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)|$ seen as curves have bidegree (b,a) and that each element of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$ is connected and with no multiple components.

Hence, we may assume that $n_1 + n_2 \geq 3$ and use induction on the integer $n_1 + n_2$. We may also use induction on the integer $n_1 + n_2$. Hence, we may assume that $Z \nsubseteq Y'$ with Y', a proper multiplrojective subspace of Y. To handle a case in which Y' has a unique factor, $\mathbb{P}^n$, we use Lemma 3.

Set $z := \deg(Z)$. Let ν' be the Segre embedding of Y. The case $d_1 = 1$ is true according to Lemma 8. Hence, we may assume that $d_1 \geq 2$ and use induction on the integer $\min\{d_1, d_2\}$.

(a) Assume that $n_2 \geq 2$. Take $H \in |\mathcal{O}_Y(0,1)|$ such that $w := \deg(Z \cap H)$ is maximal. If $Z \subset H$, then we use the inductive assumption on $n_1 + n_2$. Hence, we may assume that $\mathrm{Res}_H(Z) \neq \emptyset$. Since $h^0(\mathcal{O}_Y(0,1)) = n_2 + 1$, $w \geq n_2 \geq 2$. Consider the residual exact sequence of H:

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(d_1, d_2 - 1) \to \mathcal{I}_Z(d_1, d_2) \to \mathcal{I}_{H \cap Z, H}(d_1, d_2) \to 0 \tag{3}$$

If $h^1(H, \mathcal{I}_{Z \cap H, H}(d_1, d_2)) > 0$, then we may use the inductive assumption on $\dim Y$. Hence, we may assume that $h^1(H, \mathcal{I}_{Z \cap H, H}(d_1, d_2)) = 0$. The long cohomology exact sequence of (3) gives $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d_1, d_2 - 1)) = 0$. We have

$$\deg(\mathrm{Res}_H(Z)) = z - w \leq \min\{2d_1, 2(d_2 - 1) + 2\}.$$

Since $d_2 \geq 2$ we may apply the inductive assumption on $\min\{d_1, d_2 - 1\}$ and determine that either there is a curve of bidegree $(1,0)$ with $\deg(\mathrm{Res}_H(Z) \cap L) \geq d_1 + 2$ (concluding the proof in this case because $\mathrm{Res}_H(Z) \subset Z$) or there is a curve R of bidegree $(0,1)$ such that $\deg(R \cap \mathrm{Res}_H(Z)) \geq d_2$ or $d_2 - 1 = d_1$, $\deg(\mathrm{Res}_H(Z)) = d_1 + d_2 + 1$, and there is a connected curve of bidegree $(1,1)$ containing $\mathrm{Res}_H(Z)$. The last case is excluded, because $w > 1$.

Assume the existence of a curve R of bidegree $(0,1)$ satisfying the inequality $\deg(\mathrm{Res}_H(Z) \cap R) \geq d_2 + 1$. Since R has bidegree $(0,1)$, $\pi_{2|R} : R \to \mathbb{P}^{n_2}$ is an embedding with a line $\pi_2(R)$. Since R is the complete intersection of $n_2 - 1$ elements of $|\mathcal{O}_Y(0,1)|$, the maximality of w gives $w \geq d_2 + 1 + (n_2 - 1)$. Hence, $z - w \leq 2d_1 + 2 - d_2 - n_2 < d_2 + 1$, a contradiction.

(b) Now assume that $n_2 = 1$, and hence, $n_1 \geq 2$. The proof of step (a) works with no modifications if $d_2 = d_1$ using the first ruling instead of the second one. Hence, we may assume that $d_2 > d_1$. Since $h^1(\mathcal{I}_Z(d_1, d_2)) > 0$, we have $h^1(\mathcal{I}_Z(d_1, d_1)) > 0$. Hence, the case of the lemma just proven shows that we are in case (i), (ii), (iii), or (iv) of the lemma, with case (iv) being the first ruling. Hence, there is a line $R \subset Y$ of bidegree $(0,1)$ such that $\deg(R \cap Z) \geq d_1 + 2$. Take $M \in |\mathcal{O}_Y(1,0)|$ such that $e := \deg(Z \cap M)$ is maximal. If $Z \subset M$, then we use the inductive assumption on the integer $\dim Y$. Hence, we may assume that $Z \nsubseteq M$. Since $\pi_1(R)$ is a point and $\deg(R \cap Z) \geq d_1 + 1$, $e \geq d_1 + 1 + (n_1 - 1) = d_1 + n_1$. Hence, $\deg(\mathrm{Res}_M(Z)) \leq d_1 + n_1$. Since $d_2 \geq 2$, the inductive assumption on the minimum of the 2 partial degrees gives $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(d_1 - 1, d_2)) = 0$. Hence, the residual exact sequence of M gives $h^1(M, \mathcal{I}_{Z \cap M, M}(d_1, d_2)) > 0$. We conclude using the inductive assumption on the dimension of the biprojective space. □

Proposition 1. *Fix positive integers k, n_i, and d_i, and $1 \leq i \leq k$. Set $t := \min\{d_1, \ldots, d_k\}$. Set $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $Z \subset Y$ be a zero-dimensional scheme such that $\deg(Z) \leq 2t + 2$. We have $h^1(\mathcal{I}_Z(d_1, \ldots, d_k)) > 0$ if and only if one of the following cases occurs:*

(i) There are $i \in \{1, \ldots, k\}$ such that $\deg(Z \cap L) \geq d_i + 2$ for some curve L of multidegree ϵ_i;
(ii) There are $i, j \in \{1, \ldots, k\}$ such that $i \neq j$, $d_i = d_j = t$, $\deg(Z) = 2t + 2$, and there is a reduced and connected curve $D \subset Y$ of multidegree $\epsilon_i + \epsilon_j$ such that $Z \subset D$;
(iii) $\deg(Z) = 2t + 2$, and there is a plane contained in one of the rulings with $d_i = t$ of Y such that Z is contained in a conic (perhaps reducible or a double line) of this plane; if the ruling is the latter, then $d_2 = d_1$.

Proof. The "if" part follows from Remarks 3–5. Hence, it is sufficient to prove the "only if" part.

Set $z := \deg(Z)$. Let ν' be the Segre embedding of Y. The case $k = 1$ is true according to Lemma 3. The case $k = 2$ is true according to Lemma 9. Hence, we may assume that

$k > 2$ and that the proposition is true for multiprojective spaces with at most $k-1$ factors. For multiprojective spaces with k factors, we use induction on the integer $n := n_1 + \cdots + n_k$. Fix $i \in \{1, \dots, k\}$ and take $H \in |\mathcal{O}_Y(\epsilon_i)|$ such that $w := \deg(Z \cap H)$ is maximal. If $Z \subset H$, then we may use the inductive assumption on the integer $\dim Y$. Hence, we may assume that $\mathrm{Res}_H(Z) \neq \emptyset$. Consider the residual exact sequence of H:

$$\begin{aligned} 0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(d_1, \dots, d_k)(-\epsilon_i) \to \mathcal{I}_Z(d_1, \dots, d_k) \\ \to \mathcal{I}_{H\cap Z,H}(d_1, \dots, d_k) \to 0 \end{aligned} \tag{4}$$

If $h^1(H, \mathcal{I}_{Z\cap H,H}(d_1, \dots, d_k)) > 0$, then it is sufficient to use the inductive assumption on n. Thus, $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(1, \dots, 1)(-\epsilon_i)) > 0$. Note that we have $\deg(\mathrm{Res}_H(Z)) = z - w$ and $w \geq \dim |\mathcal{O}_Y(\epsilon_i)| = n_i$.

(a) Assume that $t = 1$. Let ν' be the Segre embedding of Y.

(a1) Assume that $d_i = 1$ for all i. The case $z \leq 2$ is true because ν' is an embedding, while the case $z = 3$ is true according to the structure of lines of the Segre variety $\nu'(Y)$. Now assume that $z = 4$. First assume that $w = 3$. Since $\mathrm{Res}_H(Z)$ is a single point and $\mathcal{O}_Y(1, \dots, 1)(-\epsilon_i)$ is globally generated, $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(1, \dots, 1)(-\epsilon_i)) = 0$, a contradiction. Now assume that $w = 2$. Since $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(1, \dots, 1)(-\epsilon_i)) > 0$, we determine that the degree 2 zero-dimensional scheme $H \cap Z$ is mapped to a point, p. Since $n_i \geq 1$, there exists $U \in |\mathcal{I}_{\mathrm{Res}_H(Z)}(\epsilon_i)|$, satisfying the inequality $\deg(U \cap Z) > \deg(\mathrm{Res}_H(Z)) = 2$, a contradiction. Now assume that $w = 1$. Since $w \geq n_i$, we get $n_i = 1$ and determine that $\pi_{i|Z} : Z \to \mathbb{P}^1$ is an embedding. In particular Z is curvilinear. Since we excluded all other cases, using the other rulings we get $Y = (\mathbb{P}^1)^k$. Since $\pi_{1|Z} : Z \to \mathbb{P}^1$ is an embedding, $\eta_{k|Z} : Z \to Y_k \cong (\mathbb{P}^1)^{k-1}$ is an embedding. Since $h^1(Y_k, \mathcal{I}_{\eta_k(Z)}(1, \dots, 1)) \geq h^1(\mathcal{I}_Z(1, \dots, 1)) > 0$. The inductive assumption on k indicates that $\eta_k(Z) \subset (\mathbb{P}^1)^k$ is in case (i) or (ii). First assume that it is in case (i) with, say, $i = 1$. The set $\eta_k^{-1}(R)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and $\deg(Z \cap \eta_k^{-1}(R)) = 3$. Take a multiprojective space U containing $\eta_k^{-1}(R)$. Since $h^1(\mathcal{I}_{\mathrm{Res}_U(Z)}(a_1, \dots, a_k)) = 0$ for all $(a_1, \dots, a_k)$, the residual exact sequence of U gives $h^1(\mathcal{I}_{Z\cap U}(1, \dots, 1)) > 0$. Hence, we are in case (i) for some L containing $Z \cup U$.

Now assume we are in case (ii). The set $T := \eta_k^{-1}(D)$ is a surface of multidegree $(1, 1, 0)$ contained in $(\mathbb{P}^1)^3$, seen as the product of the i-th, j-th, and k-th factor of Y. If $k > 3$ there is a multiprojective space strictly contained in Y and containing $(\mathbb{P}^1)^3$, concluding the proof. Now assume that $k = 3$. T is a divisor of Y, and as a divisor it has multidegree $(0, 0, 1)$. Since $h^1(\mathcal{I}_{\mathrm{Res}_T(Z)}(1, 1, 0)) = 0$, the residual exact sequence of T gives $h^1(T, \mathcal{I}_{Z\cap T,T}(1, 1, 1)) > 0$. First assume that D is smooth. Hence, $D \cong \mathbb{P}^1$ and $T \cong \mathbb{P}^1 \times \mathbb{P}^1$ with $\mathcal{O}_D(1, 1, 1) \cong \mathcal{O}_T(2, 1)$. Apply Lemma 5 to $Z \cap T$.

Now assume that D is singular, say, $D = L_1 \cup L_2$, with L_1 of multidegree $(1, 0, 0)$ and L_2 with multidegree $(0, 1, 0)$. Set $T_i = L_i$. If $\deg(Z \cap T_i) \geq 3$ for some $i = 1, 2$, then we are in case (i). Assume that $\deg(Z \cap T_i) = 2$ for all i, and hence, $Z \cap T_1 \cap T_2 = \emptyset$. The surface T_1 (resp. T_2) is an element of $|\mathcal{O}_Y(\epsilon_2)|$ (resp. $|\mathcal{O}_Y(\epsilon_1)|$). The residual exact sequence of T_2 gives $h^1(\mathcal{I}_{Z\cap T_1}(0, 1, 1)) > 0$. Hence, $Z \cap T_1$ is contained in a curve R_1 of multidegree $(1, 0, 0)$. Using T_1, we determine that $Z \cap T_2$ is contained in curve R_2 of multidegree $(0, 1, 0)$. Hence, $Z \subset R_1 \cup R_2$.

(a2) Assume that $t = 1$, but that there exists at least one $i \in \{1, \dots, 1\}$ such that $d_i \geq 2$. Since $h^1(\mathcal{I}_Z(d_1, \dots, d_k)) \leq h^1(\mathcal{I}_Z(1, \dots, 1))$, part (a1) indicates that we are in case (i), (ii), or (iii), except that for case (i) we only know that $\deg(Z \cap L) \geq 3$, which is not enough if $d_i > 1$. Assume that $\deg(Z \cap L) = 3$. Note that $\pi_j(L)$ is a point for all $j \neq i$. Since $k \geq 2$, there is $j \neq i$. Take any $M \in |\mathcal{I}_L(\epsilon_j)|$. Since $\deg(\mathrm{Res}_M(Z)) \leq 1$, $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(1, \dots, 1)(-\epsilon_j)) = 0$. Hence, the residual exact sequence of M and the inductive assumption on n show that $Z \cap M$ satisfies the lemma.

(b) According to step (a) we may assume that $t \geq 2$ and that the proposition is true for all multidegrees $(a_1, \dots, a_k)$ such that $1 \leq \min\{a_1, \dots, a_k\} \leq t-1$. Take i, H, and w as before with $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d_1, \dots, d_k)(-\epsilon_i)) > 0$. If either $w \geq 2$ or $d_i > t$, we use the inductive assumption on t. Since $w \geq n_i$, we conclude, unless $Y = (\mathbb{P}^1)^k$ and $d_i = t$ for all t. Take $U \in |\mathcal{O}_Y(1,1,0,\dots,0)|$ such that $c := \deg(U \cap Z)$ is maximal.

Assume for the moment that $h^1(U, \mathcal{I}_{U\cap Z,Z}(t, \dots, t)) = 0$. The residual exact sequence of U gives $h^1(\mathcal{I}_{\mathrm{Res}_U(Z)}(t-1, t-1, t, \dots, t)) > 0$. Since $c \geq \dim |\mathcal{O}_Y(1,1,0,\dots,0)| = 3$, the inductive assumption gives $i \in \{1, \dots, k\}$ and a curve L

Now assume that $h^1(U, \mathcal{I}_{U\cap Z,Z}(t, \dots, t)) > 0$. We have $U \cong D \times (\mathbb{P}^1)k - 3$ for some $D \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$. As in step (a) we handle the 2 cases for the possible types of reduced curves D, smooth or reducible. If D is smooth, we have $\mathcal{O}_U(1, \dots, 1) \cong \mathcal{O}_{(\mathbb{P}^1)^{k-1}}(2t, t, \dots, t)$. If D is singular, U is the union of 2 multiprojective spaces whose intersection is a codimension 1 multiprojective space. □

Remark 6. *Take Z as in Proposition 6. In case (i) we have $d_i + 2 \leq 2t + 2$. Assume we are in case (ii), and hence, $\deg(Z) = 2t + 2$. The reducedness of D may be omitted, because every curve of multidegree $\epsilon_i + \epsilon_j$ is reduced by the assumption $i \neq j$. If $n = 2$, i.e., if $Y = \mathbb{P}^1 \times \mathbb{P}^1$, every curve of bidegree $(1,1)$ is connected, but if $n > 2$, many curves $\epsilon_i + \epsilon_j$ have 2 connected components. Take D connected components of multidegree $\epsilon_i + \epsilon_j$ containing Z. Assume that D is reducible, say, $D = L \cup R$, with L of multidegree ϵ_i and R of multidegree ϵ_j. Note that D is isomorphic to a reducible conic. If we are not in case (i), then $\deg(L \cap Z) \leq t + 1$ and $\deg(R \cap Z) \leq t + 1$. Hence, $\deg(L \cap Z) = \deg(Z \cap R) = t + 1$ and $L \cap R \notin Z_{\mathrm{red}}$. The same applies in case (iii) if the conic is reducible. Now assume that the conic is a double line $2L$. If we are not in case (i), then $\deg(Z \cap L) \leq t + 1$. Hence, $\deg(L \cap Z) = t + 1$.*

4. Proofs of Theorem 1 and 2

Remark 7. *Take case (iii) of Theorem 1 with C singular. If $C = 2L$ is a double line, then $A \subset L$, and hence, this case is covered by (i). If C is reducible, then $\deg(2A \cap C) \geq 2t + 2$ implies that one of the irreducible components of C is as in case (i).*

Proof of Theorem 1. According to Lemma 2 there is a minimal scheme $Z \subset 2A$ such that $Z_{\mathrm{red}} = A$, $Z = A$ if $\dim\langle A\rangle \leq \deg(A) - 2$, and $\dim\langle \nu(Z)\rangle = \deg(Z) - 2$ if $\dim\langle A\rangle = \deg(A) - 1$. Fix such a scheme. According to Proposition 1 and Remark 6, one of the following cases occur:

(i) There are $i \in \{1, \dots, k\}$ such that $\deg(Z \cap L) \geq d_i + 2$ for some curve L of multidegree ϵ_i;
(ii) There are $i, j \in \{1, \dots, k\}$ such that $i \neq j$, $d_i = d_j = t$, $\deg(Z) = 2t + 2$, and there is a reduced and connected curve $D \subset Y$ of multidegree $\epsilon_i + \epsilon_j$ such that $Z \subset D$;
(iii) $\deg(Z) = 2t + 2$, and there is a plane contained in one of the rulings with $d_i = t$ of Y such that Z is contained in a conic (perhaps reducible or a double line) of this plane.

In Proposition 1 it is "if and only if". However, in our cases, to determine that $\nu(A) \in \mathcal{T}(X, x)$ we need to check if $h^0(\mathcal{I}_{2A}(d_1, \dots, d_k)) \neq 0$. In all cases we have $h^1(\mathcal{I}_{2A}(d_1, \dots, d_k)) > 0$ for the following reasons. In case (i) we have $\deg(L \cap 2A) \geq 2\#(A \cap L) \geq d_i + 2$. In case (ii) we have $\deg(D \cap 2A) \geq 2t + 2$. In case (iii) we have $\deg(C \cap 2A) \geq 2t + 2$. Thus, we may use Remarks 3 and 5. The condition $\langle \nu(2A)\rangle \neq \mathbb{P}^r$ is satisfied, because $\deg(2A) = (n+1)x \leq r + 1$ and $h^1(\mathcal{I}_{2A}(d_1, \dots, d_k)) > 0$. □

Proof of Theorem 2. First assume that $t = 1$, and hence, $x = 1$. Remark 4 gives $w(2S) = 0$. Thus, we may assume that $t > 1$ and use induction on the integer t. We use induction on the integer k. For a fixed k we use induction on $n := \dim Y$.

(a) Assume that $k = 1$. First assume that $n = 1$. We use the fact that

$$h^1(\mathbb{P}^1, \mathcal{I}_Z(t)) = \max\{0, \deg(Z) - t - 1\}$$

for all zero-dimensional schemes $Z \subset \mathbb{P}^1$. Hence, we may assume that $n \geq 2$ and use induction on the integer n. Fix $A \in S(\mathbb{P}^n, x)$ such that $h^1(\mathcal{I}_{2A}(t))$ is maximal, and take a hyperplane $H \subset \mathbb{P}^n$ such that $y := \#(A \cap H)$ is maximal. Since H is smooth, the definition of the residual scheme gives $\mathrm{Res}_H(2A) = 2(A \setminus A \cap H) \cup (A \cap H)$. Consider the residual exact sequence of H:

$$0 \to \mathcal{I}_{2(A\setminus A\cap H)\cup(A\cap H))}(t-1) \to \mathcal{I}_{2A}(t) \to \mathcal{I}_{H\cap 2A,A}(t) \to 0 \tag{5}$$

Assume that $t = 1$, and hence, $x = 1$. Since $\mathcal{O}_{\mathbb{P}^n}(1)$ is very ample, $h^1(\mathcal{I}_{2p}(1)) = 0$ for all $p \in \mathbb{P}^n$. Since any point of $\mathbb{P}^n$ is contained in a line, the case $t = 1$ is true. Hence, for $k = 1$ we may assume that $t > 1$ and use induction on the integer t. Remark 5 gives

$$h^2(\mathcal{I}_{2(A\setminus A\cap H)\cup(A\cap H))}(t-1)) = 0.$$

First assume that $y = x$, i.e., $A \subset H$. Since $x \leq t$, we have $h^1(\mathcal{I}_A(t-1)) = 0$ (Lemma 3). Since $h^2(\mathcal{I}_A(t-1)) = 0$, the long cohomology exact sequence of (5) gives the equality $h^1(\mathcal{I}_{2A}(t)) = h^1(H, \mathcal{I}_{2A}(t))$. Hence, it is sufficient to use the inductive assumption on n. We proved that if A is contained in a line L, then $h^1(\mathcal{I}_{2A}(t)) = \max\{0, t+1-2x\}$.

Now assume that $y < x$. Since $y < x$, the maximality condition of H implies that $H = \langle A \cap H \rangle$. Since $\dim |\mathcal{O}_{\mathbb{P}^n}(1)| = n$, we have $y \geq n \geq 2$.

(a1) Assume that $h^1(\mathcal{I}_{2(A\setminus A\cap H)\cup A\cap H}(t-1)) = 0$. The long cohomology exact sequence of (5) gives the equality $h^1(\mathcal{I}_{2A}(t)) = h^1(H, \mathcal{I}_{A\cap H,H}(t))$, because we have $h^2(\mathcal{I}_{2(A\setminus A\cap H)\cup(A\cap H))}(t-1)) = 0$ (Remark 5). Based on the inductive assumption on n, we obtain the inequality $h^1(H, \mathcal{I}_{A\cap H,H}(t)) \leq \max\{0, 2y-1\}$, contradicting the assumption $2x \geq t+2$ for part (b) in the case of x collinear points and in the case of $2x \leq t+1$, proving that $h^1(\mathcal{I}_{2A}(t)) = 0$.

(a2) Assume that $h^1(\mathcal{I}_{2(A\setminus A\cap H)\cup A\cap H}(t-1)) > 0$. According to Lemma 1 there is a zero-dimensional scheme Z such that $Z_{\mathrm{red}} \subseteq A$, each connected component of Z has at most degree 2, each point of $A \cap H$ is a connected component of Z, $h^1(\mathcal{I}_Z(t-1)) > 0$, and $h^1(\mathcal{I}_{Z'}(t-1)) = 0$ for all $Z' \subsetneq Z$. We have $\deg(Z) \leq 2(x-y) + y = 2x - y \leq 2(t-1)$. Lemma 3 indicates the existence of a line R such that $\deg(R \cap Z) \geq t+1$.

(a2.1) First assume that $R \subset H$. Since each connected component of Z has at most degree 2, and it has degree 1 if it is contained in H, a line is uniquely determined by 2 of its points and $R \subset H$ by assumption, with $\deg(Z) \leq 2$, $Z = Z_{\mathrm{red}}$. Since $\deg(Z) \geq t+1$, we get $t = 1$, a contradiction.

(a2.2) Now assume that $R \nsubseteq H$. Since $\deg(R \cap H) \leq 1$, R contains at most 1 point of the set $A \cap H$. Hence, Z_{red} contains at least $\lceil t/2 \rceil$ points of $A \setminus A \cap H$. Note that we have the inequality $\#(Z_{\mathrm{red}} \cap R) \geq \lceil t/2 \rceil + 1$. Let M be a hyperplane containing R, with $c := \#(M \cap S)$ being maximal among the hyperplanes containing R. As in step (a1) we get a contradiction if $h^1(\mathcal{I}_{2(A\setminus A\cap M)\cup A\cap M}(t-1)) = 0$. Hence, we may assume that $h^1(\mathcal{I}_{2(A\setminus A\cap M)\cup A\cap M}(t-1)) > 0$. According to Lemma 1 there is a zero-dimensional scheme Z' such that $Z'_{\mathrm{red}} \subseteq A$, each connected component of Z' has at most degree 2, each point of $A \cap M$ is a connected component of Z, $h^1(\mathcal{I}_{Z'}(t-1)) > 0$, and $h^1(\mathcal{I}_{Z''}(t-1)) = 0$ for all $Z'' \subsetneq Z'$. Note the inequality $\deg(Z') \leq 2(x-c) + c = 2x - c \leq 2(t-1)$. Lemma 3 indicates the existence of a line R' such that $\deg(R \cap Z') \geq t+1$. Since $t > 1$, Z'_{red} contains at most one point of $A \cap M$. Hence, $R' \neq R$. Since $\deg(R \cap R') \leq 1$, we have the inequality $x \geq -1 + (1 + \lceil t/2 \rceil) + (1 + \lceil t/2 \rceil) > t$, a contradiction.

(b) According to step (a) we may assume that $k \geq 2$ and use induction on k and n. Recall that $t > 1$ and that we use induction on t. Fix $j \neq i$. Since $\pi_j(L)$ is a point, there is $H \in |\mathcal{I}_L(\epsilon_j)|$. Take H such that $b := \#(A \cap H)$ is maximal. Since H is smooth, we have an equality $\mathrm{Res}_H(2A) = 2(A \setminus A \cap H) \cup (A \cap H)$ in the schemes. Consider the residual exact sequence of H:

$$0 \to \mathcal{I}_{2(A\setminus A\cap H)\cup(A\cap H))}(d_1,\dots,d_k)(-\epsilon_j) \to \mathcal{I}_{2A}(d_1,\dots,d_k) \to \mathcal{I}_{H\cap 2A,A}(d_1,\dots,d_k) \to 0 \tag{6}$$

(b1) Assume that $d_i = t$ and $A \subset L$. We have $x \leq t$ and $t \geq 2$. Thus, we obtain $h^1(\mathcal{I}_A(d_1,\dots,d_k)(-\epsilon_j)) = 0$ even if $d_j = t$ (Proposition 1). Hence, in this case, through induction on n, we have $h^1(\mathcal{I}_{2A}(t)) = \max\{0, 2x - t - 1\}$. Note that we use the case $k = 1$, because if $n_j = 1$, the multiprojective space H has only $k-1$ positive-dimensional factors.

(b2) Assume that either $d_i > t$ or $d_i = t$ and $A \nsubseteq L$. Set $a := \#(A \cap L)$. By assumption, $a \geq \lceil d_i/2 \rceil + 1 \geq \lceil t/2 \rceil$.

First assume that $A \subset H$. Proposition 1 gives $h^1(\mathcal{I}_A(d_1,\dots,d_k)(-\epsilon_j)) = 0$ even if $d_j - 1 = t - 1$. Hence, the long cohomology exact sequence of H gives the equality $h^1(\mathcal{I}_{2A}(d_1,\dots,d_k)) = h^1(H, \mathcal{I}_{H\cap 2A,H}(d_1,\dots,d_k))$. The inductive assumption gives $h^1(H, \mathcal{I}_{H\cap 2A,H}(d_1,\dots,d_k)) \leq \max\{0, 2x - t + 1\}$ with equality if $2x \geq t + 2$ if and only if there is $h \in \{1,\dots,k\}$ (with $h \neq j$ if $n_j = 1$) such that $d_h = t$ and a curve R of multidegree ϵ_h such that $A \subset R$. Since $R \neq L$ by assumption and $\#(R \cap L) \leq 1$, we get a contradiction.

Now assume that $A \nsubseteq H$. If $h^1(\mathcal{I}_{2(A\setminus A\cap H)\cup(A\cap H))}(d_1,\dots,d_k)(-\epsilon_j)) = 0$, we get a contradiction in the same way.

Now assume that $h^1(\mathcal{I}_{2(A\setminus A\cap H)\cup(A\cap H))}(d_1,\dots,d_k)(-\epsilon_j)) > 0$. According to Lemma 1 there is a zero-dimensional scheme Z such that $Z_{\mathrm{red}} \subseteq A$, each connected component of Z has at most degree 2, each point of $A \cap H$ is a connected component of Z, and the following inequality $h^1(\mathcal{I}_Z(d_1,\dots,d_k)(-\epsilon_j)) > 0$ holds. Set $b := \#(A \cap H)$. We have $b \geq a$, and hence, $2(x-b) + b = 2x - b \leq 2t - 1 - \lceil t/2 \rceil \leq 2(t-1) + 1$. According to Proposition 1 there is an integer $h \in \{1,\dots,k\}$ and a curve C of multidegree ϵ_h with $\deg(C \cap Z) \geq d_h + 2$ if $h \neq j$, and $\deg(C \cap Z) \geq d_h + 1$ if $h = j$. Since $H \supseteq L$, $C \neq L$. Hence, $\deg(L \cap C) \leq 1$. We get $x \geq -1 + \#(A \cap L) + \#(A \cap C) \geq -1 + \lceil (d_h + 1)/2 \rceil + 1 + \lceil d_i/2 \rceil > t$, a contradiction. □

5. Examples

In this section we describe the geometry and weights of the sets arising in Theorem 1 and the weight of some other set. We show that for $x = t + 1$ and $k > 1$, the description of the possible weights is very rich. Fix positive integers k, n_i, and d_i, and set $n := n_1 + \cdots + n_k$, $t := \min\{d_1,\dots,d_k\}$, and $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We allow the case $n = 1$, i.e., $Y = \mathbb{P}^1$. If $k \geq 2$, we permute the integers $d_1,\dots,d_k$ to get $d_1 \leq \cdots \leq d_k$. Hence, there is an integer e such that $d_i = t$ if and only if $i \leq e$. We write $Y = Y' \times Y''$ with Y', the product of the first e factors of Y and Y'', and the product of the last $k - e$ ones, with the convention that Y'' is a point if $e = k$. Set $m := \dim Y' = n_1 + \cdots + n_e$. Let $\pi : Y \to Y'$ denote the projection, with the convention that π is the identity map if $e = k$.

Remark 8. *Recall that $w(2S) = 0$ if $\#S = 1$ (Remark 4) and that for $k > 1$ a finite set $A \in S(Y, x)$, $x \geq 2$, is contained in a curve of multidegree ϵ_i if and only if $\#\pi_j(A) = 1$ for all $j \neq i$, and $\pi_i(A)$ is contained in a line. The latter condition if always satisfied if either $x = 2$ or $n_i = 1$. Note that if the curve exists, the integer i is uniquely determined by A.*

Example 2. *Take $n = 1$, and hence, $Y = \mathbb{P}^1$. For all positive integers t and x and all $A \in S(\mathbb{P}^1, x)$, the cohomology of line bundles on $\mathbb{P}^1$ gives the two equalities $h^0(\mathcal{I}_{2A}(t)) = \max\{0, t + 1 - 2x\}$ and $h^1(\mathcal{I}_{2A}(t)) = \max\{0, 2x - t - 1\}$.*

Example 3. *Take $t = 1$. First assume that $k = 1$. We have $h^0(\mathcal{I}_{2A}(1)) = 0$ for all $x > 0$ and all $A \in S(\mathbb{P}^n, x)$, and hence, $w(2\nu(A)) = (n+1)(x-1)$.*

Now assume that $k \geq 2$ and $x = 2$. Take $A \in S(Y, 2)$.

(a) Assume that there is no $i \leq e$ such that A is contained in a curve of multidegree ϵ_i. According to Theorem 1 (which was proven in step (a)) we have $w(2\nu(A)) = 0$.

(b) Assume that $e = k$ and that A is contained in a curve L of multidegree ϵ_i for some i. Fix $j \neq i$. Since $\#\pi_j(L) = 1$, there is $H \in |\mathcal{I}_L(\epsilon_j)|$. Since $\pi_{1|L} : L \to \mathbb{P}^{n_1}$ is an embedding, $\#A = 2$ and $A \subset L$, we have $h^1(\mathcal{I}_A(1, \dots, 0)) = 0$. Hence, $h^1(\mathcal{I}_{2A}(1, \dots, 1)) = h^0(Y_j, \mathcal{I}_{\eta_j(A)}(1, \dots, 1))$ according to Remark 5 and the long cohomology exact sequence of H. Iterating this trick we get the equality $w(2\nu(A)) = h^1(\mathbb{P}^{n_i}, \mathcal{I}_{\pi_i(A)}) = n_i$. Hence, we obtain $w(2\nu(A)) = n_i$.

(c) Assume the existence of $i \leq e$ such that A is contained in a line L of multidegree ϵ_i. First assume that $e < k$. Hence, there is $H \in |\mathcal{I}_L(\epsilon_k)|$. Since $\mathcal{O}_Y(d_1, \dots, d_k)(-\epsilon_i))$ is very ample, $h^1(\mathcal{I}_A(d_1, \dots, d_k)(-\epsilon_i)) = 0$. Hence, Remark 5 and the long cohomology exact sequence of H give the equality of the integers $h^1(\mathcal{I}_{2A}(d_1, \dots, d_k))$ and $h^1(H, \mathcal{I}_{(2A,H),H}(d_1, \dots, d_k))$. Repeating this construction we get $h^1(\mathcal{I}_{2A}(d_1, \dots, d_k)) = h^1(Y', \mathcal{I}_{2\pi(A)}(1, \dots, 1))$. Step (b) gives the equality $w(2\nu(A)) = n_i$.

Example 4. *Fix integer $t \geq 3$, $x > y > 0$ such that $\lceil t/2 \rceil + 1 \leq y \leq t$ and $x - y < \lceil t/2 \rceil$. Let $L \subset \mathbb{P}^n$, where $n \geq 2$, be a line. Fix $A \in S(\mathbb{P}^n, x)$ such that $\#(A \cap L) = y$. In this example we prove that $h^1(\mathcal{I}_{2A}(t)) = 2y - t - 1$. Set $F := A \cap L$ and $E := A \setminus F$. Since $\#(F) = y \geq \lceil t/2 \rceil + 1$, we have $h^0(L, \mathcal{I}_{L \cap 2A}(t)) = 0$, and hence, $h^1(L, \mathcal{I}_{L \cap 2A, L}(t)) = 2y - t - 1$. Thus we want to prove the equality $h^1(\mathcal{I}_{2A}(t)) = h^1(L, \mathcal{I}_{L \cap 2A, L}(t))$.*

(a) In this step we check that $h^1(\mathcal{I}_{2E \cup F}(t-1)) = 0$. Assume that $h^1((a)\mathcal{I}_{2E \cup F}(t-1)) > 0$. According to Lemma 1 there is a zero-dimensional scheme W such that $W_{\mathrm{red}} \subseteq F$, each connected component of W has at most degree 2, and $h^1(\mathcal{I}_{W \cup F}(t-1)) > 0$. Since we have $\deg(W) \leq 2(x-y)$, $\deg(W \cup F) \leq 2x - y \leq 2(t-1) + 1$. Since $h^1(\mathcal{I}_{W \cup F}(t-1)) > 0$, Lemma 3 indicates the existence of a line R such that $\deg(R \cap (W \cup F)) \geq t + 1$. Since $L \cap (W \cup F) = F$ and we assumed that $y \leq t$ and $R \neq L$, we obtain the inequality $\deg(R \cap L) \leq 1$. Thus, we get $\deg(W \cap R) \geq t$. Since each connected component of W has at most degree 2, we get $x - y \geq \lceil t/2 \rceil$, a contradiction.

(b) First assume that $n = 2$. In this case it is sufficient to use the residual sequence of L and the fact that $h^1(\mathcal{I}_{2E \cup F}(t-1)) = 0$. Now assume that $n > 2$. Take a general hyperplane $H \supset L$. We have $H \cap A = F$, and hence, $\mathrm{Res}_H(2A) = 2E \cup F$. Since $h^1(\mathcal{I}_{2E \cup F}(t-1)) = 0$, it is sufficient to use the long cohomology exact sequence of H and induction on the dimension of the projective space.

Example 5. *Fix integer $t \geq 5$ and $x > y > 0$ such that $\lceil t/2 \rceil + 1 \leq y \leq t$,*

$$(n, t) \notin \{(2, 6), (3, 6), (4, 5), (4, 6)\}$$

and $(n+1)(x-y) \leq \binom{n+t-2}{n}$. Let $L \subset \mathbb{P}^n$, where $n \geq 2$, be a line. Take $F \subset L$ such that $\#F = y$. Take a general $E \in S(\mathbb{P}^n, x - y)$ and set $A := E \cup F$. We prove the equality $h^1(\mathcal{I}_{2A}(t)) = 2y - t - 1$. To adapt the proof of Example 4, it is sufficient to prove that $h^1(\mathcal{I}_{2E \cup F}(t-1)) = 0$. Since $y \leq t$, we have $h^1(\mathcal{I}_F(t-1)) = 0$ (Lemma 3). Since $F \subset L$, the residual exact sequence of a general hyperplane containing L shows that it is sufficient to prove that $h^1(\mathcal{I}_{2E}(t-2)) = 0$. This is true according to the Alexander–Hirschowitz theorem ([9]).

In the next example we compute $w(2S)$ in case (iii) of Theorem 1 when the conic C is smooth.

Example 6. *Take $t \geq 3$. Let $C \subset \mathbb{P}^n$, where $n \geq 2$, be a smooth conic. Take $A \in S(C,x)$ with $x \leq 2t-3$. Fix $A \in S(C,x)$. In this example we check that $h^1(\mathcal{I}_{2A}(t)) = \max\{0, 2x-2t-1\}$. Since $C \cong \mathbb{P}^1$, $\deg((2A,C)) = 2x$ and $\deg(\mathcal{O}_C(t)) = 2t$, we have $h^1(C, \mathcal{I}_{2(A,C),C}(t)) = \max\{0, 2x-2t-1\}$. Since $h^1(\mathcal{I}_C(z)) = 0$ for all z, we get $h^1(\mathcal{I}_{(2A,C)}(t)) = \max\{0, 2x-2t-1\}$. Hence, Remarks 3 and 5 give $h^1(\mathcal{I}_{2A}(t)) \geq \max\{0, 2x-2t-1\}$.*

(a) Assume that $n = 2$. We have the residual exact sequence of C:

$$0 \to \mathcal{I}_A(t-2) \to \mathcal{I}_{2A}(t) \to \mathcal{I}_{(2A,C)}(t) \to 0 \tag{7}$$

If $h^1(\mathcal{I}_A(t-2)) = 0$, then it is sufficient to use Remark 1 and the long cohomology exact sequence of (7). Now assume that $h^1(\mathcal{I}_A(t-2)) > 0$. By assumption, $t \geq 3$ and $\#A \leq 2(t-2)+1$. Hence, there is a line $L \subset \mathbb{P}^2$ such that $\#(L \cap A) \geq t$. Since $t \geq 3$, and C is an irreducible conic, the theorem of Bezout gives a contradiction.

(b) Assume that $n > 2$. Taking a general hyperplane H containing the plane spanned by C, we get an exact sequence:

$$0 \to \mathcal{I}_A(t-1) \to \mathcal{I}_{2A}(t) \to \mathcal{I}_{(2A,H)}(t) \to \mathcal{I}_{(2A,H)),H}(t)) \to 0 \tag{8}$$

As in step (a) we get $h^1(\mathcal{I}_A(t-1)) = 0$. Hence, Remarks 3 and 5, with induction on n, give the equality $h^1(\mathcal{I}_{2A}(t)) = \max\{0, 2x-2t-1\}$.

Example 7. *Fix integers t and x such that $t \geq 3$ and $x \leq 2t-3$. Fix lines $L, R \subset \mathbb{P}^n$ and $n \geq 2$ such that $L \neq R$ and $L \cap R \neq \emptyset$. Take the finite set $A \subset L \cup R$ such that $L \cap R$ and the finite set $B \subset L \cup R$ such that $L \cap R \in B$. Set $a_1 := \#(L \cap A)$ and $a_2 := \#R \cap B$, and $b_1 := \#(L \cap B)$ and $b_2 := \#(R \cap B)$. Note that $\#A = a_1 + a_2$ and $\#B = b_1 + b_2 + 1$. If t is odd we take $a_1 = a_2 = (t+1)/2$. If t is even we take $b_1 = b_2 = 1 + t/2$. Note that these are the only case for sets A and B with $\#A \geq t+1$, $\#B \geq t+1$, $2\#(A \cap L) \leq t+1$, $2\#(A \cap R) \leq t+1$, $2\#(B \cap L) \leq t+1$, and $2\#(B \cap R) \leq t+1$. Set $C := L \cup R$. We have $h^0(C, \mathcal{O}_C(t)) = 2t+1$.*

(a) Assume that t odd. In this part we prove the quality $h^1(\mathcal{I}_{2A}(t)) = 1$. Note that we have $\deg(C \cap 2A) = 2a_1 + 2a_2 = 2t+2$. We have $h^0(L, \mathcal{I}_{2(A \cap L),L}(t)) = h^0(R, \mathcal{I}_{2(A \cap R),R}(t)) = 0$. A Mayer–Vietoris exact sequence gives $h^0(C, \mathcal{I}_{2A \cap C,C}(t)) = 0$. Since $\deg(C \cap 2A) = 2t+2$, we have $h^1(C, \mathcal{I}_{2A \cap C,C}(t)) = 1$. Then as in Example 6, we get $h^1(\mathcal{I}_{2A}(t)) = 1$.

(b) Assume that t is even. In this part we prove that $h^1(\mathcal{I}_{2A}(t)) = 2$. Note that this is different from the result for $x = t+1$ computed for a smooth conic in Example 6. Let $\langle C \rangle$ be the plane containing C. Since C is singular at the point $L \cap R$, $(2(L \cap R, \langle C \rangle) \subset C$. Hence, we obtain $\deg(2C \cap B) = 2t+3$. We have $h^0(L, (2(B \cap L), L)(t)) = h^0(R, \mathcal{I}_{2(B \cap R),R}(t)) = 0$. A Mayer–Vietoris exact sequence gives $h^0(C, \mathcal{I}_{2B \cap C,C}(t)) = 0$. Since $\deg(C \cap 2A) = 2t+2$, we obtain the equality $h^1(C, \mathcal{I}_{2A \cap C,C}(t)) = 2$. Then as in Example 6, we get $h^1(\mathcal{I}_{2A}(t)) = 2$.

Example 8. *Assume $t \geq 2$. In this example we take $x = t+1$ and find A such that $h^1(\mathcal{I}_{2A}(d_1, \dots, d_k)) = 1$. Assume that $k \geq 2$, with $d_1 = d_2 = t$, and (if $k > 2$) $d_i \geq t$ for all $i \geq 3$. Let $C \subset Y$ be a smooth and connected curve of multidegree $\epsilon_1 + \epsilon_i$. Fix $A \in S(C,x)$. We have $C \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_C(t, t, d_3, \dots, d_k)) = 2t$. Hence,*

$$h^1(C, \mathcal{I}_{C \cap 2A,C}(t, t, d_3, \dots, d_k)) = \max\{0, 2x-2t-1\}.$$

Note that C is contained in a unique multiprojective space, the product of all $\pi_i(C)$, and that this multiprojective space is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

(a) Assume that $k = 2$ and $n_1 = n_2 = 1$. Consider the residual exact sequence of H:

$$0 \to \mathcal{I}_A(t-1, t-1) \to \mathcal{I}_{2A}(t,t) \to \mathcal{I}_{C\cap 2A,C}(t,t) \to 0 \tag{9}$$

Since C is irreducible, every curve of bidegree $(0,1)$ or of bidegree $(1,0)$ intersects C in a degree 1 scheme. Since $\#A = t+1$, and no curve of multidegree $(1,0)$ or $(0,1)$ contains at least 2 points of A, Lemma 5 gives $h^1(\mathcal{I}_A(t-1,t-1)) = 0$. Hence, Remark 5 and the long cohomology exact sequence of (9) give $h^1(\mathcal{I}_{2A}(t,t)) = 1$.

(b) Assume that $k = 2$ and $n_1 + n_2 \geq 3$, say, $n_2 \geq 2$. According to step (a) we want use induction on the integer $n_1 + n_2$ to prove that $h^1(\mathcal{I}_{2A}(t,t)) = 1$. Take $H \in |\mathcal{O}_Y(0,1)|$ containing C. Consider the residual exact sequence of H:

$$0 \to \mathcal{I}_A(t, t-1) \to \mathcal{I}_{2A}(t,t) \to \mathcal{I}_{H\cap 2A,H}(t,t) \to 0 \tag{10}$$

The inductive assumption gives $h^1(H, \mathcal{I}_{H\cap 2A,H}(t,t)) = 0$. Note that every curve of bidegree $(1,0)$ or $(0,1)$ contains at most one point of A. Since $x = t+1$ and $t \geq 2$, Lemma 9 gives $h^1(H, \mathcal{I}_{H\cap 2A,H}(t,t)) = 0$. Hence, Remark 5 and the long cohomology exact sequence of (10) give $h^1(\mathcal{I}_{2A}(t,t)) = 1$.

(c) Assume that $k > 2$. According to steps (a) and (b), we may use induction on the integer k. Since $\#\pi_k(C) = 1$, there is $M \in |\mathcal{I}_C(\epsilon_k)|$. We have $\mathrm{Res}_M(2A) = A$. As in step (a) we see that $h^1(\mathcal{I}_A(d_1, \dots, d_k)(-\epsilon_k)) = 0$. Hence, induction on k (case $n_k = 1$), induction on n (case $n_k > 1$), and the long cohomology exact sequence of the residual sequence of M give $h^1(\mathcal{I}_{2A}(d_1, \dots, d_k)) = 1$.

Example 9. *Assume that $t \geq 3$, $k \geq 2$ and $d_1 = d_2 = t$. In this example we take $x = t+1$ and find E and B with $h^1(\mathcal{I}_{2E}(t,t,\dots,d_k)) = 1$ (case where t is odd) and $h^1(\mathcal{I}_{2B}(t,t,\dots,d_k)) = 2$ (case where t is even). Let $L \subset Y$ be a curve of multidegree ϵ_1 and $R \subset Y$ a curve of multidegree ϵ_2 such that $L \cap R \neq \emptyset$. Thus, $L \cap R$ is a single point. Set $C := L \cup R$. Note that $h^0(C, \mathcal{O}_C(t,t,d_1,\dots,d_k)) = 2t+1$. If t is odd, take $E \subset C$ such that $\#(E \cap L) = \#(R \cap L) = (t+1)/2$. Note that $L \cap R \notin E$. If t is odd, take $B \subset C$ such that $R \cap L \in B$ and $\#(E \cap L) = \#(R \cap L) = 1 + t/2$. Note that $\#E = \#B = t+1$. Since E is contained in the smooth locus of C, we have $\deg(C \cap 2E) = 2t+2$. Since $L \cap R$ is the unique singular point of C, C is nodal, and $R \cap L \in B$, we have $\deg(C \cap 2B) = 2t+3$. Set $A := E$ if t is odd and $A := B$ if t is even. Thus,*

$$h^0(L, \mathcal{I}_{L\cap 2A}(t,t,\dots,d_k)) = h^0(L, \mathcal{I}_{L\cap 2A}(t,t,\dots,d_k)) = 0.$$

Hence, a Mayer–Vietoris exact sequence gives $h^0(C, \mathcal{I}_{C\cap 2A,C}(t,t,\dots,d_k)) = 0$. Hence, we obtain the equality $h^1(C, \mathcal{I}_{C\cap 2A,C}(t,t,\dots,d_k)) = \deg(C \cap 2A) - 2t - 1$, which is 1 for odd t and 2 for even t.

- *(a) Assume that $k = 2$ and $n_1 = n_2 = 1$. To quote the proof of step (a) of Example 8, it is sufficient to prove that $h^1(\mathcal{I}_A(t-1,t-1)) = 0$. This is true according to Proposition 1, because L and R are the only lines intersecting C at at least 2 points, and $\#(A \cap L) = \#(A \cap R) = \lceil (t+1)/2 \rceil < t+1$.*
- *(b) Assume that $k = 2$ and $n_1 + n_2 > 1$. We draw the same conclusion as in case (b) of Example 8, because we proved that $h^1(\mathcal{I}_A(t-1,t-1)) = 0$.*
- *(c) Assume that $k > 2$. Since C is connected and $\#\pi_k(L)) = \#\pi_k(R) = 1$, we have $\#\pi_k(C) = 1$. We take M as in step (c) of Example 8 and draw the same conclusion.*

6. Discussion

In recent years the Terracini loci $\mathcal{T}(X,x)$ of an embedded projective variety $X \subset \mathbb{P}^r$ have been studied in several papers [1–7].In [6] Galuppi, Santarsiero, Torrance, and Turatti

studied them for many embedded varieties X. In each of their X, they computed the first integer x_0 such that the Terracini set $\mathcal{T}(X, x_0)$ is non-empty and described the geometry of the set $\mathcal{T}(X, x_0)$. The Segre–Veronese varieties are among the varieties considered in [6]. In our paper we restrict the study to the Segre–Veronese varieties and describe $\mathcal{T}(X, x)$ roughly for all $x \leq 2x_0 - 2$. For these x values we describe the maximal weight of the elements of $\mathcal{T}(X, x)$, extending some ideas in [1] to all Segre–Veronese varieties. Our proofs use a very useful and far reaching extension of the notion of critical scheme of the elements of $\mathcal{T}(X, x)$ used in [2]. We expect that it is useful not only for the Terracini loci of other varieties, but also for describing the geometry of other sets of linearly dependent zero-dimensional schemes. In the last section we give some examples to show how to use these cohomological tools.

The extension of this paper to other embedded varieties could be very interesting, for instance, Grassmannians, which are related to skew-symmetric tensors. Theorem 2 may be extended to other varieties without having an extension of Theorem 1. One only needs to know the integer x_0 appearing for that embedded variety. Looking at [6] may help. It may be possible to extend Theorem 1 to the same variety, in a multiprojective space, but for higher x, for instance, for $x \leq 4x_0$. The statement of Theorem 1 contains a multidegree $(d_1, \ldots, d_k)$. The end of the Introduction contains a roadmap for the extension in terms of the entries of the multidegree.

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