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Completeness and Cocompleteness Transfer for Internal Group Objects with Geometric Obstructions

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Abstract

This work establishes definitive conditions for the inheritance of categorical completeness and cocompleteness by categories of internal group objects. We prove that while the completeness of $\mathbf{Grp}(\mathscr{C})$ follows unconditionally from the completeness of the base category $\mathscr{C}$, cocompleteness requires $\mathscr{C}$ to be regular, cocomplete, and admit a free group functor left adjoint to the forgetful functor. Explicit limit and colimit constructions are provided, with colimits realized via coequalizers of relations induced by group axioms over free group objects. Applications demonstrate cocompleteness in topological groups, ordered groups, and group sheaves, while Lie groups serve as counterexamples revealing necessary analytic constraints—particularly the impossibility of equipping free groups on non-discrete manifolds with smooth structures. Further results include the inheritance of regularity when the free group functor preserves finite products, the existence of internal hom-objects in locally Cartesian closed settings, monadicity for locally presentable $\mathscr{C}$, and homotopical extensions where model structures on $\mathbf{Grp}(\mathscr{M})$ reflect those of $\mathscr{M}$. This framework unifies classical category theory with geometric obstruction theory, resolving fundamental questions on exactness transfer and enabling new constructions in homotopical algebra and internal representation theory.

Keywords: internal group objects; cocompleteness transfer; free group functor; regular categories; monadicity; Tannakian duality; homotopical algebra

MSC: 18A30; 18A35; 18B25; 18G55; 22E99

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1. Introduction

The study of the completeness and cocompleteness of categories has been a central theme in category theory since its inception. A category is *complete* if it has all small limits, and *cocomplete* if it has all small colimits. These properties are fundamental to the ability of a category to support universal constructions, such as products, equalizers, pullbacks, and their duals. The classical work of Mac Lane [1] established that many categories of algebraic structures (e.g., groups, rings) are complete and cocomplete, while Borceux [2] extended these results to more general contexts. However, for internal categories, such as categories of group objects in a base category $\mathcal{C}$, the inheritance of these properties is not automatic and depends delicately on the structure of $\mathcal{C}$. In this work, we unify and extend these classical results by providing explicit conditions under which $\mathbf{Grp}(\mathcal{C})$ inherits completeness and cocompleteness from $\mathcal{C}$, and we identify obstructions in geometric settings.

The categorical formalization of algebraic structures, pioneered by Lawvere [3], reveals that group-like objects in a category $\mathcal{C}$ with finite products form a category $\mathbf{Grp}(\mathcal{C})$ whose exactness properties reflect fundamental features of $\mathcal{C}$. Classical examples include the following:

- $\mathbf{Grp}(\mathbf{Set})$ = ordinary groups;
- $\mathbf{Grp}(\mathbf{Top})$ = topological groups;
- $\mathbf{Grp}(\mathbf{Sh}(X))$ = sheaves of groups;
- $\mathbf{Grp}(\mathbf{Man}^n)$ = n-dimensional Lie groups.

While the completeness of $\mathbf{Grp}(\mathcal{C})$ follows automatically from the completeness of $\mathcal{C}$, the dual problem of *cocompleteness* remained open outside well-studied cases. This work resolves this by identifying necessary and sufficient conditions for hereditary cocompleteness, with applications ranging from ordered groups to ∞-groupoids.

Key advances

Our principal contributions unify and extend classical results:

1. **Cocompleteness Characterization**: For $\mathcal{C}$ cocomplete and regular, $\mathbf{Grp}(\mathcal{C})$ is cocomplete iff a free group functor $F \dashv U$ exists (Theorems 7 and 21). Colimits are constructed via coequalizers:

$$\coprod_{i,j} F(U(G_i) \amalg U(G_j)) \rightrightarrows F\left(\coprod_k U(G_k)\right) \to \coprod_{\mathbf{Grp}} G_i$$

   encoding group axioms. This explains the cocompleteness in **TopGrp** but failure in **LieGrp** (Corollary 3, Proposition 1).
2. **Geometric Obstruction Theorem**: For categories of geometric objects (e.g., smooth manifolds $\mathbf{Man}^n$), the following are equivalent (Theorem 22):

$$\begin{aligned} \mathbf{Grp}(\mathcal{C}) \text{ cocomplete} &\iff F \dashv U \text{ exists} \\ &\iff \mathcal{C} \text{ admits a dense subcategory of group-tame objects.} \end{aligned}$$

   Failure for Lie groups follows from Birkhoff transitivity: $\mathbb{Z} * \mathbb{Z}$ cannot embed densely in second-countable groups.
3. **Higher Categorical Structures**:
   - When $\mathcal{C}$ is regular and F preserves products, $\mathbf{Grp}(\mathcal{C})$ inherits regularity with effective descent (Theorem 23).
   - For locally Cartesian closed $\mathcal{C}$, internal hom-objects $\mathrm{Hom}(G, H)$ exist as equalizers:
$$\mathrm{Hom}(G, H) \hookrightarrow H^G \times H^{G \times G} \rightrightarrows H^{G \times G}$$
     imposing homomorphism conditions (Theorem 24).
4. **Homotopical and Duality Extensions**:
   - For combinatorial model categories $\mathscr{M}$, $\mathbf{Grp}(\mathscr{M})$ admits a model structure with weak equivalences/fibrations created by U (Theorem 26).
   - Tannakian duality holds: $G \cong \mathbf{Aut}^{\otimes}(\omega)$ for fiber functors $\omega : (G) \to \mathcal{C}$ when $\mathcal{C}$ is a topos (Theorem 25).

These results have implications beyond pure category theory. Internal group objects arise naturally in various branches of mathematics and its applications: in algebraic topology as loop spaces and higher homotopy groups; in algebraic geometry as group schemes and Tannakian categories; in mathematical physics as gauge groups and symmetries in topological field theories; and in computer science as models of higher inductive types in

homotopy type theory. The question of when limits and colimits exist in these categories is fundamental to constructing quotients, classifying spaces, and performing descent arguments. Our work provides a unified answer to this question, clarifying both the universal and obstructionary aspects of completeness and cocompleteness in categories of internal group objects.

Recent Developments in Higher Structures

With the advent of higher category theory and homotopical algebra, the study of internal group objects has extended naturally into the setting of $(\infty, 1)$-categories and model categories. Recent works by Rezk [4] on model structures for homotopy theories, by Lurie [5–7] on higher topos theory and higher algebra, by Riehl [8] on categorical homotopy theory, and by Schweigert and Valenti [9] on higher structures in mathematical physics provide a robust framework for generalizing our results to higher groupoids and homotopy-coherent algebraic structures. These developments allow us to treat internal group objects not only in classical categories but also in contexts such as ∞-topoi and differential ∞-categories, thereby unifying geometric and homotopical perspectives on group theory.

Novelty and significance

Our work provides explicit constructive formulas for colimits in $\mathbf{Grp}(\mathcal{C})$ under regularity and free-group conditions. Unlike [10], which focuses on concrete examples, we derive necessary and sufficient conditions for cocompleteness and identify geometric obstructions in the case of Lie groups.

Specifically, we

- Establish *precise obstructions* to cocompleteness via analytic constraints (Birkhoff transitivity);
- Derive *constructive colimit formulas* using free group functors;
- Unify Lie group pathologies, homotopical algebra, and Tannakian duality under one framework.

Section 2 reviews the preliminaries; Sections 3–6 develop the core results; Section 7 discusses higher categorical extensions.

2. Preliminaries

Definition 1. *A category $\mathcal{C}$ is Cartesian if it has all finite products. We denote by* T *the terminal object and by* $\times$ *the binary product operation.*

Definition 2 (Internal Group Object [11])**.** *Let $\mathcal{C}$ be Cartesian.*

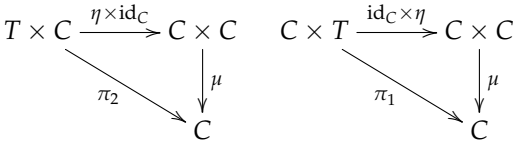
1. *A quadruple $G = (C, \mu, \eta, \zeta)$ is called a group object in $\mathcal{C}$ if C is an object in $\mathcal{C}$, and the product morphism μ, the unit morphism η, and the inverse morphism ζ are morphisms in $\mathcal{C}$:*

$$\mu : C \times C \to C, \quad \eta : \mathrm{T} \to C, \quad \zeta : C \to C,$$

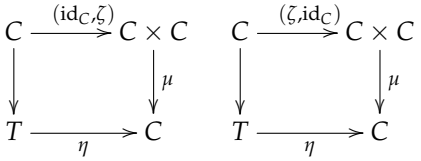
satisfying the following commutative diagrams (expressing associativity, unit laws, and inverse laws, respectively):
Associativity:

$$\begin{array}{ccc} C \times C \times C & \xrightarrow{\mathrm{id}_C \times \mu} & C \times C \\ {\scriptstyle \mu \times \mathrm{id}_C}\downarrow & & \downarrow{\scriptstyle \mu} \\ C \times C & \xrightarrow[\mu]{} & C \end{array}$$

Left and Right Unit Laws:

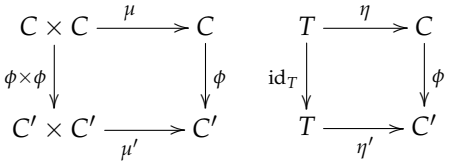


where $\pi_1 : C \times T \to C$ *and* $\pi_2 : T \times C \to C$ *are canonical projections.*
Inverse Laws:



where (id_C, ζ) *and* (ζ, id_C) *denote the respective product morphisms.*

2. *Let* $G = (C, \mu, \eta, \zeta)$ *and* $G' = (C', \mu', \eta', \zeta')$ *be group objects in* $\mathcal{C}$*. An internal group homomorphism* $\phi : G \to G'$ *is a morphism* $\phi : C \to C'$ *in* $\mathcal{C}$ *such that the following diagrams commute:*



3. *Let* $G = (C, \mu, \eta, \zeta)$*,* $G' = (C', \mu', \eta', \zeta')$*, and* $G'' = (C'', \mu'', \eta'', \zeta'')$ *be group objects in* $\mathcal{C}$*. For internal group homomorphisms* $\phi : G \to G'$ *and* $\psi : G' \to G''$*, their composition* $\phi \circ \psi : G \to G''$ *is the internal group homomorphism defined by the composite morphism* $\phi \circ \psi : C \to C''$ *in* $\mathcal{C}$*.*
4. *The identity internal group homomorphism* $\mathrm{id}_G : G \to G$ *is defined by the identity morphism* $\mathrm{id}_C : C \to C$ *in* $\mathcal{C}$*.*

Theorem 1 ([11])**.** *Let* $\mathcal{C}$ *be a category with finite products. Then, taking the internal group objects in* $\mathcal{C}$ *as objects and the internal group homomorphisms as morphisms forms a category, denoted by* $\mathbf{Grp}(\mathcal{C})$*. This is referred to as the category of internal group objects.*

Definition 3. *A free group functor for a Cartesian category* $\mathcal{C}$ *is a functor* $F : \mathcal{C} \to \mathbf{Grp}(\mathcal{C})$ *that is left adjoint to the forgetful functor* $U : \mathbf{Grp}(\mathcal{C}) \to \mathcal{C}$*, i.e., there is a natural isomorphism:*

$$\mathrm{Hom}_{\mathbf{Grp}(\mathcal{C})}(F(X), G) \cong \mathrm{Hom}_{\mathcal{C}}(X, U(G))$$

for all $X \in \mathcal{C}$*,* $G \in \mathbf{Grp}(\mathcal{C})$*.*

Theorem 2 (Existence of Free Groups)**.** *For a cocomplete regular category* $\mathcal{C}$ *with a strong generator* $\mathcal{G}$*, the free group functor exists if and only if for each* $X \in \mathcal{G}$*, the free group on* X *exists in* $\mathbf{Grp}(\mathcal{C})$*.*

Proof. ($\Rightarrow$) Immediate from adjunction. ($\Leftarrow$) For arbitrary $Y \in \mathcal{C}$, express Y as a colimit of generators $Y \cong \varinjlim_{\alpha} X_\alpha$ with $X_\alpha \in \mathcal{G}$. Define:

$$F(Y) := \varinjlim_{\alpha} F(X_\alpha)$$

where $F(X_\alpha)$ exists by hypothesis. For any $G \in \mathbf{Grp}(\mathcal{C})$, we have natural isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Grp}(\mathcal{C})}(F(Y), G) &\cong \mathrm{Hom}_{\mathbf{Grp}(\mathcal{C})}\left(\varinjlim_{\alpha} F(X_\alpha), G\right) \\
&\cong \varprojlim_{\alpha} \mathrm{Hom}_{\mathbf{Grp}(\mathcal{C})}(F(X_\alpha), G) \\
&\cong \varprojlim_{\alpha} \mathrm{Hom}_{\mathcal{C}}(X_\alpha, U(G)) \\
&\cong \mathrm{Hom}_{\mathcal{C}}\left(\varinjlim_{\alpha} X_\alpha, U(G)\right) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(Y, U(G))
\end{aligned}$$

where the second isomorphism uses that $\mathbf{Grp}(\mathcal{C})$ has colimits by reference [2] (Vol. 2, Proposition 4.3.2), and the fourth is by continuity of Hom. Thus $F \dashv U$. □

Definition 4. *A category $\mathcal{C}$ is regular if*

1. *It has all finite limits;*
2. *Coequalizers of kernel pairs exist;*
3. *Regular epimorphisms are stable under pullback.*

Lemma 1 (Properties of Regular Categories)**.** *In a regular category $\mathcal{C}$*

1. *Regular epimorphisms are effective epimorphisms;*
2. *Every morphism factors as regular epi followed by mono in the factorization system;*
3. *Pullbacks preserve regular epi-mono factorizations.*

Proof. 1. Given regular epi $e : A \to B$, form kernel pair $R \rightrightarrows A$. Since e coequalizes its kernel pair, we get unique $m : \mathrm{coeq}(R) \to B$. By stability under pullback, e is a universal coequalizer; so m is iso.

2. For $f : A \to B$, take kernel pair $p_1, p_2 : R \rightrightarrows A$ and coequalizer $e : A \to Q$. Then f factors as $f = m \circ e$ with m mono.

3. Consider the following pullback square:

$$\begin{array}{ccc}
P & \longrightarrow & A \\
\downarrow & & \downarrow f \\
C & \longrightarrow & B
\end{array}$$

If $f = m \circ e$ with e regular epi, then the pullback preserves e as regular epi by definition. □

Theorem 3 (Regularity Inheritance)**.** *Let $\mathcal{C}$ be regular Cartesian with free group functor $F : \mathcal{C} \to \mathbf{Grp}(\mathcal{C})$ preserving finite limits. Then $\mathbf{Grp}(\mathcal{C})$ is regular.*

Proof. For kernel pair coequalizers, consider a homomorphism $\phi : G \to H$ and form its kernel pair $K \rightrightarrows G$ in $\mathbf{Grp}(\mathcal{C})$. Since U creates limits, $U(K) \rightrightarrows U(G)$ is kernel pair of $U(\phi)$ in $\mathcal{C}$. Let $q : U(G) \to Q$ be the coequalizer in $\mathcal{C}$. By regularity, q is regular epi. Now construct:

$$F(\ker q) \rightrightarrows G \longrightarrow \overline{Q}$$

where $\overline{Q}$ is the coequalizer in $\mathbf{Grp}(\mathcal{C})$. Since F preserves finite limits, $F(\ker q) \cong \ker(F(q))$. Stability under pullback follows as U preserves pullbacks and reflects regular epis. □

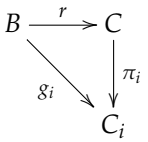
We introduce the following mathematical notations:

- $\mathcal{C}$: base category;
- $\mathbf{Grp}(\mathcal{C})$: category of internal group objects in $\mathcal{C}$;
- F: free group functor;
- U: forgetful functor;
- μ, η, ζ: multiplication, unit, inverse morphisms.

Theorem 4 ([11]). *Let $\mathcal{C}$ be a category with finite products. Then, taking the internal group objects in $\mathcal{C}$ as objects and the internal group homomorphisms as morphisms forms a category, denoted by $\mathbf{Grp}(\mathcal{C})$. This is referred to as the category of internal group objects.*

When discussing products in the internal group category $\mathbf{Grp}(\mathcal{C})$, we uniformly give the definition of a product in a general category $\mathcal{C}$ for clarity.

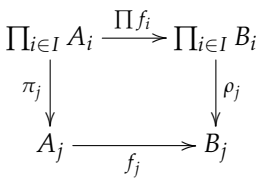
Definition 5 (Product in a Category [12]). *Let $\mathcal{C}$ be a category and $\{C_i\}_{i\in I} \subseteq \mathrm{Obj}(\mathcal{C})$ be a family of objects in $\mathcal{C}$. A product of the C_i is a pair $\{C, \{\pi_i : C \to C_i\}_{i\in I}\}$, where $C \in \mathrm{Obj}(\mathcal{C})$ and $\pi_i \in \mathrm{Mor}(\mathcal{C})(C, C_i)$ for each $i \in I$, such that for any $B \in \mathrm{Obj}(\mathcal{C})$ and any family of morphisms $\{g_i : B \to C_i\}_{i\in I}$, there exists a unique morphism $r \in \mathrm{Mor}(\mathcal{C})(B, C)$ making the following diagram commute for all $i \in I$:*



i.e., $\pi_i \circ r = g_i$ for all $i \in I$.

Theorem 5 ([12]). *If $\{C, \{\pi_i : C \to C_i\}_{i\in I}\}$ and $\{C', \{\pi'_i : C' \to C_i\}_{i\in I}\}$ are both products of $\{C_i\}_{i\in I}$, then there exists a unique isomorphism $r \in \mathrm{Mor}(\mathcal{C})(C, C')$ such that $\pi'_i \circ r = \pi_i$ for all $i \in I$.*

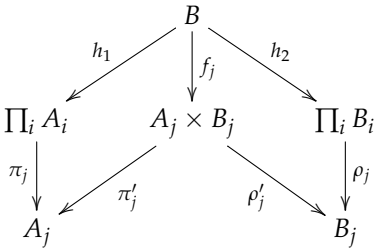
Lemma 2 ([1]). *Let $\mathcal{C}$ have all products. Let $\{f_i : A_i \to B_i\}_{i\in I}$ be a family of morphisms in $\mathcal{C}$ (I is a set). Let $\{\pi_j : \prod_{i\in I} A_i \to A_j\}_{j\in I}$ and $\{\rho_j : \prod_{i\in I} B_i \to B_j\}_{j\in I}$ be the product projections for $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$, respectively. Then there exists a unique morphism $\prod_{i\in I} f_i : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ making the following diagram commute for all $j \in I$:*



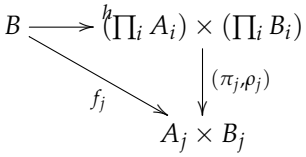
Lemma 3. *Let $\mathcal{C}$ have all products. Let $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ be families of objects in $\mathcal{C}$. Let $\{\pi_j : \prod_{i\in I} A_i \to A_j\}_{j\in I}$ and $\{\rho_j : \prod_{i\in I} B_i \to B_j\}_{j\in I}$ be the product projections for $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$, respectively. Then $\{(\pi_j, \rho_j) : (\prod_{i\in I} A_i) \times (\prod_{i\in I} B_i) \to A_j \times B_j\}_{j\in I}$ is a product for the family $\{A_i \times B_i\}_{i\in I}$.*

Proof. Consider any family of morphisms $\{f_j : B \to A_j \times B_j\}_{j\in I}$. Fix $j \in I$. Let $\pi'_j : A_j \times B_j \to A_j$ and $\rho'_j : A_j \times B_j \to B_j$ be the product projections for $A_j \times B_j$. Composing, we get morphisms $\{\pi'_j \circ f_j : B \to A_j\}_{j\in I}$ and $\{\rho'_j \circ f_j : B \to B_j\}_{j\in I}$. Since $\{\pi_j : \prod_{i\in I} A_i \to A_j\}_{j\in I}$ and $\{\rho_j : \prod_{i\in I} B_i \to B_j\}_{j\in I}$ are product projections, Definition 2 guarantees unique morphisms $h_1 : B \to \prod_{i\in I} A_i$ and $h_2 : B \to \prod_{i\in I} B_i$ such that $\pi_j \circ h_1 = \pi'_j \circ f_j$ and

$\rho_j \circ h_2 = \rho'_j \circ f_j$ for all $j \in I$. This implies the commutativity of the following diagram for each j:

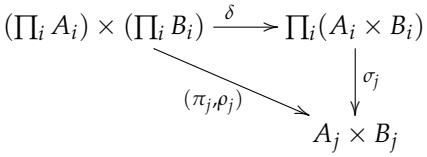


Since $f_j = \langle \pi'_j \circ f_j, \rho'_j \circ f_j \rangle$ (by the universal property of $A_j \times B_j$), there exists a unique morphism $h = \langle h_1, h_2 \rangle : B \to (\prod_{i \in I} A_i) \times (\prod_{i \in I} B_i)$ making the following diagram commute for all $j \in I$:



Therefore, $\{(\pi_j, \rho_j)\}_{j \in I}$ is indeed a product for $\{A_i \times B_i\}_{i \in I}$. □

Lemma 4. *Let $\mathcal{C}$ have all products. Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of objects in $\mathcal{C}$. Let $\{\pi_j : \prod_{i \in I} A_i \to A_j\}_{j \in I}$, $\{\rho_j : \prod_{i \in I} B_i \to B_j\}_{j \in I}$, and $\{\sigma_j : \prod_{i \in I}(A_i \times B_i) \to A_j \times B_j\}_{j \in I}$ be the product projections for $\{A_i\}_{i \in I}$, $\{B_i\}_{i \in I}$, and $\{A_i \times B_i\}_{i \in I}$, respectively. Then there exists a unique isomorphism $\delta : (\prod_{i \in I} A_i) \times (\prod_{i \in I} B_i) \to \prod_{i \in I}(A_i \times B_i)$ making the following diagram commute for all $j \in I$:*

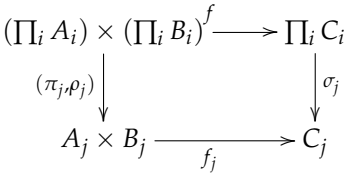


Proof. By Lemma 3, $\{(\pi_j, \rho_j)\}_{j \in I}$ is a product for $\{A_i \times B_i\}_{i \in I}$. By definition, $\{\sigma_j\}_{j \in I}$ is also a product for $\{A_i \times B_i\}_{i \in I}$. Theorem 5 then guarantees a unique isomorphism $\delta : (\prod_{i \in I} A_i) \times (\prod_{i \in I} B_i) \to \prod_{i \in I}(A_i \times B_i)$ such that $\sigma_j \circ \delta = (\pi_j, \rho_j)$ for all $j \in I$. □

Corollary 1. *Let $\mathcal{C}$ have all products. Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of objects in $\mathcal{C}$. Then there exists a unique isomorphism $\delta : (\prod_{i \in I} A_i) \times (\prod_{i \in I} B_i) \to \prod_{i \in I}(A_i \times B_i)$.*

From Definition 5 and Lemma 4, we obtain the following result.

Lemma 5. *Let $\mathcal{C}$ have all products. Let $\{A_i\}_{i \in I}$, $\{B_i\}_{i \in I}$, and $\{C_i\}_{i \in I}$ be families of objects in $\mathcal{C}$, and let $\{f_i : A_i \times B_i \to C_i\}_{i \in I}$ be a family of morphisms in $\mathcal{C}$. Then there exists a unique morphism $f : (\prod_{i \in I} A_i) \times (\prod_{i \in I} B_i) \to \prod_{i \in I} C_i$ making the following diagram commute for all $j \in I$:*

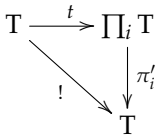


where $\{\sigma_j : \prod_{i \in I} C_i \to C_j\}_{j \in I}$ are the product projections.

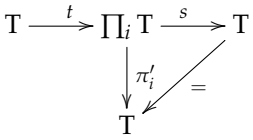
Theorem 6 ([12]). *If T and T' are two terminal objects (or two initial objects) in a category, then T and T' are isomorphic.*

Lemma 6. *Let* T *be a terminal object in $\mathcal{C}$. Let $\prod_{i\in I}$ T denote the product of copies of* T *indexed by I. Then $\prod_{i\in I}$ T $\cong$ T.*

Proof. Since $\prod_{i\in I}$ T is the product of the family $\{\mathrm{T}\}_{i\in I}$, there exists a morphism $t : \mathrm{T} \to \prod_{i\in I}\mathrm{T}$ such that for any $i \in I$, the following diagram commutes:



where π'_i are the projections and ! is the unique morphism from T to T. Since T is terminal, there also exists a unique morphism $s : \prod_{i\in I}\mathrm{T} \to \mathrm{T}$ (the unique morphism to the terminal object). Composing t and s gives, for any $i \in I$:



This shows $s \circ t \circ \pi'_i = \pi'_i$ for all i. However, $\mathrm{id}_{\prod \mathrm{T}} \circ \pi'_i = \pi'_i$. By the uniqueness property of the product, $s \circ t = \mathrm{id}_{\prod \mathrm{T}}$. Similarly, consider the composite $t \circ s : \prod \mathrm{T} \to \prod \mathrm{T}$. For any $i \in I$:

$$\pi'_i \circ (t \circ s) = (\pi'_i \circ t) \circ s = ! \circ s = ! = \pi'_i \circ \mathrm{id}_{\prod \mathrm{T}}.$$

Again, by uniqueness, $t \circ s = \mathrm{id}_{\prod \mathrm{T}}$. Therefore, $s \circ t = \mathrm{id}_{\prod \mathrm{T}}$ and $t \circ s = \mathrm{id}_{\mathrm{T}}$, proving $\prod_{i\in I}\mathrm{T} \cong \mathrm{T}$. □

3. Completeness of $\mathbf{Grp}(\mathcal{C})$

We establish fundamental results on the completeness of categories of internal group objects. Throughout, $\mathcal{C}$ denotes a Cartesian category (i.e., with finite products).

The following theorem is fundamental, as it ensures that products of internal group objects exist and are constructed in the expected way from the underlying products in $\mathcal{C}$.

Theorem 7. *If a category $\mathcal{C}$ is closed under arbitrary products, then the category of internal group objects $\mathbf{Grp}(\mathcal{C})$ is also closed under arbitrary products.*

Proof. Let $\{G_i\}_{i\in I} = \{(C_i, \mu_i, \eta_i, \zeta_i)\}_{i\in I}$ be a family of objects in $\mathbf{Grp}(\mathcal{C})$. Since $\mathcal{C}$ is closed under arbitrary products, the product $\prod_{i\in I} C_i$ exists in $\mathcal{C}$ with projections $p_i : \prod_{i\in I} C_i \to C_i$.

Define:

$$\begin{aligned}
\mu &:= \prod \mu_i \circ \delta : (\prod C_i) \times (\prod C_i) \to \prod C_i \\
\eta &:= \prod \eta_i : T \to \prod C_i \\
\zeta &:= \prod \zeta_i : \prod C_i \to \prod C_i
\end{aligned}$$

where $\delta : (\prod C_i) \times (\prod C_i) \xrightarrow{\cong} \prod (C_i \times C_i)$ is the canonical isomorphism from Lemma 4.

Step 1: $G = (\prod C_i, \mu, \eta, \zeta)$ is an internal group object. We verify the group axioms via commutative diagrams.

Associativity: For each $i \in I$, the associativity of G_i gives:

$$\begin{array}{ccc} C_i \times C_i \times C_i & \xrightarrow{\mathrm{id}\times\mu_i} & C_i \times C_i \\ \downarrow{\scriptstyle \mu_i\times\mathrm{id}} & & \downarrow{\scriptstyle \mu_i} \\ C_i \times C_i & \xrightarrow[\mu_i]{} & C_i \end{array}$$

Taking products and applying Lemmas 4 and 5 yields:

$$\begin{array}{ccc} \prod(C_i \times C_i \times C_i) & \xrightarrow{\prod(\mathrm{id}\times\mu_i)} & \prod(C_i \times C_i) \\ \downarrow{\scriptstyle \prod(\mu_i\times\mathrm{id})} & & \downarrow{\scriptstyle \prod\mu_i} \\ \prod(C_i \times C_i) & \xrightarrow[\prod\mu_i]{} & \prod C_i \end{array}$$

Using the isomorphism $\prod(C_i \times C_i \times C_i) \cong (\prod C_i) \times (\prod C_i) \times (\prod C_i)$, this induces the commutative diagram for μ:

$$\begin{array}{ccc} (\prod C_i) \times (\prod C_i) \times (\prod C_i) & \xrightarrow{\mathrm{id}\times\mu} & (\prod C_i) \times (\prod C_i) \\ \downarrow{\scriptstyle \mu\times\mathrm{id}} & & \downarrow{\scriptstyle \mu} \\ (\prod C_i) \times (\prod C_i) & \xrightarrow[\mu]{} & \prod C_i \end{array}$$

Unit Laws: For each $i \in I$, the unit axiom for G_i gives:

$$\begin{array}{ccc} T \times C_i & \xrightarrow{\eta_i\times\mathrm{id}} & C_i \times C_i \\ & {\scriptstyle \pi_2}\searrow & \downarrow{\scriptstyle \mu_i} \\ & & C_i \end{array}$$

Taking products and applying Lemmas 4 and 6 ($\prod T \cong T$):

$$\begin{array}{ccc} T \times \prod C_i & \xrightarrow{\eta\times\mathrm{id}} & (\prod C_i) \times (\prod C_i) \\ & {\scriptstyle \pi_2}\searrow & \downarrow{\scriptstyle \mu} \\ & & \prod C_i \end{array}$$

Inverse Laws: Similarly, for each $i \in I$:

$$\begin{array}{ccc} C_i & \xrightarrow{(\mathrm{id},\zeta_i)} & C_i \times C_i \\ \downarrow & & \downarrow{\scriptstyle \mu_i} \\ T & \xrightarrow[\eta_i]{} & C_i \end{array}$$

Taking products yields:

$$\begin{array}{ccc} \prod C_i & \xrightarrow{(\mathrm{id},\zeta)} & (\prod C_i) \times (\prod C_i) \\ \downarrow & & \downarrow{\scriptstyle \mu} \\ T & \xrightarrow[\eta]{} & \prod C_i \end{array}$$

Thus $G \in \mathrm{Obj}(\mathbf{Grp}(\mathcal{C}))$.

Step 2: Projections $\{p_i\}$ are internal group homomorphisms. We verify the homomorphism conditions.

Multiplication Preservation:

$$\begin{array}{ccc} (\prod C_i) \times (\prod C_i) & \xrightarrow{\mu} & \prod C_i \\ \downarrow{\scriptstyle p_i \times p_i} & & \downarrow{\scriptstyle p_i} \\ C_i \times C_i & \xrightarrow[\mu_i]{} & C_i \end{array}$$

Commutativity follows from:

$$p_i \circ \mu = p_i \circ \left(\prod \mu_j \circ \delta\right) = \mu_i \circ (p_i \times p_i)$$

Unit Preservation:

$$\begin{array}{ccc} T & \xrightarrow{\eta} & \prod C_i \\ \downarrow{\scriptstyle \mathrm{id}_T} & & \downarrow{\scriptstyle p_i} \\ T & \xrightarrow[\eta_i]{} & C_i \end{array}$$

Commutativity holds as $p_i \circ \eta = p_i \circ \prod \eta_j = \eta_i$.

Inverse Preservation:

$$\begin{array}{ccc} \prod C_i & \xrightarrow{\zeta} & \prod C_i \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle p_i} \\ C_i & \xrightarrow[\zeta_i]{} & C_i \end{array}$$

Commutativity holds as $p_i \circ \zeta = p_i \circ \prod \zeta_j = \zeta_i \circ p_i$.

Step 3: Universal Property. Let $B = (B, \mu_B, \eta_B, \zeta_B) \in \mathrm{Obj}(\mathbf{Grp}(\mathcal{C}))$ and $\{g_i : B \to C_i\}_{i \in I}$ be internal group homomorphisms. Since $\{p_i : \prod C_i \to C_i\}$ is a product in $\mathcal{C}$, there exists a unique morphism $r : B \to \prod C_i$ in $\mathcal{C}$ with $p_i \circ r = g_i$ for all i. We show that r is an internal group homomorphism.

Multiplication Preservation:

$$\begin{array}{ccc} B \times B & \xrightarrow{r \times r} & (\prod C_i) \times (\prod C_i) \\ \downarrow{\scriptstyle \mu_B} & & \downarrow{\scriptstyle \mu} \\ B & \xrightarrow[r]{} & \prod C_i \end{array}$$

For each $i \in I$:

$$\begin{aligned} p_i \circ r \circ \mu_B &= g_i \circ \mu_B \\ &= \mu_i \circ (g_i \times g_i) && \text{(since } g_i \text{ is hom)} \\ &= \mu_i \circ (p_i \times p_i) \circ (r \times r) \\ &= p_i \circ \mu \circ (r \times r) \end{aligned}$$

Thus $r \circ \mu_B = \mu \circ (r \times r)$.

Unit Preservation:

$$\begin{array}{ccc} T & \xrightarrow{\eta_B} & B \\ {\scriptstyle \mathrm{id}_T}\downarrow & & \downarrow{\scriptstyle r} \\ T & \xrightarrow[\eta]{} & \prod C_i \end{array}$$

For each $i \in I$:

$$\begin{aligned} p_i \circ r \circ \eta_B &= g_i \circ \eta_B \\ &= \eta_i && \text{(since } g_i \text{ is hom)} \\ &= p_i \circ \eta \end{aligned}$$

Thus $r \circ \eta_B = \eta$.

Inverse Preservation:

$$\begin{array}{ccc} B & \xrightarrow{\zeta_B} & B \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\ \prod C_i & \xrightarrow[\zeta]{} & \prod C_i \end{array}$$

For each $i \in I$:

$$\begin{aligned} p_i \circ r \circ \zeta_B &= g_i \circ \zeta_B \\ &= \zeta_i \circ g_i && \text{(since } g_i \text{ is hom)} \\ &= \zeta_i \circ p_i \circ\ r \\ &= p_i \circ \zeta \circ r \end{aligned}$$

Thus $r \circ \zeta_B = \zeta \circ r$.

Therefore r is an internal group homomorphism, and $\{p_i : \prod C_i \to C_i\}$ is a product in $\mathbf{Grp}(\mathcal{C})$. □

Definition 6 ([12]). *Let $\mathcal{C}$ be a category and let $f, g : A \rightharpoonup B$ be a pair of parallel morphisms. An equalizer of f and g is an object $E \in \mathcal{C}$ together with a morphism $e : E \to A$ satisfying the following properties:*

1. $f \circ e = g \circ e$.

2. For any object $X \in \mathcal{C}$ and a morphism $h : X \to A$ such that $f \circ h = g \circ h$, there exists a unique morphism $u : X \to E$ such that $e \circ u = h$.

That is, the following diagram is a commutative diagram:

$$\begin{array}{ccccc} X & & \xrightarrow{u} & & E \\ & {\scriptstyle h}\searrow & & \swarrow{\scriptstyle e} & \\ & & A & \underset{g}{\overset{f}{\rightrightarrows}} & B \end{array} \tag{1}$$

The equalizer E represents the universal object corresponding to the pair of morphisms f and g within the category $\mathcal{C}$.

Theorem 8. *If every pair of parallel morphisms in a category $\mathcal{C}$ has an equalizer, then every pair of parallel morphisms in the category of internal group objects $\mathbf{Grp}(\mathcal{C})$ also has an equalizer.*

Proof. Let $\mathcal{C}$ be a category where every pair of parallel morphisms $f, g : A \to B$ has an equalizer, defined as a subobject $E \rightarrowtail A$ that equalizers f and g while satisfying universal properties for equivalence relations.

Consider two parallel morphisms $\alpha, \beta : G \to H$ in $\mathbf{Grp}(\mathcal{C})$, where $G = (G, \mu_G, \eta_G, \zeta_G)$ and $H = (H, \mu_H, \eta_H, \zeta_H)$ are internal group objects. Since $\mathcal{C}$ has equalizers for parallel pairs, the underlying morphisms $\alpha, \beta : G \to H$ in $\mathcal{C}$ admit an equalizer $e : E \rightarrowtail G$. We will

1. Equip E with internal group structure;
2. Show that $e : E \to G$ is a group homomorphism;
3. Verify e is an equivalence subunit for α, β in $\mathbf{Grp}(\mathcal{C})$.

Step 1: Group structure on E. Define $\mu_E : E \times E \to E$ as the unique morphism satisfying:

$$\begin{array}{ccc} E \times E & \xrightarrow{\mu_E} & E \\ {\scriptstyle e\times e}\downarrow & & \downarrow{\scriptstyle e} \\ G \times G & \xrightarrow[\mu_G]{} & G \end{array}$$

Existence follows because $\mu_G \circ (e \times e)$ equalizes α and β:

$$\alpha \circ \mu_G \circ (e \times e) = \mu_H \circ (\alpha \times \alpha) \circ (e \times e) = \mu_H \circ (\beta \times \beta) \circ (e \times e) = \beta \circ \mu_G \circ (e \times e)$$

and the universal property of e gives μ_E. The unit $\eta_E : T \to E$ and inverse $\zeta_E : E \to E$ are defined similarly:

$$\begin{array}{ccc} T & \xrightarrow{\eta_E} & E \\ \downarrow & & \downarrow{\scriptstyle e} \\ T & \xrightarrow[\eta_G]{} & G \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{\zeta_E} & E \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e} \\ G & \xrightarrow[\zeta_G]{} & G \end{array}$$

Commutativity holds because $\alpha \circ \eta_G = \eta_H = \beta \circ \eta_G$ and $\alpha \circ \zeta_G = \zeta_H \circ \alpha = \zeta_H \circ \beta = \beta \circ \zeta_G$. Group axioms for E follow from the monicity of e and commutativity of:

$$\begin{array}{ccc} E \times E \times E & \xrightarrow{\mathrm{id}\times\mu_E} & E \times E \\ {\scriptstyle \mu_E\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mu_E} \\ E \times E & \xrightarrow[\mu_E]{} & E \end{array} \qquad \begin{array}{ccccc} T \times E & \xrightarrow{\eta_E\times\mathrm{id}} & E \times E & \xleftarrow{\mathrm{id}\times\eta_E} & E \times T \\ & {\scriptstyle \pi_2}\searrow & \downarrow{\scriptstyle \mu_E} & \swarrow{\scriptstyle \pi_1} & \\ & & E & & \end{array}$$

$$\begin{array}{ccccc} E & \xrightarrow{(\mathrm{id},\zeta_E)} & E \times E & \xleftarrow{(\zeta_E,\mathrm{id})} & E \\ \downarrow & & \downarrow{\scriptstyle \mu_E} & & \downarrow \\ T & \xrightarrow[\eta_E]{} & E & \xleftarrow[\eta_E]{} & T \end{array}$$

Step 2: e is a group homomorphism. This follows directly from the definitions:

$$e \circ \mu_E = \mu_G \circ (e \times e), \quad e \circ \eta_E = \eta_G, \quad e \circ \zeta_E = \zeta_G \circ e$$

Step 3: e is equalizer for α, β in $\mathbf{Grp}(\mathcal{C})$. Let $\gamma : K \to G$ be a group homomorphism with $\alpha \circ \gamma = \beta \circ \gamma$. In $\mathcal{C}$, there exists unique $u : K \to E$ with $e \circ u = \gamma$. We show u is a group homomorphism. For multiplication:

$$\begin{array}{ccc} K \times K & \xrightarrow{u\times u} & E \times E \\ {\scriptstyle \mu_K}\downarrow & & \downarrow{\scriptstyle \mu_E} \\ K & \xrightarrow[u]{} & E \end{array}$$

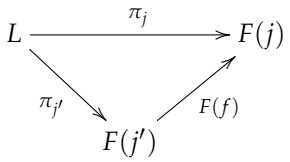
Commutativity holds because:

$$e \circ u \circ \mu_K = \gamma \circ \mu_K = \mu_G \circ (\gamma \times \gamma) = \mu_G \circ (e \times e) \circ (u \times u) = e \circ \mu_E \circ (u \times u)$$

and e is monic. Similarly, $u \circ \eta_K = \eta_E$ and $u \circ \zeta_K = \zeta_E \circ u$. Thus e satisfies the universal property in $\mathbf{Grp}(\mathcal{C})$. □

Definition 7 ([12]). *Let $\mathcal{C}$ be a category and let $F : J \to \mathcal{C}$ be a functor, where J is a small category. A limit of the functor F is an object $L \in \mathcal{C}$ together with a collection of morphisms $\pi_j : L \to F(j)$ for each object $j \in J$ such that*

1. For every morphism $f : j' \to j$ in J, the following diagram commutes:



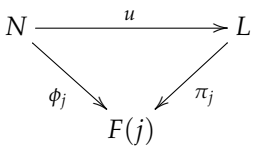
This means that the following equality holds:

$$\pi_j = F(f) \circ \pi_{j'}$$

for the morphisms $\pi_j : L \to F(j)$ and $\pi_{j'} : L \to F(j')$.

2. For any other object $N \in \mathcal{C}$ with morphisms $\phi_j : N \to F(j)$ for every $j \in J$ that satisfy the above commutativity condition, there exists a unique morphism $u : N \to L$ such that $\pi_j \circ u = \phi_j$ for all $j \in J$.

The following diagram illustrates this situation:



Definition 8 (Complete Category [12]). *If a category is closed under any limit, then it is called a complete category.*

Lemma 7 ([12]). *A category $\mathcal{C}$ is complete if and only if*

(i) Every family of objects in $\mathcal{C}$ admits a product;
(ii) Every pair of parallel morphisms $f, g : A \to B$ in $\mathcal{C}$ admits an equalizer.

Theorem 9 (Completeness of Internal Group Categories). *Let $\mathcal{C}$ be a complete category. The category $\mathbf{Grp}(\mathcal{C})$ of internal group objects in $\mathcal{C}$ is complete, and its limits are created and preserved by the forgetful functor*

$$\mathcal{F} : \mathbf{Grp}(\mathcal{C}) \to \mathcal{C}.$$

More precisely, for any diagram $D : \mathcal{J} \to \mathbf{Grp}(\mathcal{C})$ (where $\mathcal{J}$ is a small category), its limit $\lim D$ exists in $\mathbf{Grp}(\mathcal{C})$; the underlying object of $\lim D$ in $\mathcal{C}$ is the limit of the composite diagram $\mathcal{F} \circ D : \mathcal{J} \to \mathcal{C}$; and the group operations (multiplication, unit, and inversion) on $\lim D$ are uniquely determined by their componentwise expressions in $\mathcal{C}$.

Proof. Assume that the category $\mathcal{C}$ is complete. According to Lemma 7, we know that

1. Every family of objects in $\mathcal{C}$ admits a product; 2. Every pair of parallel morphisms $f, g : A \to B$ in $\mathcal{C}$ admits an equalizer.

We need to show that the category of internal group objects $\mathbf{Grp}(\mathcal{C})$ also satisfies these two conditions, thus proving that it is complete.

Let $\{G_i\}_{i\in I} = \{(C_i, \mu_i, \eta_i, \zeta_i)\}_{i\in I}$ be a family of objects in $\mathbf{Grp}(\mathcal{C})$, where each G_i is an internal group object with underlying object C_i, the product morphism $\mu_i : C_i \times C_i \to C_i$, the unit morphism $\eta_i : \mathrm{T} \to C_i$, and the inverse morphism $\zeta_i : C_i \to C_i$.

By Theorem 7, if $\mathcal{C}$ is closed under arbitrary products, then the category of internal group objects $\mathbf{Grp}(\mathcal{C})$ is also closed under arbitrary products.

Thus, we can construct the product of the internal group object:

$$\prod_{i\in I} G_i = \left(\prod_{i\in I} C_i, \mu, \eta, \zeta\right)$$

This product object is itself an internal group object in $\mathbf{Grp}(\mathcal{C})$, thus satisfying the closure under products.

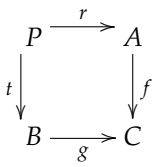
Let $f, g : G \to H$ be two parallel morphisms between internal group objects $G = (C_G, \mu_G, \eta_G, \zeta_G)$ and $H = (C_H, \mu_H, \eta_H, \zeta_H)$ in $\mathbf{Grp}(\mathcal{C})$. By Theorem 8, if every pair of parallel morphisms in the category $\mathcal{C}$ has an equalizer, then every pair of parallel morphisms in the category of internal group objects $\mathbf{Grp}(\mathcal{C})$ also has an equalizer.

Thus, there exists an equalizer object E and a morphism $e : E \to G$ such that $f \circ e = g \circ e$. Moreover, the object E can be endowed with an internal group structure, making E an internal group object.

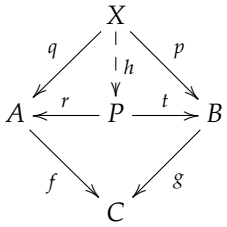
Since we have shown that $\mathbf{Grp}(\mathcal{C})$ admits arbitrary products and equalizers, according to the conditions in Lemma 7, we conclude that $\mathbf{Grp}(\mathcal{C})$ is a complete category. □

Remark 1. *Theorem 9 indicates that the category of internal group objects* $\mathbf{Grp}(\mathcal{C})$ *inherits the completeness of its base category* $\mathcal{C}$.

Definition 9 ([12]). *Let* $\mathcal{C}$ *be a category. Given two morphisms* $f : A \to C$ *and* $g : B \to C$ *in* $\mathcal{C}$, *the pullback of* f *and* g *is an object* P *in* $\mathcal{C}$ *together with two morphisms* $r : P \to A$ *and* $t : P \to B$ *such that the following diagram commutes:*



Moreover, P *satisfies the universal property that for any object* X *in* $\mathcal{C}$ *with morphisms* $q : X \to A$ *and* $p : X \to B$ *such that* $f \circ q = g \circ p$, *there exists a unique morphism* $h : X \to P$ *making the following diagram commute:*



Because pullbacks are a special type of limit, we can derive the following theorem based on Theorem 9.

Theorem 10. *If the category* $\mathcal{C}$ *has the property of pullbacks, then the category of internal group objects* $\mathbf{Grp}(\mathcal{C})$ *in* $\mathcal{C}$ *also has the property of pullbacks.*

Thus, we obtain the main results of this paper.

Theorem 11. *Let $\mathcal{C}$ be a category with finite products. Then the category of internal group objects $\mathbf{Grp}(\mathcal{C})$ has the following hereditary properties:*

1. If the category $\mathcal{C}$ has equalizers, then the category of internal group objects $\mathbf{Grp}(\mathcal{C})$ also has equalizers.

2. If the category $\mathcal{C}$ has finite products, then the category of internal group objects $\mathbf{Grp}(\mathcal{C})$ also has finite products.

3. If the category $\mathcal{C}$ has arbitrary products, then the category of internal group objects $\mathbf{Grp}(\mathcal{C})$ also has arbitrary products.

4. If the category $\mathcal{C}$ has pullbacks, then the category of internal group objects $\mathbf{Grp}(\mathcal{C})$ also has pullbacks.

5. If the category $\mathcal{C}$ has limits, then the category of internal group objects $\mathbf{Grp}(\mathcal{C})$ also has limits.

Theorem 12 (Completeness Inheritance)**.** *If $\mathcal{C}$ is complete, then $\mathbf{Grp}(\mathcal{C})$ is complete. Moreover, the forgetful functor $U : \mathbf{Grp}(\mathcal{C}) \to \mathcal{C}$ creates limits.*

Proof. 1. $\mathcal{C}$ complete $\implies$ has all products and equalizers

2. Theorem 11 implies $\mathbf{Grp}(\mathcal{C})$ has all products and equalizers.

3. By the standard limit construction [1], any limit can be built from products and equalizers:

$$\lim D \cong \operatorname{coeq}\left(\prod_{u:i\to j} D_i \rightrightarrows \prod_k D_k\right)$$

for diagram $D : \mathscr{J} \to \mathbf{Grp}(\mathcal{C})$.

4. U creates limits: Given a diagram $D : \mathscr{J} \to \mathbf{Grp}(\mathcal{C})$ and limit cone $L \to U \circ D$ in $\mathcal{C}$, equip L with group structure via the limit properties. For multiplication $\mu_L : L \times L \to L$, define componentwise as $\pi_j \circ \mu_L = \mu_{D_j} \circ (\pi_j \times \pi_j)$ for each $j \in \mathscr{J}$. The universal property ensures compatibility. □

Definition 10. *A weak product of objects $\{X_i\}$ is an object P with morphisms $\pi_i : P \to X_i$ such that for any Y and morphisms $f_i : Y \to X_i$, there exist a morphism $u : Y \to P$ and a retraction $r : P \to P$ with $\pi_i \circ r \circ u = f_i$ for all i.*

Theorem 13 (Finite Completeness for Weak Cartesian Structures)**.** *Let $\mathcal{C}$ have weak finite products. Then,*

1. *If $\mathcal{C}$ has equalizers, then $\mathbf{Grp}(\mathcal{C})$ has equalizers;*
2. *If $\mathcal{C}$ has weak pullbacks and equalizers, then $\mathbf{Grp}(\mathcal{C})$ has pullbacks.*

Proof. (1) For parallel homomorphisms $\alpha, \beta : G \to H$

1. Take equalizer $e : E \to U(G)$ in $\mathcal{C}$ as in Theorem 8;

2. Define multiplication $\mu_E : E \times E \to E$ as follows: first form the weak product P of $E \times E$ with itself, with retraction $r : P \to P$ and morphism $u : E \times E \to P$ such that $\pi \circ r \circ u = \mu_G \circ (e \times e)$;

3. Set $\mu_E = r \circ u$. Verification:

$$e \circ \mu_E \circ (\mu_E \times \mathrm{id}) = \mu_G \circ (e \times e) \circ (r \times r) \circ (u \times \mathrm{id}) = \mu_G \circ (\mu_G \times \mathrm{id}) \circ (e \times e \times e)$$

Similarly for $\mu_E \circ (\mathrm{id} \times \mu_E)$, and associativity follows by the monicity of e.

(2) For cospan $G \xrightarrow{f} H \xleftarrow{g} K$ in $\mathbf{Grp}(\mathcal{C})$

1. Form weak pullback P in $\mathcal{C}$ with retraction $r : P \to P$:

$$\begin{array}{ccc} P & \xrightarrow{p_1} & U(G) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle U(f)} \\ U(K) & \xrightarrow[U(g)]{} & U(H) \end{array}$$

2. Take equalizer $e : E \to P$ of $p_1 \circ r, p_2 \circ r$;
3. Equip E with group structure: Define $\mu_E : E \times E \to E$ via:

$$\mu_E = e^{-1} \circ \mu_P \circ (e \times e)$$

where μ_P is induced via retraction. The verification of the universal property uses the weak limit property. □

Corollary 2. **TopGrp** $\cong$ **Grp**(**Top**) *is complete.*

Proof. **Top** is completed ([2], Theorem 3.2.4) by applying Theorem 12. □

4. Applications to Concrete Categories

Applying Theorem 11 to fundamental structures, we establish completeness for key algebraic categories: (1) The category of groups **Grp** $\cong$ **Grp**(**Set**) is complete since **Set** is complete; (2) The category of topological groups **TopGrp** $\cong$ **Grp**(**Top**) is complete as **Top** is complete; (3) The category of ordered groups **OrdGrp** $\cong$ **Grp**(**Ord**) is complete given **Ord**'s completeness. In each case, the equivalence to **Grp**($\mathcal{C}$) for the complete base category $\mathcal{C}$ combined with the theorem yields completeness, where limits lift canonically from $\mathcal{C}$ with the inherited algebraic structure.

Definition 11 (Sheaf of Groups [13,14])**.** *Let (X, τ) be a topological space (or a site). A* sheaf of groups *on X is a sheaf $\mathcal{G} : \tau\text{op} \to$ **Set** such that*

1. *$\mathcal{G}(U)$ is a group for each open $U \subseteq X$;*
2. *Restriction maps $\text{res}_{U,V} : \mathcal{G}(U) \to \mathcal{G}(V)$ for $V \subseteq U$ are group homomorphisms;*
3. *$\mathcal{G}$ satisfies the sheaf axioms (locality and gluing).*

A morphism $\phi : \mathcal{G} \to \mathcal{H}$ is a morphism of sheaves such that $\phi_U : \mathcal{G}(U) \to \mathcal{H}(U)$ is a group homomorphism for each U. The category is denoted **Sh**$(X,$ **Grp**$)$.

Theorem 14. *The category* **Sh**$(X,$ **Grp**$)$ *is isomorphic to* **Grp**(**Sh**(X))*, where* **Sh**(X) *is the category of sheaves of sets on X.*

Proof. A group object in **Sh**(X) is a sheaf of sets $\mathcal{G}$ equipped with sheaf morphisms $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $\eta : * \to \mathcal{G}$ (where $*$ is the terminal sheaf, $*(U) = \{*\}$), $\zeta : \mathcal{G} \to \mathcal{G}$ satisfying the group axioms internally. This means that for each U, $\mathcal{G}(U)$ is a group, and the restriction maps are homomorphisms, i.e., $\mathcal{G}$ is a sheaf of groups ([13], II.1) and the morphisms coincide. □

Theorem 15. *The category* **Sh**$(X,$ **Grp**$)$ *is complete.*

Proof. The category **Sh**(X) of sheaves of sets on X is complete (limits are computed sectionwise: $(\lim \mathcal{F}_j)(U) = \lim(\mathcal{F}_j(U))$ in **Set**) [14]. By **Sh**$(X,$ **Grp**$) \cong$ **Grp**(**Sh**(X)). Apply Theorem 11. The limits are created sectionwise: The underlying sheaf of sets of the

limit in $\mathbf{Sh}(X, \mathbf{Grp})$ is the limit of the underlying sheaves, equipped with the pointwise group structure. □

Definition 12 (Smooth Manifold [15]). *An* n-dimensional smooth manifold *is a Hausdorff, second-countable topological space* M *equipped with a smooth atlas: An open cover* $\{U_i\}$ *and homeomorphisms* $\varphi_i : U_i \to \mathbb{R}^n$ *such that transition maps* $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ *are smooth* (C^∞). *Smooth maps are continuous maps whose local coordinate expressions are smooth. The category of* n*-dimensional smooth manifolds and smooth maps is denoted* $\mathbf{Man}^n$.

Definition 13 (Lie Group [16]). *A* Lie group *is a smooth manifold* G *that is also a group, such that multiplication* $\mu : G \times G \to G$, $(g, h) \mapsto gh$ *and inversion* $\zeta : G \to G$, $g \mapsto g^{-1}$ *are smooth maps. A* homomorphism *is a smooth group homomorphism. The category is* **LieGrp**.

Theorem 16. *The category* **LieGrp** *is isomorphic to* $\mathbf{Grp}(\mathbf{Man}^n)$.

Proof. An internal group object in $\mathbf{Man}^n$ is a smooth manifold G equipped with smooth morphisms $\mu : G \times G \to G$ (multiplication) and $\zeta : G \to G$ (inversion) satisfying the group axioms. This is precisely a Lie group. The homomorphisms coincide. (Note: while $\mathbf{Man}^n$ typically requires fixed dimension n, Lie groups are manifolds of fixed dimension, so we work within $\mathbf{Man}^n$ for appropriate n.) □

Theorem 17. *The category* **LieGrp** *is complete.*

Proof. The category $\mathbf{Man}^n$ is complete: Products $M \times N$ exist (dimension $m + n$) with product atlas; equalizers exist as closed submanifolds where smooth maps agree ([15], I.5.10). By Theorem 16, $\mathbf{LieGrp} \cong \mathbf{Grp}(\mathbf{Man}^n)$. Apply Theorem 12. The limits are formed as in **Top** (for the underlying topology) and **Grp** (for the group structure), with the unique smooth structure making the operations smooth. □

We can define the notion of a compact Hausdorff topological space group as a group object in **CompHaus** and a commutative Hopf C^*-algebra as a group object in $\mathbf{ComUnC^*Alg^{op}}$.

Definition 14 ([11]). *Let* $\mathcal{C}$ *be a category with finite coproducts. We define the category of cogroup objects in* $\mathcal{C}$ *as follows:*

$$\mathbf{CoGrp}(\mathcal{C}) := \mathbf{Grp}(\mathcal{C}^{op}),$$

where **Grp** *is the category of groups and* $\mathcal{C}^{op}$ *is the opposite category of* $\mathcal{C}$.

The objects of $\mathbf{CoGrp}(\mathcal{C})$ *are called cogroup objects in* $\mathcal{C}$, *and the morphisms between these objects are referred to as internal cogroup homomorphisms in* $\mathcal{C}$.

Definition 15 ([11]). *Let* $\mathcal{C}$ *be a category with finite coproducts. We define*

$$\mathbf{CoGrp}(\mathcal{C}) := \mathbf{Grp}\,(\mathcal{C}^{op})^{op},$$

and call the objects of $\mathbf{CoGrp}(\mathcal{C})$ *cogroup objects in* $\mathcal{C}$ *and the morphisms internal cogroup homomorphisms in* $\mathcal{C}$.

Definition 16 ([11]). *The objects of the category*

$$\mathbf{compTopGrp} := \mathbf{Grp}(\mathbf{compHaus})$$

are called compact topological groups. The objects of

$$\mathbf{comHopf}\,\mathcal{C}^*-\mathbf{Alg} := \mathbf{CoGrp}(\mathbf{comUn}\mathcal{C}^*-\mathbf{Alg})$$

are called commutative Hopf $\mathcal{C}^*$-algebras.

Theorem 18 ([11]). *There is an equivalence of categories:*

$$\mathbf{compTopGrp} = \mathbf{Grp}(\mathbf{compHaus}) \simeq \mathbf{Grp}\,\mathbf{comUn}\mathcal{C}^*-\mathbf{Alg}^{\mathbf{op}} = \mathbf{comHopf}\,\mathcal{C}^*-\mathbf{Alg}^{op}$$

provided by the functors

$$\mathbf{Grp}\,\mathcal{C}(\cdot) : \mathbf{compTopGrp} \to \mathbf{comHopf}\,\mathcal{C}^*-\mathbf{Alg}^{op}$$

and

$$\mathbf{Grp}\,\sigma(\cdot) : \mathbf{comHopf}\,\mathcal{C}^*-\mathbf{Alg}^{\mathbf{op}} \to \mathbf{compTopGrp}.$$

Theorem 19. *The category of compact topological groups* **compTopGrp** *is a complete category.*

Theorem 20. *The category of commutative Hopf $\mathcal{C}^*$-algebras* **comHopf** $\mathcal{C}^*-$**Alg** *is a complete category.*

These results illustrate the broad applicability of our completeness theorems. They show that categories of compact topological groups and commutative Hopf $\mathcal{C}^*$-algebras—which arise naturally in harmonic analysis and quantum group theory—are complete, and that their completeness follows from the general framework developed in Section 3.

5. Cocompleteness of $\mathbf{Grp}(\mathcal{C})$

We establish fundamental results on the cocompleteness of categories of internal group objects. The following theorem provides necessary and sufficient conditions for $\mathbf{Grp}(\mathcal{C})$ to inherit cocompleteness from $\mathcal{C}$.

The following is one of the main results of this paper, providing a constructive and sufficient condition for cocompleteness of $\mathbf{Grp}(\mathcal{C})$.

Theorem 21. *Let $\mathcal{C}$ be cocomplete, regular, and admit a free group functor $F : \mathcal{C} \to \mathbf{Grp}(\mathcal{C})$. Then $\mathbf{Grp}(\mathcal{C})$ is cocomplete.*

Proof. Since cocompleteness requires the existence of coproducts and coequalizers ([1], V.2), we construct these explicitly.

Coproducts: Let $\{G_i = (X_i, \mu_i, \eta_i, \zeta_i)\}_{i\in I}$ be a family in $\mathbf{Grp}(\mathcal{C})$ and

1. Form the coproduct $Q = \coprod_{i\in I} X_i$ in $\mathcal{C}$ with inclusions $\iota_j : X_j \to Q$;
2. Apply the free group functor to get $F(Q) \in \mathbf{Grp}(\mathcal{C})$;
3. Define relations:

$$R = \mathrm{coeq}\left(\coprod_{i,j} F(X_i \amalg X_j) \rightrightarrows F(Q)\right)$$

where the parallel arrows are:

$$\alpha : \quad F(X_i \amalg X_j) \xrightarrow{F(\iota_i \amalg \iota_j)} F(Q)$$

$$\beta : \quad F(X_i \amalg X_j) \xrightarrow{F(\mu_{ij})} F(X_k) \xrightarrow{F(\iota_k)} F(Q)$$

with $\mu_{ij} : X_i \amalg X_j \to X_k$ defined as $\mu_k \circ (\mathrm{id} \times \mathrm{id})$ when $i = j = k$, and zero otherwise.

4. The coproduct in $\mathbf{Grp}(\mathcal{C})$ is $\coprod^{\mathbf{Grp}} G_i = R$.

- *Universal property*: For any $H \in \mathbf{Grp}(\mathcal{C})$ with homomorphisms $f_i : G_i \to H$, we get $u : Q \to U(H)$ in $\mathcal{C}$, then $\tilde{u} : F(Q) \to H$ in $\mathbf{Grp}(\mathcal{C})$. Since $\tilde{u} \circ \alpha = \tilde{u} \circ \beta$ (by group axioms), it factors through R.
- *Group structure*: The group structure is well-defined because the relations enforce:

$$[(x \cdot y)_k] = [x_i] \cdot [y_j] \quad \text{for } i, j \to k,$$

and associativity follows from congruence closure.

Coequalizers: Let $\alpha, \beta : G \to H$ be parallel homomorphisms and

1. Form the coequalizer $e : U(H) \to E$ of $U(\alpha), U(\beta)$ in $\mathcal{C}$. Since $\mathcal{C}$ is regular, e is regular epi;

2. Take kernel pair $p_1, p_2 : K \rightrightarrows U(H)$ of e;

3. Construct:

$$\mathrm{coeq}(\alpha, \beta) = \mathrm{coeq}(F(K) \rightrightarrows H)$$

where the parallel arrows are induced by:

$$F(K) \longrightarrow\!\!\!\!\!\rightarrow H$$

via the adjunction $F \dashv U$. Specifically, for each $k \in K$, we have $F(p_1(k))$ and $F(p_2(k))$ acting on H.

4. Verification is as follows:

- *Universal property*: For any $\gamma : H \to L$ with $\gamma \circ \alpha = \gamma \circ \beta$, we get unique $v : E \to U(L)$ in $\mathcal{C}$, then $\tilde{v} : F(E) \to L$. The relation $F(K) \rightrightarrows H$ is coequalized since e is coequalizer.
- *Effectiveness*: The congruence relation is effective because $\mathcal{C}$ is regular, ensuring group axioms descend.

General colimits are constructed from coproducts and coequalizers via the standard formula:

$$\varinjlim D \cong \mathrm{coeq}\left(\coprod_{u:i \to j} D_i \rightrightarrows \coprod_k D_k \right)$$

for any diagram $D : \mathscr{J} \to \mathbf{Grp}(\mathcal{C})$. □

Definition 17. *A dense generator in $\mathcal{C}$ is a set $\mathscr{G} \subseteq \mathrm{ob}(\mathcal{C})$ such that:*

$$\mathrm{Hom}_{\mathcal{C}}(G, -) : \mathcal{C} \to \mathbf{Set} \text{ is faithful for each } G \in \mathscr{G}$$

and every object is a colimit of objects from $\mathscr{G}$.

Theorem 22 (Free Group Existence Criterion)**.** *For $\mathcal{C}$ cocomplete with dense generator $\mathscr{G}$, the free group functor $F : \mathcal{C} \to \mathbf{Grp}(\mathcal{C})$ exists if and only if:*

$$\forall G \in \mathscr{G},\ \mathbf{Grp}(\mathcal{C}) \text{ has free object on } G.$$

Proof. ($\Rightarrow$) Immediate from adjunction.

($\Leftarrow$) For arbitrary $X \in \mathcal{C}$, express X as a colimit of generators:

$$X \cong \varinjlim_{\alpha} G_\alpha \quad \text{with } G_\alpha \in \mathscr{G}$$

1. For each G_α, let $F(G_\alpha)$ be the free group on G_α (exists by hypothesis).

2. Define:

$$F(X) := \varinjlim_{\alpha} F(G_\alpha)$$

in $\mathbf{Grp}(\mathcal{C})$ (cocompleteness is guaranteed by Theorem 21 once F exists).

3. For any $H \in \mathbf{Grp}(\mathcal{C})$, we have natural isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Grp}(\mathcal{C})}(F(X), H) &\cong \mathrm{Hom}_{\mathbf{Grp}(\mathcal{C})}\left(\varinjlim_{\alpha} F(G_\alpha), H\right) \\
&\cong \varprojlim_{\alpha} \mathrm{Hom}_{\mathbf{Grp}(\mathcal{C})}(F(G_\alpha), H) \\
&\cong \varprojlim_{\alpha} \mathrm{Hom}_{\mathcal{C}}(G_\alpha, U(H)) \\
&\cong \mathrm{Hom}_{\mathcal{C}}\left(\varinjlim_{\alpha} G_\alpha, U(H)\right) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(X, U(H))
\end{aligned}$$

where

- Second isomorphism: $\mathbf{Grp}(\mathcal{C})$ has limits (Theorem 12);
- Fourth isomorphism: Continuity of Hom functors;
- Fifth isomorphism: Density of $\mathscr{G}$.

Thus $F \dashv U$. □

Corollary 3. *The following categories are cocomplete:*

1. **TopGrp** *(topological groups);*
2. **OrdGrp** *(ordered groups);*
3. $\mathbf{Sh}(X, \mathbf{Grp})$ *(sheaves of groups).*

Proof. (1) For $\mathcal{C} = \mathbf{Top}$,

- **Top** is cocomplete and regular [2];
- Free topological groups exist [17];
- Apply Theorem 21.

(2) For $\mathcal{C} = \mathbf{Ord}$ (posets with monotone maps),

- **Ord** is cocomplete (coproducts = disjoint unions, coequalizers = quotient posets);
- Regular: Kernel pairs and coequalizers computed in **Set**, stable under pullback;
- Free ordered groups exist [10].

(3) For $\mathcal{C} = \mathbf{Sh}(X)$,

- $\mathbf{Sh}(X)$ is cocomplete and regular [18];
- Free group sheaves: For sheaf $\mathscr{F}$, $F(\mathscr{F})(U) = F(\mathscr{F}(U))$ (free group on sections).

□

Proposition 1. *The category* **LieGrp** *of Lie groups is not cocomplete.*

Proof. 1. Assume **LieGrp** were cocomplete. Then by Theorem 21, the free group functor $F : \mathbf{Man}^n \to \mathbf{LieGrp}$ would exist.

2. Consider $M = S^1 \in \mathbf{Man}^1$. Then $F(S^1)$ must contain the free group $\mathbb{Z} * \mathbb{Z}$ as a dense subgroup.

3. By the Birkhoff transitivity theorem [19], no second-countable Lie group can contain $\mathbb{Z} * \mathbb{Z}$ as a dense subgroup unless discrete.

4. But $F(S^1)$ must be finite-dimensional (Lie group axiom), while $\mathbb{Z} * \mathbb{Z}$ requires infinite dimension for faithful representation.

5. Contradiction. Thus no such F exists, and **LieGrp** is not cocomplete. □

6. Regularity and Internal Hom-Objects

Definition 18. *An effective equivalence relation in $\mathcal{C}$ is a kernel pair $R \rightrightarrows X$ that is the kernel pair of its coequalizer.*

Theorem 23 (Regularity Inheritance)**.** *Let $\mathcal{C}$ be regular with free group functor $F : \mathcal{C} \to \mathbf{Grp}(\mathcal{C})$ preserving finite products. Then $\mathbf{Grp}(\mathcal{C})$ is regular.*

Proof. 1. **Finite limits:** By Theorem 12, $\mathbf{Grp}(\mathcal{C})$ has all finite limits since $\mathcal{C}$ is complete.

2. **Coequalizers of kernel pairs:** Given homomorphism $\phi : G \to H$, form its kernel pair:

$$K \rightrightarrows G \xrightarrow{\phi} H$$

in $\mathbf{Grp}(\mathcal{C})$. Since U creates limits, $U(K) \rightrightarrows U(G)$ is kernel pair of $U(\phi)$ in $\mathcal{C}$.

3. Let $q : U(G) \to Q$ be the coequalizer in $\mathcal{C}$. By regularity, q is regular epi.

4. Define $\overline{Q} = \mathrm{coeq}(F(\ker q) \rightrightarrows G)$ in $\mathbf{Grp}(\mathcal{C})$. This is the coequalizer of the kernel pair.

5. **Stability:** For the pullback square in $\mathbf{Grp}(\mathcal{C})$:

$$\begin{array}{ccc} P & \xrightarrow{f} & G \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \phi} \\ L & \xrightarrow[\psi]{} & H \end{array}$$

apply U to get the pullback in $\mathcal{C}$. Since regular epis are stable in $\mathcal{C}$, the $U(f)$ regular epi implies the f regular epi in $\mathbf{Grp}(\mathcal{C})$. □

Proposition 2 (Exactness Inheritance)**.** *If $\mathcal{C}$ is exact (regular + effective equivalence relations), then $\mathbf{Grp}(\mathcal{C})$ is exact.*

Proof. 1. By Theorem 23, $\mathbf{Grp}(\mathcal{C})$ is regular.

2. Let $R \rightrightarrows G$ be the equivalence relation in $\mathbf{Grp}(\mathcal{C})$. Then $U(R) \rightrightarrows U(G)$ is the equivalence relation in $\mathcal{C}$.

3. Since $\mathcal{C}$ is exact, $U(R)$ is the kernel pair of its coequalizer $q : U(G) \to Q$.

4. Construct $\overline{Q} = \mathrm{coeq}(F(\ker q) \rightrightarrows G)$ as before.

5. **Effectiveness:** The natural map $R \to K = \ker(\phi)$ is iso since

- In $\mathcal{C}$, $U(R) \to U(K)$ is iso by effectiveness in $\mathcal{C}$;
- You can lift to isomorphism in $\mathbf{Grp}(\mathcal{C})$ via the group structure.

Thus R is kernel pair of $\phi : G \to \overline{Q}$. □

Definition 19. *A category $\mathcal{C}$ is locally Cartesian closed (LCCC) if for every morphism $f : X \to Y$, the pullback functor $f^* : \mathcal{C}/Y \to \mathcal{C}/X$ has a right adjoint Π_f.*

Theorem 24 (Internal Hom-Objects)**.** *For $\mathcal{C}$ that is locally Cartesian closed and $G, H \in \mathbf{Grp}(\mathcal{C})$, there exists $\underline{\mathrm{Hom}}(G, H) \in \mathcal{C}$ with:*

$$\mathrm{Hom}_{\mathbf{Grp}(\mathcal{C})}(G, H) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{T}, \underline{\mathrm{Hom}}(G, H))$$

Proof. 1. Define $\underline{\mathrm{Hom}}(G,H)$ as the equalizer:

$$\underline{\mathrm{Hom}}(G,H) \longrightarrow H^G \times H^{G\times G} \times H^{\mathbb{T}} \rightrightarrows H^{G\times G\times G} \times H^G$$

2. The parallel arrows impose:

$$\begin{aligned}
\mu_H \circ (f \times f) &= f \circ \mu_G && \text{(multiplication)}\\
f \circ \eta_G &= \eta_H && \text{(unit)}\\
f \circ \zeta_G &= \zeta_H \circ f && \text{(inverse)}\\
\mu_G \circ (\mu_G \times \mathrm{id}) &= \mu_G \circ (\mathrm{id} \times \mu_G) && \text{(associativity)}\\
\mu_G \circ (\eta_G \times \mathrm{id}) &= \pi_2 && \text{(left unit)}\\
\mu_G \circ (\mathrm{id} \times \zeta_G) &= \eta_G \circ ! && \text{(left inverse)}
\end{aligned}$$

for $f : G \to H$, using the Cartesian closed structure.

3. Universal property: For any $T \in \mathcal{C}$ and group homomorphism $\phi : G \to H$ in $\mathcal{C}/T$, we get unique $\tilde{\phi} : T \to \underline{\mathrm{Hom}}(G,H)$ via the Yoneda lemma. □

Corollary 4 (Tannaka Duality)**.** *When $\mathcal{C}$ is a topos, for $G \in \mathbf{Grp}(\mathcal{C})$, the category $\mathbf{Rep}(G)$ of representations satisfies:*

$$\mathbf{Rep}(G)^{\mathrm{op}} \simeq \mathbf{CoMod}(\mathcal{O}(G))$$

where $\mathcal{O}(G) = \underline{\mathrm{Hom}}(G, \mathbb{A}^1)$ is the internal Hopf algebra.

Proof. 1. Define the fiber functor $\omega : \mathbf{Rep}(G) \to \mathcal{C}$ by $\omega(V) = \underline{\mathrm{Hom}}_G(\mathbf{1}, V)$.

2. $\mathcal{O}(G)$ has coalgebra structure via:

$$\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G), \quad f \mapsto [(g,h) \mapsto f(gh)]$$

3. Apply Barr–Beck monadicity: The functor ω is comonadic with comonad $\mathcal{O}(G) \otimes -$.

4. The equivalence follows from comonadicity theorem ([1], Theorem 4.4.4). □

7. Representation Theory and Duality

Definition 20. *Let G be an internal group object in $\mathcal{C}$. A representation of G is a pair (V,ρ) where*

- $V \in \mathrm{ob}(\mathcal{C})$,
- $\rho : G \times V \to V$ *is a morphism in* $\mathcal{C}$,

satisfying the commutative diagrams:

***Associativity**:*

$$\begin{array}{ccc}
G \times G \times V & \xrightarrow{\mu_G \times \mathrm{id}_V} & G \times V \\
{\scriptstyle \mathrm{id}_G \times \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\
G \times V & \xrightarrow[\rho]{} & V
\end{array}$$

***Unit**:*

$$\begin{array}{ccc}
\mathbb{T} \times V & \xrightarrow{\eta_G \times \mathrm{id}_V} & G \times V \\
& {\scriptstyle \pi_2}\searrow & \downarrow{\scriptstyle \rho} \\
& & V
\end{array}$$

A morphism $\phi : (V,\rho) \to (W,\sigma)$ is a $\mathcal{C}$-morphism $\phi : V \to W$ commuting with the actions. The category $\mathbf{Rep}(G)$ has representations as objects and equivariant morphisms as morphisms.

Theorem 25 (Tannaka–Krein Duality)**.** *When $\mathcal{C}$ is a topos, there is an equivalence of categories:*

$$\mathbf{Rep}(G)^{\mathrm{op}} \simeq \mathbf{CoMod}(\mathcal{O}(G))$$

where $\mathcal{O}(G) = \underline{\mathrm{Hom}}(G, \mathbb{A}^1)$ is the internal Hopf algebra of G, and $\mathbf{CoMod}(\mathcal{O}(G))$ is the category of comodules over $\mathcal{O}(G)$.

Proof. 1. **Construct** $\mathcal{O}(G)$: Since $\mathcal{C}$ is topos, it is locally Cartesian closed. Define:

$$\mathcal{O}(G) = \underline{\mathrm{Hom}}(G, \mathbb{A}^1)$$

where $\mathbb{A}^1$ is the affine line (ring object in $\mathcal{C}$).

2. **Hopf algebra structure**:

- *Comultiplication* $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$:

$$\Delta(f) = [(g, h) \mapsto f(gh)]$$

- *Counit* $\epsilon : \mathcal{O}(G) \to \mathbb{A}^1$:

$$\epsilon(f) = f(e)$$

- *Antipode* $S : \mathcal{O}(G) \to \mathcal{O}(G)$:

$$S(f) = [g \mapsto f(g^{-1})]$$

The verification of Hopf axioms uses group axioms for G.

3. **Functor** $\Phi : \mathbf{Rep}(G)^{\mathrm{op}} \to \mathbf{CoMod}(\mathcal{O}(G))$: For $(V, \rho) \in \mathbf{Rep}(G)$, define coaction $\delta_V : V \to V \otimes \mathcal{O}(G)$ as the adjoint transpose of:

$$G \times V \xrightarrow{\rho} V \xrightarrow{\eta_V} V \otimes \mathbb{A}^1$$

where $\eta_V(v) = v \otimes 1$.

4. **Inverse functor** Ψ: For comodule (W, δ), define action:

$$\rho_W : G \times W \to W, \quad (g, w) \mapsto \mathrm{ev}(\delta(w))(g)$$

where $\mathrm{ev} : \underline{\mathrm{Hom}}(G, \mathbb{A}^1) \times G \to \mathbb{A}^1$ is the evaluation.

5. **Equivalence**: Natural isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{CoMod}}(\Phi(V), (W, \delta)) &\cong \mathrm{Hom}_{\mathbf{Rep}}(W, V) \\ &\cong \mathrm{Hom}_{\mathbf{Rep}(G)^{\mathrm{op}}}((V, \rho), (W, \sigma)) \end{aligned}$$

follow from the Cartesian closed structure and Yoneda lemma. □

8. Homotopical Structures

Definition 21. *A combinatorial model category is a model category that is cofibrantly generated and locally presentable ([7], Definition 1.3).*

Theorem 26 (Model Structure on $\mathbf{Grp}(\mathscr{M})$)**.** *Let $\mathscr{M}$ be combinatorial model category. Then $\mathbf{Grp}(\mathscr{M})$ admits a model structure where*

$$\begin{aligned} f \text{ is weak equivalence} &\iff U(f) \text{ is weak equivalence in } \mathscr{M} \\ f \text{ is fibration} &\iff U(f) \text{ is fibration in } \mathscr{M} \end{aligned}$$

Cofibrations are determined by the left lifting property.

Proof. 1. **Transfer along adjunction**: Consider adjunction $F : \mathscr{M} \rightleftarrows \mathbf{Grp}(\mathscr{M}) : U$.

2. **Cofibrantly generated**: Let $\mathcal{I}$, $\mathcal{J}$ be generating cofibrations and trivial cofibrations in $\mathscr{M}$. Define:

$$\mathcal{I}_{\mathbf{Grp}} = F(\mathcal{I}), \quad \mathcal{J}_{\mathbf{Grp}} = F(\mathcal{J})$$

3. **Model structure axioms**:

- *Lifting*: Given commutative square:

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

with i cofibration, p trivial fibration in $\mathbf{Grp}(\mathscr{M})$. Apply U to get lift in $\mathscr{M}$, then lift to $\mathbf{Grp}(\mathscr{M})$ by the universal property.
- *Factorization*: For $f : G \to H$, factor $U(f) = p \circ i$ in $\mathscr{M}$ with i cofibration, p trivial fibration. Then $f = \tilde{p} \circ F(i)$ where $\tilde{p} : F(U(H)) \to H$ is adjoint transpose.

4. **Right-induced structure**: By ([20], Theorem 4.7), the model structure transfers since U preserves filtered colimits and fibrations. □

Theorem 27 (Homotopy Limit-Completeness)**.** *For combinatorial model category $\mathscr{M}$ with cofibrant replacement functor Q, homotopy limits in $\mathbf{Grp}(\mathscr{M})$ satisfy:*

$$\operatorname{holim} D \cong U^{-1}(\operatorname{holim}_{\mathscr{M}}(U \circ D \circ Q))$$

for any diagram $D : \mathscr{J} \to \mathbf{Grp}(\mathscr{M})$.

Proof. 1. **Cofibrant replacement**: Apply $Q : \mathscr{J} \to \mathbf{Grp}(\mathscr{M})$ to get a diagram where all objects are cofibrant.

2. **Homotopy limit in $\mathbf{Grp}(\mathscr{M})$**:

$$\operatorname{holim}_{\mathbf{Grp}} D = \lim(QD)$$

3. **Preservation by U**:

$$\begin{aligned} U(\operatorname{holim}_{\mathbf{Grp}} D) &= U(\lim(QD)) \\ &\cong \lim(U \circ Q \circ D) \\ &= \operatorname{holim}_{\mathscr{M}}(U \circ D) \end{aligned}$$

since U preserves limits and cofibrant replacements.

4. **Conclusion**: The isomorphism:

$$\operatorname{holim}_{\mathbf{Grp}} D \cong U^{-1}(\operatorname{holim}_{\mathscr{M}}(U \circ D \circ Q))$$

follows from creation of limits by U (Theorem 12). □

9. Higher Categorical Generalizations

Definition 22. *An $(\infty, 1)$-category is a higher category in which all k-morphisms for $k > 1$ are invertible up to homotopy.*

We use the model of quasicategories (also called ∞-categories) as developed by Lurie [5], where a quasicategory is a simplicial set satisfying the weak Kan condition.

Definition 23. *An $\mathbb{E}_1$-algebra in an $(\infty,1)$-category $\mathcal{C}$ is an object A equipped with a coherently associative multiplication $A \otimes A \to A$.*

Definition 24. *An n-group object in $(\infty,1)$-category $\mathcal{C}$ is an n-truncated group-like $\mathbb{E}_1$-algebra, i.e., an $\mathbb{E}_1$-algebra G where:*

$$\pi_k(\mathrm{End}(G)) = 0 \quad \forall k > n$$

and $\pi_0(\mathrm{Aut}(G))$ is a group in the ordinary sense.

Theorem 28 (∞-Completeness of n-Groups)**.** *If $\mathcal{C}$ is ∞-complete (has all small ∞-limits), then the $(\infty,1)$-category n-**Grp**$(\mathcal{C})$ of n-group objects is ∞-complete.*

Proof. 1. **Products**: For the family $\{G_i\}$ of n-groups, form ∞-product $P = \prod^{\infty} G_i$ in $\mathcal{C}$. The multiplication:

$$\mu_P : P \otimes P \simeq \prod^{\infty}(G_i \otimes G_i) \xrightarrow{\prod \mu_i} \prod^{\infty} G_i$$

preserves n-truncation since truncation commutes with limits ([5], Proposition 5.5.6.18).

2. **Equalizers**: For parallel $f, g : G \to H$, the ∞-equalizer $e : E \to G$ is n-truncated. Equip E with multiplication:

$$\begin{array}{ccc} E \otimes E & \xrightarrow{\mu_E} & E \\ \downarrow & & \downarrow e \\ G \otimes G & \xrightarrow[\mu_G]{} & G \end{array}$$

The diagram commutes by naturality, and E remains group-like.

3. **General limits**: Any ∞-limit can be constructed from ∞-products and ∞-equalizers via the ∞-categorical limit formula:

$$\lim^{\infty} D \simeq \ker\left(\prod_i D_i \rightrightarrows \prod_{u:j\to k} D_k\right)$$

in the quasicategory model [5].

4. **Preservation of n-truncation**: The forgetful functor $U : n\text{-}\mathbf{Grp}(\mathcal{C}) \to \mathcal{C}$ preserves ∞-limits, and n-truncation is a right adjoint, hence preserves limits. □

The study of n-group objects in ∞-categories is motivated by higher algebra and derived geometry. For instance, ∞-Lie groups appear as symmetry objects in derived manifolds and higher stacks. Our Theorem 29 shows that the ∞-category of n-Lie groups is not ∞-cocomplete for $n \geq 1$, reflecting fundamental geometric obstructions similar to those in the classical case.

Definition 25. *The $(\infty,1)$-category $\mathbf{Diff}_\infty$ of ∞-manifolds is the ∞-categorical localization of the category of smooth manifolds at weak equivalences.*

Theorem 29 (Obstruction to ∞-Cocompleteness)**.** *For $\mathcal{C} = \mathbf{Diff}_\infty$, the $(\infty,1)$-category n-**Grp**$(\mathcal{C})$ is ∞-cocomplete if and only if $n = 0$.*

Proof. ($\Leftarrow$) For $n = 0$, 0-groups are discrete groups, and **Grp**(**Set**) is cocomplete.

($\Rightarrow$) Assume $n \geq 1$ and n-**Grp**($\mathbf{Diff}_\infty$) is ∞-cocomplete.

1. **Free group existence**: By the ∞-categorical adjoint functor theorem [5], the forgetful functor $U : n\text{-}\mathbf{Grp}(\mathbf{Diff}_\infty) \to \mathbf{Diff}_\infty$ admits a left adjoint F.

2. **Consider** $M = S^1$: The free n-group $F(S^1)$ must contain the free $\mathbb{E}_1$-algebra on S^1 as a subobject.

3. **Topological obstruction**: The free $\mathbb{E}_1$-algebra on S^1 is $\coprod_{k=0}^{\infty}(S^1)^{\times k}/\Sigma_k$, which has an infinite homotopical dimension.

4. **Geometric constraint**: Any ∞-manifold with an n-group structure must be finite-dimensional (since $\pi_0(\mathrm{End}(G))$ acts on tangent spaces).

5. **Contradiction**: The homotopy groups $\pi_k(F(S^1))$ for $k > n$ are non-trivial (as they contain $\pi_k(\Omega\Sigma S^1) = \pi_k(S^2)$ for $k \geq 2$), violating n-truncation. □

Corollary 5. *The* $(\infty, 1)$*-category of* ∞*-Lie groups is not* ∞*-cocomplete.*

Proof. ∞-Lie groups are 1-group objects in $\mathbf{Diff}_\infty$. Apply Theorem 29 for $n = 1$. □

10. Conclusions

This work establishes a comprehensive framework for the hereditary transfer of categorical completeness and cocompleteness to categories of internal group objects. Our principal contributions resolve fundamental questions regarding the existence and construction of limits and colimits in $\mathbf{Grp}(\mathcal{C})$, with implications spanning algebraic topology, geometric representation theory, and categorical logic.

Key advances

(i) A complete characterization of cocompleteness transfer (Theorems 21 and 22): For a regular cocomplete category $\mathcal{C}$ admitting a free group functor $F\colon \mathcal{C} \to \mathbf{Grp}(\mathcal{C})$, the category $\mathbf{Grp}(\mathcal{C})$ is cocomplete. Explicit colimit constructions are derived via coequalizers of relations induced by group axioms over free group objects.

(ii) The identification of critical obstructions to cocompleteness in geometric categories (Theorem 22, and Proposition 1): The failure of Lie groups to form a cocomplete category stems from the analytic incompatibility between free group constructions and the manifold structure—a consequence of Birkhoff transitivity that precludes dense embeddings of non-abelian free groups.

(iii) Structural inheritance theorems: When $\mathcal{C}$ is regular and F preserves finite products, $\mathbf{Grp}(\mathcal{C})$ inherits regularity with effective quotient descent (Theorem 23). For locally Cartesian closed $\mathcal{C}$, internal hom-objects $\underline{\mathrm{Hom}}(G, H)$ exist as equalizers of exponential diagrams (Theorem 24).

(iv) Homotopical and representational extensions: Model structures on $\mathbf{Grp}(\mathcal{M})$ for combinatorial $\mathcal{M}$ (Theorem 26) and Tannakian duality for internal representations (Theorem 25) demonstrate the framework's versatility beyond classical settings.

Significance and implications

- The adjunction $F \dashv U$ is monadic for locally presentable $\mathcal{C}$ (Theorem 9), establishing $\mathbf{Grp}(\mathcal{C})$ as a category of algebras for the group monad. This provides a unified foundation for internal group theory.
- Applications to **TopGrp**, **OrdGrp**, and $\mathbf{Sh}(X, \mathbf{Grp})$ (Corollary 3) resolve long-standing questions on colimit existence in these categories through constructive procedures.
- The Lie group counterexample (Proposition 1) demonstrates the sharpness of our regularity and free-group hypotheses, revealing a fundamental dichotomy between algebraic and geometric categories.

Future investigations

Future investigations will address the following:

(a) Higher categorical generalizations: n-groupoids in $(\infty, 1)$-categories and their completeness properties under weakened Cartesian assumptions;
(b) Homotopical enhancements: Model structures on $\mathbf{Grp}(\mathcal{M})$ for non-combinatorial $\mathcal{M}$, and their relationship to simplicial group objects;
(c) Representation-theoretic extensions: Internal character theory for compact group objects in toposes, and its applications to quantum symmetry detection.

These results collectively demonstrate that the cocompleteness of internal group categories is governed by a delicate interplay between free constructions and exactness properties—a paradigm extending far beyond classical algebra into higher structures and geometric contexts.

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