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First-Order Impulses for an Impulsive Stochastic Differential Equation System

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Abstract

We consider first-order impulses for impulsive stochastic differential equations driven by fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$ involving a nonlinear ϕ-Laplacian operator. The system incorporates both state and derivative impulses at fixed time instants. First, we establish the existence of at least one mild solution under appropriate conditions in terms of nonlinearities, impulses, and diffusion coefficients. We achieve this by applying a nonlinear alternative of the Leray–Schauder fixed-point theorem in a generalized Banach space setting. The topological structure of the solution set is established, showing that the set of all solutions is compact, closed, and convex in the function space considered. Our results extend existing impulsive differential equation frameworks to include fractional stochastic perturbations (via fBm) and general ϕ-Laplacian dynamics, which have not been addressed previously in tandem. These contributions provide a new existence framework for impulsive systems with memory and hereditary properties, modeled in stochastic environments with long-range dependence.

Keywords: stochastic differential equation; energy and industry; fractional Brownian motion; impulsive differential equations; matrix; generalized Banach space; iterative methods

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1. Introduction

In some problems, partial differential equations are reduced to ordinary differential equations. In special cases, when incorporating randomness, equations can be reduced to systems of stochastic differential equations (SDEs). Differential equations for random processes are described based on which one or more of the terms are stochastic, meaning involving a random process. These equations are used to model systems that evolve over time with inherent randomness, such as financial markets, physical systems with thermal noise, or population dynamics with random events. In [1], generalizations of these equations have been considered in the context of a simplified heart, which is considered a constant-volume mixing chamber in some simple and special cases. Randomly generated

differential equations across various types and difficulty levels can be seen, for instance, in works by [2–5]. For basic stochastic differential equation analysis, please see the results in [6–8].

Impulsive differential equations are used to model systems that experience sudden, instantaneous changes at certain moments in time. These are common in fields like control theory, population dynamics, pharmacokinetics, and physics, where a system evolves continuously but undergoes abrupt changes due to external or internal impulses. The general form of an impulsive differential equation is given by

$$\frac{d\varpi}{dt} = g(\varpi, t),\ t \neq t_k,$$

with impulse (jump) condition at certain moments $t = t_k$:

$$\varpi(t_k^+) = \varpi(t_k^-) + I_k\varpi(t_k^-),$$

where

1. $\varpi(t_k^+)$ is the state just after the impulse.
2. $\varpi(t_k^-)$ is the state just before the impulse.
3. $I_k(\varpi)$ is the impulse function (which can depend on the state and/or time).
4. t_k is the sequence of impulse times.

This theory describes the processes of a sudden change in state at certain moments (see, for example, [9–13]).

Here, we consider impulsive connection with stochastic differential equations. This allows us to construct a generalization of a wide class of processes, which play an important role in physical applications (in the theory of Brownian motion, in particular), as well as in cases with finite jumps.

In this paper, we conduct existence analysis on the IVP with impulse effects.

$$\begin{cases} (g(\varpi'(t)(t)))' & = f_1(t,\varpi(t),\varkappa(t))dt + \eta_1(t)d^-B^H(t),\ t \in \mathcal{K} - \{t_k, k \in [1,m]_{\mathbb{N}}\}, \\ (g(\varkappa'(t)))' & = f_2(t,\varpi(t),\varkappa(t))dt + \eta_2(t)d^-B^H(t),\ t \in \mathcal{K} - \{t_k, k \in [1,m]_{\mathbb{N}}\}, \\ \Delta\varpi(t) & = I_k(\varpi(t_k),\varkappa(t_k)) \quad \Delta\varpi'(t) = I_k^1(\varpi^{'}(t_k),\varkappa'(t_k)), \quad t = t_k \quad k \in [1,m]_{\mathbb{N}}, \\ \Delta\varkappa(t) & = \bar{I}_k(\varpi(t_k),\varkappa(t_k)) \quad ,\Delta\varkappa'(t) = \bar{I}_k^2(\varpi'(t_k),\varkappa'(t_k)), \quad t = t_k \quad k \in [1,m]_{\mathbb{N}}, \\ \varpi(0) & = U_0, \quad \varkappa(0) = V_0, \\ \varpi'(0) & = U_1, \quad \varkappa'(0) = V_1, \end{cases} \tag{1}$$

Here, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, where $\mathcal{K} := [0,T]$, $[1,m]_{\mathbb{N}} = \{1,2,\ldots,m\}$, $f_1, f_2 : \mathcal{J} \times \mathbb{R}^2 \to \mathbb{R}$, $\eta_1, \eta_2 : \mathcal{K} \to \mathbb{R}$ are measurable functions and the processes $B^H(t)$ stand for the independent fBms, where the Hurst parameter $H \in (\frac{1}{2}, 1)$, and $g : \mathbb{R} \to \mathbb{R}$ is a suitable monotone homeomorphism, $I_k^1, \bar{I}_k^1, \bar{I}_k^2, I_k^2 \in \mathcal{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $k \in [1,m]_{\mathbb{N}}$ and $U_j, V_j \in \mathbb{R}$, for each $j \in \{0,1\}$,

$$\begin{aligned} \Delta\varpi|_{t=t_k} &= \varpi(t_k^+) - \varpi(t_k^-), \\ \Delta\varkappa|_{t=t_k} &= \varkappa(t_k^+) - \varkappa(t_k^-), \end{aligned}$$

and

$$\begin{aligned} \Delta\varpi'|_{t=t_k} &= \varpi'(t_k^+) - \varpi'(t_k^-), \\ \Delta\varkappa'|_{t=t_k} &= \varkappa'(t_k^+) - \varkappa'(t_k^-). \end{aligned}$$

The notations

$$\varkappa(t_k^+) = \lim_{h \to 0^+} \varkappa(t_k + h),$$

and

$$\varkappa(t_k^-) = \lim_{h \to 0^+} \varkappa(t_k - h),$$

denote the left and right limits of the function $\varkappa$ at $t = t_k$, respectively.

So far, a considerable number of theories and techniques have been introduced for exploring stochastic systems, the most important of which relates to approximation theory, through which a simplified system is introduced to replace the original, requiring a certain relationship between their solutions (see [14–16]).

We structured our work as follows. In Section 2, the necessary background and tools on stochastic theory and iterative methods are introduced. In Section 3, we state and show certain new results regarding the existence and compactness of solution sets for the main problem. Some additional topological properties are given for these sets.

2. Background and Preliminaries

In this section, we briefly introduce some basic notations and certain useful tools (see [17,18]).

To define and work with stochastic processes, let us introduce the space $(\Omega, \mathcal{F}, \mathbb{P})$, which is the usual complete probability space, and let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, such that $(\mathcal{F} = \mathcal{F}_t)_{t\geq 0}$, satisfying

1. Right continuous: $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$.
2. The completeness: each $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$).

Then, the tuple $(\Omega, \mathcal{F}, \mathcal{F}_t)_{t\geq 0}, \mathbb{P}$ is called a filtered probability space. This framework is necessary to define the stochastic process $\varpi(\cdot,\cdot) : [0,T] \times \Omega \to \mathbb{R}^n$ which will be written, for simplicity, as $\varpi(t)$ (see [4,19,20]).

Definition 1. *Let $H \in (0,1)$ be the Hurst parameter. The fractional Brownian motion $B^H = (B_t^H)_{t\geq 0}$ with Hurst index H is a centered self-similar Gaussian process with*

(i) $B^H(0) = 0$;

(ii) *Zero mean:*

$$E(B_t^H) = 0, t \in \mathbb{R}^+.$$

(iii) *Covariance function:*

$$E(B_t^H B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} + |t-s|^{2H}), t, s \in \mathbb{R}^+.$$

Remark 1.

1. *When $H = \frac{1}{2}$, fractional Brownian motion reduces to the standard equivalent.*
2. *Note that the fractional Brownian motion is not a semi-martingale unless $H = \frac{1}{2}$. In this case, Itô calculus does not apply directly.*
3. *Self-similarity:*

$$V_{ct}^H = c^H V_t^H,\ t \geq 0, c > 0.$$

4. *Stationary increments:*

$$V_{t+s}^H - V_s^H \equiv V_t^H\ t, s \geq 0.$$

5. *Long-range dependence:*
   - *(a)* *If $H > \frac{1}{2}$: Positive correlation (persistence) can be applied in Riemann–Stieltjes integrals.*

(b) *If $H < \frac{1}{2}$, negative correlation (anti-persistence) can be applied in rough path theory, especially when $H \in (\frac{1}{4}, \frac{1}{2})$.*

Let $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ be Borel measurable and $\frac{1}{2} \leq H < 1$. $\varpi \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ be given as

$$\varpi(r, \tau) = H(2H-1)|r-\tau|^{2H-2}, r, \tau > 0,$$

So, we have

$$L^2_\varpi = \{f : |f|^2_\varpi = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(t)f(\tau)\varpi(t,\tau)d\tau dt < \infty\}.$$

equipped with the following inner product

$$\langle f_1, f_2\rangle_\varpi = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f_1(t)f_2(\tau)\varpi(t,\tau)d\tau dt$$

then $L^2_\varpi(\mathbb{R}^+)$ will be a separable Hilbert space.

Let us now define $\mathcal{S}$ as a set of smooth and cylindrical random variables as

$$F = f(B^H(\psi_1), B^H(\psi_2), \ldots, B^H(\psi_n)),\ n \geq 1,$$

Here, $f \in C^\infty_b(\mathbb{R}^n)$, $\psi_i \in \mathcal{H}$, $\mathcal{H}$ is a Hilbert space; for more details, please see [21–23].

The derivative operator D^H_t of F is given as the $\mathcal{H}$-valued random variable:

$$D^H_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}((B^H(\psi_1), B^H(\psi_2), \ldots, B^H(\psi_n))\psi_i.$$

As in [24], the Malliavin ϖ-derivative of F is

$$D^\varpi_t F = \int_{\mathbb{R}^+} \varpi(t,\tau)D^H_\tau F d\tau.$$

Definition 2. *Let $u(t)$ be a stochastic process with integrable trajectories.*

(1) The symmetric integral of $u(t)$ with respect to $B^H(t)$ is given as the limit in the probability as $\epsilon \to 0$ of

$$\frac{1}{2\epsilon}\int_0^T u(s)\big(B^H(s+\epsilon) - B^H(s-\epsilon)\big)ds,$$

provided that this limit exists in probability, and is denoted by

$$\int_0^T u(s)d^\circ B^H(s).$$

(2) The forward integral of $u(t)$ with respect to B^H is introduced as the limit in the probability when $\epsilon \to 0$ of

$$\frac{1}{\epsilon}\int_0^T u(s)\left(\frac{B^H(s+\epsilon) - B^H(s)}{\epsilon}\right)ds,$$

provided that this limit exists in probability, and is denoted by

$$\int_0^T u(s)d^- B^H(s).$$

(3) The backward integral of $u(t)$ with respect to B^H is introduced as the limit in probability as $\epsilon \to 0$ of

$$\frac{1}{\epsilon}\int_0^T u(s)\left(\frac{B^H(s-\epsilon) - B^H(s)}{\epsilon}\right)ds,$$

provided that this limit exists in probability, and is denoted by

$$\int_0^T u(s)d^+B^H(s).$$

Remark 2 ([24]). *Let $\mathcal{L}_g(\mathcal{K})$ of integrands be given as the family of stochastic processes y on $\mathcal{K}$ where $y \in \mathcal{L}_g(\mathcal{K})$ if $\mathbb{E}|y(t)|_g^2 < \infty$. Suppose that y is a stochastic process in $L(\mathcal{K})$ with*

$$\iint_{\mathcal{K}\times\mathcal{K}} |D_\tau^H y(r)||r-\tau|^{2H-2}d\tau dr < \infty.$$

Thus, the symmetric integral exists and

$$\int_0^T y(s)d^\circ B^H(s) = \int_0^T y(s) \diamond dB^H(s) + \int_0^T (D_s^g y(t))ds, \tag{2}$$

where $\diamond$ is the Wick product, $\frac{1}{2} < H < 1$.

Remark 3 ([24]). *Let $\mathcal{L}_\varpi(0,T)$ of integrands be defined on $[0,T]$, where*

$$y(t) \in \mathcal{L}_\varpi(0,T),\ E|y(t)|_\varpi^2 < \infty.$$

Suppose that $y(t)$ is a stochastic process in $L(0,T)$ and

$$\int_0^T \int_0^T |D_\tau^H y(t)||t-\tau|^{2H-2}d\tau dt < \infty.$$

Then, there is the symmetric integral and the next relationship is true:

$$\int_0^T y(\tau)d^\circ B^H(\tau) = \int_0^T y(\tau) \diamond dB^H(\tau) + \int_0^T (D_\tau^\varpi y(t))d\tau,$$

where $\diamond$ represents the Wick product, $\frac{1}{2} < H < 1$.

To introduce the principle of random means (averaging), we need the lemma.

Lemma 1 ([25]). *Let $\vartheta(s)$ be a stochastic process in $\mathcal{L}_\varpi(0,T)$ and $B^H(t)(H > \frac{1}{2})$ be a fractional Brownian motion. Then, $\forall T \in (0,\infty)$, $\exists C(H,T)$, such that*

$$\int_0^T \Big(\vartheta(s)d^\circ B^H(s)\Big)^2 \le 2C(H,T)E\Big(\Big(\int_0^T |\vartheta(s)|^2 ds\Big) + 4CT^2 \tag{3}$$

where $C(H,T) = HT^{2H-1}$.

A vector metric space and generalized Banach spaces shall be introduced. Let $\varpi, \varkappa \in \mathbb{R}^n$, $\varpi = (\varpi_1, \ldots, \varpi_n)$, $\varkappa = (\varkappa_1, \ldots, \varkappa_n)$; we note that

- For $\varpi \in \mathbb{R}^n$, $(\varpi)_i = \varpi_i$, $i \in [1,n]_{\mathbb{N}}$;
- If $\varpi \le \varkappa$ we mean $\varpi_i \le \varkappa_i$ for all $i \in [1,n]_{\mathbb{N}}$;
- $|\varpi| = (|\varpi_1|, \ldots, |\varpi_n|)$ and $\max(\varpi, \varkappa) = \max(\max(\varpi_1, \varkappa_1), \ldots, \max(\varpi_n, \varkappa_n))$;
- Let $k \in \mathbb{R}$, so $\varpi \le k$ indicates that $\varpi_i \le k$, $i \in [1,n]_{\mathbb{N}}$.

Definition 3. *Set E as a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We define the vector-valued norm on E by the map $\|.\| : E \to \mathbb{R}_+^n$ where*

(i) We have $\|\varpi\| \ge 0, \forall \varpi \in E$, if $\|\varpi\| = 0$, then $\varpi = 0$.

(*ii*) *We have* $\|\lambda\varpi\| = |\lambda|\|\varpi\|, \forall\varpi \in E$ *and* $\lambda \in \mathbb{K}$.
(*iii*) *We have* $\|\varpi + \varkappa\| \leq \|\varpi\| + \|\varkappa\|, \forall\varpi, \varkappa \in E$.

Lemma 2 ([23]). *Let N be some a matrix of non-negative numbers. Then, we have*

(*i*) *The matrix* $N \to 0$;
(*ii*) *The matrix* $I - N$ *is non-singular and*

$$(I - N)^{-1} = \sum_{k\geq 0} N^k,$$

(*iii*) *We have* $\|\lambda\| < 1$, $\forall\lambda \in \mathbb{C}$ *where* $det(N - \lambda I) = 0$;
(*iv*) *The matrix* $(I - N)$ *is non-singular and* $(I - N)^{-1}$ *has non-negative elements.*

3. Solution

Let

$$\mathcal{K}_k = (t_k, t_{k+1}],\ k \in [1, m]_{\mathbb{N}},$$

we introduce the spaces

$$H_2([0,T], L^2(\Omega,\mathbb{R})) = \left\{ \begin{array}{c} \varpi : \mathcal{K} \to L^2(\Omega,\mathbb{R}) : \varpi\chi_{(t_k,t_{k+1}]} \in \mathcal{C}((t_k,t_{k+1}], L^2(\Omega,\mathbb{R})),\ k \in [1,m]_{\mathbb{N}} \\ \text{and } \exists\varpi(t_k^+), \text{for } k \in [1,m]_{\mathbb{N}} \end{array} \right\}$$

and

$$H_2^1([0,T], L^2(\Omega,\mathbb{R})) = \left\{ \begin{array}{c} \varpi : \mathcal{K} \to L^2(\Omega,\mathbb{R}) : \varpi\chi_{(t_k,t_{k+1}]} \in \mathcal{C}^1((t_k,t_{k+1}], L^2(\Omega,\mathbb{R})),\ k \in [1,m]_{\mathbb{N}} \\ \text{and } \exists\varpi(t_k^+), \text{for } k \in [1,m]_{\mathbb{N}} \end{array} \right\}$$

The space $H_2([0,T], L^2(\Omega,\mathbb{R}))$ is equipped with

$$\|\varpi\|_{H_2} = \sup_{s\in\mathcal{K}} \sqrt{\mathbb{E}|\varpi(s,.)|^2}.$$

It is not hard to see that H_2^1 is a Banach space with

$$\|\varpi\|_{H_2^1} = \|\varpi\|_{H_2} + \|\varpi^{(1)}\|_{H_2}$$

We define the space

$$\mathcal{PC} = \{\varpi : [0,T] \to L^2(\Omega,\mathbb{R}) : \varpi\chi_{\mathcal{K}} \in H_2^1 \text{ and } \sup_{t\in\mathcal{K}} \mathbb{E}|\varpi(t,.)|^2 < \infty \text{ a.e}\},$$

equipped with

$$\|\varpi\|_{\mathcal{PC}} = \sup_{0\leq s\leq T} (\mathbb{E}|\varpi(s,.)|^2)^{\frac{1}{2}}.$$

It is easy to see that $\mathcal{PC}$ is a Banach space with $\|.\|_{\mathcal{PC}}$.

To prove the existence of a solution, we will need the next assumptions to use iterative methods with some appropriate topological properties.

$(H1)$ $f_i : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with

$$E\left(|g^{-1}(\varpi)|^2\right) \leq g^{-1}(E|\varpi|^2),$$

and

$$\varpi \in \mathbb{R},\ I_k, \bar{I}_k \in \mathcal{C}(\mathbb{R},\mathbb{R}).$$

$(H2)$ Suppose that $\exists \overline{d}_i, \overline{e}_i, c_i \in \mathbb{R}^+$, where

$$|f_i(t,\varpi,\varkappa)|^2 \leq \overline{d}_i|\varpi|^2 + \overline{e}_i|\varkappa|^2 + c_i,\ i=1,\ \forall \varpi,\varkappa \in \mathbb{R},\ t \in \mathcal{K}.$$

$(H3)$ Suppose that $\exists \overline{c}_i \in \mathbb{R}^+$ with

$$|\eta_i(t)|^2 \leq \overline{c}_i, \quad i \in \{1,2\}, \forall \varpi,\varkappa \in \mathbb{R},\ t \in \mathcal{K}.$$

Theorem 1. *Under Assumptions* $(H1)$–$(H3)$*, the system* (1) *admits at least one solution* $(\varpi,\varkappa)$ *in a compact set* S_c*, where*

$$S_c = \{(\varpi,\varkappa) \in PC \times PC : (\varpi,\varkappa) \text{ is a solution of } (1)\}.$$

Proof. Our proof is based on the next steps.

Step 1: Let the system

$$\begin{cases} (g(\varpi'(t)))' &= f_1(t,\varpi(t),\varkappa(t))dt + \eta_1(t)d^-B^H(t),\ 0 \leq t \leq t_1 \\ (g(\varkappa'(t)))' &= f_2(t,\varpi(t),\varkappa(t))dt + \eta_2(t)d^-B^H(t),\ 0 \leq t \leq t_1 \\ \varpi(0) = U_0 \quad , & \quad \varkappa(0) = V_0 \\ \varpi'(0) = U_1 \quad , & \quad \varkappa'(0) = V_1, \end{cases} \tag{4}$$

Let

$$\widehat{C}_{\tau_0} = \left\{\varpi : [0,t_1] \to L^2(\Omega,\mathbb{R}) : \varpi\chi_{[0,t_1]} \in \mathcal{C}^1([0,t_1], L^2(\Omega,\mathbb{R})) \text{ and there exist } \varpi(t_1^+)\right\}$$

with

$$C_{\tau_0} = \{\varpi : [0,t_1] \to L^2(\Omega,\mathbb{R}) : \varpi\chi_{[0,t_1]} \in \widehat{C}_{\tau_0} \quad \sup_{0\leq t\leq t_1} \mathbb{E}|\varpi(t,.)|^2 < \infty\},$$

Define

$$G^0 : C_{\tau_0} \times C_{\tau_0} \to C_{\tau_0} \times C_{\tau_0}$$

given by

$$G^0(\varpi,\varkappa) = (G_1^0(\varpi,\varkappa), G_2^0(\varpi,\varkappa)),\ (\varpi,\varkappa) \in C_{\tau_0} \times C_{\tau_0}$$

where

$$\begin{cases} G_1^0(\varpi,\varkappa) &= U_0 + \int_0^t g^{-1}\big(g(U_1) + \int_0^s f_1(\tau,\varpi(\tau),\varkappa(\tau))d\tau + \\ & \quad + \int_0^s \eta_1(\tau)d^-B^H(\tau)\big)ds,\ 0 \leq t \leq t_1, \\ G_2^0(\varpi,\varkappa) &= V_0 + \int_0^t g^{-1}\big(g(V_1) + \int_0^s f_2(\tau,\varpi(\tau),\varkappa(\tau))d\tau + \\ & \quad + \int_0^s \eta_2(\tau)d^-B^H(\tau)\big)ds,\ 0 \leq t \leq t_1. \end{cases}$$

Clearly, the fixed points of $G^0 = (G_1^0, G_2^0)$ are solutions of the problem (4).

To use the Leray–Schauder-type nonlinear variant, we prove that G^0 is strictly continuous, supporting our subsequent claims.

Claim 1: G^0 sends bounded sets into bounded sets in $C_{\tau_0} \times C_{\tau_0}$. In fact, it suffices to prove that $\forall r > 0$, $\exists \alpha >$, where

$$(\varpi,\varkappa) \in V_r = \{(\varpi,\varkappa) \in C_{\tau_0} \times C_{\tau_0} : \sup_{0\leq t\leq t_1} \mathbb{E}|\varpi(t,\cdot)|^2 \leq r,\ \sup_{0\leq t\leq t_1} \mathbb{E}|\varpi(t,\cdot)|^2 \leq r\}.$$

So,

$$\|G^0(\varpi,\varkappa)\| \leq \alpha = (\alpha_1,\alpha_2).$$

Thus,

$$\begin{aligned}\mathbb{E}|G_1^0(\varpi,\varkappa)|^2 &\leq 2\mathbb{E}|U_0|^2+2\int_0^t\mathbb{E}\Big(g^{-1}\Big(g(U_1)+\int_0^s f_1(\tau,\varpi(\tau),\varkappa(\tau))d\tau\\ &\quad+\int_0^s\eta_2(\tau)d^-B^H(\tau)\Big)ds\Big)^2,\ 0\leq t\leq t_1.\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}&\mathbb{E}\Big(g(U_1)+\int_0^s f_1(\tau,\varpi(\tau),\varkappa(\tau))d\tau+\int_0^s\eta_1(\tau)d^-B^H(\tau)\Big)^2\\ \leq\ & 3|g(U_1)|_X^2+3t_1\int_0^s(\bar{d}_1|\varpi(\tau)|_X^2+\bar{b}_1|\varkappa(\tau)|^2+c_1)d\tau\\ &+6C(H,t_1)\int_0^s\bar{c}_1d\tau+12Ct_1^2,\end{aligned}$$

Then,

$$\mathbb{E}\Big(g(U_1)+\int_0^s f_1(s,\varpi(s),\varkappa(s))ds+\int_0^s\eta_1(s)d^-B^H(s)\Big)^2\in\bar{e}(0,\ell_1),$$

where

$$\begin{aligned}\ell_1 &= 3\mathbb{E}|g(U_1)|^2+3t_1\int_0^s(\bar{d}_1\mathbb{E}|\varpi(\tau)|^2+\bar{e}_1\mathbb{E}|\varkappa(\tau)|^2+c_1)d\tau\\ &\quad+6C(H,t_1)\bar{c}_1t_1+12Ct_1^2.\end{aligned}$$

As g^{-1} is continuous, we have

$$\sup_{\beta_1\in\bar{e}(0,\ell_1)}|g^{-1}(\beta_1)|<\infty.$$

So,

$$\mathbb{E}|G_1^0(\varpi,\varkappa)|^2\leq 2\mathbb{E}|U_0|^2+2t_1\sup_{\beta_1\in\bar{e}(0,\ell_1)}|g^{-1}(\beta_1)|:=\alpha_1$$

and

$$\mathbb{E}|G_2^0(\varpi,\varkappa)|^2\leq 2\mathbb{E}|V_0|^2+2t_1\sup_{\beta_2\in\bar{e}(0,\ell_2)}|g^{-1}(\beta_2)|:=\alpha_2$$

here

$$\begin{aligned}\ell_2 &= 3\mathbb{E}|g(V_1)|^2+3t_1\int_0^s(\bar{d}_2\mathbb{E}|\varpi(\tau)|^2+\bar{e}_2\mathbb{E}|\varkappa(\tau)|^2+c_2)d\tau\\ &\quad+6C(H,t_1)\bar{c}_2t_1+12Ct_1^2\end{aligned}$$

As g^{-1} is continuous, we have

$$\sup_{\beta_1\in\bar{e}(0,\ell_1)}|g^{-1}(\beta_1)|<\infty.$$

Claim 2: G^0 maps bounded sets to equicontinuous sets. Take $\ell_1,\ell_2\in[0,t_1]$, $\ell_1<\ell_2$, and let $V_r\subset C_{\tau_0}\times C_{\tau_0}$, as in Claim 1. Let $(\varpi,\varkappa)\in V_r$. Then,

$$\begin{aligned}\mathbb{E}|(G_1^0(\varpi,\varkappa))'^2 &= E\Big(g^{-1}\Big(g(U_1)+\int_0^t f_1(s,\varpi(s),\varkappa(s))d\tau\\ &+\int_0^t \eta_1(s)d^-B^H(s)\Big)-g^{-1}(g(U_1))\Big)^2\\ &\le \mathbb{E}\Big(g^{-1}\Big(g(U_1)+\int_0^t f_1(s,\varpi(s),\varkappa(s))d\tau\\ &+\int_0^t \eta_1(s)d^-B^H(s)\Big)\Big)^2+E(U_1)^2\\ &\le \sup_{\beta_1\in\bar{e}(0,\ell_1)}|g^{-1}(\beta_1)|+E|U_1|=\tau'.\end{aligned}$$

By the classical mean value theorem, we have

$$\mathbb{E}|(G_1^0(\varpi,\varkappa))(\ell_2)-(G_1^0(\varpi,\varkappa))(\ell_1)| = \mathbb{E}|(G_1^0(\varpi,\varkappa))'(\xi,\bar{\xi})(\ell_2-\ell_1)| \le r'|\ell_2-\ell_1|. \quad (5)$$

Since $\ell_2\to\ell_1$, the RHS tends (5) to 0 and

$$\begin{aligned}\mathbb{E}|(G_2^0(\varpi,\varkappa))'(t)|_X^2 &= \mathbb{E}\Big|g^{-1}\Big(g(V_1)+\int_0^t f_2(s,\varpi(s),\varkappa(s))d\tau\\ &+\int_0^t \eta_2(s)d^-B^H(s)\Big)-g^{-1}(g(V_1))\Big|^2\\ &\le \Big|g^{-1}\Big(g(V_1)+\int_0^t f_2(s,\varpi(s),\varkappa(s))d\tau\\ &+\int_0^t \eta_2(s)d^-B^H(s)\Big)\Big|^2+\Big|V_1\Big|^2\\ &\le \sup_{\beta_2\in\bar{e}(0,\ell_2)}|g^{-1}(\beta_2)|+|V_1|=\tau.\end{aligned}$$

Using the mean value theorem, we get

$$\mathbb{E}|(G_2^0(\varpi,\varkappa))(\ell_2)-(G_2^0(\varpi,\varkappa))(\ell_1)| = \mathbb{E}|(G_2^0(\varpi,\varkappa))'(\xi,\bar{\xi})(\ell_2-\ell_1)| \le r'|\ell_2-\ell_1|. \quad (6)$$

As $\ell_2\to\ell_1$, the RHS of (6) tends to 0.

Claim 3: We have G^0, which is continuous. Let $(\varpi_n,\varkappa_n)_{n\in\mathbb{N}}$ be a sequence where

$$(\varpi_n,\varkappa_n)\to(\varpi,\varkappa),\ in\ C_{\tau_0}\times C_{\tau_0}.$$

Then, for $r>0$, where

$$\sup_{0\le t\le t_1}\mathbb{E}|\varpi_n(t,\cdot)|^2\le r,$$

$$\sup_{0\le t\le t_1}\mathbb{E}|\varkappa_n(t,\cdot)|^2\le r\le r,\ \forall n\in\mathbb{N},$$

and

$$\sup_{0\le t\le t_1}\mathbb{E}|\varpi(t,\cdot)|^2\le r,$$

$$\sup_{0\le t\le t_1}\mathbb{E}|\varkappa(t,\cdot)|^2\le r,$$

$(\varpi_n,\varkappa_n)\in V_r$ and $(\varpi,\varkappa)\in V_r$. Then, for $0\le t\le t_1$, we get

$$\begin{aligned}\mathbb{E}|G_1^0(\varpi_n,\varkappa_n)-G_1^0(\varpi,\varkappa)|_X^2 \quad\leq\quad & \int_0^t \mathbb{E}\Big|g^{-1}\Big(g(U_1)+\int_0^s f_1(\tau,\varpi(\tau),\varkappa(\tau))d\tau\\ & +\int_0^s \eta_1(v)d^-B^H(v)\Big)-g^{-1}\Big(g(U_1)+\int_0^s f_1(v,\varpi(\tau),\varkappa(\tau))dv\\ & +\int_0^s \eta_1(\tau)d^-B^H(v)\Big)\Big|^2 ds\end{aligned}$$

Using the dominated convergence theorem, we obtain

$$E\Big|g(U_1)+\int_0^s f_1(v,\varpi_n(s),\varkappa_n(s))dv+\int_0^s \eta_1(v)d^-B^H(v)$$

$$-g(U_1)-\int_0^s f(s,\varpi(s),\varkappa(s))ds-\int_0^s \eta_1(v)d^-B^H(v)\Big|_X^2\to 0 \quad \text{as } n\to\infty,$$

by the continuity of g^{-1}. Thus, using the dominated convergence theorem, we obtain

$$\begin{aligned}&\sup_{0\leq t\leq t_1}\mathbb{E}|G_1^0(\varpi_n,\varkappa_n)-G_1^0(\varpi,\varkappa)|^2\\ &\leq\int_0^{t_1} E|g^{-1}[g(B)+\int_0^s f_1(\tau,\varpi_n,\varkappa_n)d\tau+\int_0^s \eta_1(\tau)d^-B^H(\tau)]\\ &-g^{-1}[g(B)+\int_0^s f_1(\tau,\varpi,\varkappa)d\tau+\int_0^s \eta_1(\tau)d^-B^H(\tau)]^2 ds\to 0,\ n\to\infty.\end{aligned}$$

Thus, G_1^0 is continuous.

Similarly,

$$\begin{aligned}&\sup_{0\leq t\leq t_1}\mathbb{E}|G_2^0(\varpi_n,\varkappa_n)-G_2^0(\varpi,\varkappa)|^2\\ &\leq\int_0^{t_1}\mathbb{E}|g^{-1}[g(B)+\int_0^s f_2(\tau,\varpi_n,\varkappa_n)d\tau+\int_0^s \eta_2(\tau)d^-B^H(\tau)]\\ &-g^{-1}[g(B)+\int_0^s f_2(\tau,\varpi,\varkappa)d\tau+\sum_{l=1}^{\infty}\int_0^s \eta_2(\tau)d^-B^H(\tau)]|^2 ds\to 0,\ n\to\infty,\end{aligned}$$

Then, G_2^0 is continuous.

Claim 4: A priori estimate. We should prove that $\exists A_0>0$, where

$$\sup_{0\leq t\leq t_1}\mathbb{E}|\varpi(t,\cdot)|_X^2\leq A_0,$$

where $(\varpi,\varkappa)$ is a solution of (4). Let $(\varpi,\varkappa)$ be a solution of (4)

$$\begin{cases}\varpi(t) &= U_0+\int_0^t g^{-1}\Big(g(U_1)+\int_0^s f_1(\tau,\varpi(\tau),\varkappa(\tau))d\tau+\\ &\quad +\int_0^s \eta_1(\tau)d^-B^H(\tau)\Big)ds,\ 0\leq t\leq t_1.\\ \varkappa(t) &= V_0+\int_0^t g^{-1}\Big(g(V_1)+\int_0^s f_2(\tau,\varpi(\tau),\varkappa(\tau))d\tau+\\ &\quad +\int_0^s \eta_2(\tau)d^-B^H(\tau)\Big)ds,\ 0\leq t\leq t_1.\end{cases}$$

By Lemma 1, we have

$$\begin{aligned}\mathbb{E}|\varpi(t)|^2 &\leq \mathbb{E}|U_0|^2 + \int_0^t \mathbb{E}\Big|g^{-1}\Big(g(U_1) + \int_0^s f_1(s,\varpi(s),\varkappa(s))ds + \int_0^s \eta_1(s)d^-B^H(s)\Big)\Big|^2 ds \\ &\leq 2\mathbb{E}|U_0|^2 + 2t_1 \sup_{\beta_1 \in \bar{e}(0,\ell_1)} |g^{-1}(\beta_1)| = A_0.\end{aligned}$$

Here,

$$\begin{aligned}\ell_1 = 3E|g(U_1)|^2 + 3t_1 \int_0^s (\bar{d}_1\mathbb{E}|\varpi(\tau)|_{\varpi}^2 + \bar{e}_1\mathbb{E}|\varkappa(\tau)|^2 + c_1)d\tau \\ +6C(H,t_1)\bar{c}_1 t_1 + 12Ct_1^2\end{aligned}$$

Thus,

$\sup_{0\leq t\leq t_1} \mathbb{E}|\varpi(t)|^2 \leq A_0$, and

$$\begin{aligned}\mathbb{E}|\varkappa(t)|^2 &\leq \mathbb{E}|V_0|^2 + \int_0^t \mathbb{E}\Big|g^{-1}\Big(g(V_1) + \int_0^s f_2(\tau,\varpi(\tau),\varkappa(\tau))d\tau + \int_0^s \eta_2(\tau)d^-B^H(\tau)\Big)\Big|^2 ds \\ &\leq 2\mathbb{E}|V_0|^2 + 2t_1 \sup_{\beta_2 \in \bar{e}(0,l_2)} |g^{-1}(\beta_2)| = A_0.\end{aligned}$$

with

$$\begin{aligned}\ell_2 = 3\mathbb{E}|g(U_1)|^2 + 3t_1 \int_0^s (\bar{d}_1\mathbb{E}|\varpi(\tau)|^2 + \bar{e}_1\mathbb{E}|\varkappa(\tau)|^2 + c_1)d\tau \\ +6C(H,t_1)\bar{c}_2 t_1 + 12Ct_1^2\end{aligned}$$

Thus,

$$\sup_{0\leq t\leq t_1} \mathbb{E}|\varkappa(t)|^2 \leq A_0.$$

Let

$$U = \{\varkappa \in C([0,t_1],\mathbb{R}) : \sup_{0\leq t\leq t_1} \mathbb{E}|\varpi(t)|^2 < A_0 + 1, \sup_{0\leq t\leq t_1} \mathbb{E}|\varkappa(t)|^2 < A_0 + 1\}.$$

From Claim 1–Claim 4, and owing to the Ascoli–Arzelà theorem, we find that the map $G^0 : \overline{U} \to C_{\tau_0} \times C_{\tau_0}$ is compact. We choose U where there is no $(\varpi,\varkappa) \in \partial U$ so that

$$(\varpi,\varkappa) = \lambda G^0(\varpi,\varkappa),\ \forall \lambda \in (0,1).$$

By the use of the nonlinear Leray–Schauder alternative, we find that G^0 admits a fixed point $(\varpi_0,\varkappa_0) \in \overline{U}$, which represents a solution of (4).

Step 2: Let us consider the system

$$\begin{cases} (g(\varpi'(t)))' & = f_1(t,\varpi(t),\varkappa(t))dt + \eta_1(\tau)d^-B^H(\tau),\ t_1 < t \leq t_2, \\ (g(\varkappa'(t)))' & = f_2(t,\varpi(t),\varkappa(t))dt + \eta_2(\tau)d^-B^H(\tau),\ t \in t_1 < t \leq t_2, \\ \varpi(t_1^+) & = \varpi_0(t_1^-) + I_1(\varpi_0(t_1^-),\varkappa_0(t_1^-)) \quad ,\varpi'(t_1^+) = \varpi_0'(t_1^-) + I_1^1(\varpi_0'(t_1^-),\varkappa_0'(t_1^-)) \\ \varkappa(t_1^+) & = \varkappa_0(t_1^-) + \overline{I}_1(\varpi_0(t_1^-),\varkappa_0(t_1^-)) \quad ,\varkappa'(t_1^+) = \varkappa_0'(t_1^-) + \overline{I}_1^2(\varpi_0'(t_1^-),\varkappa_0'(t_1^-)) \end{cases} \tag{7}$$

Let

$$\widehat{C}_{\tau_1} = \{\varpi : (t_1,t_2] \to L^2(\Omega,\mathbb{R}) \quad ,\varpi\,|_{(t_1,t_2]} \in C^1((t_1,t_2],L^2(\Omega,\mathbb{R})),\quad k = 1,2,..,m \quad \text{and} \quad \exists \varpi(t_2^+)\}$$

and

$$D_{t_1} = \{\varpi : (t_1,t_2] \to L^2(\Omega,\mathbb{R}),\ \varpi\,|_{(t_1,t_2]} \in \widehat{C}_{\tau_1} : \sup_{t_1<t\leq t_2} \mathbb{E}|\varpi(t,.)|^2 < \infty\},$$

Taking

$$C_1 = C_{\tau_0} \cap D_{t_1}$$

Define $G^1 : C_1 \times C_1 \to C_1 \times C_1$ by

$$G^1(\varpi, \varkappa) = (G_1^1(\varpi, \varkappa), G_2^1(\varpi, \varkappa)),\ (\varpi, \varkappa) \in C_1 \times C_1$$

We observe that any solutions of (7) are considered fixed points of

$$P_i^1 : C_1 \times C_1 \to C_1,\ i = 1, 2,$$

given as

$$\begin{cases} G_1^1(\varpi, \varkappa) &= U_3 + \int_{t_1}^t g^{-1}\Big(g(U_4) + \int_{t_1}^s f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \\ &\quad + \int_0^s \eta_1(\tau)d^-B^H(\tau)\Big)ds,\ 0 \le t \le t_1\,. \\ G_2^1(\varpi, \varkappa) &= V_3 + \int_{t_1}^t g^{-1}\Big(g(V_4) + \int_{t_1}^s f_2(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \\ &\quad + \int_0^s \eta_2(\tau)d^-B^H(\tau)\Big)ds,\ 0 \le t \le t_1\,. \end{cases}$$

and

$$\begin{aligned} U_3 &= \varpi_1(t_1) + I_1(\varpi_1(t_1), \varkappa_1(t_1)), \\ U_4 &= \varpi_1'(t_1^-) + I_1^1(\varpi_1'(t_1^-), \varkappa_1'(t_1^-)), \\ V_3 &= \varkappa_1(t_1) + \bar{I}_1(\varpi_1(t_1), \varkappa_1(t_1)), \\ V_4 &= \varkappa_1'(t_1^-) + \bar{I}_1^2(\varpi_1'(t_1^-), \varkappa_1'(t_1^-)), \end{aligned}$$

Similar to the first step, we may show that G^1 admits at least one fixed point that represents a solution of (7).

Step 3: We complete this process using the fact that

$$(\varpi_m, \varkappa_m) := (\varpi\big|_{(t_m,T]}, \varkappa\big|_{(t_m,T]}),$$

is a solution to

$$\begin{cases} (g(\varpi'(t)))' &= f_1(t, \varpi(t), \varkappa(t))dt + \eta_1(t)d^-B^H(t),\ t_m < t \le T, \\ (g(\varkappa'(t)))' &= f_2(t, \varpi(t), \varkappa(t))dt + \eta_1(t)d^-B^H(t),\ t_m < t \le T, \\ \varpi(t_m^+) &= \varpi_{m-1}(t_m^-) + I_m(\varpi_0(t_{m-1}^-)) \\ \varpi'(t_m^+) &= \varpi_{m-1}^{'}(t_m^-) + I_m^1(\varpi_{m-1}^{'}(t_m^-)) \\ \varkappa(t_m^+) &= \varkappa_{m-1}(t_m^-) + \bar{I}_m(\varpi_0(t_{m-1}^-)) \\ \varkappa'(t_m^+) &= \varkappa_{m-1}^{'}(t_m^-) + \bar{I}_m^2(\varkappa_{m-1}^{'}(t_m^-)) \end{cases} \tag{8}$$

A solution $(\varpi, \varkappa)$ of (8) is given as

$$(\varpi(t), \varkappa(t)) = \begin{cases} (\varpi_0(t), \varkappa_0(t)), & \text{if } 0 \le t \le t_1, \\ (\varpi_1(t), \varkappa_1(t)), & \text{if } 0 < t \le t_2, \\ \dots \\ (\varpi_m(t), \varkappa_m(t)), & \text{if } t_m < t \le T. \end{cases}$$

Step 4: We prove the compaction of

$$S_c = \{(\varpi, \varkappa) \in PC \times PC : (\varpi, \varkappa) \text{ is a solution of (1)}\}$$

Let $(\varpi_n, \varkappa_n)_{n\in\mathbb{N}}$ be a sequence in S_c. Let

$$B = \{(\varpi_n, \varkappa_n) : n \in \mathbb{N}\} \subseteq PC \times PC.$$

Then, from the previous parts of the proof of this theorem, we deduce that B is finite and isotropically continuous. Then, from the Ascoli–Arzelà theorem, we find that B is compact.

We have $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$, $k = 1, \ldots, m$; then,

• $(\varpi_n, \varkappa_n)|_{J_0}$ has a subsequence

$$(\varpi_{n_m}, \varkappa_{n_m})_{n_m\in\mathbb{N}} \subset S_{c_1} = \{(\varpi, \varkappa) \in C_{\tau_0} \times C_{\tau_0} : (\varpi, \varkappa) \textit{ is a solution of } (4)\}$$

where $(\varpi_{n_m}, \varkappa_{n_m}) \to (\varpi, \varkappa)$. Let

$$\begin{cases} z_0(t) &= U_0 + \int_0^t g^{-1}\Big(g(U_1) + \int_0^s f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \\ & \quad + \int_0^s \eta_1(\tau) d^- B^H(\tau)\Big) ds, \ 0 \le t \le t_1 . \\ \bar{z}_0(t) &= V_0 + \int_0^t g^{-1}\Big(g(V_1) + \int_0^s f_2(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \\ & \quad + \int_0^s \eta_2(\tau) d^- B^H(\tau)\Big) ds, \ 0 \le t \le t_1 . \end{cases} \tag{9}$$

$$\begin{aligned} \mathbb{E}\Big|\varpi_{n_m}(t) - z_0(t)\Big|^2 &\le \int_0^t E\Big|g^{-1}\Big(g(U_1) + \int_0^s f_1(\tau, \varpi_{n_m}(\tau), \varkappa_{n_m}(\tau))d\tau \\ &+ \int_0^s \eta_1(\tau) d^- B^H(\tau)\Big) - g^{-1}\Big(g(U_1) + \int_0^s f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \\ &+ \int_0^s \eta_1(\tau) d^- B^H(\tau)\Big)\Big|^2 ds. \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\Big|\varkappa_{n_m}(t) - \bar{z}_0(t)\Big|^2 &\le \int_0^t E\Big|g^{-1}\Big(g(V_1) + \int_0^s f_1(\tau, \varpi_{n_m}(\tau), \varkappa_{n_m}(\tau))d\tau \\ &+ \int_0^s \eta_1(\tau) d^- B^H(\tau)\Big) - g^{-1}\Big(g(U_1) + \int_0^s f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \\ &+ \int_0^s \eta_1(\tau) d^- B^H(\tau)\Big)\Big|^2 ds. \end{aligned}$$

As $n_m \to +\infty$, $(\varpi_{n_m}, \varkappa_{n_m}) \to (z_0(t), \bar{z}_0(t))$, and then

$$\varpi(t) = U_0 + \int_0^t g^{-1}\Big(g(U_1) + \int_0^s f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_0^s \eta_1(\tau) d^- B^H(\tau)\Big) ds,$$

and

$$\varkappa(t) = V_0 + \int_0^t g^{-1}\Big(g(V_1) + \int_0^s f_2(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_0^s \eta_2(\tau) d^- B^H(\tau)\Big) ds.$$

• $(\varpi_n, \varkappa_n)|_{J_1}$ has a subsequence relabeled as $(\varpi_{n_m}, \varkappa_{n_m}) \subset S_{c_2} \to (\varpi, \varkappa)$ in $C_1 \times C_1$ where

$$S_{c_2} = \{(\varpi, \varkappa) \in C_1 \times C_1 : (\varpi, \varkappa) : \textit{ is a solution of } (7)\}.$$

Taking

$$z_1(t) = U_3 + \int_{t_1}^{t} g^{-1}\Big(g(U_4) + \int_{t_1}^{s} f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_{t_1}^{s} \eta_1(\tau)d^- B^H(\tau)\Big)ds,$$

and

$$\bar{z}_1(t) = V_3 + \int_{t_1}^{t} g^{-1}\Big(g(V_4) + \int_{t_1}^{s} f_2(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_{t_1}^{s} \eta_2(\tau)d^- B^H(\tau)\Big)ds.$$

$$\begin{aligned}\mathbb{E}\Big|\varpi_{n_m}(t) - z_1(t)\Big|^2 \leq \int_{t_1}^{t} E\Big|g^{-1}\Big(g(U_4) + \int_{t_1}^{s} f_1(\tau, \varpi_{n_m}(\tau), \varkappa_{n_m}(\tau))d\tau \\ + \int_{t_1}^{s} \eta_1(\tau)d^- B^H(\tau)\Big) - g^{-1}\Big(g(U_4) + \int_{t_1}^{s} f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \\ + \int_{t_1}^{s} \eta_1(\tau)d^- B^H(\tau)\Big)\Big|^2 ds.\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\Big|\varkappa_{n_m}(t) - \bar{z}_1(t)\Big|^2 \leq \int_{t_1}^{t} E\Big|g^{-1}\Big(g(V_4) + \int_{t_1}^{s} f_1(\tau, \varpi_{n_m}(\tau), \varkappa_{n_m}(\tau))d\tau \\ + \eta_1(\tau)d^- B^H(\tau) \int_{t_1}^{s} \eta_1(\tau)d^- B^H(\tau)\Big) - g^{-1}\Big(g(U_4) + \int_{t_1}^{s} f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \\ + \int_{t_1}^{s} \eta_1(\tau)d^- B^H(\tau)\Big)\Big|^2 ds.\end{aligned}$$

As $n_m \to +\infty$, $(\varpi_{n_m}(t), \varkappa_{n_m}(t)) \to (z_1(t), \bar{z}_1(t))$, and then

$$\varpi(t) = U_3 + \int_{t_1}^{t} g^{-1}\Big(g(U_4) + \int_{t_1}^{s} f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_{t_1}^{s} \eta_1(\tau)d^- B^H(\tau)\Big)ds,$$

and

$$\varkappa(t) = V_3 + \int_{t_1}^{t} g^{-1}\Big(g(V_4) + \int_{t_1}^{s} f_2(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_{t_1}^{s} \eta_2(\tau)d^- B^H(\tau)\Big)ds.$$

• We complete this process to find that $\{(\varpi_n, \varkappa_n) \mid n \in \mathbb{N}\}$ admits a subsequence that converges to

$$z_m(t) = U_{m+2} + \int_{t_m}^{t} g^{-1}\Big(g(U_{m+3}) + \int_{t_m}^{s} f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_{t_m}^{s} \eta_1(\tau)d^- B^H(\tau)\Big)ds,$$

and

$$\bar{z}_m(t) = V_{m+2} + \int_{t_m}^{t} g^{-1}\Big(g(V_{m+3}) + \int_{t_m}^{s} f_2(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_{t_m}^{s} \eta_2(\tau)d^- B^H(\tau)\Big)ds.$$

This ensures the compaction of S_c. □

Now, assumptions $(H2)$ and $(H3)$ in Theorem 1 will be replaced by $(H3)'$. There exists a function $p_i \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi_i : [0, \infty) \to [0, \infty)$ with

$$\mathbb{E}|f_i(t, \varpi, \varkappa)|^2 \leq p_i(t)\psi_i(\mathbb{E}(|x|^2 + |\varkappa|^2)), i = 1, 2,$$

and

$$\mathbb{E}|\eta_i(t)|^2 \leq p_i$$

Theorem 2. *Let (H3)′ hold. Then, the system* (1) *admits at least one solution in a compact set.*

Proof. Similar to the proof of Theorem 1, one may prove that the system (1) admits at least one solution using the same technique.

- Let $0 \leq t \leq t_1$, and we get

$$\begin{cases} \varpi(t) &= U_0 + \int_0^t g^{-1}\Big(g(U_1) + \int_0^s f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \\ & \quad + \int_0^s \eta_1(\tau) d^- B^H(\tau)\Big)ds, \;\; 0 \leq t \leq t_1\,. \\ \varkappa(t) &= V_0 + \int_0^t g^{-1}\Big(g(V_1) + \int_0^s f_2(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \\ & \quad + \int_0^s \eta_2(\tau) d^- B^H(\tau)\Big)ds, \;\; 0 \leq t \leq t_1\,. \end{cases}$$

Thus,

$$\mathbb{E}|\varpi(t)|^2 \leq 2\mathbb{E}|U_0|^2 + 2\int_0^t \mathbb{E}\Big|g^{-1}\Big(g(U_1) + \int_0^s f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_0^s \eta_1(\tau) d^- B^H(\tau)\Big)\Big|^2 ds$$

Let the functions $\nu, \overline{\nu}$ defined for $0 \leq \tau \leq t_1$ as

$$\begin{aligned} \nu(\tau) &= \sup_{0\leq s\leq \tau}\{\mathbb{E}|\varpi(s)|^2\}, \\ \overline{\nu}(\tau) &= \sup_{0\leq s\leq \tau}\{\mathbb{E}|\varkappa(s)|^2\}. \end{aligned}$$

Using Lemma 1, we get

$$\begin{aligned} &\mathbb{E}\Big|g(U_1) + \int_0^s f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_0^s \eta_1(\tau) d^- B^H(\tau)\Big|^2 \\ &\leq 3\mathbb{E}|g(U_1)|^2_{\varpi} + 3\int_0^s G_1(\tau)\psi_1(\mathbb{E}(|\varpi(\tau)|^2 + |\varkappa(\tau)|^2))d\tau \\ &+6C(H,t_1)\|G_1\|_{L^1} + 12Ct_1^2 \\ &\leq 3\mathbb{E}|g(U_1)|^2 + ||\overline{p}||_{L^1}\psi_1(\nu(s) + \overline{\nu}(s)) \end{aligned}$$

Here,

$$||\overline{p}||_{L^1} = 3t_1\|G_1\|_{L^1}$$

$$\lambda_1 = 3\mathbb{E}|g(U_1)|^2 + 6C(H,t_1)\|G_1\|_{L^1} + 12Ct_1^2$$

and, consequently,

$$\nu(t) \leq 2\mathbb{E}|U_0|^2 + \int_0^t \widehat{\psi}_1(\nu(s) + \overline{\nu}(s)), \quad 0 \leq t \leq t_1,$$

for

$$\widehat{\psi}_1 = (g^{-1} \circ \widetilde{\psi}_1),$$

$$\widetilde{\psi}_1(u) = \lambda_1 + ||\overline{p}_1||_{L^1}\psi_1(u).$$

Similarly,

$$\overline{\nu}(t) \leq 2\mathbb{E}|V_0|^2 + \int_0^t \widehat{\psi}_2(\nu(s) + \overline{\nu}(s))ds, \quad 0 \leq t \leq t_1,$$

$$\lambda_2 = 3\mathbb{E}|g(V_1)|^2 + 6C(H,t_1)\|G_2\|_{L^1} + 12Ct_1^2$$

and where

$$||\overline{p}||_{L^1} = 3t_1\|G_2\|_{L^1}$$

where

$$\widehat{\psi}_2 = (g^{-1} \circ \widetilde{\psi}_2),$$

and

$$\widetilde{\psi}_2(u) = \lambda_2 + ||\overline{p}||_{L^1}\psi_1(u).$$

By $\nu(t)$ and $\overline{\nu}(t)$, we have

$$\nu(t) + \overline{\nu}(t) \leq 2\mathbb{E}|U_0|^2 + 2\mathbb{E}|V_0|^2 + \sum_{i=1}^{2}\int_0^t \widehat{\psi}_i(\nu(s) + \overline{\nu}(s))ds, \quad 0 \leq t \leq t_1,$$

Using the well-known nonlinear Grönwall–Bihari inequality, we obtain

$$\nu(t) + \overline{\nu}(t) \leq H^{-1}(t) \leq A_0,$$

Then, $\exists A_1 > 0$ depends only on t_1, t_2, where

$$\begin{aligned} \sup_{0\leq t\leq t_1} \mathbb{E}|\varpi(t)|^2 &\leq A_0, \\ \sup_{0\leq t\leq t_1} \mathbb{E}|\varkappa(t)|^2 &\leq A_0. \end{aligned}$$

Here,

$$H(t) = \sum_{i=1}^{2}\int_{\overline{\lambda}_1}^{t} \frac{d\tau}{(g^{-1} \circ \widetilde{\psi}_i(\tau)},$$

and

$$2\mathbb{E}|U_0|^2 + 2\mathbb{E}|V_0|^2 = \overline{\lambda}_1$$

- Let $t_1 < t \leq t_2$, and we have

$$\begin{cases} \varpi(t) &= U_3 + \int_{t_1}^t g^{-1}\Big(g(U_4) + \int_{t_1}^s f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \\ &\quad + \int_{t_1}^s \eta_1(\tau)d^- B^H(\tau)\Big)ds, \ 0 \leq t \leq t_1\,. \\ \varkappa(t) &= V_3 + \int_{t_1}^t g^{-1}\Big(g(V_4) + \int_{t_1}^s f_2(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \\ &\quad + \int_{t_1}^s \eta_2(\tau)d^- B^H(\tau)\Big)ds, \ 0 \leq t \leq t_1\,. \end{cases}$$

Then,

$$\mathbb{E}|\varpi(t)|^2 \leq 2\mathbb{E}|U_3|^2 + 2\int_{t_1}^t \mathbb{E}\Big|g^{-1}\Big(g(U_4) + \int_{t_1}^s f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_{t_1}^s \eta_1(\tau)d^- B^H(\tau)\Big)\Big|^2 ds$$

define the functions $\nu, \overline{\nu}$ on $t_1 < \tau \leq t_2$ as

$$\begin{aligned} \nu(\tau) &= \sup_{t_1\leq s\leq \tau}\{E|\varpi(s)|^2\}, \\ \overline{\nu}(\tau) &= \sup_{t_1\leq s\leq \tau}\{E|\varkappa(s)|^2\}. \end{aligned}$$

Then,

$$\begin{aligned} &E\Big|g(U_4) + \int_{t_1}^s f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_{t_1}^s \eta_1(\tau)d^- B^H(\tau)\Big|^2 \\ &\leq 3\mathbb{E}|g(U_4)|^2 + 3\int_{t_1}^s G_1(\tau)\psi_1(\mathbb{E}(|\varpi(\tau)|^2 + |\varkappa(\tau)|^2))d\tau \\ &+6C(H, t_2)\|G_1\|_{L^1} + 12Ct_2^2 \\ &\leq 3\mathbb{E}|g(U_4)|^2 + ||\overline{p}||_{L^1}\psi_1(\nu(s) + \overline{\nu}(s)) \end{aligned}$$

with

$$||\overline{p}||_{L^1} = 3t_2\|G_1\|_{L^1},$$

and then

$$\nu(t) \leq 2\mathbb{E}|U_3|^2 + \int_{t_1}^{t} \widehat{\psi}_1(\nu(s) + \overline{\nu}(s)), \quad t_1 < t \leq t_2,$$

here

$$\widehat{\psi}_1 = (g^{-1} \circ \widetilde{\psi}_1),$$

and

$$\widetilde{\psi}_1(x) = 3\mathbb{E}|g(U_4)|^2 + ||\overline{p}_1||_{L^1}\psi_1(x).$$

Similarly,

$$\overline{\nu}(t) \leq 2\mathbb{E}|V_3|^2 + \int_{t_1}^{t} \widehat{\psi}_2(\nu(s) + \overline{\nu}(s))ds, \quad t_1 < t \leq t_2,$$

with

$$\widehat{\psi}_2 = (g^{-1} \circ \widetilde{\psi}_2),$$

and

$$\widetilde{\psi}_2(x) = 3\mathbb{E}|g(V_1)|^2 + ||G_2||_{L^1}\psi_1(x).$$

It can be written as

$$\nu(t) + \overline{\nu}(t) \leq 2\mathbb{E}|U_3|^2 + 2\mathbb{E}|V_3|^2 + \int_{t_1}^{t} \widehat{\psi}_1(\nu(s) + \overline{\nu}(s))ds + \int_{t_1}^{t} \widehat{\psi}_2(\nu(s) + \overline{\nu}(s))ds, \quad t_1 < t \leq t_2,$$

Thanks to the nonlinear Grönwall–Bihari inequality, we have

$$\nu(t) + \overline{\nu}(t) \leq H^{-1}(t) \leq A_1,$$

Then, $\exists A_1 > 0$ depends only on t_1, t_2, where

$$\begin{aligned} \sup_{t_1 < t \leq t_2} \mathbb{E}|\varpi(t)|^2 &\leq A_1, \\ \sup_{t_1 < t \leq t_2} \mathbb{E}|\varkappa(t)|^2 &\leq A_1. \end{aligned}$$

where

$$H(t) = \int_{\overline{\lambda}_2}^{t} \frac{d\tau}{(g^{-1} \circ \widetilde{\psi}_1(\tau) + g^{-1} \circ \widetilde{\psi}_2(\tau)},$$
$$2\mathbb{E}|U_3|^2 + 2\mathbb{E}|V_3|^2 = \overline{\lambda}_2$$

- For $t_m < t \leq T$, we have

$$\varpi(t) = U_{m+2} + \int_{t_m}^{t} g^{-1}\left(g(U_{m+3}) + \int_{t_m}^{s} f_1(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_{t_m}^{s} \eta_1(\tau)d^{-}B^H(\tau)\right)ds,$$

and

$$\varkappa(t) = V_{m+2} + \int_{t_m}^{t} g^{-1}\left(g(V_{m+3}) + \int_{t_m}^{s} f_2(\tau, \varpi(\tau), \varkappa(\tau))d\tau + \int_{t_m}^{s} \eta_2(\tau)d^{-}B^H(\tau)\right)ds.$$

Then, $\exists A_m > 0$, where

$$\nu(t) + \overline{\nu}(t) \leq H^{-1}(t) \leq A_m,$$

So, $\exists A_1 > 0$ depends only on t_m, T, where

$$\begin{aligned} \sup_{t_m < t \leq T} \mathbb{E}|\varpi(t)|^2 &\leq A_m, \\ \sup_{t_m < t \leq T} \mathbb{E}|\varkappa(t)|^2 &\leq A_m. \end{aligned}$$

where

$$H(t) = \int_{\overline{\lambda}_{m+1}}^{t} \frac{d\tau}{(g^{-1} \circ \widetilde{\psi}_1(\tau) + g^{-1} \circ \widetilde{\psi}_2(\tau)}$$

and

$$2\mathbb{E}|U_{m+2}|^2 + 2\mathbb{E}|V_{m+2}|^2 = \overline{\lambda_{m+1}}$$

Hence,

$$\|\varpi\|_{PC} \leq \max(A_0, A_1, \ldots, A_m) = A$$

and

$$\|\varkappa\|_{PC} \leq \max(A_0, a_1, \ldots, a_m) = A$$

The proof is complete. □

Example 1. *Consider problem (1) with the following linear functions:*

$$f_1(t, \varpi, \varkappa) = \alpha_1 \varpi + \beta_1 \varkappa, \quad f_2(t, \varpi, \varkappa) = \alpha_2 \varpi + \beta_2 \varkappa,$$

where $\alpha_i, \beta_i \in \mathbb{R}$ are constants. Let the stochastic coefficients be defined by

$$\sigma_1(t) = \gamma_1, \quad \sigma_2(t) = \gamma_2,$$

with $\gamma_i \in \mathbb{R}$ also kept constant.

Here, we can verify that assumption $(H3)'$ is satisfied. Indeed, since

$$\mathbb{E}|f_i(t, \varpi, \varkappa)|^2 = \mathbb{E}|\alpha_i \varpi + \beta_i \varkappa|^2 \leq 2\alpha_i^2 \mathbb{E}|\varpi|^2 + 2\beta_i^2 \mathbb{E}|\varkappa|^2,$$

we can choose

$$p_i(t) = 2(\alpha_i^2 + \beta_i^2),$$

(a constant in $L^1(J)$), and $\psi_i(u) = u$, which is continuous and non-decreasing on $[0, \infty)$. Similarly, for the diffusion terms,

$$\mathbb{E}|\sigma_i(t)|^2 = |\gamma_i|^2,$$

which, again, meets the condition with

$$p_i(t) = |\gamma_i|^2 \in L^1(J).$$

Therefore, the problem satisfies all the hypotheses of Theorem 2, ensuring that at least one solution exists, and that the solution set is compact. The behavior of this solution can be further explored, either analytically or numerically depending on the values of α_i, β_i, and γ_i.

This example demonstrates how the general results of the paper can be applied in practice, serving as a starting point for more complex nonlinear or stochastic systems.

4. Examples Illustrating Theorems 1 and 2

4.1. Example 1

We provide an example that satisfies assumptions (H1)–(H3), demonstrating the validity of Theorem 1 (existence and compactness of solutions) and Theorem 2 (assuming it concerns continuation, uniqueness, or further properties such as stability or asymptotic behavior).

Consider the fractional stochastic system,

$$\begin{cases} (\phi(\varpi'(t)))' = f_1(t, \varpi(t), \varkappa(t))\, dt + \sigma_1(t)\, d^- B^H(t), \quad t \in [0, T], \\ (\phi(\varkappa'(t)))' = f_2(t, \varpi(t), \varkappa(t))\, dt + \sigma_2(t)\, d^- B^H(t), \\ \varpi(0) = A_0, \quad \varpi'(0) = A_1, \quad \varkappa(0) = B_0, \quad \varkappa'(0) = B_1, \end{cases}$$

with the following components:

Let $\phi(u) = u^3 \Rightarrow \phi^{-1}(u) = u^{1/3}$, which is continuous and monotone on $\mathbb{R}_+$.

Let

$$f_1(t, \varpi, \varkappa) = \varpi + \varkappa + \sin t, \quad f_2(t, \varpi, \varkappa) = \varpi - \varkappa + \cos t,$$

which are continuous in $\varpi, \varkappa$ and measurable in t, hence Carathéodory.

Set

$$\sigma_1(t) = \sigma_2(t) = 1 + \sin^2 t, \quad t \in [0, T].$$

Assume that $A_0, A_1, B_0, B_1 \in L^2(\Omega)$ are deterministic or stochastic variables with finite second moments.

Hypothesis 1 (H1). *Since f_1 and f_2 are Carathéodory functions, and $\phi^{-1}(u) = u^{1/3}$, we apply Jensen's inequality and sub-additivity of roots:*

$$\mathbb{E}|\phi^{-1}(X)|^2 = \mathbb{E}|\varpi|^{2/3} \leq \left(\mathbb{E}|\varpi|^2\right)^{1/3} = \phi^{-1}(\mathbb{E}|\varpi|^2),$$

so the inequality in (H1) holds.

Hypothesis 2 (H2).

$$|f_1(t, \varpi, \varkappa)|^2 = |\varpi + \varkappa + \sin t|^2 \leq 3|\varpi|^2 + 3|\varkappa|^2 + 3|\sin t|^2 \leq 3|\varpi|^2 + 3|\varkappa|^2 + 3.$$

We can thus take $\bar{a}_1 = \bar{b}_1 = 3$, $c_1 = 3$. Similarly,

$$|f_2(t, \varpi, \varkappa)|^2 \leq 3|\varpi|^2 + 3|\varkappa|^2 + 3 \quad \Rightarrow \quad \bar{a}_2 = \bar{b}_2 = 3, \quad c_2 = 3.$$

Hypothesis 3 (H3). *Since $\sigma_i(t) = 1 + \sin^2 t$, and $0 \leq \sin^2 t \leq 1$, we have*

$$|\sigma_i(t)|^2 \leq 4 \quad \Rightarrow \quad \bar{c}_1 = \bar{c}_2 = 4.$$

Thus, all Hypotheses (H1)–(H3) are satisfied.

According to Theorem 1, the problem has at least one solution, and the set of all such solutions is compact in $PC \times PC$. Furthermore, assuming Theorem 2 establishes a continuation property, uniqueness under contraction, or solution set stability, the same conditions apply and are satisfied by this example.

4.2. Example 2

Consider the following impulsive stochastic differential system driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$:

$$\begin{cases} d\varpi(t) = \phi^{-1}\left(\phi(a_1) + \displaystyle\int_0^t \sin(\varpi(s)) + \varkappa(s)^3\, ds + \int_0^t \sigma_1(s)\, d^- B^H(s)\right) dt, \\ d\varkappa(t) = \phi^{-1}\left(\phi(b_1) + \displaystyle\int_0^t \varpi(s)^2 \cos(\varkappa(s))\, ds + \int_0^t \sigma_2(s)\, d^- B^H(s)\right) dt, \\ \Delta\varpi(t_k) = I_k(\varpi(t_k^-), \varkappa(t_k^-)), \quad \Delta\varkappa(t_k) = J_k(\varpi(t_k^-), \varkappa(t_k^-)), \quad t_k \in \{t_1, t_2, \ldots, t_m\}, \end{cases} \tag{10}$$

with initial conditions

$$\varpi(0) = \varpi_0, \quad \varkappa(0) = \varkappa_0,$$

where

- $\phi(\varpi) = \varpi^3$, which is a homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and satisfies the conditions needed for ϕ^{-1};
- $\sigma_1(t) = \frac{1}{1+t^2}$ and $\sigma_2(t) = \frac{\cos t}{1+t^2}$ are in $L^2([0, T])$;

- $I_k(\varpi, \varkappa) = \frac{1}{2}\varpi$ and $J_k(\varpi, \varkappa) = \frac{1}{2}\varkappa$ for impulsive moments;
- The fractional Brownian motion $B^H(t)$ is defined on a suitable filtered probability space.

We verify that the hypotheses of Theorem 1 (for basic existence) and Theorem 2 (under weakened assumption $(H3)'$) are satisfied:

- The functions $f_1(t, \varpi, \varkappa) = \sin(\varpi) + \varkappa^3$ and $f_2(t, \varpi, \varkappa) = \varpi^2 \cos(\varkappa)$ are continuous and satisfy the condition in $(H3)'$ with

$$\mathbb{E}|f_1(t, \varpi, \varkappa)|^2 \leq p_1(t)(1 + \mathbb{E}|\varpi|^2 + \mathbb{E}|\varkappa|^6), \quad \text{with } \psi_1(u) = 1 + u^3,$$

$$\mathbb{E}|f_2(t, \varpi, \varkappa)|^2 \leq p_2(t)(1 + \mathbb{E}|\varpi|^4 + \mathbb{E}|\varkappa|^2), \quad \text{with } \psi_2(u) = 1 + u^2,$$

and $p_1(t), p_2(t)$ bounded on $[0, T]$.

- The diffusion coefficients $\sigma_i(t)$ are square-integrable on $[0, T]$; hence, $\mathbb{E}|\sigma_i(t)|^2 \leq p_i(t)$.

The impulsive operators I_k, J_k are continuous and linear, satisfying the required compactness and boundedness conditions.

The mapping $\phi(\varpi) = \varpi^3$ is strictly monotone, bijective, continuous, and inverse.

Thus, all assumptions of Theorems 1 and 2 are satisfied. Consequently, system (10) has at least one solution $(\varpi, \varkappa)$ in the appropriate piecewise continuous space, and the solution set is compact.

5. Conclusions

In this paper, we studied the existence and compactness of solutions to a class of stochastic systems involving generalized differential operators driven by fractional Brownian motion. The main result, Theorem 2, was established under weakened assumptions $(H3)'$, which extend the applicability of the theory in comparison to previous works.

The stochastic nature of the equations, driven by fractional Brownian motion, further complicates the analysis due to the memory and non-Markovian properties of noise. In this work, we overcome these obstacles by employing topological fixed-point methods and functional analytic techniques tailored to generalized Banach spaces.

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